COMMUN. MATH. SCI.
Vol. 17, No. 2, pp. 393–411

SOLVING THE YANG-BAXTER-LIKE MATRIX EQUATION WITH NON-DIAGONALIZABLE ELEMENTARY MATRICES*

DUANMEI ZHOU†, GUOLIANG CHEN‡, JIU DING§, AND HAIYAN TIAN¶

Abstract. Let $A=I-uv^T$, where u and v are two n-dimensional complex vectors with $v^Tu=0$. Thus A is not diagonalizable. We find all solutions of the quadratic matrix equation $AXA=XAX$. This is a continuation of the work [Computers Math. Appl., 72(6):1541–1548, 2016] from the case of diagonalizable elementary matrices to non-diagonalizable ones.

Keywords. Nonlinear matrix equation; Elementary matrix; Spectral perturbation.

AMS subject classifications. 15A18.

1. Introduction

In a recent paper [4], all solutions of the nonlinear matrix equation

$$AXA=XAX, \tag{1.1}$$

where A is a diagonalizable elementary matrix, have been obtained. The above homogeneous quadratic equation, with A a given $n\times n$ complex matrix, is called the *Yang-Baxter-like matrix equation*. Its origin is the classical Yang-Baxter equation [1,13]. The original Yang-Baxter equation has found many connections and applications to knot theory, braid group theory, integrable systems, quantum theory, and statistical physics in mathematical and physical sciences; see [10,11,14]. Solving (1.1) has been a research topic of linear algebra in the past several years, and a few results on its solutions have been obtained for various classes of matrices A with different approaches; see, e.g., [3,5]. Most solutions obtained so far are commuting ones, that is, solutions X of (1.1) such that $AX=XA$. Although all commuting solutions have been constructed for diagonalizable matrices A in [8], nilpotent matrices A in [9], and general matrices in [12], it is still challenging to find non-commuting solutions when the given matrix A is arbitrary.

There have been some efforts toward finding all solutions of (1.1) when A has a special structure, e.g., [2,15]. In a recent paper [4], all solutions have been constructed if A is a diagonalizable elementary matrix. This paper continues the study of [4] by dropping the assumption of A being diagonalizable. As will be seen later, finding all solutions with A not diagonalizable is much more tedious with many cases to consider. The tediousness is compatible to our experience of solving polynomial equations.

If we multiply out the both sides of (1.1), solving the Yang-Baxter matrix equation is equivalent to solving a system of n^2 quadratic polynomial equations of n^2 variables with n the size of the matrix A, and the solution set of this system is a disconnected nonlinear manifold in general. Thus, unlike solving linear matrix equations, it seems that there is no effective way to express all the solutions in terms of a basis of elementary

*Received: January 28, 2018; Accepted (in revised form): December 6, 2018. Communicated by Shi Jin.
†College of Mathematics and Computer Science, Gannan Normal University, Ganzhou 341000, China (gzzdm2008@163.com).
‡Department of Mathematics, East China Normal University, Shanghai 200241, China (glchen@math.ecnu.edu.cn).
§Department of Mathematics, University of Southern Mississippi, Hattiesburg, MS 39406, USA (jiu.ding@usm.edu).
¶Department of Mathematics, University of Southern Mississippi, Hattiesburg, MS 39406, USA (haiyan.tian@usm.edu).

solutions such as some eigenvectors, even if A is symmetric. As a simple example, when A is the 2×2 Jordan block J_1 given by (2.1) in the next section, all the solutions of the corresponding matrix Equation (1.1) are the two trivial solutions $X=0$ and $X=J_1$, together with the one-parameter nontrivial solutions

$$X=\begin{bmatrix} x & (x-1)^2 \\ -1 & 2-x \end{bmatrix}, \forall x,$$

none of which is a commuting solution.

To overcome the difficulty in finding non-commuting solutions of (1.1), our strategy here is to use the Jordan decomposition of A to obtain a simplified Yang-Baxter matrix equation with A replaced by a simple 2×2 block diagonal matrix, and then we solve a system of four matrix equations for the smaller sized solution blocks. Because of the special structure of the new system, we are lucky to be able to apply some spectral perturbation results proved about ten years ago for rank-one or rank-two updated matrices, resulting in all the solutions of the original Yang-Baxter matrix equation.

We shall establish the equivalent system of four matrix equations and present the result for commuting solutions in Section 2. The next three sections are devoted to finding all solutions of (1.1) under different assumptions. We conclude with Section 6.

2. All commuting solutions of the equation

Throughout the paper we let the given matrix in the Equation (1.1) be $A=I-uv^T$, where u and v are two nonzero n dimensional complex column vectors such that $v^Tu=0$. Let $v_1,\dots,v_{n-1}$ be linearly independent vectors in $\mathbf{C}^n$ such that $v^Tv_j=0$ for $j=1,\dots,n-1$. Then $Av_j=v_j$ for each j, from which 1 is an eigenvalue of A with eigenvectors $v_1,\dots,v_{n-1}$. As usual, we use $R(B)$ and $N(B)$ in the following to denote the range and null space of a matrix B, respectively.

Since $Au=(1-v^Tu)u=u$, clearly u belongs to the $(n-1)$-dimensional subspace spanned by $v_1,\dots,v_{n-1}$. Since $A-I=-uv^T\neq0$ and $(A-I)^2=(-uv^T)(-uv^T)=(v^Tu)uv^T=0$, the minimal polynomial of A is $\phi(x)=(x-1)^2$. The fact that $x=1$ is the double zero of ϕ ensures that the only eigenvalue of A is 1 with geometric multiplicity $n-1$ while its algebraic multiplicity is n, so A is not diagonalizable. In this paper, any eigenvalue, whose algebraic multiplicity is one more than its geometric multiplicity, will be said to be of deficiency -1.

To find the Jordan form J of A and the corresponding similarity matrix W, from $w\equiv(A-I)\bar{v}=-\|v\|^2u\neq0$ and $(A-I)w=(A-I)^2\bar{v}=0$, we see $A[w,\bar{v}]=[w,\bar{v}]J_1$ with

$$J_1=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{2.1}$$

the 2×2 Jordan block. Since $v^Tw=0$, we may let $v_{n-1}=w$ in the basis $\{v_1,\dots,v_{n-1}\}$ of the eigenspace $N(A-I)$, getting the Jordan form decomposition $A=WJW^{-1}$, where

$$W=[v_1,\dots,v_{n-2},w,\bar{v}] \text{ and } J=\operatorname{diag}(I_{n-2},J_1).$$

Solving (1.1) is equivalent to solving the simplified Yang-Baxter-like matrix equation

$$JYJ=YJY \tag{2.2}$$

with the relation of solutions X to (1.1) and those Y to (2.2) given by $X=WYW^{-1}$. We solve (2.2) by partitioning Y as J into a 2×2 block matrix

$$Y=\begin{bmatrix} Z & H \\ K^T & T \end{bmatrix}, \tag{2.3}$$

where the $(n-2)\times 2$ sub-matrices $H=[h,\hat{h}]$ and $K=[k,\hat{k}]$, and the 2×2 matrix

$$T=T(t,s,c,d)\equiv\begin{bmatrix} t & s \\ c & d \end{bmatrix}. \tag{2.4}$$

In the remainder of the paper, whenever the matrix Y appears, it always has the block structure given by (2.3). Then (2.2) can be written as the system of four equations

$$\begin{cases} Z^2-Z = -HJ_1K^T, \\ ZH \quad = HJ_1(I-T), \\ K^TZ \quad = (I-T)J_1K^T, \\ K^TH \quad = J_1TJ_1-TJ_1T, \end{cases} \tag{2.5}$$

which, after multiplying out, is equivalent to

$$\begin{cases} Z^2-Z = -[hk^T+(h+\hat{h})\hat{k}^T], \\ Zh \quad = (1-t-c)h-c\hat{h}, \\ Z\hat{h} \quad = (1-s-d)h+(1-d)\hat{h}, \\ k^TZ \quad = (1-t)k^T+(1-t-s)\hat{k}^T, \\ \hat{k}^TZ \quad = -ck^T+(1-c-d)\hat{k}^T, \\ k^Th \quad = (t+c)(1-t)-sc, \\ k^T\hat{h} \quad = (1-s-d)(t+s)+s^2+c+d, \\ \hat{k}^Th \quad = c(1-t-c-d), \\ \hat{k}^T\hat{h} \quad = (c+d)(1-d)-sc. \end{cases} \tag{2.6}$$

Note that if $K^TH=0$, then the last equation of (2.5) is just a 2×2 Yang-Baxter-like matrix equation $J_1TJ_1=TJ_1T$ with all the solutions

$$T=0,J_1, \text{ and } T(t,(1-t)^2,-1,2-t), \ \forall t$$

given by (2.4), after a direct computation. This fact will often be used when we solve (2.5) under different situations of H and K.

We first find all commuting solutions of (2.5) by requiring further that

$$\begin{bmatrix} I_{n-2} & 0 \\ 0 & J_1 \end{bmatrix}\begin{bmatrix} Z & H \\ K^T & T \end{bmatrix}=\begin{bmatrix} Z & H \\ K^T & T \end{bmatrix}\begin{bmatrix} I_{n-2} & 0 \\ 0 & J_1 \end{bmatrix}.$$

This additional equation implies $H=HJ_1, K^T=J_1K^T$, and $J_1T=TJ_1$. It follows that $H=[0,\hat{h}], K=[k,0]$, and $T=T(t,s,0,t)$ for any numbers t and s. Therefore, $HJ_1K^T=0, HJ_1(I-T)=(1-t)H, (I-T)J_1K^T=(1-t)K^T$, and

$$K^TH=\begin{bmatrix} 0 & k^T\hat{h} \\ 0 & 0 \end{bmatrix}, \ J_1TJ_1-TJ_1T=\begin{bmatrix} t(1-t) & s+2t-2ts-t^2 \\ 0 & t(1-t) \end{bmatrix}.$$

Hence $t=0$ or $t=1$ from the last equation of (2.5), from which $T=T(0,s,0,0)$ or $T(1,s,0,1)$ for any number s, respectively. Substituting the two values $t=0$ or $t=1$ with the corresponding T expressions respectively into (2.5) gives the following two systems of equations for the remaining unknowns with any number s:

$$\begin{cases} Z^2-Z = 0, \\ Z\hat{h} \quad = \hat{h}, \\ k^TZ \quad = k^T, \\ k^T\hat{h} \quad = s \end{cases} \quad \text{or} \quad \begin{cases} Z^2-Z = 0, \\ Z\hat{h} \quad = 0, \\ k^TZ \quad = 0^T, \\ k^T\hat{h} \quad = 1-s. \end{cases}$$

A matrix whose square equals itself is a projection. It is well-known that a projection is diagonalizable with exactly two eigenvalues 1 and 0, except for the zero matrix with the only eigenvalue 0 and the identity matrix with the unique eigenvalue 1. We therefore have obtained the following result for commuting solutions.

