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TIME PERIODIC SOLUTIONS TO THE FULL HYDRODYNAMIC MODEL TO SEMICONDUCTORS*

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Abstract. In this paper, a full hydrodynamic semiconductor model with a time periodic external force is concerned. First, we regularize the system under consideration and prove the existence of time periodic solutions to the linearized approximate system by applying Tychonoff fixed point theorem combined with the energy method and the decay estimates. This idea is from the Massera-type criteria for linear periodic evolution equations. Then, the existence of a strong time periodic solution under some smallness assumptions is established by using the topological degree theory and an approximation scheme. The uniqueness of time periodic solutions is proved basing on the energy estimates. Also, the existence of the stationary solution is obtained.

Keywords. Hydrodynamic model to semiconductors; Time periodic solutions; Fixed point theorem and topological degree.

AMS subject classifications. 35Q31; 35B10; 76N15; 82D37.

1. Introduction

The paper concerns the full hydrodynamic model to semiconductors:

$$\bar{\rho}_t + \mathrm{div}(\bar{\rho}\bar{u}) = 0, \tag{1.1}$$

$$\bar{\rho}(\bar{u}_t + \bar{u}\cdot\nabla\bar{u}) + \nabla(\bar{\rho}\bar{\theta}) + \bar{\rho}\bar{u} = \bar{\rho}\bar{E} + \bar{\rho}f, \tag{1.2}$$

$$\frac{3}{2}\bar{\rho}(\bar{\theta}_t + \bar{u}\cdot\nabla\bar{\theta}) + \bar{\rho}\bar{\theta}\mathrm{div}\bar{u} = \triangle\bar{\theta} + \kappa\bar{\rho}|\bar{u}|^2 - \bar{\rho}(\bar{\theta}-1), \tag{1.3}$$

$$\bar{E} = \nabla\phi, \tag{1.4}$$

$$\triangle\phi = \bar{\rho} - b(x), \tag{1.5}$$

where $x \in \Omega := \prod_{i=1}^{3}(0,L_i) \subset \mathbb{R}^3$, κ is a constant, $\bar{\rho},\bar{u},\bar{\theta}$ denote the current density, the average velocity and the absolute temperature, respectively, $\bar{E}$ is the gradient of electrostatic potential, $b(x)$ is a given function which describes the prescribed background ion density, f is an external force.

System (1.1)-(1.5) is a nonisentropic hydrodynamic model. It describes that electron flow transport in the semiconductor devices, as in submicron devices or in the occurrence of high field phenomena. The corresponding drift-diffusion model works well under some assumptions of low carrier densities and small electronic fields. This model can be derived from the Boltzmann equation by a moment method. For the physical background of semiconductor devices, we refer to [1–3] and the references therein. In the past few years, the study of semiconductor devices has attracted a lot of attention from the mathematical point of view. In [4], Tao *et al.* established the global existence of smooth solutions to the Cauchy problem for the one-dimensional isentropic hydrodynamic model for semiconductors with small initial data. They found that these solutions converge to the stationary ones of the drift-diffusion equations. The model

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in one-dimensions has been concerned in the literature. One can see [5–9]. In multi-space dimensions, Hsiao and Wang considered a full hydrodynamic semiconductor model in [10]. They proved the global existence and large-time behavior of smooth solutions. The model in multi-dimensions has been studied in many directions, see, for example [11–13].

In the present paper, we are devoted to studying time periodic solutions to semiconductor models. To the best of our knowledge, this is the first result giving a proof of existence of the time periodic solutions to system (1.1)-(1.5). Different from some previous results, we don't need the symmetry of the external force. We can estimate the L^2 norm of solutions by the structure of the system directly. Also, the regularized assumption of the external force is more general because we adopt a new method to solve the linearized system and make careful energy estimates. The basic idea of the proof is the following. First, we reformulate the system. Next, we give the energy estimates of the linearized system which is regularized. The existence of time periodic solutions to the linearized system is obtained by the Tychonoff fixed point theorem. This idea is from the Massera-type criteria for periodic evolution equations [21–23]. From the argument, one can see that the initial data of time periodic solutions to the linearized system lies in a convex hull. Then, inspired by [18, 19] and [20], we use the Leray-Schauder degree theory and approximation scheme to establish the existence of time periodic solutions. Moreover, the uniqueness of time periodic solutions is proved.

As is known, one of the interesting phenomena in fluid mechanics is time periodic flow. It has been studied theoretically and numerically in the last decades. In particular, there has been a lot of interest in the study of time periodic compressible Navier-Stokes equations. Feireisl *et al.* [14, 15] established the existence of time periodic weak solutions by the Faedo-Galerkin method and the vanishing viscosity method. In [16], Valli considered the existence and asymptotic stability of small strong solutions in a bounded domain with nonslip boundary condition. He obtained the existence of a time periodic solution by following the approach of Serrin [17] concerned with time periodic solutions of incompressible Navier-Stokes equations. In 2015, Jin and Yang [18, 19] studied the problem in three-dimensional space when the external force satisfied the oddness condition. They obtained the existence of a small amplitude periodic solution around a positive constant state by topological degree theory and energy method. With similar ideas in [18, 19], Tan *et al.* [20] considered the existence and uniqueness of time periodic solutions to compressible Euler equations with damping. They employed the continuity and momentum equations to overcome the difficulty of the lower order dissipation of damping when deriving the highest order energy estimates.

Our aim is to investigate the time periodic solution around $(1,0,1)$. Denote $\bar\rho = 1+\rho, \bar u = u, \bar\theta = 1+\theta$ with (ρ, u, θ) being small. Then, system (1.1)-(1.5) is reformulated as follows:

$$\rho_t + \mathrm{div}u = -\mathrm{div}(\rho u), \tag{1.6}$$

$$u_t + u - E + \nabla\theta + \nabla\rho + u\cdot\nabla u = \Big(\frac{\rho-\theta}{1+\rho}\Big)\nabla\rho + f, \tag{1.7}$$

$$\frac{3}{2}\theta_t + \theta + \mathrm{div}u - \triangle\theta = -\frac{3}{2}u\cdot\nabla\theta - \theta\mathrm{div}u + \kappa|u|^2 - \Big(\frac{\rho}{1+\rho}\Big)\triangle\theta, \tag{1.8}$$

$$E = \nabla\phi, \tag{1.9}$$

$$\triangle\phi = \rho + 1 - b(x). \tag{1.10}$$

Throughout the paper, we assume that $b(x)=1$. The same argument works when $b(x) = 1+\gamma b_1(x)$ with a sufficiently small constant γ and a smooth function $b_1(x)$. The functions (ρ,u,θ,ϕ) are periodic in each x_i of period L_i, $i=1,2,3$. $f(x,t)$ is a given external force, which is periodic both in space and in time, respectively. In addition, we assume that $\int_\Omega \phi dx=0$ to assure uniqueness.

Before stating the main result, we explain notations and conventions. We will omit variables t,x of functions for simplicity if it does not cause any confusion. In addition, we abbreviate $\int_\Omega$ by $\int$ for convenience. C is used to denote a generic positive constant which may vary in different estimates. $C_{a,b}$ denotes a generic positive constant which depends on a,b. We use H^s to denote the usual Sobolev space defined over Ω equipped with the norm $||\cdot||_{H^s}$, and L^p $(1\le p\le\infty)$ to denote the usual Lebesgue space defined over Ω equipped with the norm $||\cdot||_p$. Especially, the norm and the inner product in L^2 are denoted by $||\cdot||$ and $\langle\cdot,\cdot\rangle$, respectively.

Now, our main result is stated as follows.

THEOREM 1.1. *Assume that $m\ge 2$, and the $T-$periodic external force $f\in L^2(0,T;H^{m+1}(\Omega))$ satisfies*

$$\int_0^T ||f||^2_{H^{m+1}}dt<\eta,$$

for some small constant $\eta>0$. Then, system (1.6)-(1.10) has a unique $T-$periodic solution (ρ,u,θ) such that

$$(\rho,u,\theta)\in X_\delta,$$

where X_δ is defined in Section 2, $\delta<1$ is an appropriately small constant.

REMARK 1.1. From the uniqueness of time periodic solutions, one can see that there is no small nontrivial time periodic solution to system (1.6)-(1.10) without external force.

REMARK 1.2. Furthermore, if we assume that f is independent of t, then f is periodic of any period $T\ge 0$. By virtue of Theorem 1.1, there exists a time periodic solution (ρ_1,u_1,θ_1) of period 1. On the other hand, there is a time periodic solution (ρ_2,u_2,θ_2) of period $\frac{1}{2}$. By the property of uniqueness, it should be $(\rho_1,u_1,\theta_1)=(\rho_2,u_2,\theta_2)$. Repeating this process, it concludes that (ρ_1,u_1,θ_1) is constant for any rational number t. From a continuity argument, we have that (ρ_1,u_1,θ_1) is independent of t. Thus, (ρ_1,u_1,θ_1) is a unique small stationary solution of the following system:

$$\begin{aligned}
&\mathrm{div}((\rho+1)u)=0,\\
&u-E+\nabla\theta+(\frac{1+\theta}{1+\rho})\nabla\rho+u\cdot\nabla u=f,\\
&\theta+\mathrm{div}u-\frac{1}{1+\rho}\triangle\theta=-\frac{3}{2}u\cdot\nabla\theta-\theta\mathrm{div}u+\kappa|u|^2,\\
&E=\nabla\phi,\ \triangle\phi=\rho.
\end{aligned}$$

The rest of this paper is organized as follows. We are going to regularize system (1.6)-(1.10). In Section 2, we derive some energy estimates of higher and lower order derivatives of the linearized system, respectively. The existence of time periodic solutions is given in Section 3. The uniqueness of time periodic solutions is shown in the last section.

2. Energy estimate of linearized system

For $m \geq 2$, we define the following suitable function spaces:

$$X = \Big\{(\rho,u,\theta) \in L^\infty(0,T;H^{m+1}(\Omega)); \theta \in L^2(0,T;H^{m+2}(\Omega));$$
$$(\rho,u,\theta) \text{ is periodic in space and in time, and } \int \rho dx = 0\Big\}$$

and

$$X_\delta = \Big\{(\rho,u,\theta) \in X \Big| ||(\rho,u,\theta)||_X^2 := \sup_{t\in[0,T]} ||(\rho,u,\theta)||_{H^{m+1}}^2 + \int_0^T ||\theta||_{H^{m+2}}^2 dt \leq \delta^2\Big\}.$$

In order to grant the uniqueness, the condition $\int \rho dx = 0$ is necessary. And we require that ϕ is periodic in each x_i of period $L_i (i=1,2,3)$, $\int \phi dx = 0$.

We regularize system (1.6)-(1.10):

$$\rho_t + \mathrm{div} u - \epsilon \triangle \rho = -\tau \mathrm{div}(\rho u), \tag{2.1}$$
$$u_t + u - E + \nabla\theta + \nabla\rho + \tau u\cdot\nabla u - \epsilon\triangle u = \tau(\frac{\rho-\theta}{1+\rho})\nabla\rho + \tau f, \tag{2.2}$$
$$\frac{3}{2}\theta_t + \theta + \mathrm{div} u - \triangle\theta = \tau\Big(-\frac{3}{2}u\cdot\nabla\theta - \theta\mathrm{div} u + \kappa|u|^2 - (\frac{\rho}{1+\rho})\triangle\theta\Big), \tag{2.3}$$
$$E = \nabla\phi, \tag{2.4}$$
$$\triangle\phi = \rho, \tag{2.5}$$

for $\tau \in [0,1]$.

Now, we consider the following linearized system:

$$\rho_t + \mathrm{div} u - \epsilon\triangle\rho = -\tau\mathrm{div}(\rho' u'), \tag{2.6}$$
$$u_t + u - E + \nabla\theta + \nabla\rho + \tau u'\cdot\nabla u - \epsilon\triangle u = \tau(\frac{\rho'-\theta'}{1+\rho'})\nabla\rho' + \tau f, \tag{2.7}$$
$$\frac{3}{2}\theta_t + \theta + \mathrm{div} u - \triangle\theta = \tau\Big(-\frac{3}{2}u'\cdot\nabla\theta' - \theta'\mathrm{div} u' + \kappa|u'|^2 - (\frac{\rho'}{1+\rho'})\triangle\theta'\Big), \tag{2.8}$$
$$E = \nabla\phi, \tag{2.9}$$
$$\triangle\phi = \rho, \tag{2.10}$$

for any given $T-$periodic function $(\rho',u',\theta') \in X_\delta$.

Since $\sup_{t\in[0,T]}||\rho'||_\infty \leq C\sup_{t\in[0,T]}||\rho'||_{H^{m+1}} \leq \delta$ for δ small enough, we assume that $|\rho'(x,t)| \leq \frac{1}{2}$ for all $(x,t) \in \Omega\times[0,T]$ without loss of generality. Now, the energy estimates of lower and higher order derivatives are obtained respectively.