THEOREM 2.1. *All commuting solutions of (1.1) are* $X=WYW^{-1}$, *in which* Z *is any projection,* $H=[0,\hat{h}]$ *and* $K=[k,0]$, *and either*

(i) $T=T(0,s,0,0)$ *for any* s, $\hat{h}\in R(Z)$, *and* $k\in R(Z^T)$ *with* $k^T\hat{h}=s$, *or*

(ii) $T=T(1,s,0,1)$ *for any* s, $\hat{h}\in N(Z)$, *and* $k\in N(Z^T)$ *with* $k^T\hat{h}=1-s$.

We tabulate the above result in Table 2.1 where Z,H,K^T, and T are the four submatrices of the partitioned 2×2 block matrix (2.3) of the solution matrix Y to the simplified Yang-Baxter-like matrix equation $JYJ=YJY$, from which the solutions of the original Yang-Baxter-like matrix Equation (1.1) are $X=WYW^{-1}$.

Z	H	K	T	Conditions
$Z=Z^2$	$[0,\hat{h}]$	$[k,0]$	$\begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix}$	$\hat{h}\in R(Z),k\in R(Z^T),k^T\hat{h}=s$
$Z=Z^2$	$[0,\hat{h}]$	$[k,0]$	$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$	$\hat{h}\in N(Z),k\in N(Z^T),k^T\hat{h}=1-s$

TABLE 2.1. *All Commuting Solutions*

In the remaining sections, we will find all non-commuting solutions of (1.1).

3. Solutions with one of H and K zero

From now on we look for all solutions of (2.5) to obtain all solutions of (1.1), which is a tedious process. Let $r(B)$ denote the rank of matrix B. We divide our analysis into three non-overlapping assumptions on H and K in the solutions Y of (2.5), based on their ranks: (i) $r(H)=0$ or $r(K)=0$; (ii) $r(H)=r(K)=1$; (iii) $r(H)=2$ and $K\neq 0$ or $r(K)=2$ and $H\neq 0$.

Suppose $H=0$ or $K=0$. Then the second or third equation of (2.5) is satisfied, and the first and last ones are $Z^2=Z$ and $J_1TJ_1=TJ_1T$. All solutions of the above two equations are projections Z and $T=0,J_1$, or $T(t,(1-t)^2,-1,2-t)$ for any number t. We next determine the structures of K,H and T from the remaining equation of (2.5) for various situations. We have two cases to investigate.

Case 1: $H=0$. If $K=0$, then all solutions of (2.5) are $\mathrm{diag}(Z,T)$. Suppose $K=[k,0]$ with $k\neq 0$. If $T=0$ or J_1, then the third equation of (2.5) is $k^TZ=k^T$ or $k^TZ=0^T$, respectively. If $T=T(t,(1-t)^2,-1,2-t)$, then $k=0$, a contradiction.

Suppose $K=[0,\hat{k}]$ with $\hat{k}\neq 0$. If $T=0,J_1$, or $T(t,(1-t)^2,-1,2-t)$ with $t\neq 0,1$, then the third equation of (2.5) implies $\hat{k}=0$, a contradiction. If $T=T(0,1,-1,2)$ or $T(1,0,-1,1)$, then the third equation of (2.5) is $\hat{k}^TZ=0^T$ or $\hat{k}^TZ=\hat{k}^T$, respectively.

Suppose $k\neq 0$ and $\hat{k}\neq 0$. If $T=0$ or J_1, then the third equation of (2.5) gives

$$k^TZ=k^T+\hat{k}^T \text{ and } \hat{k}^TZ=\hat{k}^T, \text{ or } k^TZ=-\hat{k}^T \text{ and } \hat{k}^TZ=0^T,$$

respectively, from which $\hat{k}=0$, a contradiction. If $T=T(t,(1-t)^2,-1,2-t)$, then the third equation of (2.5) becomes

$$k^TZ=(1-t)k^T+t(1-t)\hat{k}^T \text{ and } \hat{k}^TZ=k^T+t\hat{k}^T,$$

from which $[k+(t-1)\hat{k}]^T Z = 0^T$.

Case 2: $K=0$. Suppose $H=[0,\hat{h}]$ with $\hat{h}\neq 0$. If $T=0$ or J_1, then the second equation of (2.5) is $Z\hat{h}=\hat{h}$ or $Z\hat{h}=0$, respectively. If $T=T(t,(1-t)^2,-1,2-t)$, then the second equation of (2.5) leads to $\hat{h}=0$, a contradiction.

Suppose $H=[h,0]$ with $h\neq 0$. If $T=0,J_1$, or $T(t,(1-t)^2,-1,2-t)$ with $t\neq 1,2$, then the second equation of (2.5) implies $h=0$, a contradiction. If $T=T(1,0,-1,1)$ or $T(2,1,-1,0)$, then the second equation of (2.5) is $Zh=h$ or $Zh=0$, respectively.

Suppose $h\neq 0$ and $\hat{h}\neq 0$. If $T=0$ or J_1, then the second equation of (2.5) leads to $h=0$, a contradiction. If $T=T(t,(1-t)^2,-1,2-t)$, then the same equation becomes

$$Zh=(2-t)h+\hat{h} \quad \text{and} \quad Z\hat{h}=(t-1)(2-t)h+(t-1)\hat{h},$$

so $Z[(t-1)h-\hat{h}]=0$.

In summary, we have the following theorem.

THEOREM 3.1. *All solutions $X=WYW^{-1}$ of (1.1) under the assumption that either $H=0$ or $K=0$, are such that Z is any projection and*

(i) $Y=\operatorname{diag}(Z,T)$ *with* $T=0,J_1$, *or* $T(t,(1-t)^2,-1,2-t)$ *for any number* t;

(ii) $H=0,\hat{k}=0$, *and either* $k\neq 0\in R(Z^T)$ *and* $T=0$ *or* $k\neq 0\in N(Z^T)$ *and* $T=J_1$;

(iii) $H=0,k=0$, *and either* $\hat{k}\neq 0\in N(Z^T)$ *and* $T=T(0,1,-1,2)$ *or* $\hat{k}\neq 0\in R(Z^T)$ *and* $T=T(1,0,-1,1)$;

(iv) $H=0,k,\hat{k}\neq 0,T=T(t,(1-t)^2,-1,2-t)$, *and* $k+(t-1)\hat{k}\in N(Z^T)$ *for any* t;

(v) $K=0,h=0$, *and either* $\hat{h}\neq 0\in R(Z)$ *and* $T=0$ *or* $\hat{h}\neq 0\in N(Z)$ *and* $T=J_1$;

(vi) $K=0,\hat{h}=0$, *and either* $h\neq 0\in R(Z)$ *and* $T=T(1,0,-1,1)$ *or* $h\neq 0\in N(Z)$ *and* $T=T(2,1,-1,0)$;

(vii) $K=0,h,\hat{h}\neq 0,T=T(t,(1-t)^2,-1,2-t)$, *and* $(t-1)h-\hat{h}\in N(Z)$ *for any* t.

The above result is listed in Table 3.1, in which T is given by (2.4) and t represents any number. Among all solutions (Z,H,K^T,T) such that at least one of H and K^T is a zero matrix, the matrix Z is any projection.

4. Solutions with $r(H)=r(K)=1$

In this section, we will find all solutions of (1.1) under the condition that both ranks of H and K are 1. This can be done by solving (2.5) under different situations. To reach our results that will be summarized into three theorems, we need the following rank-one spectral perturbation result (Theorem 2.1 of [6]).

LEMMA 4.1. *Let M be an $m\times m$ matrix with eigenvalues $\mu_1,\mu_2,\ldots,\mu_m$, counting algebraic multiplicity, and let x and y be two m-dimensional column vectors such that x is an eigenvector of M or y is a left eigenvector of M associated with μ_1. Then the eigenvalues of $M+xy^T$ are $\mu_1+y^Tx,\mu_2,\ldots,\mu_m$, counting algebraic multiplicity.*

For the sake of simpler presentation, we divide our discussion into several situations. First we assume that only one column of both H and K is zero, which has four cases.

H	K	T	Conditions
$[0,0]$	$[0,0]$	$0, J_1, \begin{bmatrix} t & (1-t)^2 \\ -1 & 2-t \end{bmatrix}$	
$[0,0]$	$[k,0]$	0	$k \neq 0 \in R(Z^T)$
$[0,0]$	$[k,0]$	J_1	$k \neq 0 \in N(Z^T)$
$[0,0]$	$[0,\hat{k}]$	$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$	$\hat{k} \neq 0 \in N(Z^T)$
$[0,0]$	$[0,\hat{k}]$	$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$	$\hat{k} \neq 0 \in R(Z^T)$
$[0,0]$	$[k,\hat{k}]$	$\begin{bmatrix} t & (1-t)^2 \\ -1 & 2-t \end{bmatrix}$	$k \neq 0 \neq \hat{k}, k+(t-1)\hat{k} \in N(Z^T)$
$[0,\hat{h}]$	$[0,0]$	0	$\hat{h} \neq 0 \in R(Z)$
$[0,\hat{h}]$	$[0,0]$	J_1	$\hat{h} \neq 0 \in N(Z)$
$[h,0]$	$[0,0]$	$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$	$h \neq 0 \in R(Z)$
$[h,0]$	$[0,0]$	$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$	$h \neq 0 \in N(Z)$
$[h,\hat{h}]$	$[0,0]$	$\begin{bmatrix} t & (1-t)^2 \\ -1 & 2-t \end{bmatrix}$	$h \neq 0 \neq \hat{h}, (t-1)h-\hat{h} \in N(Z)$

TABLE 3.1. *All Solutions with $H=0$ or $K=0$*

Case 1. $H=[h,0]$ and $K=[k,0]$ with $h\neq 0$ and $k\neq 0$. Then (2.6) becomes

$$\begin{cases} Z^2-Z = -hk^T, \\ Zh = (1-t-c)h, \\ 0 = (1-s-d)h, \\ k^TZ = (1-t)k^T, \\ 0^T = -ck^T, \\ k^Th = (t+c)(1-t)-sc, \\ 0 = (1-s-d)(t+s)+s^2+c+d, \\ 0 = c(1-t-c-d), \\ 0 = (c+d)(1-d)-sc. \end{cases} \tag{4.1}$$

In the above system, the third and fifth equations imply $s+d=1$ and $c=0$, so $s^2+d=0$ and $d(1-d)=0$ from the seventh and ninth equations. But the equalities $s+d=1, s^2+d=0$, and $d(1-d)=1$ are contradictory. Hence, system (4.1) has no solution.