LEMMA 2.1. *There exists a large enough constant $M_1 > 0$ such that*

$$\frac{d}{dt}(||\rho||^2 + ||u||^2 + \frac{3}{2}||\theta||^2 - M_1\langle \mathrm{div} u, \rho\rangle) + (\epsilon||\nabla\rho||^2 + \frac{1}{2}||u||^2 + \epsilon||\nabla u||^2$$
$$+ \frac{M_1}{4}||\rho||_{H^1}^2 + \frac{1}{2}||\theta||_{H^1}^2)$$
$$\leq C\Big(||\nabla\rho'||^2||u'||_{H^2}^2 + ||\rho'||_{H^2}^2||\mathrm{div} u'||^2 + ||\mathrm{div} u'||_{H^2}||u||^2 + (||\theta'||_{H^2}^2 + ||\rho'||_{H^2}^2)||\rho'||_{H^2}^2$$
$$+ ||f||^2 + ||u'||_{H^2}^2||\nabla\theta'||^2 + ||\theta'||_{H^2}^2||\mathrm{div} u'||^2 + ||u'||_4^4 + M_1\big(||\nabla\theta||^2 + ||\mathrm{div} u||^2$$

$$+||u'||^2_{H^2}||\nabla u||^2+(||\theta'||_{H^2}+||\rho'||^2_{H^2})||\nabla\rho'||^2+||f||^2+||\nabla\rho'||^2||u'||^2_{H^2}$$
$$+||\mathrm{div}u'||^2||\rho'||^2_{H^2}+\epsilon||\triangle\rho||^2+\epsilon||\triangle u||^2)\Big). \tag{2.11}$$

Proof. Multiplying Equations (2.6)-(2.8) by ρ,u,θ, respectively, integrating them over Ω by parts, we have

$$\frac{1}{2}\frac{d}{dt}||\rho||^2+\int\mathrm{div}u\rho dx+\epsilon||\nabla\rho||^2=-\tau\int\mathrm{div}(\rho'u')\rho dx, \tag{2.12}$$

$$\frac{1}{2}\frac{d}{dt}||u||^2+||u||^2-\int E\cdot udx+\int\nabla\theta\cdot udx+\int\nabla\rho\cdot udx+\tau\int(u'\cdot\nabla)u\cdot udx+\epsilon||\nabla u||^2$$
$$=\int\tau(\frac{\rho'-\theta'}{1+\rho'})\nabla\rho'\cdot udx+\int\tau f\cdot udx, \tag{2.13}$$

$$\frac{3}{4}\frac{d}{dt}||\theta||^2+||\theta||^2+\int\mathrm{div}u\theta dx+\int|\nabla\theta|^2dx$$
$$=-\tau\int\frac{\rho'}{1+\rho'}\triangle\theta'\theta dx-\tau\int\frac{3}{2}(u'\cdot\nabla)\theta'\theta dx-\tau\int\theta'\mathrm{div}u'\theta dx+\tau\int\kappa|u'|^2\theta dx. \tag{2.14}$$

Note that

$$-\int\frac{\rho'}{1+\rho'}\triangle\theta'\theta dx=\int\frac{\rho'}{1+\rho'}\nabla\theta\nabla\theta'dx+\int\theta\nabla(\frac{\rho'}{1+\rho'})\nabla\theta'dx$$
$$\leq C||\nabla\theta||||\nabla\theta'||||\rho'||_{H^2}+C||\nabla\theta||||\nabla\theta'||_{H^1}||\nabla\rho'||,$$

$$\int(u'\cdot\nabla)u\cdot udx=-\frac{1}{2}\int\mathrm{div}u'|u|^2dx\leq C||\mathrm{div}u'||_\infty||u||^2.$$

Then, summing up Equations (2.12)-(2.14), and using Hölder's and Young's inequalities, we have

$$\frac{d}{dt}(||\rho||^2+||u||^2+\frac{3}{2}||\theta||^2)+(\epsilon||\nabla\rho||^2+\frac{1}{2}||u||^2+\epsilon||\nabla u||^2+\frac{1}{2}||\theta||^2+\frac{1}{2}||\nabla\theta||^2)$$
$$\leq C\big(||\nabla\rho'||^2||u'||^2_{H^2}+||\rho'||^2_{H^2}||\mathrm{div}u'||^2+||\mathrm{div}u'||_{H^2}||u||^2+||\rho||^2+||\nabla\theta'||^2||\rho'||^2_{H^2}$$
$$+||\nabla\theta'||^2_{H^1}||\nabla\rho'||^2+(||\theta'||^2_{H^2}+||\rho'||^2_{H^2})||\nabla\rho'||^2+||f||^2+||u'||^2_{H^2}||\nabla\theta'||^2$$
$$+||\theta'||^2_{H^2}||\mathrm{div}u'||^2+\kappa||u'||^4_4\big). \tag{2.15}$$

On the other hand, multiplying Equation (2.7) by $\nabla\rho$ and integrating it over Ω by parts yields

$$-\langle\mathrm{div}\partial_tu,\rho\rangle+||\nabla\rho||^2+\langle\nabla\theta,\nabla\rho\rangle-\langle\mathrm{div}u,\rho\rangle-\langle E,\nabla\rho\rangle+\tau\langle u'\cdot\nabla u,\nabla\rho\rangle-\epsilon\langle\triangle u,\nabla\rho\rangle$$
$$=\tau\int(\frac{\rho'-\theta'}{1+\rho'})\nabla\rho'\cdot\nabla\rho dx+\tau\int f\nabla\rho dx.$$

Since

$$\langle\mathrm{div}\partial_tu,\rho\rangle=\frac{d}{dt}\langle\mathrm{div}u,\rho\rangle-\langle\mathrm{div}u,\rho_t\rangle$$
$$=\frac{d}{dt}\langle\mathrm{div}u,\rho\rangle+||\mathrm{div}u||^2+\langle\mathrm{div}u,\tau\mathrm{div}(\rho'u')\rangle-\langle\mathrm{div}u,\epsilon\triangle\rho\rangle,$$
$$\langle E,\nabla\rho\rangle=\langle\nabla\phi,\nabla\rho\rangle=-||\rho||^2,$$

one deduces that

$$\begin{aligned}&\frac{1}{2}||\rho||_{H^1}^2-\frac{d}{dt}\langle \mathrm{div}u,\rho\rangle\\ \le& C\big(||\nabla\theta||^2+||\mathrm{div}u||^2+||u'||_{H^2}^2||\nabla u||^2+(||\theta'||_{H^2}+||\rho'||_{H^2}^2)||\nabla\rho'||^2+||f||^2\\ &+||\nabla\rho'||^2||u'||_{H^2}^2+||\mathrm{div}u'||^2||\rho'||_{H^2}^2+\epsilon||\triangle\rho||^2+\epsilon||\triangle u||^2\big).\end{aligned}\tag{2.16}$$

Therefore, we can multiply inequality (2.16) by a constant $M_1>0$ large enough to absorb the term $||\rho||$ in (2.15). It completes the proof. ∎

LEMMA 2.2. *There exists a small enough constant $\epsilon'>0$ such that*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||\mathrm{div}u||^2+||\nabla\rho||^2+||\rho||^2+\frac{3}{2}||\nabla\theta||^2+||\mathrm{curl}u||^2)+\frac{1}{2}||\mathrm{div}u||^2+\frac{1}{2}||\mathrm{curl}u||^2\\ &+||\nabla\theta||^2+\frac{1}{2}||\triangle\theta||^2+\epsilon(||\nabla\rho||^2+||\triangle\rho||^2+||\nabla\mathrm{div}u||^2+||\nabla\mathrm{curl}u||^2)\\ \le& C\Big(||\nabla u'||_{H^2}||\nabla u||^2+||\mathrm{div}f||^2+C||\mathrm{curl}f||^2+(||\theta'||_{H^2}^2+||\rho'||_{H^2}^2)||\triangle\rho'||^2\\ &+(||\theta'||_{H^2}^2+||\rho'||_{H^2}^2)||\nabla\rho'||_{H^2}^2||\nabla\rho'||^2+(||\nabla\theta'||_{H^2}^2+||\nabla\rho'||_{H^2}^2)||\nabla\rho'||^2\Big)+\epsilon'||\rho||_{H^1}^2\\ &+C_{\epsilon'}||\rho'||_{H^2}^2||u'||_{H^2}^2+C\Big(||u'||_{H^2}^2||\nabla\theta'||^2+||\theta'||_{H^2}^2||\mathrm{div}u'||^2+||\rho'||_{H^2}^2||\triangle\theta'||^2+||u'||_4^4\Big).\end{aligned}\tag{2.17}$$

Proof. Multiplying Equation (2.7) by $-\nabla\mathrm{div}u$ and integrating it over Ω by parts, we infer that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\mathrm{div}u||^2-\langle\nabla\theta,\nabla\mathrm{div}u\rangle-\langle\nabla\rho,\nabla\mathrm{div}u\rangle+||\mathrm{div}u||^2+\langle E,\nabla\mathrm{div}u\rangle+\epsilon||\nabla\mathrm{div}u||^2\\ =&\tau\langle u'\cdot\nabla u,\nabla\mathrm{div}u\rangle+\tau\int\mathrm{div}f\,\mathrm{div}u\,dx+\tau\int\mathrm{div}\Big(\frac{\rho'-\theta'}{1+\rho'}\nabla\rho'\Big)\mathrm{div}u\,dx.\end{aligned}$$

Note that

$$-\langle u'\cdot\nabla u,\nabla\mathrm{div}u\rangle=\int\partial_j u_i'\partial_i u_j\partial_k u_k dx-\frac{1}{2}\int\partial_i u_i'|\partial_j u_j|^2dx\le C||\nabla u'||_\infty||\nabla u||^2,$$

$$\langle E,\nabla\mathrm{div}u\rangle=-\langle\rho,-\rho_t-\tau\mathrm{div}(\rho'u')+\epsilon\triangle\rho\rangle,$$

where we have used the representation of $\mathrm{div}u$ from Equation (2.6).

Hence, using Hölder's, Sobolev's and Young's inequalities, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\mathrm{div}u||^2-\langle\nabla\theta,\nabla\mathrm{div}u\rangle-\langle\nabla\rho,\nabla\mathrm{div}u\rangle+\frac{1}{2}||\mathrm{div}u||^2+\frac{1}{2}\frac{d}{dt}||\rho||^2+\epsilon||\nabla\rho||^2\\ &+\epsilon||\nabla\mathrm{div}u||^2\\ \le& C||\nabla u'||_{H^2}||\nabla u||^2+C||\mathrm{div}f||^2+C(||\theta'||_{H^2}^2+||\rho'||_{H^2}^2)||\triangle\rho'||^2+\epsilon'||\rho||^2\\ &+C(||\theta'||_{H^2}^2+||\rho'||_{H^2}^2)||\nabla\rho'||_{H^2}^2||\nabla\rho'||^2+C(||\nabla\theta'||_{H^2}^2+||\nabla\rho'||_{H^2}^2)||\nabla\rho'||^2\\ &+C_{\epsilon'}(||\nabla\rho'||^2||u'||_{H^2}^2+||\rho'||_{H^2}^2||\mathrm{div}u'||^2).\end{aligned}\tag{2.18}$$

Taking ∇ to Equation (2.6), multiplying by $\nabla\rho$ and integrating it over Ω by parts, we find that

$$\frac{1}{2}\frac{d}{dt}||\nabla\rho||^2+\langle\nabla\mathrm{div}u,\nabla\rho\rangle+\epsilon||\triangle\rho||^2=\tau\int-\nabla\mathrm{div}(\rho'u')\nabla\rho dx$$

$$\leq \epsilon' ||\nabla\rho||^2 + C_{\epsilon'} ||\rho'||_{H^2}^2 ||u'||_{H^2}^2. \tag{2.19}$$

Now, multiplying (2.8) by $-\triangle\theta$ and integrating it over Ω by parts, we have

$$\frac{3}{4}\frac{d}{dt}||\nabla\theta||^2 + ||\nabla\theta||^2 + \langle\nabla \mathrm{div} u, \nabla\theta\rangle + \frac{1}{2}||\triangle\theta||^2 \leq C||\tau G_1(\rho', u', \theta')||^2, \tag{2.20}$$

where $G_1(\rho', u', \theta') = -\frac{3}{2}u' \cdot \nabla\theta' - \theta' \mathrm{div} u' + \kappa|u'|^2 - \frac{\rho'}{1+\rho'}\triangle\theta'$.

Combining the above inequalities (2.18)-(2.20), we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\mathrm{div} u||^2 + ||\nabla\rho||^2 + ||\rho||^2 + \frac{3}{2}||\nabla\theta||^2) + \frac{1}{2}||\mathrm{div} u||^2 + ||\nabla\theta||^2 + \frac{1}{2}||\triangle\theta||^2 + \epsilon(||\nabla\rho||^2 \\
&\quad + ||\triangle\rho||^2 + ||\nabla \mathrm{div} u||^2) \\
&\leq C||\nabla u'||_{H^2}||\nabla u||^2 + C||\mathrm{div} f||^2 + C(||\theta'||_{H^2}^2 + ||\rho'||_{H^2}^2)||\triangle\rho'||^2 + C||\rho'||_{H^2}^2||\triangle\theta'||^2 \\
&\quad + C(||\theta'||_{H^2}^2 + ||\rho'||_{H^2}^2)||\nabla\rho'||_{H^2}^2||\nabla\rho'||^2 + C(||\nabla\theta'||_{H^2}^2 + ||\nabla\rho'||_{H^2}^2)||\nabla\rho'||^2 + \epsilon'||\rho||_{H^1}^2 \\
&\quad + C_{\epsilon'}||\rho'||_{H^2}^2||u'||_{H^2}^2 + C||u'||_{H^2}^2||\nabla\theta'||^2 + C||\theta'||_{H^2}^2||\mathrm{div} u'||^2 + C||u'||_4^4.
\end{aligned} \tag{2.21}$$

On the other hand, applying curl to Equation (2.7), multiplying by curl$u$ and integrating it over Ω by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\mathrm{curl} u||^2 + ||\mathrm{curl} u||^2 + \tau\int \mathrm{curl}(u' \cdot \nabla u) \cdot \mathrm{curl} u dx + \epsilon||\nabla \mathrm{curl} u||^2 \\
=&\tau\int \mathrm{curl}\Big(\frac{\rho' - \theta'}{1+\rho'}\nabla\rho'\Big) \cdot \mathrm{curl} u dx + \tau\int \mathrm{curl} f \mathrm{curl} u dx.
\end{aligned}$$

Focusing on one component of the term $\tau\int \mathrm{curl}(u' \cdot \nabla u) \cdot \mathrm{curl} u dx$, we obtain that

$$\begin{aligned}
&\int (u_i'\partial_i\partial_2 u_3(\partial_2 u_3 - \partial_3 u_2) - u_i'\partial_i\partial_3 u_2(\partial_2 u_3 - \partial_3 u_2))dx \\
\leq&\frac{1}{2}\int u_i\partial_i[(\partial_2 u_3)^2 + (\partial_3 u_2)^2] + \int \partial_i u_i \partial_3 u_2 \partial_2 u_3 \leq C||\nabla u'||_\infty ||\nabla u||^2.
\end{aligned}$$

The argument for other components is similar. Hence, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\mathrm{curl} u||^2 + \frac{1}{2}||\mathrm{curl} u||^2 + \epsilon||\nabla \mathrm{curl} u||^2 \\
&\leq C\Big(||\nabla u'||_\infty ||\nabla u||^2 + \Big\|\mathrm{curl}\big(\frac{\rho' - \theta'}{1+\rho'}\nabla\rho'\big)\Big\|^2 + ||\mathrm{curl} f||^2\Big) \\
&\leq C\big(||\nabla u'||_{H^2}||\nabla u||^2 + (||\theta'||_{H^2}^2 + ||\rho'||_{H^2}^2)||\triangle\rho'||^2 + (||\theta'||_{H^2}^2 + ||\rho'||_{H^2}^2)||\nabla\rho'||_{H^2}^2||\nabla\rho'||^2 \\
&\quad + (||\nabla\theta'||_{H^2}^2 + ||\nabla\rho'||_{H^2}^2)||\nabla\rho'||^2 + ||\mathrm{curl} f||^2\big).
\end{aligned}$$

Combining with (2.21) implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\mathrm{div} u||^2 + ||\nabla\rho||^2 + ||\rho||^2 + \frac{3}{2}||\nabla\theta||^2 + ||\mathrm{curl} u||^2) + \frac{1}{2}||\mathrm{div} u||^2 + \frac{1}{2}||\mathrm{curl} u||^2 \\
&\quad + ||\nabla\theta||^2 + \frac{1}{2}||\triangle\theta||^2 + \epsilon(||\nabla\rho||^2 + ||\triangle\rho||^2 + ||\nabla \mathrm{div} u||^2 + ||\nabla \mathrm{curl} u||^2) \\
&\leq C||\nabla u'||_{H^2}||\nabla u||^2 + C||\mathrm{div} f||^2 + C||\mathrm{curl} f||^2 + C(||\theta'||_{H^2}^2 + ||\rho'||_{H^2}^2)||\triangle\rho'||^2 \\
&\quad + C(||\theta'||_{H^2}^2 + ||\rho'||_{H^2}^2)||\nabla\rho'||_{H^2}^2||\nabla\rho'||^2 + C(||\nabla\theta'||_{H^2}^2 + ||\nabla\rho'||_{H^2}^2)||\nabla\rho'||^2 \\
&\quad + C\big(||u'||_{H^2}^2||\nabla\theta'||^2 + ||\theta'||_{H^2}^2||\mathrm{div} u'||^2 + ||u'||_4^4 + ||\rho'||_{H^2}^2||\triangle\theta'||^2\big)
\end{aligned}$$

$$+C_{\epsilon'}||\rho'||^2_{H^2}||u'||^2_{H^2}+\epsilon'||\rho||^2_{H^1},$$

which completes the proof. □

LEMMA 2.3. *There exists a large enough constant $M_2\gg M_1>0$ such that*

$$\begin{aligned}&\frac{d}{dt}\Big((1+M_2)||\rho||^2+M_2||\nabla\rho||^2+||u||^2+M_2||\mathrm{div}u||^2+M_2||\mathrm{curl}u||^2+\frac{3}{2}||\theta||^2\\&+\frac{3M_2}{2}||\nabla\theta||^2-M_1\langle\mathrm{div}u,\rho\rangle\Big)+\epsilon((1+\frac{M_2}{2})||\nabla\rho||^2+\frac{M_2}{2}||\triangle\rho||^2+||\nabla u||^2\\&+\frac{M_2}{2}||\nabla\mathrm{div}u||^2+\frac{M_2}{2}||\nabla\mathrm{curl}u||^2)+\frac{1}{4}||u||^2+\frac{M_2}{4}(||\mathrm{div}u||^2+||\mathrm{curl}u||^2)\\&+\frac{M_1}{4}||\rho||^2_{H^1}+\frac{1}{2}||\theta||^2_{H^1}+\frac{M_2}{2}(||\nabla\theta||^2+||\triangle\theta||^2)\\\le&C\Big(||f||^2_{H^1}+(||\theta'||^2_{H^3}+||\rho'||^2_{H^3})||\rho'||^2_{H^2}+(||\theta'||^2_{H^2}+||\rho'||^2_{H^2})||\rho'||^2_{H^3}||\rho'||^2_{H^1}\\&+||\rho'||^2_{H^2}||u'||^2_{H^2}+||u'||^2_{H^2}||\theta'||^2_{H^2}+||u'||^4_{H^1}\Big).\end{aligned}$$

Proof. We will multiply inequality (2.17) by $M_2>0$ large enough to absorb the terms $||\nabla\theta||^2,||\mathrm{div}u||^2,\epsilon||\triangle\rho||^2,\epsilon||\triangle u||^2$ on the right-hand side of (2.11). Then, choosing $\epsilon'=\frac{M_1}{2M_2}$, we have

$$\begin{aligned}&\frac{d}{dt}\Big((1+M_2)||\rho||^2+M_2||\nabla\rho||^2+||u||^2+M_2||\mathrm{div}u||^2+M_2||\mathrm{curl}u||^2+\frac{3}{2}||\theta||^2\\&+\frac{3M_2}{2}||\nabla\theta||^2-M_1\langle\mathrm{div}u,\rho\rangle\Big)+\epsilon((1+\frac{M_2}{2})||\nabla\rho||^2+\frac{M_2}{2}||\triangle\rho||^2+||\nabla u||^2\\&+\frac{M_2}{2}||\nabla\mathrm{div}u||^2+\frac{M_2}{2}||\nabla\mathrm{curl}u||^2)+\frac{1}{2}||u||^2+\frac{M_2}{2}(||\mathrm{div}u||^2+||\mathrm{curl}u||^2)+\frac{M_1}{4}||\rho||^2_{H^1}\\&+\frac{1}{2}||\theta||^2_{H^1}+\frac{M_2}{2}(||\nabla\theta||^2+||\triangle\theta||^2)\\\le&C\Big(||f||^2_{H^1}+(||\theta'||^2_{H^3}+||\rho'||^2_{H^3})||\rho'||^2_{H^2}+(||\theta'||^2_{H^2}+||\rho'||^2_{H^2})||\rho'||^2_{H^3}||\rho'||^2_{H^1}\\&+||\rho'||^2_{H^2}||u'||^2_{H^2}+||u'||^2_{H^2}||\theta'||^2_{H^2}+||u'||^4_{H^1}+||\mathrm{div}u'||_{H^2}||u||^2+M_1||u'||^2_{H^2}||\nabla u||^2\\&+M_2||\nabla u'||_{H^2}||\nabla u||^2\Big).\end{aligned}$$

Note that

$$||\nabla u||\sim||\mathrm{div}u||+||\mathrm{curl}u||\text{ and }(\rho',u',\theta')\in X_\delta.$$

It completes the proof providing that $\delta>0$ small enough. □

Next, we will derive the energy estimate on higher order derivatives of ρ,u,θ.

LEMMA 2.4. *Let $k=1,\dots,m$, $m\ge2$. There exists a large enough constant $M_3>0$ such that*

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}||\nabla^{k+1}\rho||^2+\frac{3}{4}||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\mathrm{div}u||^2+\frac{1}{2}||\nabla^k\rho||^2+\frac{1}{2}||\nabla^k\mathrm{curl}u||^2\\&-\frac{1}{M_3}\langle\nabla^k\mathrm{div}u,\nabla^k\rho\rangle\Big)+\frac{\epsilon}{4}(||\nabla^k\triangle\rho||^2+||\nabla^{k+1}\rho||^2)+\frac{1}{4M_3}(||\nabla^{k+1}\rho||^2+||\nabla^k\rho||^2)\\&+\frac{1}{4}||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\triangle\theta||^2+\frac{1}{8}(||\nabla^k\mathrm{div}u||^2+||\nabla^k\mathrm{curl}u||^2)+\frac{\epsilon}{4}(||\nabla^{k+1}\mathrm{div}u||^2\end{aligned}$$

$$
\begin{aligned}
&+||\nabla^{k+1}\mathrm{curl}u||^2)\\
\le & C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2}+||\rho'||^2_{H^2}||u'||^2_{H^{m+1}})+C\big(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2}+||\theta'||^2_{H^2}||u'||^2_{H^{m+1}}\\
&+||\rho'||^2_{H^m}||\theta'||^2_{H^{m+2}}+||u'||^4_{H^m}+||\nabla^k\mathrm{div}f||^2+||\rho'||^2_{H^m}||u'||^2_{H^m}+||u'||^2_{H^{m+1}}||u||^2_{H^{m+1}}\big)\\
&+C_\epsilon\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}+||\rho'||^2_{H^m}\big)||\nabla\rho'||^2_{H^m}+C||\nabla^k\mathrm{curl}f||^2+C||\nabla^kf||^2.
\end{aligned}
$$

Proof. Let $k=1,\ldots,m$, $m\ge 2$. Applying ∇^{k+1} to Equations (2.6) and (2.8), multiplying $\nabla^{k+1}\rho$ and $\nabla^{k+1}\theta$, and integrating it over Ω by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla^{k+1}\rho||^2+\langle\nabla^{k+1}\mathrm{div}u,\nabla^{k+1}\rho\rangle+\epsilon||\nabla^k\triangle\rho||^2=-\tau\int\nabla^{k+1}\mathrm{div}(\rho'u')\nabla^{k+1}\rho dx,\\
&\frac{3}{4}\frac{d}{dt}||\nabla^{k+1}\theta||^2+||\nabla^{k+1}\theta||^2+||\nabla^k\triangle\theta||^2+\langle\nabla^{k+1}\mathrm{div}u,\nabla^{k+1}\theta\rangle\\
=&-\frac{3\tau}{2}\int\nabla^{k+1}(\theta'\mathrm{div}u')\nabla^{k+1}\theta dx-\tau\int\nabla^{k+1}(u'\cdot\nabla\theta')\nabla^{k+1}\theta dx\\
&+\tau\int\kappa\nabla^{k+1}|u'|^2\nabla^{k+1}\theta dx-\int\tau\nabla^{k+1}\big(\frac{\rho'}{1+\rho'}\triangle\theta'\big)\nabla^{k+1}\theta dx.
\end{aligned}
$$

Then, from Proposition 3.6 in [24], there holds

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla^{k+1}\rho||^2+\langle\nabla^{k+1}\mathrm{div}u,\nabla^{k+1}\rho\rangle+\epsilon||\nabla^k\triangle\rho||^2\\
\le&\frac{\epsilon}{2}||\nabla^k\triangle\rho||^2+C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2}+||u'||^2_{H^{m+1}}||\rho'||^2_{H^2}),
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
&\frac{3}{4}\frac{d}{dt}||\nabla^{k+1}\theta||^2+||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\triangle\theta||^2+\langle\nabla^{k+1}\mathrm{div}u,\nabla^{k+1}\theta\rangle\\
\le&C||\rho'||^2_{H^m}||\triangle\theta'||^2_{H^m}+C(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2}+||\theta'||^2_{H^2}||u'||^2_{H^{m+1}})+C||u'||^4_{H^m}.
\end{aligned}
\tag{2.23}
$$

Now, multiplying Equation (2.7) by $\nabla^{2k+1}\mathrm{div}u$ and integrating it over Ω by parts, we get

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}||\nabla^k\mathrm{div}u||^2-||\nabla^k\mathrm{div}u||^2-\langle\nabla^kE,\nabla^{k+1}\mathrm{div}u\rangle+\langle\nabla^{k+1}\theta,\nabla^{k+1}\mathrm{div}u\rangle\\
&+\langle\nabla^{k+1}\rho,\nabla^{k+1}\mathrm{div}u\rangle+\tau\langle\nabla^k(u'\cdot\nabla u),\nabla^{k+1}\mathrm{div}u\rangle-\epsilon||\nabla^{k+1}\mathrm{div}u||^2\\
=&-\tau\langle\nabla^k(\frac{\theta'-\rho'}{1+\rho'}\nabla\rho'),\nabla^{k+1}\mathrm{div}u\rangle+\tau\langle\nabla^kf,\nabla^{k+1}\mathrm{div}u\rangle.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\langle\nabla^kE,\nabla^{k+1}\mathrm{div}u\rangle=&-\langle\nabla^k\rho,\nabla^k\mathrm{div}u\rangle=-\langle\nabla^k\rho,\nabla^k(-\rho_t+\epsilon\triangle\rho-\tau\mathrm{div}(\rho'u'))\rangle\\
=&\frac{1}{2}\frac{d}{dt}||\nabla^k\rho||^2+\epsilon||\nabla^{k+1}\rho||^2+\tau\langle\nabla^k\rho,\nabla^k\mathrm{div}(\rho'u')\rangle.
\end{aligned}
$$

For $m\ge 3$, we have

$$
\begin{aligned}
&\langle\nabla^k(u'\cdot\nabla u),\nabla^{k+1}\mathrm{div}u\rangle\\
=&\int u'\cdot\nabla^{k+1}u\nabla^{k+1}\mathrm{div}udx+\sum_{l=1}^{k}C_k^l\int\nabla^lu'\nabla^{k-l}\nabla u\nabla^{k+1}\mathrm{div}udx\\
\le&C||\nabla u'||_{H^2}||\nabla^{k+1}u||^2+C(||\nabla^2u'||_{H^2}||\nabla u||_{H^{m-1}}+||\nabla^2u'||_{H^{m-1}}||\nabla u||_{H^2})||\nabla^{k+1}u||
\end{aligned}
$$

$$+C(||\nabla u'||_{H^2}||\nabla^2 u||_{H^{m-1}}+||\nabla u'||_{H^{m-1}}||\nabla^2 u||_{H^2})||\nabla^{k+1}u||.$$

For $k=m=2$, we have

$$\begin{aligned}&\langle\nabla^2(u'\cdot\nabla u),\nabla^3\text{div}u\rangle\\ =&\int u'\cdot\nabla^3 u\nabla^3\text{div}u dx+C_2^1\int\nabla u'\nabla^2 u\nabla^3\text{div}u dx+C_2^2\int\nabla^2 u'\nabla u\nabla^3\text{div}u dx\\ \le& C||\nabla u'||_{H^2}||u||_{H^3}^2.\end{aligned}$$

Therefore, there exists a small constant $\epsilon'>0$ such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\nabla^k\text{div}u||^2+\frac{1}{2}||\nabla^k\text{div}u||^2+\frac{1}{2}\frac{d}{dt}||\nabla^k\rho||^2+\epsilon||\nabla^{k+1}\rho||^2+\frac{\epsilon}{2}||\nabla^{k+1}\text{div}u||^2\\ &-\langle\nabla^{k+1}\theta,\nabla^{k+1}\text{div}u\rangle-\langle\nabla^{k+1}\rho,\nabla^{k+1}\text{div}u\rangle\\ \le& C||u'||_{H^{m+1}}||u||_{H^{m+1}}^2+C||\nabla^k\text{div}f||^2+C_{\epsilon'}||\rho'||_{H^m}||u'||_{H^m}+\epsilon'||\nabla^{k+1}\rho||\\ &+C_\epsilon\big(||\theta'||_{H^m}^2+||\theta'||_{H^m}^2||\rho'||_{H^m}^2+||\rho'||_{H^m}^2\big)||\nabla\rho'||_{H^m}^2.\end{aligned}\tag{2.24}$$

Applying ∇^kcurl to Equation (2.7), multiplying by $\nabla^k\text{curl}u$ and integrating it over Ω by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\nabla^k\text{curl}u||^2+||\nabla^k\text{curl}u||^2+\epsilon||\nabla^{k+1}\text{curl}u||^2\\ =&-\tau\langle\nabla^k\text{curl}(u'\cdot\nabla u),\nabla^k\text{curl}u\rangle\\ &-\tau\langle\nabla^k\text{curl}(\frac{\theta'-\rho'}{1+\rho'}\nabla\rho'),\nabla^k\text{curl}u\rangle+\tau\langle\nabla^k\text{curl}f,\nabla^k\text{curl}u\rangle.\end{aligned}$$