Case 2. $H=[h,0]$ and $K=[0,\hat{k}]$ with $h\neq 0$ and $\hat{k}\neq 0$. Now (2.6) is

$$\begin{cases} Z^2-Z = -h\hat{k}^T, \\ Zh = (1-t-c)h, \\ 0 = (1-s-d)h, \\ 0^T = (1-t-s)\hat{k}^T, \\ \hat{k}^TZ = (1-c-d)\hat{k}^T, \\ 0 = (t+c)(1-t)-sc, \\ 0 = (1-s-d)(t+s)+s^2+c+d, \\ \hat{k}^Th = c(1-t-c-d), \\ 0 = (c+d)(1-d)-sc. \end{cases} \tag{4.2}$$

In the above system, the third and fourth equations ensure $s+d=1$ and $t+s=1$, so $d=t$, and then the seventh and ninth equations imply $s^2+c+d=0$ and $d(1-d)=0$. Consequently $c=-1$ with either $d=t=1$ and $s=0$ or $d=t=0$ and $s=1$. Hence (4.2) is reduced to the systems $Z^2-Z=-h\hat{k}^T$, $Zh=h$, $\hat{k}^TZ=\hat{k}^T$, $\hat{k}^Th=0$ or $Z^2-Z=-h\hat{k}^T$, $Zh=2h$, $\hat{k}^TZ=2\hat{k}^T$, $\hat{k}^Th=-2$, which can be shortened to

$$Z^2-Z=-h\hat{k}^T,\ \ Zh=h,\ \ \hat{k}^TZ=\hat{k}^T$$

or

$$Z^2-Z=-h\hat{k}^T,\ \ Zh=2h,\ \ \hat{k}^TZ=2\hat{k}^T,$$

corresponding to $T=T(1,0,-1,1)$ or $T=T(0,1,-1,0)$, respectively.

Case 3. $H=[0,\hat{h}]$ and $K=[k,0]$ with $\hat{h}\neq 0$ and $k\neq 0$. Then (2.6) is simplified to

$$\begin{cases} Z^2-Z = 0, \\ 0 = -c\hat{h}, \\ Z\hat{h} = (1-d)\hat{h}, \\ k^TZ = (1-t)k^T, \\ 0^T = -ck^T, \\ 0 = (t+c)(1-t)-sc, \\ k^T\hat{h} = (1-s-d)(t+s)+s^2+c+d, \\ 0 = c(1-t-c-d), \\ 0 = (c+d)(1-d)-sc. \end{cases}$$

Clearly $c=0$. From the first equation above, Z is a projection, so its eigenvalues are either 1 or 0. This implies that d and t are 1 or 0. The possibilities that d and t take different values are excluded since otherwise we would obtain contradictory systems $Z\hat{h}=0, k^TZ=k^T, k^T\hat{h}=1$ or $Z\hat{h}=\hat{h}, k^TZ=0, k^T\hat{h}=1$. Thus, we obtain the system

$$Z^2=Z,\ \ Z\hat{h}=\hat{h},\ \ k^TZ=k^T,\ \ k^T\hat{h}=s$$

or the system

$$Z^2=Z,\ \ Z\hat{h}=0,\ \ k^TZ=0,\ \ k^T\hat{h}=1-s,$$

corresponding to $T=T(0,s,0,0)$ or $T=T(1,s,0,1)$ for any number s, respectively.

Case 4. $H=[0,\hat{h}]$ and $K=[0,\hat{k}]$ with $\hat{h}\neq 0$ and $\hat{k}\neq 0$. Then (2.6) can be written as

$$\begin{cases} Z^2-Z = -\hat{h}\hat{k}^T, \\ 0 = -c\hat{h}, \\ Z\hat{h} = (1-d)\hat{h}, \\ 0^T = (1-t-s)\hat{k}^T, \\ \hat{k}^TZ = (1-c-d)\hat{k}^T, \\ 0 = (t+c)(1-t)-sc, \\ 0 = (1-s-d)(t+s)+s^2+c+d, \\ 0 = c(1-t-c-d), \\ \hat{k}^T\hat{h} = (c+d)(1-d)-sc. \end{cases}$$

Consequently, $c=0$ and $t+s=1$, so $s(1-s)=0$ and $s^2-s+1=0$, which is impossible.

In summary, the key system (2.6) has solutions only in Cases 2 and 3. Using Lemma 4.1 and the same technique as in [4,8], we can prove the following theorem.

THEOREM 4.1. *All solutions* $X=WYW^{-1}$ *of (1.1) under the assumption that* $r(H)=r(K)=1$ *with exactly one zero column from* H *and* K *each are such that*

(i) $\hat{h}=0$ *and* $k=0$. *Either* $T=T(1,0,-1,1)$ *and* Z *has eigenvalue* 1 *of deficiency* -1 *with corresponding eigenvector* h *and left eigenvector* $\hat{k}^T$ *satisfying* $\hat{k}^Th=0$, *and the other possible eigenvalue of* Z *is* 0 *that is semi-simple, or* $T=T(0,1,-1,0)$, *and* Z *is diagonalizable and has one simple eigenvalue* 2 *with corresponding eigenvector* h *and left eigenvector* $\hat{k}^T$ *satisfying* $\hat{k}^Th=-2$, *and the other eigenvalues in* $\{0,1\}$;

(ii) $h=0,\hat{k}=0$, *and* Z *is any projection. Either* $T=T(0,s,0,0),\hat{h}\in R(Z)$, *and* $k\in R(Z^T)$ *with* $k^T\hat{h}=s$ *or* $T=T(1,s,0,1),\hat{h}\in N(Z)$, *and* $k\in N(Z^T)$ *with* $k^T\hat{h}=1-s$, *for any* s.

Proof. The proof of the first part of (i) is exactly the same as that for Proposition 2.5 (iii) of [4], and the second part of (i) can be shown the same way as Theorem 4.1 in [8]. The conclusion of (ii) is obvious. ◻

REMARK 4.1. Compared with Theorem 2.1, all solutions from Theorem 4.1(ii) are exactly the commuting ones.

There are also four cases when exactly one of the four columns of $[H\,K]$ is zero.

Case (i): $h\neq 0,\hat{h}=\alpha h$ with $\alpha\neq 0,k=0$, and $\hat{k}\neq 0$. Then we can express (2.6) as

$$\begin{cases} Z^2-Z &= -(1+\alpha)h\hat{k}^T,\\ Zh &= (1-t-c-c\alpha)h,\\ Zh &= \left(\frac{1-s-d}{\alpha}+1-d\right)h,\\ 0^T &= (1-t-s)\hat{k}^T,\\ \hat{k}^TZ &= (1-c-d)\hat{k}^T,\\ 0 &= (t+c)(1-t)-sc,\\ 0 &= (1-s-d)(t+s)+s^2+c+d,\\ \hat{k}^Th &= c(1-t-c-d),\\ \hat{k}^Th &= \frac{(c+d)(1-d)-sc}{\alpha}.\end{cases}$$

Again $t+s=1$. Then $t(1-t)=0$ from the sixth equation above. So $t=0$ and $s=1$, or $t=1$ and $s=0$. So $c=-1$ by the seventh equation. Thus the above system is split to

$$\begin{cases} Z^2-Z &= -(1+\alpha)h\hat{k}^T,\\ Zh &= (2+\alpha)h,\\ Zh &= \left(1-d-\frac{d}{\alpha}\right)h,\\ \hat{k}^TZ &= (2-d)\hat{k}^T,\\ \hat{k}^Th &= d-2,\\ \hat{k}^Th &= \frac{d(2-d)}{\alpha}\end{cases} \quad\text{or}\quad \begin{cases} Z^2-Z &= -(1+\alpha)h\hat{k}^T,\\ Zh &= (1+\alpha)h,\\ Zh &= (1-d)\left(1+\frac{1}{\alpha}\right)h,\\ \hat{k}^TZ &= (2-d)\hat{k}^T,\\ \hat{k}^Th &= d-1,\\ \hat{k}^Th &= -\frac{(d-1)^2}{\alpha}.\end{cases} \tag{4.3}$$

We first solve (4.3) with $\alpha=-1$. Then Z is a projection, so the equation $\hat{k}^TZ=(2-d)\hat{k}^T$ implies $d=1$ or 2. If $d=1$, then (4.3) becomes

$$\begin{cases} Z^2-Z &= 0,\\ Zh &= h,\\ \hat{k}^TZ &= \hat{k}^T,\\ \hat{k}^Th &= -1\end{cases} \quad\text{or}\quad \begin{cases} Z^2-Z &= 0,\\ Zh &= 0,\\ \hat{k}^TZ &= \hat{k}^T,\end{cases}$$

corresponding to $T=T(0,1,-1,1)$ or $T(1,0,-1,1)$, respectively. If $d=2$, then the respective two systems of (4.3) are

$$\begin{cases} Z^2-Z &= 0,\\ Zh &= h,\\ \hat{k}^T Z &= 0 \end{cases} \quad \text{or} \quad \begin{cases} Z^2-Z &= 0,\\ Zh &= 0,\\ \hat{k}^T Z &= 0,\\ \hat{k}^T h &= 1, \end{cases}$$

corresponding to $T=T(0,1,-1,2)$ or $T(1,0,-1,2)$, respectively.

Next we assume $\alpha \neq -1$. Then $d=-\alpha\neq 1$ in the left system of (4.3) and $d=1-\alpha\neq 2$ in the right one of (4.3), and the two systems are reduced to

$$\begin{cases} Z^2-Z &= -(\alpha+1)h\hat{k}^T,\\ Zh &= (2+\alpha)h,\\ \hat{k}^T Z &= (2+\alpha)\hat{k}^T \end{cases} \quad \text{or} \quad \begin{cases} Z^2-Z &= -(\alpha+1)h\hat{k}^T,\\ Zh &= (1+\alpha)h,\\ \hat{k}^T Z &= (1+\alpha)\hat{k}^T \end{cases}$$

since the last equation in (4.3) is now redundant, corresponding to $T=T(0,1,-1,-\alpha)$ or $T(1,0,-1,1-\alpha)$, respectively.