We focus on the first component of the term $\langle\nabla^k\text{curl}(u'\cdot\nabla u),\nabla^k\text{curl}u\rangle$. The estimate of other components is obtained similarly. The first component is

$$\int\nabla^k(\partial_2(u_i'\partial_i u_3)-\partial_3(u_i'\partial_i u_2))\nabla^k(\partial_2 u_3-\partial_3 u_2)dx.$$

The hard term is

$$\begin{aligned}&\int(u_i'\partial_2\nabla^k\partial_i u_3-u_i'\partial_3\nabla^k\partial_i u_2)(\nabla^k\partial_2 u_3-\nabla^k\partial_3 u_2)dx\\ =&\int\frac{u_i'}{2}\partial_i(\nabla^k\partial_2 u_3)^2dx+\frac{u_i'}{2}\partial_i(\nabla^k\partial_3 u_2)^2dx-\int u_i'\partial_i\nabla^k\partial_2 u_3\nabla^k\partial_3 u_2 dx\\ &-\int u_i'\partial_i\nabla^k\partial_3 u_2\nabla^k\partial_2 u_3 dx.\end{aligned}$$

Note that

$$\begin{aligned}&-\int u_i'\partial_i\nabla^k\partial_2 u_3\nabla^k\partial_3 u_2 dx-\int u_i'\partial_i\nabla^k\partial_3 u_2\nabla^k\partial_2 u_3 dx\\ =&\int\partial_i(u_i'\nabla^k\partial_3 u_2)\nabla^k\partial_2 u_3 dx-\int u_i'\partial_i\nabla^k\partial_3 u_2\nabla^k\partial_2 u_3 dx=\int\partial_i u_i'\nabla^k\partial_3 u_2\nabla^k\partial_2 u_3 dx.\end{aligned}$$

Hence, there is no derivative of u at the $m+2$ order. With a similar argument as above, we have

$$\frac{1}{2}\frac{d}{dt}||\nabla^k\text{curl}u||^2+\frac{1}{2}||\nabla^k\text{curl}u||^2+\frac{\epsilon}{2}||\nabla^{k+1}\text{curl}u||^2\le C||u'||_{H^{m+1}}||u||_{H^{m+1}}^2$$

$$+C_\epsilon\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}+||\rho'||^2_{H^m}\big)||\nabla\rho'||^2_{H^m}+C||\nabla^k\mathrm{curl}f||^2. \tag{2.25}$$

It concludes from (2.22)-(2.25) that

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}||\nabla^{k+1}\rho||^2+\frac{3}{4}||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\mathrm{div}u||^2+\frac{1}{2}||\nabla^k\rho||^2+\frac{1}{2}||\nabla^k\mathrm{curl}u||^2\Big)\\&+\epsilon(||\nabla^k\triangle\rho||^2+||\nabla^{k+1}\rho||^2)+||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\triangle\theta||^2+\frac{1}{4}(||\nabla^k\mathrm{div}u||^2+||\nabla^k\mathrm{curl}u||^2)\\&+\frac{\epsilon}{2}(||\nabla^{k+1}\mathrm{div}u||^2+||\nabla^{k+1}\mathrm{curl}u||^2)\\\le&C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2}+||\rho'||^2_{H^2}||u'||^2_{H^{m+1}})+C(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2}+||\theta'||^2_{H^2}||u'||^2_{H^{m+1}})\\&+C\big(||\rho'||^2_{H^m}||\theta'||^2_{H^{m+2}}+||u'||^4_{H^m}+||\nabla^k\mathrm{div}f||^2+||u'||_{H^{m+1}}||u||^2_{H^{m+1}}+||\nabla^k\mathrm{curl}f||^2\big)\\&+C_\epsilon\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}+||\rho'||^2_{H^m}\big)||\nabla\rho'||^2_{H^m}+C_{\epsilon'}||\rho'||^2_{H^m}||u'||^2_{H^m}\\&+\epsilon'||\nabla^{k+1}\rho||^2.\end{aligned} \tag{2.26}$$

On the other hand, applying ∇^k to Equation (2.7), multiplying it by $\nabla^{k+1}\rho$ and integrating it over Ω, we have

$$\begin{aligned}&-\langle\nabla^k\mathrm{div}u_t,\nabla^k\rho\rangle+\langle\nabla^k u,\nabla^{k+1}\rho\rangle-\langle\nabla^k E,\nabla^{k+1}\rho\rangle-\epsilon\langle\nabla^k\triangle u,\nabla^{k+1}\rho\rangle\\&\quad+\langle\nabla^{k+1}\theta,\nabla^{k+1}\rho\rangle+\langle\nabla^{k+1}\rho,\nabla^{k+1}\rho\rangle+\tau\langle\nabla^k(u'\cdot\nabla u),\nabla^{k+1}\rho\rangle\\&=-\tau\langle\nabla^k\Big(\frac{\theta'-\rho'}{1+\rho'}\nabla\rho'\Big),\nabla^{k+1}\rho\rangle+\tau\langle\nabla^k f,\nabla^{k+1}\rho\rangle.\end{aligned}$$

Note that

$$\begin{aligned}&-\langle\nabla^k E,\nabla^{k+1}\rho\rangle=||\nabla^k\rho||^2,\\&||\nabla^k\Big(\frac{\theta'-\rho'}{1+\rho'}\nabla\rho'\Big)||^2\le C\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}+||\rho'||^2_{H^m}\big)||\nabla\rho'||^2_{H^m},\\&-\langle\nabla^k\mathrm{div}u_t,\nabla^k\rho\rangle=-\frac{d}{dt}\langle\nabla^k\mathrm{div}u,\nabla^k\rho\rangle+\langle\nabla^k\mathrm{div}u,\nabla^k\rho_t\rangle\\&\qquad=-\frac{d}{dt}\langle\nabla^k\mathrm{div}u,\nabla^k\rho\rangle+\langle\nabla^k\mathrm{div}u,\nabla^k(-\mathrm{div}u+\epsilon\triangle\rho-\tau\mathrm{div}(\rho'u'))\rangle,\end{aligned}$$

where we have used the representation of ρ_t in Equation (2.6).

Therefore, we obtain

$$\begin{aligned}&\frac{1}{2}||\nabla^{k+1}\rho||^2+\frac{1}{2}||\nabla^k\rho||^2-\frac{d}{dt}\langle\nabla^k\mathrm{div}u,\nabla^k\rho\rangle\\\le&C||\nabla^k\mathrm{div}u||^2+C||\nabla^{k+1}\theta||^2+C||u'||^2_{H^m}||\nabla u||^2_{H^m}+C\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}\\&+||\rho'||^2_{H^m}\big)||\nabla\rho'||^2_{H^m}+C||\nabla^k f||^2+C(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2}+||\rho||^2_{H^2}||u'||^2_{H^{m+1}})\\&+\epsilon||\nabla^k\triangle\rho||^2+\epsilon||\nabla^k\triangle u||^2.\end{aligned} \tag{2.27}$$

To absorb the terms $||\nabla^k\mathrm{div}u||^2,||\nabla^{k+1}\theta||^2,\epsilon||\nabla^k\triangle\rho||^2,\epsilon||\nabla^k\triangle u||^2$, we choose $M_3>0$ large enough and $\epsilon'=\frac{1}{4M_3}$. Multiplying (2.27) by $\frac{1}{M_3}$, it concludes from (2.26) that

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}||\nabla^{k+1}\rho||^2+\frac{3}{4}||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\mathrm{div}u||^2+\frac{1}{2}||\nabla^k\rho||^2+\frac{1}{2}||\nabla^k\mathrm{curl}u||^2\\&-\frac{1}{M_3}\langle\nabla^k\mathrm{div}u,\nabla^k\rho\rangle\Big)+\frac{\epsilon}{4}(||\nabla^k\triangle\rho||^2+||\nabla^{k+1}\rho||^2)+\frac{1}{4M_3}(||\nabla^{k+1}\rho||^2+||\nabla^k\rho||^2)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\triangle\theta||^2+\frac{1}{4}(||\nabla^k\mathrm{div}u||^2+||\nabla^k\mathrm{curl}u||^2)\\
&+\frac{\epsilon}{4}(||\nabla^{k+1}\mathrm{div}u||^2+||\nabla^{k+1}\mathrm{curl}u||^2)\\
\le& C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2}+||\rho'||^2_{H^2}||u'||^2_{H^{m+1}})+C\big(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2}+||\theta'||^2_{H^2}||u'||^2_{H^{m+1}}\\
&+||\rho'||^2_{H^m}||\theta'||^2_{H^{m+2}}+||u'||^4_{H^m}+||\nabla^k\mathrm{div}f||^2+||\rho'||^2_{H^m}||u'||^2_{H^m}+||u'||^2_{H^{m+1}}||u||^2_{H^{m+1}}\big)\\
&+C_\epsilon\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}+||\rho'||^2_{H^m}\big)||\nabla\rho'||^2_{H^m}+C||\nabla^k\mathrm{curl}f||^2+C||\nabla^kf||^2,
\end{aligned}
$$

which completes the proof. □

PROPOSITION 2.1. *Assume that $(\rho',u',\theta')\in X_\delta$ for $\delta>0$ small enough, (ρ,u,θ) is the solution of system (2.6)-(2.10) with initial data (ρ_0,u_0,θ_0). Then, there exists a constant $C_1>0$ independent of ϵ such that*

$$
\begin{aligned}
&||(\rho,u,\theta)||^2_{H^{m+1}}\\
\le& Ce^{-C_1t}||(\rho_0,u_0,\theta_0)||^2_{H^{m+1}}+\int_0^t e^{C_1(s-t)}\big(C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2}+||\rho'||^2_{H^2}||u'||^2_{H^{m+1}})\\
&+C(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2}+||\theta'||^2_{H^2}||u'||^2_{H^{m+1}}+||\rho'||^2_{H^m}||\theta'||^2_{H^{m+2}}+||u'||^4_{H^m}+||f||^2_{H^{m+1}})\\
&+C_\epsilon\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}+||\rho'||^2_{H^m}\big)||\rho'||^2_{H^{m+1}}+C||\rho'||^2_{H^m}||u'||^2_{H^m}\big)ds.
\end{aligned}
$$

Proof. From Lemma 2.4, summing up with k, we obtain that

$$
\begin{aligned}
&\sum_{k=1}^m\frac{d}{dt}\Big(\frac{1}{2}||\nabla^{k+1}\rho||^2+\frac{3}{4}||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\mathrm{div}u||^2+\frac{1}{2}||\nabla^k\rho||^2+\frac{1}{2}||\nabla^k\mathrm{curl}u||^2\\
&-\frac{1}{M_3}\langle\nabla^k\mathrm{div}u,\nabla^k\rho\rangle\Big)+\sum_{k=1}^m\frac{\epsilon}{4}(||\nabla^k\triangle\rho||^2+||\nabla^{k+1}\rho||^2)+\sum_{k=1}^m\Big(\frac{1}{4M_3}(||\nabla^{k+1}\rho||^2\\
&+||\nabla^k\rho||^2)+\frac{1}{4}(||\nabla^{k+1}\theta||^2+||\nabla^k\triangle\theta||^2)+\frac{1}{8}(||\nabla^k\mathrm{div}u||^2+||\nabla^k\mathrm{curl}u||^2)\\
&+\frac{\epsilon}{4}(||\nabla^{k+1}\mathrm{div}u||^2+||\nabla^{k+1}\mathrm{curl}u||^2)\\
\le& C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2}+||\rho'||^2_{H^2}||u'||^2_{H^{m+1}})+C(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2}+||\theta'||^2_{H^2}||u'||^2_{H^{m+1}})\\
&+C||\rho'||^2_{H^m}||\theta'||^2_{H^{m+2}}+C||u'||^4_{H^m}+C_\epsilon\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}+||\rho'||^2_{H^m}\big)||\nabla\rho'||^2_{H^m}\\
&+C||f||^2_{H^{m+1}}+C||\rho'||^2_{H^m}||u'||^2_{H^m},
\end{aligned}
$$

provided $\delta>0$ small enough.

Then, from Lemma 2.3, there exists a constant $C_1>0$ independent of ϵ such that

$$
\begin{aligned}
&\frac{d}{dt}\Big((1+M_2)||\rho||^2+M_2||\nabla\rho||^2+||u||^2+M_2||\mathrm{div}u||^2+M_2||\mathrm{curl}u||^2+\frac{3}{2}||\theta||^2\\
&+\frac{3M_2}{2}||\nabla\theta||^2-M_1\langle\mathrm{div}u,\rho\rangle\Big)+\epsilon((1+\frac{M_2}{2})||\nabla\rho||^2+\frac{M_2}{2}||\triangle\rho||^2+||\nabla u||^2\\
&+\frac{M_2}{2}||\nabla\mathrm{div}u||^2+\frac{M_2}{2}||\nabla\mathrm{curl}u||^2)+C_1\Big((1+M_2)||\rho||^2+M_2||\nabla\rho||^2+||u||^2\\
&+M_2||\mathrm{div}u||^2+M_2||\mathrm{curl}u||^2+\frac{3}{2}||\theta||^2+\frac{3M_2}{2}||\nabla\theta||^2-M_1\langle\mathrm{div}u,\rho\rangle\Big)+\frac{M_2}{2}||\triangle\theta||^2\\
\le& C\Big(||f||^2_{H^1}+(||\theta'||^2_{H^3}+||\rho'||^2_{H^3})||\rho'||^2_{H^2}+(||\theta'||^2_{H^2}+||\rho'||^2_{H^2})||\rho'||^2_{H^3}||\rho'||^2_{H^1}\\
&+||\rho'||^2_{H^2}||u'||^2_{H^2}+||u'||^2_{H^2}||\theta'||^2_{H^2}+||u'||^4_{H^1}\Big),
\end{aligned}
$$

and

$$\begin{aligned}&\sum_{k=1}^{m}\frac{d}{dt}\Big(||\nabla^{k+1}\rho||^2+\frac{3}{2}||\nabla^{k+1}\theta||^2+||\nabla^{k}\mathrm{div}u||^2+||\nabla^{k}\rho||^2+||\nabla^{k}\mathrm{curl}u||^2\\&-\frac{1}{M_3}\langle\nabla^{k}\mathrm{div}u,\nabla^{k}\rho\rangle\Big)+\frac{1}{8}\sum_{k=1}^{m}(\epsilon(||\nabla^{k}\triangle\rho||^2+||\nabla^{k+1}\rho||^2)+||\nabla^{k}\triangle\theta||^2+\epsilon(||\nabla^{k+1}\mathrm{div}u||^2\\&+||\nabla^{k+1}\mathrm{curl}u||^2))+C_1\sum_{k=1}^{m}\Big(||\nabla^{k+1}\rho||^2+\frac{3}{2}||\nabla^{k+1}\theta||^2+||\nabla^{k}\mathrm{div}u||^2+||\nabla^{k}\rho||^2\\&+||\nabla^{k}\mathrm{curl}u||^2-\frac{1}{M_3}\langle\nabla^{k}\mathrm{div}u,\nabla^{k}\rho\rangle\Big)\\\le&C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2}+||\rho'||^2_{H^2}||u'||^2_{H^{m+1}})+C(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2}+||\theta'||^2_{H^2}||u'||^2_{H^{m+1}})\\&+C||\rho'||^2_{H^m}||\theta'||^2_{H^{m+2}}+C||u'||^4_{H^m}+C_\epsilon\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}+||\rho'||^2_{H^m}\big)||\rho'||^2_{H^{m+1}}\\&+C||\rho'||^2_{H^m}||u'||^2_{H^m}+C||f||^2_{H^{m+1}}.\end{aligned}$$