Case (ii): $h\neq 0, \hat{h}=\alpha h$ with $\alpha\neq 0, k\neq 0$, and $\hat{k}=0$. Then (2.6) is simplified to

$$\begin{cases} Z^2-Z &= -hk^T,\\ Zh &= (1-t-c-c\alpha)h,\\ Zh &= \left(\frac{1-s-d}{\alpha}+1-d\right)h,\\ k^T Z &= (1-t)k^T,\\ 0^T &= -ck^T,\\ k^T h &= (t+c)(1-t)-sc,\\ k^T h &= \frac{(1-s-d)(t+s)+s^2+c+d}{\alpha},\\ 0 &= c(1-t-c-d),\\ 0 &= (c+d)(1-d)-sc. \end{cases}$$

We have $c=0$, so $d=0$ or 1 by the last equation above. Thus,

$$\begin{cases} Z^2-Z &= -hk^T,\\ Zh &= (1-t)h,\\ Zh &= \left(\frac{1-s}{\alpha}+1\right)h,\\ k^T Z &= (1-t)k^T,\\ k^T h &= t(1-t),\\ k^T h &= \frac{t(1-s)+s}{\alpha} \end{cases} \quad \text{or} \quad \begin{cases} Z^2-Z &= -hk^T,\\ Zh &= (1-t)h,\\ Zh &= -\frac{s}{\alpha}h,\\ k^T Z &= (1-t)k^T,\\ k^T h &= t(1-t),\\ k^T h &= \frac{1-ts}{\alpha}. \end{cases} \tag{4.4}$$

The first three equations of the left or right system in (4.4) imply $k^T h=(s-1)(1-t)/\alpha$ or $k^T h=-ts/\alpha$, respectively, which contradicts respectively the last equation. Hence neither system in (4.4) has a solution.

Case (iii): $h=0, \hat{h}\neq 0, k\neq 0$, and $\hat{k}=\beta k$ with $\beta\neq 0$. Simplifying (2.6) leads to $c=0$ and $t=0$, or $c=0$ and $t=1$, with their respective systems

$$\begin{cases} Z^2-Z &= -\beta\hat{h}k^T,\\ Z\hat{h} &= (1-d)\hat{h},\\ k^T Z &= [1+\beta(1-s)]k^T,\\ k^T Z &= (1-d)k^T,\\ k^T\hat{h} &= s+d(1-s),\\ k^T\hat{h} &= \frac{d(1-d)}{\beta} \end{cases} \quad \text{or} \quad \begin{cases} Z^2-Z &= -\beta\hat{h}k^T,\\ Z\hat{h} &= (1-d)\hat{h},\\ k^T Z &= -\beta s k^T,\\ k^T Z &= (1-d)k^T,\\ k^T\hat{h} &= 1-ds,\\ k^T\hat{h} &= \frac{d(1-d)}{\beta}. \end{cases} \tag{4.5}$$

The first, third, and fourth equations of the left or right system in (4.5) imply $k^T\hat{h}=(s-1)(1-d)$ or $k^T\hat{h}=-ds$, respectively, which contradicts the respective fifth equation. Hence neither system in (4.5) has a solution.

Case (iv): $h\neq 0, \hat{h}=0, k\neq 0$, and $\hat{k}=\beta k$ with $\beta\neq 0$. In this case we have

$$\begin{cases} Z^2-Z &= -(1+\beta)hk^T, \\ Zh &= (2-t)h, \\ k^TZ &= [1-(1+\beta)t]k^T, \\ k^TZ &= \left(\frac{1}{\beta}+2\right)k^T, \\ k^Th &= t(2-t), \\ k^Th &= \frac{t-2}{\beta} \end{cases} \quad \text{or} \quad \begin{cases} Z^2-Z &= -(1+\beta)hk^T, \\ Zh &= (2-t)h, \\ k^TZ &= (1+\beta)(1-t)k^T, \\ k^TZ &= \left(\frac{1}{\beta}+1\right)k^T, \\ k^Th &= -(t-1)^2, \\ k^Th &= \frac{t-1}{\beta} \end{cases} \tag{4.6}$$

with $s=1, c=-1$, and $d=0$, or $s=0, c=-1$, and $d=1$, respectively. The next step is similar to case (i). Assume $\beta=-1$ first. Since Z is a projection, the second equation of both systems in (4.6) implies $t=1$ or 2. If $t=1$, then (4.6) is reduced to

$$\begin{cases} Z^2-Z &= 0, \\ Zh &= h, \\ k^TZ &= k^T, \\ k^Th &= 1 \end{cases} \quad \text{or} \quad \begin{cases} Z^2-Z &= 0, \\ Zh &= h, \\ k^TZ &= 0^T, \end{cases}$$

corresponding to $T=T(1,1,-1,0)$ or $T(1,0,-1,1)$, respectively. On the other hand, if $t=2$, then we have

$$\begin{cases} Z^2-Z &= 0, \\ Zh &= 0, \\ k^TZ &= k^T \end{cases} \quad \text{or} \quad \begin{cases} Z^2-Z &= 0, \\ Zh &= 0, \\ k^TZ &= 0^T, \\ k^Th &= -1, \end{cases}$$

corresponding to $T=T(2,1,-1,0)$ or $T(2,0,-1,1)$, respectively.

Suppose now $\beta\neq -1$. Then $t=-1/\beta\neq 1$ in the left system of (4.6) and $t=1-1/\beta\neq 2$ in its right system, so (4.6) is

$$\begin{cases} Z^2-Z &= -(\beta+1)hk^T, \\ Zh &= \left(2+\frac{1}{\beta}\right)h, \\ k^TZ &= \left(2+\frac{1}{\beta}\right)k^T \end{cases} \quad \text{or} \quad \begin{cases} Z^2-Z &= -(\beta+1)hk^T, \\ Zh &= \left(1+\frac{1}{\beta}\right)h, \\ k^TZ &= \left(1+\frac{1}{\beta}\right)k^T, \end{cases}$$

after removing the redundant last two equations therein, corresponding to $T=T(-1/\beta,1,-1,0)$ or $T(1-1/\beta,0,-1,1)$, respectively.

To summarize, the system (2.6) has a solution only for Cases (i) and (iv) above, and the resulting theorem is as follows.

THEOREM 4.2. *All solutions $X=WYW^{-1}$ of (1.1) under the assumption that $r(H)=r(K)=1$ with exactly one zero column from $[H\ K]$ are such that*

(i) *$h\neq 0, \hat{h}=-h, k=0, \hat{k}\neq 0$, and Z is any projection.*

a. $T=T(0,1,-1,1), h\in R(Z)$, and $\hat{k}\in R(Z^T)$ with $\hat{k}^Th=-1$.

b. $T=T(1,0,-1,1), h\in N(Z)$, and $\hat{k}\in R(Z^T)$.

c. $T=T(0,1,-1,2), h\in R(Z)$, and $\hat{k}\in N(Z^T)$.

d. $T=T(1,0,-1,2), h\in N(Z)$, and $\hat{k}\in N(Z^T)$ with $\hat{k}^Th=1$.

(ii) *$h\neq 0, \hat{h}=\alpha h$ for any $\alpha\neq 0,-1, k=0$, and $\hat{k}\neq 0$. Either $T=T(0,1,-1,\alpha)$, and Z is diagonalizable and has one simple eigenvalue $2+\alpha$ with corresponding eigenvector h and left eigenvector $\hat{k}^T$ satisfying $\hat{k}^T h=-(2+\alpha)$, or $T=T(1,0,-1,1-\alpha)$, and Z is diagonalizable and has one simple eigenvalue $1+\alpha$ with corresponding eigenvector h and left eigenvector $\hat{k}^T$ satisfying $\hat{k}^T h=-\alpha$. The other eigenvalues of Z are in $\{0,1\}$.*

(iii) *$h\neq 0, \hat{h}=0, k\neq 0, \hat{k}=-k$, and Z is any projection.*

a. $T=T(1,1,-1,0), h\in R(Z)$, and $k\in R(Z^T)$ with $k^T h=1$.

b. $T=T(1,0,-1,1), h\in R(Z)$, and $k\in N(Z^T)$.

c. $T=T(2,1,-1,0), h\in N(Z)$, and $k\in R(Z^T)$.

d. $T=T(2,0,-1,1), h\in N(Z)$, and $k\in N(Z^T)$ with $k^T h=1$.

(iv) *$h\neq 0, \hat{h}=0, k\neq 0$, and $\hat{k}=\beta k$ for any $\beta\neq 0,-1$. Either $T=T(-1/\beta,1,-1,0)$, and Z is diagonalizable and has one simple eigenvalue $2+1/\beta$ with corresponding eigenvector h and left eigenvector k^T satisfying $k^T h=-(2\beta+1)/\beta^2$, or $T=T(1-1/\beta,0,-1,1)$, and Z is diagonalizable and has one simple eigenvalue $1+1/\beta$ with corresponding eigenvector h and left eigenvector k^T satisfying $k^T h=-1/\beta^2$. The other eigenvalues of Z are in $\{0,1\}$.*

The remaining situation is $h\neq 0, \hat{h}=\alpha h$ for any $\alpha\neq 0, k\neq 0$, and $\hat{k}=\beta k$ for any $\beta\neq 0$. Then (2.6) becomes

$$\begin{cases} Z^2-Z &= -[1+(1+\alpha)\beta)]hk^T, \\ Zh &= [1-t-c(1+\alpha)]h, \\ Zh &= \left(\frac{1-d-s}{\alpha}+1-d\right)h, \\ k^T Z &= [1-t+\beta(1-t-s)]k^T, \\ k^T Z &= \left(-\frac{c}{\beta}+1-c-d\right)k^T, \\ k^T h &= (t+c)(1-t)-sc, \\ k^T h &= \frac{(1-s-d)(t+s)+s^2+c+d}{\alpha}, \\ k^T h &= \frac{c(1-t-c-d)}{\beta}, \\ k^T h &= \frac{(c+d)(1-d)-sc}{\alpha\beta}. \end{cases} \tag{4.7}$$

Since $1-t-c(1+\alpha)$ and its square are an eigenvalue of Z and Z^2 respectively, the first two equations of (4.7) and Lemma 4.1 imply

$$[1-t-c(1+\alpha)]^2=1-t-c(1+\alpha)-[1+(1+\alpha)\beta][t(1-t)+c(1-t-s)].$$

The second and fourth equations of (4.7) guarantee $1-t-c(1+\alpha)=1-t+\beta(1-t-s)$, so $1-t-s=-c(1+\alpha)/\beta$. It follows that

$$[1-t-c(1+\alpha)]^2=1-t-c(1+\alpha)-[1+(1+\alpha)\beta]\left[t(1-t)-\frac{c^2(1+\alpha)}{\beta}\right],$$

which can be simplified to

$$(\alpha+1)(c-\beta t)(c-\beta t+\beta)=0.$$

Therefore, $\alpha=-1, c=t\beta$, or $c=\beta(t-1)$. We analyze each case as follows.

Case I: $\alpha=-1$. Then $1-d-s=0$ and (4.7) becomes

$$\begin{cases} Z^2-Z &= -hk^T, \\ Zh &= (1-t)h, \\ Zh &= sh, \\ k^TZ &= (1-t)k^T, \\ k^TZ &= \left(-\frac{c}{\beta}+1-c-d\right)k^T, \\ k^Th &= t(1-t), \\ k^Th &= t(s-1)-s-c, \\ k^Th &= \frac{c(1-t-c-d)}{\beta}, \\ k^Th &= \frac{c(s-1)-d(1-c-d)}{\beta}, \end{cases}$$

from which

$$\begin{cases} 1-t-s &= 0, \\ 1+t\beta &= 0, \\ t(1-t)+t(1-s)+s+c &= 0, \\ c(1-t-c-d)+c(1-s)+d(1-c-d) &= 0. \end{cases}$$

All solutions of the above system are

$$c=-1,\ d=1,\ s=1+\frac{1}{\beta},\ t=-\frac{1}{\beta} \quad \text{or} \quad c=-1,\ d=2,\ s=\frac{1}{\beta},\ t=1-\frac{1}{\beta}.$$

We therefore have two respective systems to solve:

$$\begin{cases} Z^2-Z &= -hk^T, \\ Zh &= \left(1+\frac{1}{\beta}\right)h, \\ k^TZ &= \left(1+\frac{1}{\beta}\right)k^T \end{cases} \quad \text{or} \quad \begin{cases} Z^2-Z &= -hk^T, \\ Zh &= \frac{1}{\beta}h, \\ k^TZ &= \frac{1}{\beta}k^T. \end{cases} \tag{4.8}$$

Here we have dropped the redundant equations of each system.

We first solve the left system of (4.8) with $\beta=-1$. Then it becomes $Z^2-Z=-hk^T, Zh=0, k^TZ=0$, and Z cannot be diagonalizable since otherwise $Z^2-Z=0$ from the fact that all eigenvalues of Z^2-Z are zero. In this case, $T=T(1,0,-1,1)$. Similarly, when $\beta\neq-1$, the corresponding $T=T(-1/\beta,1+1/\beta,-1,1)$.

By the same token, for the right system of (4.8), if $\beta=1$, then the corresponding $T=T(0,1,-1,2)$, and if $\beta\neq1$, then the corresponding $T=T(1-1/\beta,1/\beta,-1,2)$.

Case II: $c=t\beta$. The second and fourth equations of (4.7) give $1-t-c(1+\alpha)=1-t+\beta(1-t-s)$, so $s=1+t\alpha$. Equating the second and fifth equations gives $1-t-c(1+\alpha)=1-c-d-t$, from which $d=t\alpha\beta$. Then comparing the sixth and seventh equations sets $\alpha t(1-t)+\alpha c(1-t-s)=t(1-s)+s+c+d(1-t-s)$, which implies $c=t\beta=-1$. Substituting the expressions $t=-1/\beta, s=1-\alpha/\beta, c=-1, d=-\alpha$ into (4.7) and removing those equations that depend on others, we obtain the simplified equivalent system

$$\begin{cases} Z^2-Z &= -[1+(1+\alpha)\beta]hk^T, \\ Zh &= \left(2+\alpha+\frac{1}{\beta}\right)h, \\ k^TZ &= \left(2+\alpha+\frac{1}{\beta}\right)k^T. \end{cases}$$

By the same idea as for Case I, we reach the following conclusions:

(1) If $(2+\alpha)\beta \neq -1 \neq (1+\alpha)\beta$ so that $2+\alpha+1/\beta \neq 0$ or 1, then $T = T(-1/\beta, 1-\alpha/\beta, -1, -\alpha)$.

(2) If $(2+\alpha)\beta = -1$, then $T = T(2+\alpha, (1+\alpha)^2, -1, -\alpha)$.

(3) If $(1+\alpha)\beta = -1$, then $T = T(1+\alpha, (1+\alpha)^2 - \alpha, -1, -\alpha)$.

Case III: $c = (t-1)\beta$. Now $s = (t-1)\alpha$ from equating the second and fourth equations of (4.7). Comparing the second and fifth equations gives $d = 1 + (t-1)\alpha\beta$. Then the sixth and seventh equations imply $c = (t-1)\beta = -1$. Putting $t = 1 - 1/\beta, s = -\alpha/\beta, c = -1, d = 1-\alpha$ into (4.7) and removing the redundant equations, we have

$$\begin{cases} Z^2 - Z = -[1+(1+\alpha)\beta]hk^T, \\ Zh \quad\;\; = \left(1+\alpha+\frac{1}{\beta}\right)h, \\ k^T Z \quad = \left(1+\alpha+\frac{1}{\beta}\right)k^T. \end{cases}$$

This gives the following parallel assertions:

(a) If $(1+\alpha)\beta \neq -1 \neq \alpha\beta$, then $T = T(1-1/\beta, -\alpha/\beta, -1, 1-\alpha)$.

(b) If $(1+\alpha)\beta = -1$, then $T = T(2+\alpha, \alpha(1+\alpha), -1, 1-\alpha)$.

(c) If $\alpha\beta = -1$, then $T = T(1+\alpha, \alpha^2, -1, 1-\alpha)$.

In summary, the system (2.6) has solutions in all the three cases, which are obtained by Lemma 4.1 and the same technique as above.

THEOREM 4.3. *All solutions $X = WYW^{-1}$ of (1.1) under the assumption that $r(H) = r(K) = 1$ with no zero columns in $[H\, K]$ are such that*

(i) $h \neq 0, \hat{h} = -h, k \neq 0$**, and** $\hat{k} = \beta k$ **for any** $\beta \neq 0$**.**

a. $\beta = -1, T = T(1,0,-1,1)$, and Z has eigenvalue 0 of deficiency -1 with corresponding eigenvector h and left eigenvector k^T satisfying $k^T h = 0$. The other possible eigenvalue of Z is 1, which is semi-simple.

b. $\beta \neq -1, T = T(-1/\beta, 1+1/\beta, -1, 1)$, and Z is diagonalizable and has one simple eigenvalue $1+1/\beta$ with corresponding eigenvector h and left eigenvector k^T satisfying $k^T h = -(\beta+1)/\beta^2$. The other eigenvalues of Z are in $\{0,1\}$.

c. $\beta = 1, T = T(0,1,-1,2)$, and Z has eigenvalue 1 of deficiency -1 with corresponding eigenvector h and left eigenvector k^T satisfying $k^T h = 0$. The other possible eigenvalue of Z is 0, which is semi-simple.

d. $\beta \neq 1, T = T(1-1/\beta, 1/\beta, -1, 2)$, and Z is diagonalizable and has one simple eigenvalue $1/\beta$ with corresponding eigenvector h and left eigenvector k^T satisfying $k^T h = (\beta-1)/\beta^2$. The other eigenvalues of Z are in $\{0,1\}$.

(ii) $h \neq 0, \hat{h} = \alpha h, k \neq 0$**, and** $\hat{k} = \beta k$ **for any** $\alpha \neq 0$ **and** $\beta \neq 0$**, and** $c = \beta t$**.**

a. $(\alpha+2)\beta \neq -1, (\alpha+1)\beta \neq -1, T = T(-1/\beta, 1-\alpha/\beta, -1, -\alpha)$, and Z is diagonalizable with simple eigenvalue $2+\alpha+1/\beta$, corresponding eigenvector h and left eigenvector k^T satisfying $k^T h = -[(2+\alpha)\beta+1]/\beta^2$, and other eigenvalues in $\{0,1\}$.

b. $(\alpha+2)\beta = -1, T = T(2+\alpha, (1+\alpha)^2, -1, -\alpha)$, and Z has eigenvalue 0 of deficiency -1 with corresponding eigenvector h and left eigenvector k^T satisfying $k^T h = 0$. The other possible eigenvalue of Z is 1, which is semi-simple.

c. $(\alpha+1)\beta = -1, T = T(1+\alpha, (1+\alpha)^2 - \alpha, -1, -\alpha)$, and Z has eigenvalue 1 of deficiency -1 with corresponding eigenvector h and left eigenvector k^T satisfying $k^T h = \alpha + 1$. The other possible eigenvalue of Z is 0, which is semi-simple.

(iii) $h \neq 0, \hat{h} = \alpha h, k \neq 0$**, and** $\hat{k} = \beta k$ **for any** $\alpha \neq 0$ **and** $\beta \neq 0$**, and** $c = \beta(t-1)$**.**

a. $(\alpha+1)\beta\neq-1,\alpha\beta\neq-1,T=T(1-1/\beta,-\alpha/\beta,-1,1-\alpha)$*, and* Z *is diagonalizable and has one simple eigenvalue* $1+\alpha+1/\beta$ *with corresponding eigenvector* h *and left eigenvector* k^T *satisfying* $k^Th=-(\alpha\beta+1)/\beta^2$*. The other eigenvalues of* Z *are in* $\{0,1\}$*.*

b. $(\alpha+1)\beta=-1,T=T(2+\alpha,\alpha(1+\alpha),-1,1-\alpha)$*, and* Z *has eigenvalue* 0 *of deficiency* -1 *with corresponding eigenvector* h *and left eigenvector* k^T *satisfying* $k^Th=-\alpha-1$*. The other possible eigenvalue of* Z *is* 1*, which is semi-simple.*

c. $\alpha\beta=-1,T=T(1+\alpha,\alpha^2,-1,1-\alpha)$*, and* Z *has eigenvalue* 1 *of deficiency* -1 *with corresponding eigenvector* h *and left eigenvector* k^T *satisfying* $k^Th=0$*. The other possible eigenvalue of* Z *is* 0*, which is semi-simple.*

Table 4.1 below combines the results of Theorems 4.1-4.3, in which $\sigma=\sigma(Z)$ is the set of all eigenvalues of Z, and α and β are any numbers but 0 or -1. Also when $h,\hat{h},k$, or $\hat{k}$ appears, it is assumed nonzero.