Combining the above inequalities, multiplying by e^{C_1t} and integrating from 0 to t, there holds

$$\begin{aligned}&e^{C_1s}\big(||\rho||^2_{H^{m+1}}+||u||^2+||\mathrm{div}u||^2_{H^m}+||\mathrm{curl}u||^2_{H^m}+||\theta||^2_{H^{m+1}}\big)\big|_0^t\\\le&\int_0^t e^{C_1s}\big(C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2}+||\rho'||^2_{H^2}||u'||^2_{H^{m+1}})+C(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2}\\&+||\theta'||^2_{H^2}||u'||^2_{H^{m+1}}+||\rho'||^2_{H^m}||\theta'||^2_{H^{m+2}}+||u'||^4_{H^m})+C_\epsilon\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}\\&+||\rho'||^2_{H^m}\big)||\rho'||^2_{H^{m+1}}+C||\rho'||^2_{H^m}||u'||^2_{H^m}+C||f||^2_{H^{m+1}}\big)ds.\end{aligned}$$

Hence, we get

$$\begin{aligned}&||(\rho,u,\theta)||^2_{H^{m+1}}\\\le&Ce^{-C_1t}||(\rho_0,u_0,\theta_0)||^2_{H^{m+1}}+\int_0^t e^{C_1(s-t)}\big(C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2}+||\rho'||^2_{H^2}||u'||^2_{H^{m+1}})\\&+C(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2}+||\theta'||^2_{H^2}||u'||^2_{H^{m+1}}+||\rho'||^2_{H^m}||\theta'||^2_{H^{m+2}}+||u'||^4_{H^m}+||f||^2_{H^{m+1}})\\&+C_\epsilon\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}+||\rho'||^2_{H^m}\big)||\rho'||^2_{H^{m+1}}+C||\rho'||^2_{H^m}||u'||^2_{H^m}\big)ds,\end{aligned}$$

which completes the proof. □

3. Existence

3.1. Existence and uniqueness of time periodic solutions to the linearized system.

LEMMA 3.1. *Assume that $m\ge2$, and the $T-$periodic external force $f\in L^2(0,T;H^{m+1}(\Omega))$ satisfies*

$$\int_0^T||f||^2_{H^{m+1}}dt<\eta,$$

for some small constant $\eta>0$. Let $\delta>0$ be an appropriately small constant. Then, for any $(\rho',u',\theta')\in X_\delta$, $\tau\in[0,1]$, linearized system (2.6)-(2.10) admits a unique $T-$periodic solution (ρ,u,θ) such that

$$(\rho,u,\theta)\in X.$$

Proof. We rewrite the linearized system in vector sense:

$$U_t = \mathcal{A}U + G(W) + F,$$

where

$$\mathcal{A}U = \begin{pmatrix} -\lambda_1 v'\nabla\sigma - \gamma \text{div} v \\ -\gamma\nabla\sigma + \bar{\mu}\triangle v + \bar{\nu}\nabla\text{div} v \frac{\lambda_2}{\bar{\rho}}(\nabla\times M)\times\bar{B} \\ \lambda_1\nabla\times(v\times\bar{B}) + \triangle M - \frac{1}{\bar{\rho}}\nabla\times\big((\nabla\times M)\times\bar{B}\big) \end{pmatrix},$$

and

$$U = (\sigma, v, M), W = (\sigma', v', M'), F = (0, \lambda_2 f, 0),$$

$$G(W) = (-\lambda_1\sigma'\text{div}v', G_1(\sigma', v', M'), G_2(\sigma', v', M')).$$

First, the linearized system with trivial initial data is global well-posed by the classical parabolic equations theory. Let $\overline{U}(x,t)$ be the solution of linearized system (2.6)-(2.10) with initial data $(0,0,0)$. Then, for any $t \geq 0$, there exists a non-negative integer n such that $t \in [nT, (n+1)T)$. Notice that $(\rho', u', \theta') \in X_\delta$ is $T-$periodic. From Proposition 2.1, we have

$$\begin{aligned}
&||\overline{U}(x,t)||^2_{H^{m+1}} \\
\leq &\int_0^t e^{C_1(s-t)}\Big(C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2} + ||\rho'||^2_{H^2}||u'||^2_{H^{m+1}}) + C(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2}) \\
&+ C||\rho'||^2_{H^m}||\theta'||^2_{H^{m+2}} + C||u'||^4_{H^m} + C_\epsilon\big(||\theta'||^2_{H^m} + ||\theta'||^2_{H^m}||\rho'||^2_{H^m} + ||\rho'||^2_{H^m}\big)||\rho'||^2_{H^{m+1}} \\
&+ C||\rho'||^2_{H^m}||u'||^2_{H^m} + C||f||^2_{H^{m+1}} + C||\theta'||^2_{H^2}||u'||^2_{H^{m+1}}\Big)ds \\
\leq &\int_0^T e^{C_1(s-t)}\Big(C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2} + ||\rho'||^2_{H^2}||u'||^2_{H^{m+1}}) + C(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2} \\
&+ ||\theta'||^2_{H^2}||u'||^2_{H^{m+1}}) + C||\rho'||^2_{H^m}||\theta'||^2_{H^{m+2}} + C||u'||^4_{H^m} + C_\epsilon\big(||\theta'||^2_{H^m} + ||\theta'||^2_{H^m}||\rho'||^2_{H^m} \\
&+ ||\rho'||^2_{H^m}\big)||\rho'||^2_{H^{m+1}} + C||\rho'||^2_{H^m}||u'||^2_{H^m} + C||f||^2_{H^{m+1}}\Big)ds \\
&+ \sum_{i=0}^{n-1} e^{C_1((i+1)T-t)}\int_0^T \Big(C_\epsilon(||\rho'||^2_{H^{m+1}}||u'||^2_{H^2} + ||\rho'||^2_{H^2}||u'||^2_{H^{m+1}}) + C||f||^2_{H^{m+1}} \\
&+ C(||\theta'||^2_{H^{m+1}}||u'||^2_{H^2} + ||\theta'||^2_{H^2}||u'||^2_{H^{m+1}}) + C||\rho'||^2_{H^m}||\theta'||^2_{H^{m+2}} + C||u'||^4_{H^m} \\
&+ C_\epsilon\big(||\theta'||^2_{H^m} + ||\theta'||^2_{H^m}||\rho'||^2_{H^m} + ||\rho'||^2_{H^m}\big)||\rho'||^2_{H^{m+1}} + C||\rho'||^2_{H^m}||u'||^2_{H^m}\Big)ds \\
\leq &C_\epsilon(\sup_{t\in[0,T]}||u'||^2_{H^2}\int_0^T||\rho'||^2_{H^{m+1}}ds + \sup_{t\in[0,T]}||\rho'||^2_{H^2}\int_0^T||u'||^2_{H^{m+1}}ds) \\
&+ C(\sup_{t\in[0,T]}||u'||^2_{H^2}\int_0^T||\theta'||^2_{H^{m+1}}ds + \sup_{t\in[0,T]}||\theta'||^2_{H^2}\int_0^T||u'||^2_{H^{m+1}}ds) \\
&+ C\sup_{t\in[0,T]}||\rho'||_{H^m}\int_0^T||\theta'||^2_{H^{m+2}}ds + C\sup_{t\in[0,T]}||u'||^4_{H^m} + C_\epsilon\sup_{t\in[0,T]}\big(||\theta'||^2_{H^m} \\
&+ ||\theta'||^2_{H^m}||\rho'||^2_{H^m} + ||\rho'||^2_{H^m}\big)\int_0^T||\rho'||^2_{H^{m+1}}ds + C\sup_{t\in[0,T]}||\rho'||^2_{H^m}\sup_{t\in[0,T]}||u'||^2_{H^m}
\end{aligned}$$

$$+C\int_0^T ||f||^2_{H^{m+1}}ds,$$

where we have used the fact

$$\sum_{i=0}^{n-1} e^{-C_1(t-(i+1)T)} \le e^{-C_1 t}\sum_{i=0}^{n-1} e^{C_1(i+1)T} \le e^{-C_1 t}e^{C_1 nT}\frac{1}{1-e^{-C_1 T}} \le \frac{1}{1-e^{-C_1 T}}.$$

Consequently,

$$||\overline{U}||_{H^{m+1}} \le C_\epsilon \delta^4 + C\eta \le C_\epsilon \delta^2,$$

for $\delta = \eta^{\frac{1}{4}}$ and η is suitably small.

Define

$$S_1 := \{\overline{U}(kT), k=0,1,\ldots\}.$$

It is obvious that S_1 is nonempty and bounded in $H^{m+1}(\Omega)$. Therefore, $\overline{Co}S_1$ is compact in $H^m(\Omega)$.

Define

$$P(\phi) := U(T,\phi), \text{ for } \phi \in \overline{Co}S_1,$$

where $U(T,\phi)$ is the solution of linearized system (2.6)-(2.10) with the initial data ϕ at time T. From the uniqueness of Cauchy problem, we can see $P(x)\in S_1$, for $x\in S_1$. Then, we assert that the map

$$P: \overline{Co}S_1 \mapsto \overline{Co}S_1$$

is continuous.

In fact, for any given $y \in Co\{\overline{U}(kT), k=0,1,\ldots\}$, there exist $\theta\in[0,1]$ and $x_1, x_2 \in \{\overline{U}(kT), k=0,1,\ldots\}$ such that $y=\theta x_1 + (1-\theta)x_2$. Hence,

$$\begin{aligned}
P(y) &= P(\theta x_1 + (1-\theta)x_2) = e^{-\mathcal{A}T}(\theta x_1 + (1-\theta)x_2) + \int_0^T e^{\mathcal{A}(\tau - T)}(G(W)+F)d\tau \\
&= \theta(e^{-\mathcal{A}T}x_1 + \int_0^T e^{\mathcal{A}(\tau-T)}(G(W)+F)d\tau) + (1-\theta)(e^{-\mathcal{A}T}x_2 + \int_0^T e^{\mathcal{A}(\tau-T)}(G(W)+F)d\tau) \\
&= \theta P(x_1) + (1-\theta)P(x_2).
\end{aligned}$$

The continuity of P is from the continuous dependence of initial data.

Therefore, from Tychonoff fixed point theorem [25], there exists $U^*\in\overline{Co}S_1$ such that $P(U^*)=U^*$. That is, there exists a $T-$periodic solution $U(t,U^*)$ of linearized system (2.6)-(2.10). Since $||U^*||_{H^{m+1}} \le C_\epsilon\delta^2$, there holds

$$\begin{aligned}
\sup_{t\in[0,T]} ||U(t,U^*)||^2_{H^{m+1}} &\le C||U^*||^2_{H^{m+1}} + C_\epsilon(\sup_{t\in[0,T]}||u'||^2_{H^2}\int_0^T ||\rho'||^2_{H^{m+1}}ds \\
&+ \sup_{t\in[0,T]}||\rho'||^2_{H^2}\int_0^T ||u'||^2_{H^{m+1}}ds) + C(\sup_{t\in[0,T]}||u'||^2_{H^2}\int_0^T||\theta'||^2_{H^{m+1}}ds \\
&+ \sup_{t\in[0,T]}||\theta'||^2_{H^2}\int_0^T||u'||^2_{H^{m+1}}ds) + C\sup_{t\in[0,T]}||\rho'||_{H^m}\int_0^T||\theta'||^2_{H^{m+2}}ds + C\sup_{t\in[0,T]}||u'||^4_{H^m}
\end{aligned}$$

$$+C\sup_{t\in[0,T]}||\rho'||^2_{H^m}\sup_{t\in[0,T]}||u'||^2_{H^m}+C_\epsilon\sup_{t\in[0,T]}\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}$$
$$+||\rho'||^2_{H^m}\big)\int_0^T||\rho'||^2_{H^{m+1}}ds+C\int_0^T||f||^2_{H^{m+1}}ds\le C_\epsilon\delta^2. \tag{3.1}$$

On the other hand, note that $U(t,U^*):=(\rho,u,\theta)$ is a $T-$periodic solution of linearized system (2.6)-(2.10). From Lemma 2.3 and 2.4, integrating from 0 to T, it yields

$$\epsilon\Big(\int_0^T||\rho||^2_{H^{m+2}}dt+\int_0^T||u||^2_{H^{m+2}}dt\Big)+\int_0^T||\theta||^2_{H^{m+2}}dt+\int_0^T||\rho||^2_{H^{m+1}}dt+\int_0^T||u||^2_{H^{m+1}}dt$$
$$\le C_\epsilon(\sup_{t\in[0,T]}||u'||^2_{H^2}\int_0^T||\rho'||^2_{H^{m+1}}ds+\sup_{t\in[0,T]}||\rho'||^2_{H^2}\int_0^T||u'||^2_{H^{m+1}}ds)$$
$$+C(\sup_{t\in[0,T]}||u'||^2_{H^2}\int_0^T||\theta'||^2_{H^{m+1}}ds+\sup_t||\theta'||^2_{H^2}\int_0^T||u'||^2_{H^{m+1}}ds)$$
$$+C\sup_{t\in[0,T]}||\rho'||_{H^m}\int_0^T||\theta'||^2_{H^{m+2}}ds+C\sup_{t\in[0,T]}||u'||^4_{H^m}+C\sup_{t\in[0,T]}||\rho'||^2_{H^m}\sup_{t\in[0,T]}||u'||^2_{H^m}$$
$$+C_\epsilon\sup_{t\in[0,T]}\big(||\theta'||^2_{H^m}+||\theta'||^2_{H^m}||\rho'||^2_{H^m}+||\rho'||^2_{H^m}\big)\int_0^T||\rho'||^2_{H^{m+1}}ds+C\int_0^T||f||^2_{H^{m+1}}ds$$
$$\le C_\epsilon\delta^2. \tag{3.2}$$

We are going to prove the uniqueness of time periodic solutions for linearized system (2.6)-(2.10). Assume that U_1 and U_2 are two solutions of linearized system (2.6)-(2.10). Denote $U_1-U_2=(\underline{\rho},\underline{u},\underline{\theta})$. With a similar argument as above, we get

$$\int_0^T(||\underline{\theta}||^2_{H^{m+2}}+||\underline{\rho}||^2_{H^{m+1}}+||\underline{u}||^2_{H^{m+1}})dt=0.$$

Thus, we obtain that $\underline{\rho}=\underline{u}=\underline{\theta}=0$, i.e., $U_1=U_2$. □

3.2. Existence of time periodic solutions to the regularized system. Now, we introduce an operator:

$$\mathcal{L}:X_\delta\times[0,1]\to X,$$
$$\big((\rho',u',\theta'),\tau\big)\to(\rho,u,\theta).$$

From the argument in Section 4.1, the operator $\mathcal{L}$ is well defined. The following lemmas show that the operator $\mathcal{L}$ is completely continuous.

LEMMA 3.2. *The operator $\mathcal{L}$ is compact.*

Proof. Multiplying Equations (2.6)-(2.8) by ρ_t,u_t,θ_t, respectively, and integrating them over $[0,T]\times\Omega$ by parts, we have

$$\int_0^T\int|\rho_t|^2dxdt+\int_0^T\int\text{div}u\rho_tdxdt=-\tau\int_0^T\int\text{div}(\rho'u')\rho_tdxdt,$$
$$\int_0^T\int|u_t|^2dxdt-\int_0^T\int Eu_tdxdt+\int_0^T\int\nabla\theta u_tdxdt+\int_0^T\int\nabla\rho u_tdxdt$$
$$+\tau\int_0^T\int u'\cdot\nabla uu_tdxdt=\tau\int_0^T\int(\frac{\rho'-\theta'}{1+\rho'})\nabla\rho'u_tdxdt+\tau\int_0^T\int fu_tdxdt,$$

$$\int_0^T\int|\theta_t|^2dxdt+\int_0^T\int\text{div}u\theta_tdxdt=-\frac{3\tau}{2}\int_0^T u'\cdot\nabla\theta' u_tdxdt-\tau\int_0^T\int\theta'\text{div}u'\theta_tdxdt$$
$$-\tau\int_0^T\int\Big(\frac{\rho'}{1+\rho'}\triangle\theta'\Big)\theta_tdxdt+\tau\int_0^T\int\kappa|u'|^2\theta_tdxdt.$$

By using Hölder's, Sobolev's and Young's inequalities, we infer that

$$\begin{aligned}&\int_0^T\int[|\rho_t|^2+|u_t|^2+|\theta_t|^2]dxdt\\&\le C\sup_{t\in[0,T]}\Big(||\text{div}u||^2+||\text{div}(\rho'u')||^2+||E||^2+||\nabla\theta||^2+||\nabla\rho||^2+||u'\cdot\nabla u||^2\\&\quad+||\frac{\rho'-\theta'}{1+\rho'}\nabla\rho'||^2+||f||^2+||u'\cdot\nabla\theta'||^2+||\theta'\text{div}u'||^2+||\frac{\rho'}{1+\rho'}\triangle\theta'||^2+||u'||_4^4\Big)\\&\le C_\epsilon\delta^2.\end{aligned}$$

Let $\beta=1,\dots,m-1$. Applying ∇^β to Equations (2.6) and (2.8), multiplying by $\nabla^\beta\rho_t$, $\nabla^\beta\theta_t$, respectively, and integrating them over $[0,T]\times\Omega$ by parts, there holds

$$\int_0^T\int|\nabla^\beta\rho_t|^2dxdt+\int_0^T\int\nabla^\beta\text{div}u\nabla^\beta\rho_tdxdt=-\tau\int_0^T\int\nabla^\beta\text{div}(\rho'u')\nabla^\beta\rho_tdxdt,$$
$$\int_0^T\int|\nabla^\beta\theta_t|^2dxdt+\int_0^T\int\nabla^\beta\text{div}u\nabla^\beta\theta_tdxdt=-\tau\int_0^T\int\nabla^\beta(\theta'\text{div}u')\nabla^\beta\theta_tdxdt$$
$$-\frac{3\tau}{2}\int_0^T\int\nabla^\beta(u'\cdot\nabla\theta')\nabla^\beta\theta_tdxdt-\tau\int_0^T\int\nabla^\beta\Big(\frac{\rho'}{1+\rho'}\triangle\theta'\Big)\nabla^\beta\theta_tdxdt$$
$$+\tau\int_0^T\int\kappa\nabla^\beta|u'|^2\nabla^\beta\theta_tdxdt.$$

Therefore, from inequality (3.1), we have

$$\begin{aligned}\int_0^T\int|\nabla^\beta\rho_t|^2dxdt&\le C\int_0^T\int|\nabla^\beta\text{div}u|^2dxdt+C\int_0^T\int|\nabla^\beta\text{div}(\rho'u')|^2dxdt\\&\le C\sup_{t\in[0,T]}(||u||^2_{H^m}+||\rho'||^2_{H^m}||u'||^2_m)\le C_\epsilon\delta^2,\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}\int_0^T\int|\nabla^\beta\theta_t|^2dxdt&\le C\sup_{t\in[0,T]}(||u||^2_{H^m}+||\theta'||^2_{H^m}||u'||^2_m+||u'||^4_{H^m}+||\rho'||^2_{H^m}||\theta||^2_{H^{m+1}})\\&\le C_\epsilon\delta^2.\end{aligned}\tag{3.4}$$

Multiplying Equation (2.7) by $\nabla^{2\beta-1}\text{div}u_t$ and integrating it over $[0,T]\times\Omega$ by parts, we have

$$\int_0^T\int|\nabla^{\beta-1}\text{div}u_t|^2dxdt+\int_0^T\int\nabla^{\beta-1}\text{div}E\nabla^{\beta-1}\text{div}u_tdxdt$$
$$+\int_0^T\int\nabla^{\beta-1}\text{div}\nabla\theta\nabla^{\beta-1}\text{div}u_tdxdt+\int_0^T\int\nabla^{\beta-1}\text{div}\nabla\rho\nabla^{\beta-1}\text{div}u_tdxdt$$
$$-\tau\int_0^T\int\nabla^{\beta-1}\text{div}(u'\cdot\nabla u)\nabla^{\beta-1}\text{div}u_tdxdt$$

$$=-\tau\int_0^T\int\nabla^{\beta-1}\mathrm{div}\Big(\frac{\theta'-\rho'}{1+\rho'}\nabla\rho'\Big)\nabla^{\beta-1}\mathrm{div}u_t dxdt+\tau\int_0^T\int\nabla^{\beta-1}\mathrm{div}f\nabla^{\beta-1}\mathrm{div}u_t dxdt.$$

Hence, by using Hölder's, Sobolev's and Young's inequalities, we obtain

$$\begin{aligned}&\int_0^T\int|\nabla^{\beta-1}\mathrm{div}u_t|^2dxdt\\ \le& C\sup_{t\in[0,T]}\Big(||\theta||^2_{H^m}+||\rho||^2_{H^m}+||u'||^2_{H^m}||u||^2_{H^{m+1}}+(||\theta'||^2_{H^m}+||\rho'||^2_{H^m})||\rho'||^2_{H^m}+||f||^2_{H^m}\Big)\\ \le& C_\epsilon\delta^2.\end{aligned}\tag{3.5}$$

On the other hand, applying $\nabla^{\beta-1}$curl to Equation (2.7), multiplying by $\nabla^\beta\mathrm{curl}u_t$ and integrating it over $[0,T]\times\Omega$ by parts, we have

$$\begin{aligned}&\int_0^T\int|\nabla^{\beta-1}\mathrm{curl}u_t|^2dxdt-\tau\int_0^T\int\nabla^{\beta-1}\mathrm{curl}(u'\cdot\nabla u)\nabla^{\beta-1}\mathrm{curl}u_t dxdt\\ =&-\tau\int_0^T\int\nabla^{\beta-1}\mathrm{curl}\Big(\frac{\theta'-\rho'}{1+\rho'}\nabla\rho'\Big)\nabla^{\beta-1}\mathrm{curl}u_t dxdt+\tau\int_0^T\int\nabla^{\beta-1}\mathrm{curl}f\nabla^{\beta-1}\mathrm{curl}u_t dxdt.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_0^T\int|\nabla^{\beta-1}\mathrm{curl}u_t|^2dxdt\\ \le& C\sup_{t\in[0,T]}\Big(||u'||^2_{H^m}||u||^2_{H^{m+1}}+(||\theta'||^2_{H^m}+||\rho'||^2_{H^m})||\rho'||^2_{H^m}+||f||^2_{H^m}dt\Big)\\ \le& C_\epsilon\delta^2.\end{aligned}\tag{3.6}$$

Combining inequalities (3.1)-(3.6) with the strong compactness of L^p space, yields the result. ∎

LEMMA 3.3. *The operator $\mathcal{L}$ is continuous.*

Proof. Assume that $(\rho'_n,u'_n,\theta'_n)\in X_\delta,(\rho',u',\theta')\in X_\delta,\tau_n\in[0,1]$, and $\tau\in[0,1]$ satisfy

$$\lim_{n\to\infty}\tau_n=\tau,$$

and

$$\lim_{n\to\infty}\sup_{t\in[0,T]}||(\rho'_n-\rho',u'_n-u',\theta'_n-\theta')||^2_{H^{m+1}}+\int_0^T||\theta'_n-\theta'||^2_{H^{m+2}}dt=0.$$

Let $(\rho_n,u_n,\theta_n)=\mathcal{L}(\rho'_n,u'_n,\theta'_n),(\rho,u,\theta)=\mathcal{L}(\rho',u',\theta')$. Then $(\rho_n-\rho,u_n-u,\theta_n-\theta)$ is a time periodic solution of the following system:

$$\begin{aligned}&(\rho_n-\rho)_t+\mathrm{div}(u_n-u)-\epsilon\triangle(\rho_n-\rho)=(\tau-\tau_n)\mathrm{div}(\rho'u')-\tau_n\mathrm{div}\big((\rho'_n-\rho')u'+\rho'_n(u'_n-u')\big),\\ &(u_n-u)_t+(u_n-u)-(E_n-E)+\nabla(\theta_n-\theta)+\nabla(\rho_n-\rho)+(\tau_n-\tau)u'\cdot\nabla u+\tau_n\big((u'_n-u')\nabla u\\ &\quad+u'_n\cdot\nabla(u_n-u)\big)-\epsilon\triangle(u_n-u)\\ &=(\tau_n-\tau)\frac{\rho'-\theta'}{1+\rho'}\nabla\rho'+\tau_n\big(\frac{\rho'_n-\theta'_n}{1+\rho'_n}-\frac{\rho'-\theta'}{1+\rho'}\big)\nabla\rho'_n+\tau\big(\frac{\rho'-\theta'}{1+\rho'}\big)\nabla(\rho'_n-\rho'),\\ &\quad\frac{3}{2}(\theta_n-\theta)_t+(\theta_n-\theta)+\mathrm{div}(u_n-u)-\triangle(\theta_n-\theta)\end{aligned}$$

$$
\begin{aligned}
&=-\frac{3}{2}(\tau_n-\tau)u'\cdot\nabla\theta'-\frac{3\tau_n}{2}((u_n'-u')\nabla\theta'\\
&+u_n'\nabla(\theta_n'-\theta'))-(\tau_n-\tau)\theta'\mathrm{div}u'-\tau_n((\theta_n'-\theta')\mathrm{div}u'+\theta_n'\mathrm{div}(u_n'-u'))+(\tau_n-\tau)\kappa|u'|^2\\
&+\tau_n\kappa(|u_n'|^2-|u'|^2)+(\tau-\tau_n)\frac{\rho'}{1+\rho'}\triangle\theta'+\tau_n(\frac{\rho'}{1+\rho'}\triangle(\theta'-\theta_n')-(\frac{\rho'}{1+\rho'}-\frac{\rho_n'}{1+\rho_n'})\triangle\theta_n'),\\
&E_n-E=\nabla(\phi_n-\phi),\\
&\triangle(\phi_n-\phi)=\rho_n-\rho.
\end{aligned}
$$

Similar to the argument in Section 3, we obtain that

$$
\lim_{n\to\infty}\sup_{t\in[0,T]}||(\rho_n-\rho,u_n-u,\theta_n-\theta)||^2_{H^{m+1}}+\int_0^T||\theta_n-\theta||^2_{H^{m+2}}dt=0.
$$

Therefore, the continuity of the operator $\mathcal{L}$ is proved. ∎

3.3. Approximated solution. In this part, we will prove the existence of time periodic solutions to system (2.1)-(2.5) by Leray-Schauder degree theory. For applying the property of Leray-Schauder degree, some high order energy estimates of ρ,u,θ will be derived.

LEMMA 3.4. *Assume that $m\geq 2,$, $k=1,\dots,m$, $|\rho|\leq\frac{1}{2}$, $|\theta|<\frac{1}{2}$. Let (ρ,u,θ) be the solution of system (2.1)-(2.5). Then*

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}||\nabla^{k+1}\rho||^2+\frac{3}{4}||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\mathrm{div}u||^2+\frac{1}{2}||\nabla^k\rho||^2+\frac{1}{2}||\nabla^k\mathrm{curl}u||^2\Big)\\
&-\frac{1}{M_3}\langle\nabla^k\mathrm{div}u,\nabla^k\rho\rangle+\frac{\epsilon}{4}(||\nabla^k\triangle\rho||^2+||\nabla^{k+1}\rho||^2)+\frac{1}{2M_3}(||\nabla^{k+1}\rho||^2+||\nabla^k\rho||^2)\\
&+\frac{1}{2}||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\triangle\theta||^2+\frac{1}{2}(||\nabla^k\mathrm{div}u||^2+||\nabla^k\mathrm{curl}u||^2)+\frac{\epsilon}{2}(||\nabla^{k+1}\mathrm{div}u||^2\\
&+||\nabla^{k+1}\mathrm{curl}u||^2)\\
\leq&\frac{1}{2}\frac{d}{dt}\int\frac{\rho}{1+\tau\rho}|\nabla^{k+1}\rho|^2dx+C\epsilon||\rho||_{H^{m+1}}||\rho||^2_{H^{m+2}}+C||u||_{H^m}||\rho||^2_{H^{m+1}}\\
&+C||\rho||^4_{H^{m+1}}+C||\rho||^6_{H^{m+1}}+C(||\theta||^2_{H^{m+1}}||u||^2_{H^2}+||\theta||^2_{H^2}||u||^2_{H^{m+1}})\\
&+C||\rho||^2_{H^m}||\theta||^2_{H^{m+2}}+C||u||^4_{H^m}+C||u||^3_{H^{m+1}}+C||f||^2_{H^{m+1}}+C||\rho||^2_{H^m}||u||^2_{H^m}\\
&+C(||\theta||^2_{H^{m+1}}||\rho||^2_{H^{m+1}}+||\rho||^4_{H^{m+1}})+\frac{1}{2}\frac{d}{dt}\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k\mathrm{div}u|^2dx\\
&+C(||u||_{m+1}+||\rho||_{H^{m+1}}||u||_{H^{m+1}}+||\theta||_{H^{m+2}}+||u||_{H^{m+1}}||\theta||_{H^{m+1}}+||u||^2_{H^m})||u||^2_{H^{m+1}}\\
&+C\Big[(||\theta||^2_{H^{m+1}}+||\rho||^2_{H^{m+1}})^2(||u||^2_{H^{m+1}}+||\rho||^2_{H^{m+1}}+||\theta||^2_{H^{m+1}}+||u||^4_{H^{m+1}}+||f||^2_{H^{m+1}}\\
&+||\rho||^4_{H^{m+1}}+||\theta||^2_{H^m}||\rho||^2_{H^{m+1}}+\epsilon||u||^2_{H^{m+2}})+(||\theta||^2_{H^{m+1}}+||\rho||^2_{H^{m+1}})(||u||^2_{H^{m+1}}\\
&+||\rho||^2_{H^2}||\theta||^2_{H^{m+2}}+||\rho||^2_{H^m})\Big]+C\epsilon(||\theta||_{H^2}+||\rho||_{H^2})(1+||\rho||_{H^{m+1}}+||\theta||_{H^{m+1}})||u||^2_{H^{m+2}}\\
&+||u||^2_{H^{m+1}}||\rho||_{H^{m+2}}+C||\theta||^2_{H^m}||\rho||^2_{H^{m+1}}||\rho||^2_{H^{m+1}}\\
&+\frac{1}{2}\frac{d}{dt}\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k\mathrm{curl}u|^2dx.
\end{aligned}\tag{3.7}
$$

Proof. In fact, similar to the argument in Section 3, we want to replace (ρ',u',θ') by (ρ,u,θ).