5. One of H and K is full ranked

It is time to study the last assumption that one of H and K is full column ranked and the other is nonzero. Suppose $r(H)=2$. Then the first two equations of (2.5) and the fact that HJ_1 is of full column rank give rise to $K^TH=(I-T)[I-J_1(I-T)]$. Since

$$J_1TJ_1-TJ_1T=\begin{bmatrix}(t+c)(1-t)-sc & (1-t)(d+s-1)-ds+c+1\\ c(1-t-c-d) & (c+d)(1-d)-cs\end{bmatrix}$$

and

$$(I-T)[I-J_1(I-T)]=\begin{bmatrix}(t+c)(1-t)-sc & (1-t)(d+s-1)-ds\\ c(1-t-c-d) & (c+d)(1-d)-cs\end{bmatrix},$$

the last equation of (2.5) is also satisfied if and only if $c=-1$. Similarly, if $r(K)=2$, then $c=-1$. Thus, (2.5) is reduced to

$$\begin{cases}Z^2-Z=-HJ_1K^T,\\ ZH\quad=HJ_1(I-T), & T=T(t,s,-1,d)\\ K^TZ\quad=(I-T)J_1K^T,\end{cases}$$

since the last equation of (2.5) can be removed.

We need a rank-2 spectral perturbation result (Theorem 2.2 in [7]) as follows.

LEMMA 5.1. *Let* M *be an* $m\times m$ *matrix with eigenvalues* $\mu_1,\mu_2,\ldots,\mu_m$*, counting algebraic multiplicity, and let* U *and* V *be two* $m\times 2$ *matrices such that* $MU=U\Lambda$ *or* $V^TM=\Lambda V^T$ *for a* 2×2 *matrix* Λ *and the eigenvalues of* Λ *are* μ_1,μ_2*. If* U *or* V *is full column ranked, then the eigenvalues of* $M+UV^T$ *are* $\eta_1,\eta_2,\mu_3,\ldots,\mu_m$*, counting algebraic multiplicity, where* η_1,η_2 *are the eigenvalues of* $\Pi=\Lambda+V^TU$*.*

To apply the above lemma to our problem, we let $M=Z,U=H,V^T=-J_1K^T$, and

$$\Lambda=J_1(I-T)=\begin{bmatrix}2-t & 1-d-s\\ 1 & 1-d\end{bmatrix}.$$

Then

$$\Pi=J_1(I-T)-J_1K^TH=\Lambda^2=\begin{bmatrix}(t-2)^2+1-d-s & (3-d-t)(1-d-s)\\ 3-d-t & (d-2)(d-1)-s\end{bmatrix}.$$

The characteristic polynomial of Λ is

$$\phi(\lambda)=|\lambda I-\Lambda|=\lambda^2+(d+t-3)\lambda+(1-d)(2-t)+d+s-1,$$

Z	H	K	T	Conditions
$\sigma\subset\{0,1\}$ 0 is semi-simple	$[h,0]$ $h\neq 0$	$[0,\hat{k}]$ $\hat{k}\neq 0$	$\begin{bmatrix}1&0\\-1&1\end{bmatrix}$	$h\in N(Z-I),\hat{k}\in N(Z^T-I)$ $\hat{k}^Th=0$, 1 has deficiency -1
diagonalizable $\sigma\subset\{0,1,2\}$	$[h,0]\neq 0$	$[0,\hat{k}]\neq 0$	$\begin{bmatrix}0&1\\-1&0\end{bmatrix}$	$h\in N(Z-2I),\hat{k}\in N(Z^T-2I)$ 2 is simple, $\hat{k}^Th=-2$
projection	$[0,\hat{h}]\neq 0$	$[k,0]\neq 0$	$\begin{bmatrix}0&s\\0&0\end{bmatrix}$	$\hat{h}\in R(Z),k\in R(Z^T),k^T\hat{h}=s$
projection	$[0,\hat{h}]\neq 0$	$[k,0]\neq 0$	$\begin{bmatrix}1&s\\0&1\end{bmatrix}$	$\hat{h}\in N(Z),k\in N(Z^T),k^T\hat{h}=1-s$
projection	$[h,-h]$	$[0,\hat{k}]$	$\begin{bmatrix}0&1\\-1&1\end{bmatrix}$	$0\neq h\in R(Z),0\neq\hat{k}\in R(Z^T),\hat{k}^Th=-1$
projection	$[h,-h]$	$[0,\hat{k}]$	$\begin{bmatrix}1&0\\-1&1\end{bmatrix}$	$0\neq h\in N(Z),0\neq\hat{k}\in R(Z^T)$
projection	$[h,-h]$	$[0,\hat{k}]$	$\begin{bmatrix}0&1\\-1&2\end{bmatrix}$	$0\neq h\in R(Z),0\neq\hat{k}\in N(Z^T)$
projection	$[h,-h]$	$[0,\hat{k}]$	$\begin{bmatrix}1&0\\-1&2\end{bmatrix}$	$0\neq h\in N(Z),0\neq\hat{k}\in N(Z^T),\hat{k}^Th=1$
diagonalizable $\sigma\subset\{0,1,2+\alpha\}$	$[h,\alpha h]\neq 0$ $\alpha\neq -1,0$	$[0,\hat{k}]\neq 0$	$\begin{bmatrix}0&1\\-1&\alpha\end{bmatrix}$	$h\in N(Z-(2+\alpha)I)$, $2+\alpha$ is simple $\hat{k}\in N(Z^T-(2+\alpha)I),\hat{k}^Th=-(2+\alpha)$
diagonalizable $\sigma\subset\{0,1,1+\alpha\}$	$[h,\alpha h]\neq 0$ $\alpha\neq -1,0$	$[0,\hat{k}]\neq 0$	$\begin{bmatrix}1&0\\-1&1-\alpha\end{bmatrix}$	$h\in N(Z-(1+\alpha)I)$, $1+\alpha$ is simple $\hat{k}\in N(Z^T-(1+\alpha)I),\hat{k}^Th=-\alpha$
projection	$[h,0]\neq 0$	$[k,-k]\neq 0$	$\begin{bmatrix}1&1\\-1&0\end{bmatrix}$	$h\in R(Z),k\in R(Z^T),k^Th=1$
projection	$[h,0]\neq 0$	$[k,-k]\neq 0$	$\begin{bmatrix}1&0\\-1&1\end{bmatrix}$	$h\in R(Z),k\in N(Z^T)$
projection	$[h,0]\neq 0$	$[k,-k]\neq 0$	$\begin{bmatrix}2&1\\-1&0\end{bmatrix}$	$h\in N(Z),k\in R(Z^T)$
projection	$[h,0]\neq 0$	$[k,-k]\neq 0$	$\begin{bmatrix}2&0\\-1&1\end{bmatrix}$	$h\in N(Z),k\in N(Z^T),k^Th=1$
diagonalizable $\sigma\subset\{0,1,2+\frac{1}{\beta}\}$	$[h,0]\neq 0$	$[k,\beta k]\neq 0$ $\beta\neq -1$	$\begin{bmatrix}-\frac{1}{\beta}&1\\-1&0\end{bmatrix}$	$h\in N(Z-(2+\frac{1}{\beta})I)$, $2+\frac{1}{\beta}$ is simple $k\in N(Z^T-(2+\frac{1}{\beta})I),k^Th=\frac{2\beta+1}{-\beta^2}$
diagonalizable $\sigma\subset\{0,1,1+\frac{1}{\beta}\}$	$[h,0]\neq 0$	$[k,\beta k]\neq 0$ $\beta\neq -1,0$	$\begin{bmatrix}1-\frac{1}{\beta}&0\\-1&1\end{bmatrix}$	$h\in N(Z-(1+\frac{1}{\beta})I)$, $1+\frac{1}{\beta}$ is simple $k\in N(Z^T-(1+\frac{1}{\beta})I),k^Th=\frac{-1}{\beta^2}$
$\sigma\subset\{0,1\}$ 1 is semi-simple	$[h,-h]\neq 0$	$[k,-k]\neq 0$	$\begin{bmatrix}1&0\\-1&1\end{bmatrix}$	$h\in N(Z)$, 0 has deficiency -1 $k\in N(Z^T),k^Th=0$
$\sigma\subset\{0,1,1+\frac{1}{\beta}\}$ diagonalizable	$[h,-h]\neq 0$	$[k,\beta k]\neq 0$ $\beta\neq -1$	$\begin{bmatrix}-\frac{1}{\beta}&1+\frac{1}{\beta}\\-1&1\end{bmatrix}$	$h\in N(Z-(1+\frac{1}{\beta})I)$, $1+\frac{1}{\beta}$ is simple $k\in N(Z^T-(1+\frac{1}{\beta})I),k^Th=\frac{\beta+1}{-\beta^2}$
$\sigma\subset\{0,1\}$ 0 is semi-simple	$[h,-h]\neq 0$	$[k,k]\neq 0$	$\begin{bmatrix}0&1\\-1&2\end{bmatrix}$	$h\in N(Z-I)$, 1 has deficiency -1 $k\in N(Z^T-I),k^Th=0$
$\sigma\subset\{0,1,\frac{1}{\beta}\}$ diagonalizable	$[h,-h]\neq 0$	$[k,\beta k]\neq 0$ $\beta\neq 1$	$\begin{bmatrix}1-\frac{1}{\beta}&\frac{1}{\beta}\\-1&2\end{bmatrix}$	$h\in N(Z-\frac{1}{\beta}I)$, $\frac{1}{\beta}$ is simple $k\in N(Z^T-\frac{1}{\beta}I),k^Th=\frac{\beta-1}{\beta^2}$
$\sigma\subset\{0,1,2+\alpha+\frac{1}{\beta}\}$ diagonalizable	$[h,\alpha h]\neq 0$ $\beta\neq\frac{-1}{\alpha+2}$	$[k,\beta k]\neq 0$ $\beta\neq\frac{-1}{\alpha+1}$	$\begin{bmatrix}-\frac{1}{\beta}&1-\frac{\alpha}{\beta}\\-1&-\alpha\end{bmatrix}$	$h\in N(Z-(2+\alpha+\frac{1}{\beta})I)$, $2+\alpha+\frac{1}{\beta}$ is simple $k\in N(Z^T-(2+\alpha+\frac{1}{\beta})I),k^Th=\frac{(2+\alpha)\beta+1}{-\beta^2}$
$\sigma\subset\{0,1\}$ 1 is semi-simple	$[h,\alpha h]$ $h\neq 0$	$[k,\frac{-1}{\alpha+2}k]$ $k\neq 0$	$\begin{bmatrix}2+\alpha&(1+\alpha)^2\\-1&-\alpha\end{bmatrix}$	$h\in N(Z)$, 0 has deficiency -1 $k\in N(Z^T),k^Th=0,$
$\sigma\subset\{0,1\}$ 0 is semi-simple	$[h,\alpha h]$ $h\neq 0$	$[k,\frac{-1}{\alpha+1}k]$ $k\neq 0$	$\begin{bmatrix}1+\alpha&1+\alpha+\alpha^2\\-1&-\alpha\end{bmatrix}$	$h\in N(Z-I)$, 1 has deficiency -1 $k\in N(Z^T-I),k^Th=1+\alpha$
$\sigma\subset\{0,1,1+\alpha+\frac{1}{\beta}\}$ diagonalizable	$[h,\alpha h]\neq 0$ $\beta\neq -\frac{1}{\alpha}$	$[k,\beta k]\neq 0$ $\beta\neq -\frac{1}{\alpha+1}$	$\begin{bmatrix}1-\frac{1}{\beta}&-\frac{\alpha}{\beta}\\-1&1-\alpha\end{bmatrix}$	$h\in N(Z-(1+\alpha+\frac{1}{\beta})I)$, $1+\alpha+\frac{1}{\beta}$ is simple $k\in N(Z^T-(1+\alpha+\frac{1}{\beta})I),k^Th=\frac{\alpha\beta+1}{-\beta^2}$
$\sigma\subset\{0,1\}$ 1 is semi-simple	$[h,\alpha h]$	$[k,\frac{-1}{\alpha+1}k]$	$\begin{bmatrix}2+\alpha&\alpha(1+\alpha)\\-1&1-\alpha\end{bmatrix}$	$h\in N(Z)$, 0 has deficiency -1 $k\in N(Z^T),k^Th=-\alpha-1$
$\sigma\subset\{0,1\}$ 0 is semi-simple	$[h,\alpha h]\neq 0$	$[k,-\frac{1}{\alpha}k]\neq 0$	$\begin{bmatrix}1+\alpha&\alpha^2\\-1&1-\alpha\end{bmatrix}$	$h\in N(Z-I)$, 1 has deficiency -1 $k\in N(Z^T-I),k^Th=0$