Since some terms on the right-hand side of the estimates involving the derivatives of ρ,u at the $m+2$ order and the constant C_ϵ depending on ϵ, we cannot bound these terms as usual. To overcome this difficulty, we use the continuity and momentum equations.

For $k=1,\dots,m$, we focus on the following terms:

$$-\int \nabla^{k+1}\mathrm{div}(\rho u)\nabla^{k+1}\rho dx, -\int \nabla^{k}\mathrm{div}(\frac{\theta-\rho}{1+\rho}\nabla\rho)\nabla^{k}\mathrm{div}u dx,$$
$$-\int \nabla^{k}\mathrm{curl}(\frac{\theta-\rho}{1+\rho}\nabla\rho)\nabla^{k}\mathrm{curl}u dx.$$

First, note that

$$-\int \nabla^{k+1}\mathrm{div}(\rho u)\nabla^{k+1}\rho dx=-\int \nabla^{k+1}(u\cdot\nabla\rho)\nabla^{k+1}\rho dx-\int \nabla^{k+1}(\rho\mathrm{div}u)\nabla^{k+1}\rho dx.$$

We estimate each term on the right-hand side:

$$\begin{aligned}
-\int \nabla^{k+1}(u\cdot\nabla\rho)\nabla^{k+1}\rho dx &= -\int (u\nabla^{k+1}\nabla\rho+\sum_{l=1}^{k+1}C_{k+1}^{l}\nabla^{l}u\nabla^{k+1-l}\nabla\rho)\nabla^{k+1}\rho dx\\
=&\frac{1}{2}\int \mathrm{div}u|\nabla^{k+1}\rho|^2dx-\int\sum_{l=1}^{k+1}C_{k+1}^{l}\nabla^{l}u\nabla^{k+1-l}\nabla\rho\nabla^{k+1}\rho dx\\
\le& C\Big(||\mathrm{div}u||_\infty||\rho||_{H^{k+1}}^2+(||\nabla u||_\infty||\nabla\rho||_{H^k}+||\nabla u||_{H^k}||\nabla\rho||_\infty)||\nabla^{k+1}\rho||\Big)\\
\le& C||u||_{H^{m+1}}||\rho||_{H^{m+1}}^2+C_{\epsilon_1}||\rho||_{H^{m+1}}^4+\epsilon_1||u||_{H^{m+1}}^2,
\end{aligned}$$

and

$$\begin{aligned}
-\int \nabla^{k+1}(\rho\mathrm{div}u)\nabla^{k+1}\rho dx &= -\int \Big(\rho\nabla^{k+1}\mathrm{div}u+\sum_{l=1}^{k+1}C_{k+1}^{l}\nabla^{l}\rho\nabla^{k+1-l}\mathrm{div}u\Big)\nabla^{k+1}\rho dx\\
\le& -\int \rho\nabla^{k+1}\mathrm{div}u\nabla^{k+1}\rho dx+C(||\nabla\rho||_\infty||\mathrm{div}u||_{H^k}+||\nabla\rho||_{H^k}||\mathrm{div}u||_\infty)||\nabla^{k+1}\rho||\\
\le& -\int \rho\nabla^{k+1}\mathrm{div}u\nabla^{k+1}\rho dx+C_{\epsilon_1}||\rho||_{H^{m+1}}^4+\epsilon_1||u||_{H^{m+1}}^2,
\end{aligned} \tag{3.8}$$

where ϵ_1 is a small constant to be determined later.

One can see the term $-\int\rho\nabla^{k+1}\mathrm{div}u\nabla^{k+1}\rho dx$ involving the derivatives of u at $k+2$ order. To overcome this difficulty, we use the representation of $\mathrm{div}u$ from Equation (2.6). A full use of the equation will be applied frequently to overcome the similar difficulties in the later part of the paper.

We rewrite the term $-\int\rho\nabla^{k+1}\mathrm{div}u\nabla^{k+1}\rho dx$ as follows:

$$\begin{aligned}
-\int \rho\nabla^{k+1}\mathrm{div}u\nabla^{k+1}\rho dx=&\int \rho\nabla^{k+1}(\frac{\rho_t-\epsilon\triangle\rho+\tau u\cdot\nabla\rho}{1+\tau\rho})\nabla^{k+1}\rho dx\\
=&\int\frac{\rho}{1+\tau\rho}(\nabla^{k+1}\rho_t-\epsilon\nabla^{k+1}\triangle\rho+\tau u\nabla^{k+2}\rho)\nabla^{k+1}\rho dx\\
&+\int\sum_{l=1}^{k+1}C_{k+1}^{l}\rho\nabla^{l}(\frac{1}{1+\tau\rho})\nabla^{k+1-l}(\rho_t-\epsilon\triangle\rho)\nabla^{k+1}\rho dx\\
&+\tau\int\sum_{l=1}^{k+1}C_{k+1}^{l}\rho\nabla^{l}(\frac{u}{1+\tau\rho})\nabla^{k+1-l}\nabla\rho\nabla^{k+1}\rho dx
\end{aligned}$$

$$:= I_1 + I_2 + I_3.$$

Since $|\rho| \le \frac{1}{2}$, by Hölder's, Sobolev's and Young's inequalities, we derive that

$$\begin{aligned}
I_1 =& \frac{1}{2}\frac{d}{dt}\int \frac{\rho}{1+\tau\rho}|\nabla^{k+1}\rho|^2 dx - \frac{1}{2}\int \frac{1}{(1+\tau\rho)^2}\rho_t|\nabla^{k+1}\rho|^2 dx \\
&+ \epsilon\int \frac{1}{(1+\tau\rho)^2}\nabla\rho\nabla^{k+2}\rho\nabla^{k+1}\rho dx + \epsilon\int \frac{\rho}{1+\tau\rho}|\nabla^{k+2}\rho|^2 dx \\
&- \frac{\tau}{2}\int \Big(\frac{1}{(1+\tau\rho)^2}u\cdot\nabla\rho + \frac{1}{1+\tau\rho}\mathrm{div}u\Big)|\nabla^{k+1}\rho|^2 dx \\
\le& \frac{1}{2}\frac{d}{dt}\int \frac{\rho}{1+\tau\rho}|\nabla^{k+1}\rho|^2 dx + C\epsilon||\rho||_{H^{m+1}}||\rho||^2_{H^{m+2}} \\
&+ C||\rho||_{H^{m+1}}||u||_{H^{m+1}}||\rho||^2_{H^{m+1}} + C||u||_{H^{m+1}}||\rho||^2_{H^{m+1}},
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
I_2 \le& C\Big(||\nabla(\frac{1}{1+\tau\rho})||_\infty||\rho_t - \epsilon\triangle\rho||_{H^k} + ||\nabla(\frac{1}{1+\tau\rho})||_{H^k}||\rho_t - \epsilon\triangle\rho||_\infty\Big)||\nabla^{k+1}\rho|| \\
\le& C_{\epsilon_1}||\rho||^4_{H^{m+1}} + C_{\epsilon_1}||\rho||^6_{H^{m+1}} + \epsilon_1||u||^2_{H^{m+1}},
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
I_3 \le& C||\nabla^{k+1}\rho||(||\nabla(\frac{u}{1+\tau\rho})||_\infty||\nabla\rho||_{H^k} + ||\nabla(\frac{u}{1+\tau\rho})||_{H^k}||\nabla\rho||_\infty) \\
\le& C_{\epsilon_1}||\rho||^4_{H^{m+1}} + C_{\epsilon_1}||\rho||^6_{H^{m+1}} + \epsilon_1||u||^2_{H^{m+1}}.
\end{aligned} \tag{3.11}$$

Substituting (3.9)-(3.11) into (3.8), we have

$$\begin{aligned}
\int \nabla^{k+1}(\rho\mathrm{div}u)\nabla^{k+1}\rho dx \le& \frac{1}{2}\frac{d}{dt}\int \frac{\rho}{1+\tau\rho}|\nabla^{k+1}\rho|^2 dx + C||u||_{H^{m+1}}||\rho||^2_{H^{m+1}} + C_{\epsilon_1}||\rho||^4_{H^{m+1}} \\
&+ C_{\epsilon_1}||\rho||^6_{H^{m+1}} + C\epsilon||\rho||_{H^{m+1}}||\rho||^2_{H^{m+2}} + 3\epsilon_1||u||^2_{H^{m+1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int \nabla^{k+1}\mathrm{div}(\rho u)\nabla^{k+1}\rho dx \le& \frac{1}{2}\frac{d}{dt}\int \frac{\rho}{1+\tau\rho}|\nabla^{k+1}\rho|^2 dx + C||u||_{H^{m+1}}||\rho||^2_{H^{m+1}} + C_{\epsilon_1}||\rho||^4_{H^{m+1}} \\
&+ C_{\epsilon_1}||\rho||^6_{H^{m+1}} + C\epsilon||\rho||_{H^{m+1}}||\rho||^2_{H^{m+2}} + 4\epsilon_1||u||^2_{H^{m+1}}.
\end{aligned}$$

Next, we estimate the term $\int \nabla^k\mathrm{div}(\frac{\theta-\rho}{1+\rho}\nabla\rho)\nabla^k\mathrm{div}u dx$. The argument of the term $\int \nabla^k\mathrm{curl}(\frac{\theta-\rho}{1+\rho}\nabla\rho)\nabla^k\mathrm{curl}u dx$ is similar and we omit the details.

Note that

$$\begin{aligned}
&-\int \nabla^k\mathrm{div}(\frac{\theta-\rho}{1+\rho}\nabla\rho)\nabla^k\mathrm{div}u dx = -\int \Big[\nabla^k(\frac{\theta-\rho}{1+\rho}\triangle\rho + \nabla(\frac{\theta-\rho}{1+\rho})\nabla\rho)\Big]\nabla^k\mathrm{div}u dx \\
=& -\int \Big[\frac{\theta-\rho}{1+\rho}\nabla^k\triangle\rho + \sum_{l=1}^k C_k^l\nabla^l(\frac{\theta-\rho}{1+\rho})\nabla^{k-l}\triangle\rho + \sum_{l=0}^k C_k^l\nabla^{l+1}(\frac{\theta-\rho}{1+\rho})\nabla^{k-l+1}\rho\Big]\nabla^k\mathrm{div}u dx.
\end{aligned}$$

We will estimate each term on the right-hand side above:

$$\int \sum_{l=1}^k C_k^l\nabla^l(\frac{\theta-\rho}{1+\rho})\nabla^{k-l}\triangle\rho\nabla^k\mathrm{div}u dx$$

$$\begin{aligned}
&\le C(||\nabla(\frac{\theta-\rho}{1+\rho})||_\infty||\triangle\rho||_{H^{m-1}}+||\nabla(\frac{\theta-\rho}{1+\rho})||_{H^{m-1}}||\nabla\rho||_\infty)||\nabla^k\mathrm{div}u||\\
&\le C_{\epsilon_1}(||\theta||^2_{H^{m+1}}||\rho||^2_{H^{m+1}}+||\rho||^4_{H^{m+1}})+\epsilon_1||\nabla^k\mathrm{div}u||^2,
\end{aligned}$$

$$\begin{aligned}
&\int\sum_{l=0}^{k}C_k^l\nabla^{l+1}(\frac{\theta-\rho}{1+\rho})\nabla^{k-l+1}\rho\nabla^k\mathrm{div}udx\\
\le& C(||\nabla(\frac{\theta-\rho}{1+\rho})||_{H^k}||\nabla\rho||_\infty+||\nabla(\frac{\theta-\rho}{1+\rho})||_\infty||\nabla\rho||_{H^k})||\nabla^k\mathrm{div}u||\\
\le& C_{\epsilon_1}(||\theta||^2_{H^{m+1}}||\rho||^2_{H^{m+1}}+||\rho||^4_{H^{m+1}})+\epsilon_1||\nabla^k\mathrm{div}u||^2.
\end{aligned}$$

The most difficult term to estimate is $-\int\frac{\theta-\rho}{1+\rho}\nabla^k\triangle\rho\nabla^k\mathrm{div}udx$. Since $|\theta|<\frac{1}{2}$, we will use the following representation of $\nabla\rho$ from Equation (2.2):

$$\nabla\rho=\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}(E+\tau f-u-u_t-\tau u\cdot\nabla u-\nabla\theta+\epsilon\triangle u).$$

We have

$$\begin{aligned}
&-\int\frac{\theta-\rho}{1+\rho}\nabla^k\triangle\rho\nabla^k\mathrm{div}udx\\
=&-\int\frac{\theta-\rho}{1+\rho}\nabla^k\mathrm{div}(\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}(-u_t-u-\tau u\cdot\nabla u-\nabla\theta+E+\tau f+\epsilon\triangle u))\nabla^k\mathrm{div}udx\\
:=&I_4+I_5+I_6+I_7+I_8+I_9+I_{10}.
\end{aligned}$$