TABLE 4.1. *All Solutions with* $r(H)=r(K)=1$

so the eigenvalues of Λ are

$$\mu,\nu=\frac{3-d-t\pm\sqrt{(1+d-t)^2+4(1-d-s)}}{2}, \tag{5.1}$$

in which μ and ν correspond to the plus and minus signs, respectively. It is easy to see that $\mu,\nu\neq 0$ if and only if $s\neq(d-1)(1-t)$ and $\mu,\nu\neq 1$ if and only if $s\neq 1-dt$. Also $\mu\neq\nu$ if and only if $s\neq 1-d+(1+d-t)^2/4$.

By Lemma 5.1 with $\mu_1=\mu$ and $\mu_2=\nu$, the eigenvalues of Z^2 are $\eta_1,\eta_2,\mu_3,\ldots,\mu_{n-2}$, where $\eta_1=\mu^2$ and $\eta_2=\nu^2$. On the other hand, the eigenvalues of Z^2 are the squares of those of Z, so $\mu_j=0$ or 1 for $j=3,\ldots,n-2$.

There are several cases on the eigenvalues of Λ.

Case 1): $\mu=0$ and $\nu=0$. Then $\phi(\lambda)=\lambda^2$ gives $(d+t-3)\lambda+(1-d)(2-t)+d+s-1\equiv 0$, so $d+t=3$ and $s=(d-1)(1-t)$. Thus $t=3-d$ and $s=(d-1)(d-2)$, and $T=T(3-d,(d-1)(d-2),-1,d)$.

Case 2): $\mu=0$ and $\nu=1$ or $\mu=1$ and $\nu=0$. Then $d+t=2$ and $s=(d-1)(1-t)$ from $\phi(\lambda)=\lambda(\lambda-1)$. So $t=2-d$ and $s=(d-1)^2$, thus $T=T(2-d,(d-1)^2,-1,d)$.

Case 3): $\mu=1$ and $\nu=1$. Then $\phi(\lambda)=(\lambda-1)^2$, so $d+t=1$ and $s=(d-1)(1-t)+1$. Therefore $t=1-d$ and $s=d(d-1)+1$. Hence $T=T(1-d,d(d-1)+1,-1,d)$.

Case 4): $\mu=0$ and $\nu\neq 0$ or 1. In this case $\nu=3-d-t\neq 0$ or 1, thus $t\neq 3-d$ or $2-d$, and $s=(d-1)(1-t)$. Consequently $T=T(t,(d-1)(1-t),-1,d)$.

Case 5): $\mu=1$ and $\nu\neq 0$ or 1. Now $t\neq 2-d$ or $1-d$, and $s=1-dt$. It follows that $T=T(t,1-dt,-1,d)$.

Case 6): $\mu\neq 0$ or 1, $\nu\neq 0$ or 1, and $\mu\neq\nu$. From $\phi(0)\neq 0,\phi(1)\neq 0$, and the equivalent condition for $\mu\neq\nu$, we see that $T=T(t,s,-1,d)$ such that

$$s\neq(d-1)(1-t),\ s\neq 1-dt,\ \text{and}\ s\neq 1-d+\frac{(1+d-t)^2}{4}.$$

Case 7): $\mu=\nu\neq 0$ or 1. Then $\mu=\nu=(3-d-t)/2$ by (5.1), so $t\neq 3-d$ or $1-d$, and $s=1-d+(1+d-t)^2/4$. Now $T=T(t,1-d+(1+d-t)^2/4,-1,d)$.

From the above analysis, and using the same arguments as in the proofs of Proposition 2.5 (iii) in [4] and Theorem 4.1 in [8], we obtain the following result.

THEOREM 5.1. *All solutions $X=WYW^{-1}$ of (1.1) under the assumption that $r(H)=2$ and $K\neq 0$ or $r(K)=2$ and $H\neq 0$ are such that*

(i) *$T=T(3-d,(d-1)(d-2),-1,d)$ for any d. And Z has eigenvalue 0 of deficiency -1 with respective eigenvector $H\xi$ and generalized eigenvector $H\hat{\xi}$, and left eigenvector $\zeta^TJ_1K^T$ and generalized left eigenvector $\hat{\zeta}^TJ_1K^T$, where 0 is the eigenvalue of $J_1(I-T)$ of deficiency -1 with respective eigenvector ξ and generalized eigenvector $\hat{\xi}$, and left eigenvectors ζ^T and generalized left eigenvector $\hat{\zeta}^T$, satisfying*

$$K^TH=\begin{bmatrix} d-2 & (d-1)(2-d) \\ 1 & 1-d \end{bmatrix}.$$

The other possible eigenvalue of Z is 1.

(ii) *$T=T(2-d,(d-1)^2,-1,d)$ for any d. And Z is a projection since it is diagonalizable with eigenvalues 0 and 1, with respective eigenvectors $H\xi,H\hat{\xi}$ and left eigenvectors $\zeta^TK^T,\hat{\zeta}^TK^T$, where 0 and 1 are eigenvalues of $J_1(I-T)$ with respective eigenvectors $\xi,\hat{\xi}$ and also eigenvalues of $(I-T)J_1$ with respective left eigenvectors $\zeta^T,\hat{\zeta}^T$, satisfying*

$$K^TH=0.$$

(iii) *$T=T(1-d,d(d-1)+1,-1,d)$ for any d. And Z has eigenvalue 1 of deficiency -1 with respective eigenvector $H\xi$ and generalized eigenvector $H\hat{\xi}$, and left eigenvector*

$\zeta^T J_1 K^T$ and generalized left eigenvector $\hat{\zeta}^T J_1 K^T$, where 1 is the eigenvalue of $J_1(I-T)$ of deficiency -1 with respective eigenvector ξ and generalized eigenvector $\hat{\xi}$, and left eigenvectors ζ^T and generalized left eigenvector $\hat{\zeta}^T$, satisfying

$$K^T H = \begin{bmatrix} 1-d & d(d-1) \\ -1 & d \end{bmatrix}.$$

The other possible eigenvalue of Z is 0.

(iv) *$T = T(t,(d-1)(1-t),-1,d)$ for any t and d satisfying $t \neq 3-d$. And Z is diagonalizable and has eigenvalues 0 and $3-d-t$, with respective eigenvectors $H\xi, H\hat{\xi}$ and left eigenvectors $\zeta^T K^T, \hat{\zeta}^T K^T$, where 0 and $3-d-t$ are eigenvalues of $J_1(I-T)$ with respective eigenvectors $\xi, \hat{\xi}$ and also eigenvalues of $(I-T)J_1$ with respective left eigenvectors $\zeta^T, \hat{\zeta}^T$, satisfying*

$$K^T H = \begin{bmatrix} (1-t)(d+t-2) & (d-1)(t-1)(d+t-2) \\ d+t-2 & (1-d)(d+t-2) \end{bmatrix}.$$

The other possible eigenvalue of Z is 1.

(v) *$T = T(t, 1-dt, -1, d)$ for any t and d satisfying $t \neq 2-d$. And Z is diagonalizable and has eigenvalues 1 and $2-d-t$, with respective eigenvectors $H\xi, H\hat{\xi}$ and left eigenvectors $\zeta^T K^T, \hat{\zeta}^T K^T$, where 1 and $2-d-t$ are eigenvalues of $J_1(I-T)$ with respective eigenvectors $\xi, \hat{\xi}$ and also eigenvalues of $(I-T)J_1$ with respective left eigenvectors $\zeta^T, \hat{\zeta}^T$, satisfying*

$$K^T H = \begin{bmatrix} t(2-d-t) & dt(d+t-2) \\ d+t-2 & d(2-d-t) \end{bmatrix}.$$

The other possible eigenvalue of Z is 0.

(vi) *$T = T(t, 1-d+(1+d-t)^2/4, -1, d)$ for any t and d satisfying $t \neq 3-d$ or $1-d$. And Z has eigenvalue $(3-d-t)/2$ of algebraic multiplicity 2 and deficiency -1 with respective eigenvector $H\xi$ and generalized eigenvector $H\hat{\xi}$, and left eigenvector $\zeta^T J_1 K^T$ and generalized left eigenvector $\hat{\zeta}^T J_1 K^T$, where $(3-d-t)/2$ is the eigenvalue of $J_1(I-T)$ of deficiency -1 with respective eigenvector ξ and generalized eigenvector $\hat{\xi}$, and left eigenvector ζ^T and generalized left eigenvector $\hat{\zeta}^T$, satisfying*

$$K^T H = \begin{bmatrix} 1-d+\frac{(1+d-t)^2}{4}-(1-t)^2 & d(d-1)+\frac{(1-d-t)(1+d-t)}{4} \\ d+t-2 & d(1-d)+\frac{(1+d-t)^2}{4} \end{bmatrix}.$$

The other eigenvalues of Z are in $\{0,1\}$.

(vii) *$T = T(t,s,-1,d)$ for any t, s, and d satisfying $s \neq 1-d+(1+d-t)^2/4, (d-1)(1-t)$, or $1-dt$. And Z is diagonalizable and has eigenvalues $\mu \neq \nu$ given by (5.1), other than 0 and 1, with respective eigenvectors $H\xi, H\hat{\xi}$ and left eigenvectors $\zeta^T K^T, \hat{\zeta}^T K^T$, where μ_1 and μ_2 are eigenvalues of $J_1(I-T)$ with respective eigenvectors $\xi, \hat{\xi}$ and also eigenvalues of $(I-T)J_1$ with respective left eigenvectors $\zeta^T, \hat{\zeta}^T$, satisfying*

$$K^T H = \begin{bmatrix} s-(1-t)^2 & (1-t)(d+s-1)-ds \\ d+t-2 & s-(1-d)^2 \end{bmatrix}.$$

The other eigenvalues of Z are in $\{0,1\}$.

In Table 5.1 below, μ and ν are given by (5.1). We denote $e \equiv 3-d-t, f \equiv 2-d-t, g \equiv 1-d-t, o \equiv s+dt-1, p \equiv (d-1)(1-t), q \equiv 1-d+(1+d-t)^2/4$.

Z	T	K^TH	Λ Conditions	H Conditions	K Conditions
$\sigma\subset\{0,1\}$	$\begin{bmatrix} 3-d & (d-1)(d-2) \\ -1 & d \end{bmatrix}$	$\begin{bmatrix} d-2 & (d-1)(2-d) \\ 1 & 1-d \end{bmatrix}$	$\xi\in N(\Lambda),\hat{\xi}\in N(\Lambda^2)$ $\zeta\in N(\Lambda^T),\hat{\zeta}\in N((\Lambda^T)^2)$	$H\xi\in N(Z)$ $H\hat{\xi}\in N(Z^2)$	$KJ_1^T\zeta\in N(Z^T)$ $KJ_1^T\hat{\zeta}\in N((Z^T)^2)$
projection	$\begin{bmatrix} 2-d & (d-1)^2 \\ -1 & d \end{bmatrix}$	0	$\xi\in N(\Lambda),\hat{\xi}\in N(\Lambda-I)$ $\zeta\in N(\Lambda^T),\hat{\zeta}\in N((\Lambda^T-I))$	$H\xi\in N(Z)$ $H\hat{\xi}\in N(Z-I)$	$KJ_1^T\zeta\in N(Z^T)$ $KJ_1^T\hat{\zeta}\in N(Z^T-I)$
$\sigma\subset\{0,1\}$	$\begin{bmatrix} 1-d & d(d-1)+1 \\ -1 & d \end{bmatrix}$	$\begin{bmatrix} 1-d & d(d-1) \\ -1 & d \end{bmatrix}$	$\xi\in N(\Lambda-I),\hat{\xi}\in N((\Lambda-I)^2)$ $\zeta\in N(\Lambda^T-I),\hat{\zeta}\in N((\Lambda^T-I)^2)$	$H\xi\in N(Z-I)$ $H\hat{\xi}\in N((Z-I)^2)$	$KJ_1^T\zeta\in N(Z^T-I)$ $KJ_1^T\hat{\zeta}\in N((Z^T-I)^2)$
diagonalizable $\sigma\subset\{1,e\},e\neq0$	$\begin{bmatrix} t & (d-1)(1-t) \\ -1 & d \end{bmatrix}$	$\begin{bmatrix} (1-t)(d+t-2) & (d-1)(t-1)(d+t-2) \\ d+t-2 & (1-d)(d+t-2) \end{bmatrix}$	$\xi\in N(\Lambda),\hat{\xi}\in N(\Lambda-eI)$ $\zeta\in N(\Lambda^T),\hat{\zeta}\in N(\Lambda^T-eI)$	$H\xi\in N(Z)$ $H\hat{\xi}\in N(Z-eI)$	$KJ_1^T\zeta\in N(Z^T)$ $KJ_1^T\hat{\zeta}\in N(Z^T-eI)$
diagonalizable $\sigma\subset\{0,f\},f\neq0$	$\begin{bmatrix} t & 1-dt \\ -1 & d \end{bmatrix}$	$\begin{bmatrix} t(2-d-t) & dt(d+t-2) \\ d+t-2 & d(2-d-t) \end{bmatrix}$	$\xi\in N(\Lambda-I),\hat{\xi}\in N(\Lambda-fI)$ $\zeta\in N(\Lambda^T-I),\hat{\zeta}\in N(\Lambda^T-fI)$	$H\xi\in N(Z-I)$ $H\hat{\xi}\in N(Z-fI)$	$KJ_1^T\zeta\in N(Z^T-I)$ $KJ_1^T\hat{\zeta}\in N(Z^T-fI)$
$\sigma\subset\{0,1,\frac{e}{2}\}$ $g\neq0,e\neq0$	$\begin{bmatrix} t & 1-d+\frac{(1+d-t)^2}{4} \\ -1 & d \end{bmatrix}$	$\begin{bmatrix} 1-d+\frac{(1+d-t)^2}{4}-(1-t)^2 & d(d-1)+\frac{(1-d-t)(1+d-t)}{4} \\ d+t-2 & d(1-d)+\frac{(1+d-t)^2}{4} \end{bmatrix}$	$\xi\in N(\Lambda-\frac{e}{2}I),\hat{\xi}\in N((\Lambda-\frac{e}{2}I)^2)$ $\zeta\in N(\Lambda^T-\frac{e}{2}I),\hat{\zeta}\in N((\Lambda^T-\frac{e}{2}I)^2)$	$H\xi\in N(Z-\frac{e}{2}I)$ $H\hat{\xi}\in N((Z-\frac{e}{2}I)^2)$	$KJ_1^T\zeta\in N(Z^T-\frac{e}{2}I)$ $KJ_1^T\hat{\zeta}\in N((Z^T-\frac{e}{2}I)^2)$
diagonalizable $\sigma\subset\{0,1,\mu,\nu\}$ $o,p,q\neq0$	$\begin{bmatrix} t & s \\ -1 & d \end{bmatrix}$	$\begin{bmatrix} s-(1-t)^2 & (1-t)(d+s-1)-ds \\ d+t-2 & s-(1-d)^2 \end{bmatrix}$	$\xi\in N(\Lambda-\mu I),\hat{\xi}\in N(\Lambda-\nu I)$ $\zeta\in N(\Lambda^T-\mu I),\hat{\zeta}\in N(\Lambda^T-\nu I)$	$H\xi\in N(Z-\mu I)$ $H\hat{\xi}\in N(Z-\nu I)$	$KJ_1^T\zeta\in N(Z^T-\mu I)$ $KJ_1^T\hat{\zeta}\in N(Z^T-\nu I)$

TABLE 5.1. *All Solutions with $r(H)=2$ and $K\neq0$ or $r(K)=2$ and $H\neq0$*

6. Conclusions

We have found all the solutions X of the Yang-Baxter-like matrix Equation (1.1) when A is a non-diagonalizable elementary matrix, after all the solutions of the equation for diagonalizable elementary matrices were obtained in [4]. Because of the simple and special structure of the Jordan form for A and with the help of spectral perturbation results for rank-1 and rank-2 perturbations of matrices, we were able to completely solve the quadratic matrix equation for general elementary matrices.

Our approach may be extended to solve the same matrix equation for more general matrices A, which will be further explored in the future work.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (Nos. 11861008, 11501126, 11471122, 11661007, 11661008), the Research Fund of Gannan Normal University(Nos.YJG-2018-11, 18zb04), and the Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice.

REFERENCES

[1] R. Baxter, *Partition function of the eight-vertex lattice model*, Ann. Phys., 281(1–2):187–222, 2000. 1

[2] A. Cibotarica, J. Ding, J. Kolibal, and N. Rhee, *Solutions of the Yang-Baxter matrix equation for an idempotent*, Numer. Algb. Control Optim., (2):347–352, 2013. 1

[3] J. Ding and N. Rhee, *Spectral solutions of the Yang-Baxter matrix equation*, J. Math. Anal. Appl., 402:567–573, 2013. 1

[4] J. Ding and H. Tian, *Solving the Yang-Baxter-like matrix equation for a class of elementary matrices*, Comput. Math. Appl., 72(6):1541–1548, 2016. 1, 1, 4, 4, 5, 6

[5] J. Ding and C. Zhang, *On the structure of the spectral solutions of the Yang-Baxter matrix equation*, Appl. Math. Lett., 35:86–89, 2014. 1

[6] J. Ding and A. Zhou, *Eigenvalues of rank–one updated matrices with some applications*, Appl. Math. Lett., 20:1223–1226, 2007. 4

[7] J. Ding and A. Zhou, *Characteristic polynomials of some perturbed matrices*, Appl. Math. Comput., 199:631–636, 2008. 5

[8] Q. Dong and J. Ding, *Commuting solutions of the Yang–Baxter–like matrix equation for diagonalizable matrices*, Comput. Math. Appl., 72(1):194–201, 2016. 1, 4, 4, 5

[9] Q. Dong, J. Ding, and Q. Huang, *Commuting solutions of a quadratic matrix equation for nilpotent matrices*, Algb. Collo., 25(1):31–44, 2018. 1

[10] F. Felix, *Nonlinear Equations, Quantum Groups and Duality Theorems: A Primer on the Yang-Baxter Equation*, VDM Verlag, 2009. 1

[11] M. Jimbo (ed.), *Yang–Baxter Equation in Integrable Systems*, Adv. Series in Math. Phys., World Scientific, 10, 1990. 1

[12] D. Shen, M. Wei, and Z. Jia, *On commuting solutions of the Yang-Baxter-like matrix equation*, J. Math. Anal. Appl., 462:665–696, 2018. 1

[13] C. Yang, *Some exact results for the many–body problem in one dimension with repulsive delta–function interaction*, Phys. Rev. Lett., 19:1312–1315, 1967. 1

[14] C. Yang and M. Ge, *Braid Group, Knot Theory, and Statistical Mechanics*, World Scientific, 1989. 1

[15] D. Zhou, G. Chen, and J. Ding, *Solving the Yang–Baxter–like matrix equation for rank–two matrices*, J. Comput. Appl. Math., 313:142–151, 2017. 1