Now, we estimate each term.

$$\begin{aligned}
I_4=&\int\frac{\theta-\rho}{1+\rho}\nabla^k(\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}\mathrm{div}u_t+u_t\nabla(\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}))\nabla^k\mathrm{div}udx\\
=&\int\frac{\theta-\rho}{1+\rho}(\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}\nabla^k\mathrm{div}u_t+\sum_{l=1}^{k}C_k^l\nabla^l\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}\nabla^{k-l}\mathrm{div}u_t\\
&+\sum_{l=0}^{k}C_k^l\nabla^lu_t\nabla^{k-l}\nabla\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta})\nabla^k\mathrm{div}udx\\
:=&I_{41}+I_{42}+I_{43}.
\end{aligned}\tag{3.12}$$

Note that

$$\begin{aligned}
I_{41}=&\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}\nabla^k\mathrm{div}u_t\nabla^k\mathrm{div}udx\\
=&\frac{1}{2}\frac{d}{dt}\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k\mathrm{div}u|^2dx-\frac{1}{2}\int|\nabla^k\mathrm{div}u|^2\frac{d}{dt}(\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta})dx\\
=&\frac{1}{2}\frac{d}{dt}\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k\mathrm{div}u|^2dx-\frac{1}{2}\int|\nabla^k\mathrm{div}u|^2(\frac{-1-\theta}{(1+\rho-\tau\rho+\tau\theta)^2}\rho_t\\
&+\frac{1+\rho}{(1+\rho-\tau\rho+\tau\theta)^2}\theta_t).
\end{aligned}$$

From Equations (2.1)-(2.5), it concludes that

$$||\rho_t||_\infty=||\epsilon\triangle\rho-\mathrm{div}u-\tau\mathrm{div}(\rho u)||_\infty\le C\epsilon||\rho||_{H^4}+C||\mathrm{div}u||_{H^2}+C||\rho||_{H^3}||u||_{H^3},$$

$$
\begin{aligned}
||\theta_t||_\infty =&\frac{2}{3}||\theta+\mathrm{div}u-\triangle\theta+\frac{3\tau}{2}u\cdot\nabla\theta+\tau\theta\mathrm{div}u-\tau\kappa|u|^2+\tau\frac{\rho}{1+\rho}\triangle\theta||_\infty\\
\le& C(||\theta||_{H^4}+||\mathrm{div}u||_{H^2}+||u||_{H^3}||\theta||_{H^3}+||u||_{H^2}^2),\\
||\mathrm{div}u_t||_{H^{k-1}} =&||\mathrm{div}(u-E+\nabla\theta+\nabla\rho+\tau u\cdot\nabla u-\epsilon\triangle u-\tau\frac{\rho-\theta}{1+\rho}\nabla\rho-\tau f)||_{H^{k-1}}\\
\le& C\big(||u||_{H^m}+||\rho||_{H^{m+1}}+||u||_{H^m}||u||_{H^{m+1}}+||\rho||_{H^m}||\rho||_{H^{m+1}}\\
&+||\theta||_{H^m}||\rho||_{H^{m+1}}+||\theta||_{H^{m+1}}+||f||_{H^m}\big)+C\epsilon||u||_{H^{m+2}},\\
||\mathrm{div}u_t||_\infty \le& C\big(||u||_{H^3}+||\rho||_{H^4}+||u||_{H^3}||u||_{H^4}+||\rho||_{H^3}||\rho||_{H^4}+||\theta||_{H^3}||\rho||_{H^4}+||f||_{H^3}\\
&+||\theta||_{H^4}\big)+C\epsilon||u||_{H^5}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I_{41}\le&\frac{1}{2}\frac{d}{dt}\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k\mathrm{div}u|^2dx+||\nabla^k\mathrm{div}u||^2||\rho_t||_\infty+||\nabla^k\mathrm{div}u||^2||\theta_t||_\infty\\
\le&\frac{1}{2}\frac{d}{dt}\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k\mathrm{div}u|^2dx+||\nabla^k\mathrm{div}u||^2\big(C_{\epsilon_1}\epsilon||\rho||_{H^4}^2+C(||\mathrm{div}u||_{H^2}\\
&+||\rho||_{H^3}||u||_{H^3}+||\theta||_{H^4}+||u||_{H^3}||\theta||_{H^3}+||u||_{H^2}^2)\big)+\epsilon_1||\nabla^k\mathrm{div}u||^2, \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
I_{42}\le&\epsilon_1||\nabla^k\mathrm{div}u||^2+C_{\epsilon_1}(||\theta||_{H^2}^2+||\rho||_{H^2}^2)(||\rho||_{H^3}^2+||\theta||_{H^3}^2)(||u||_{H^m}^2\\
&+||\rho||_{H^{m+1}}^2+||\theta||_{H^{m+1}}^2+||u||_{H^m}^2||u||_{H^{m+1}}^2+||f||_{H^m}^2+||\rho||_{H^m}^2||\rho||_{H^{m+1}}^2\\
&+||\theta||_{H^m}^2||\rho||_{H^{m+1}}^2+\epsilon||u||_{H^{m+2}}^2), \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
I_{43}\le&\epsilon_1||\nabla^k\mathrm{div}u||^2+C_{\epsilon_1}(||\theta||_\infty^2+||\rho||_\infty^2)(||\nabla\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}||_\infty||u_t||_{H^k}\\
&+||\nabla\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}||_{H^k}||u_t||_\infty)^2\\
\le&\epsilon_1||\nabla^k\mathrm{div}u||^2+C_{\epsilon_1}(||\theta||_{H^2}^2+||\rho||_{H^2}^2)\big[(||\rho||_{H^3}^2+||\theta||_{H^3}^2)(||u||_{H^m}^2\\
&+||\rho||_{H^{m+1}}^2+||\theta||_{H^{m+1}}^2+||u||_{H^m}^2||u||_{H^{m+1}}^2+||f||_{H^m}^2+||\rho||_{H^m}^2||\rho||_{H^{m+1}}^2\\
&+||\theta||_{H^m}^2||\rho||_{H^{m+1}}^2+\epsilon||u||_{H^{m+2}}^2)+(||\rho||_{H^{m+1}}^2+||\theta||_{H^{m+1}}^2)(||u||_{H^2}^2+||\rho||_{H^3}^2\\
&+||u||_{H^2}^2||u||_{H^3}^2+||\rho||_{H^2}^2||\rho||_{H^4}^3+||\theta||_{H^3}^2+||f||_{H^2}^2+\epsilon||u||_{H^4}^2)\big]. \qquad (3.15)
\end{aligned}
$$

Substituting (3.13)-(3.15) into (3.12), we have

$$
\begin{aligned}
I_4\le&\frac{1}{2}\frac{d}{dt}\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k\mathrm{div}u|^2dx+||\nabla^k\mathrm{div}u||^2\big(C_{\epsilon_1}\epsilon||\rho||_{H^4}^2+C(||\mathrm{div}u||_{H^2}\\
&+||\rho||_{H^3}||u||_{H^3}+||\theta||_{H^4}+||u||_{H^3}||\theta||_{H^3}+||u||_{H^2}^2)\big)+3\epsilon_1||\nabla^k\mathrm{div}u||^2\\
&+C_{\epsilon_1}(||\theta||_{H^2}^2+||\rho||_{H^2}^2)(||\rho||_{H^{m+1}}^2+||\theta||_{H^{m+1}}^2)(||u||_{H^m}^2+||\rho||_{H^{m+1}}^2+||\theta||_{H^{m+1}}^2\\
&+||u||_{H^m}^2||u||_{H^{m+1}}^2+||f||_{H^m}^2+||\rho||_{H^m}^2||\rho||_{H^{m+1}}^2+||\theta||_{H^m}^2||\rho||_{H^{m+1}}^2+\epsilon||u||_{H^{m+2}}^2),\\
I_5\le& C_{\epsilon_1}(||\theta||_{H^2}^2+||\rho||_{H^2}^2)\big[(||\rho||_{H^{m+1}}^2+||\theta||_{H^{m+1}}^2)||u||_{H^2}^2+(||\rho||_{H^2}^2+||\theta||_{H^2}^2)||u||_{H^{m+1}}^2\big]\\
&+\epsilon_1||\nabla\mathrm{div}u||^2,\\
I_6\le&\epsilon_1||\nabla^k\mathrm{div}u||^2+C_{\epsilon_1}\big[(||\theta||_{H^3}^2+||\rho||_{H^3}^2)||u||_{H^3}^2+(||\theta||_{H^2}^2+||\rho||_{H^2}^2)(||\rho||_{H^{m+1}}^2\\
&+||\theta||_{H^{m+1}}^2)||u||_{H^m}^2||u||_{H^{m+1}}^2\big].
\end{aligned}
$$

We also have

$$I_7 \le \epsilon_1||\nabla^k \mathrm{div} u||^2 + C(||\theta||_\infty^2 + ||\rho||_\infty^2)\big[||\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}||_\infty^2||\nabla\theta||_{H^{k+1}}^2 + ||\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}||_{H^{k+1}}^2||\nabla\theta||_\infty^2\big]$$
$$\le \epsilon_1||\nabla^k \mathrm{div} u||^2 + C_{\epsilon_1}(||\theta||_{H^2}^2 + ||\rho||_{H^2}^2)\big[||\rho||_{H^2}^2||\theta||_{H^{m+2}}^2 + (||\rho||_{H^{m+1}}^2 + ||\theta||_{H^{m+1}}^2)||\theta||_{H^3}^2\big],$$

$$I_8 = -\int \frac{\theta-\rho}{1+\rho}\Big(\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}\nabla^k \mathrm{div} E + \sum_{l=1}^k C_k^l \nabla^l\big(\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}\big)\nabla^{k-l}\mathrm{div} E + \sum_{l=0}^k C_k^l \nabla^l E \nabla^{k-l}\nabla\frac{1+\rho}{1+\rho-\tau\rho+\tau\theta}\Big)\nabla^k \mathrm{div} u dx$$
$$\le \epsilon_1||\nabla^k \mathrm{div} u||^2 + C_{\epsilon_1}(||\theta||_{H^2}^2 + ||\rho||_{H^2}^2)||\rho||_{H^m}^2 + C_{\epsilon_1}(||\theta||_{H^2}^2 + ||\rho||_{H^2}^2)\big[(||\rho||_{H^3}^2 + ||\theta||_{H^3}^2)||\rho||_{H^m}^2 + (||\rho||_{H^m}^2 + ||\theta||_{H^m}^2)||\rho||_{H^2}^2\big],$$

$$I_9 \le \epsilon_1||\nabla^k \mathrm{div} u||^2 + C_{\epsilon_1}(||\theta||_{H^2}^2 + ||\rho||_{H^2}^2)(||\rho||_{H^2}^2||f||_{H^{m+1}}^2 + (||\rho||_{H^{m+1}}^2 + ||\theta||_{H^{m+1}}^2)||f||_{H^2}^2).$$

$$I_{10} \le C\epsilon(||\theta||_{H^2} + ||\rho||_{H^2})(1 + ||\rho||_{H^{m+1}} + ||\theta||_{H^{m+1}})||u||_{H^{m+2}}^2.$$

Thus,

$$-\int \nabla^k \mathrm{div}(\frac{\theta-\rho}{1+\rho}\nabla\rho)\nabla^k \mathrm{div} u dx$$
$$\le C(||\theta||_{H^{m+1}}^2||\rho||_{H^{m+1}}^2 + ||\rho||_{H^{m+1}}^4) + \frac{1}{2}\frac{d}{dt}\int \frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k \mathrm{div} u|^2 dx$$
$$+ C(||u||_{m+1} + ||\rho||_{H^{m+1}}||u||_{H^{m+1}} + ||\theta||_{H^{m+2}} + ||u||_{H^{m+1}}||\theta||_{H^{m+1}} + ||u||_{H^m}^2)||\nabla^k \mathrm{div} u||^2 + C_{\epsilon_1}\Big[(||\theta||_{H^{m+1}}^2 + ||\rho||_{H^{m+1}}^2)^2(||u||_{H^{m+1}}^2 + ||\rho||_{H^{m+1}}^2 + ||\theta||_{H^{m+1}}^2$$
$$+ ||u||_{H^{m+1}}^4 + ||f||_{H^{m+1}}^2 + ||\rho||_{H^{m+1}}^4 + ||\theta||_{H^m}^2||\rho||_{H^{m+1}}^2 + \epsilon||u||_{H^{m+2}}^2) + (||\theta||_{H^{m+1}}^2$$
$$+ ||\rho||_{H^{m+1}}^2)(||u||_{H^{m+1}}^2 + ||\rho||_{H^2}^2||\theta||_{H^{m+2}}^2 + ||\rho||_{H^m}^2)\Big] + C\epsilon(||\theta||_{H^2} + ||\rho||_{H^2})(1$$
$$+ ||\rho||_{H^{m+1}} + ||\theta||_{H^{m+1}})||u||_{H^{m+2}}^2 + C\epsilon||u||_{H^{m+1}}^2||\rho||_{H^{m+2}} + 10\epsilon_1||\nabla^k \mathrm{div} u||^2.$$

Therefore, it concludes that

$$\frac{1}{2}\frac{d}{dt}||\nabla^k\rho||^2 + \langle\nabla^{k+1}\mathrm{div} u, \nabla^{k+1}\rho\rangle + \epsilon||\nabla^k\triangle\rho||^2$$
$$\le \frac{1}{2}\frac{d}{dt}\int \frac{\rho}{1+\tau\rho}|\nabla^{k+1}\rho|^2 dx + C\epsilon||\rho||_{H^{m+1}}||\rho||_{H^{m+2}}^2 + 3\epsilon_1||u||_{H^{m+1}}^2 + C||u||_{H^m}||\rho||_{H^{m+1}}^2$$
$$+ C_{\epsilon_1}||\rho||_{H^{m+1}}^4 + C_{\epsilon_1}||\rho||_{H^{m+1}}^6,$$

and

$$\frac{1}{2}\frac{d}{dt}||\nabla^k \mathrm{div} u||^2 + \frac{1}{2}||\nabla^k \mathrm{div} u||^2 + \frac{1}{2}||\nabla^k\rho||^2 + \epsilon||\nabla^{k+1}\rho||^2 + \epsilon||\nabla^{k+1}\mathrm{div} u||^2$$
$$-\langle\nabla^{k+1}\theta, \nabla^{k+1}\mathrm{div} u\rangle - \langle\nabla^{k+1}\rho, \nabla^{k+1}\mathrm{div} u\rangle$$

$$
\begin{aligned}
&\le C||\nabla u||_{H^2}||u||_{H^{m+1}}^2+C||f||_{H^{m+1}}^2+C_{\epsilon'}||\rho||_{H^m}^2||u||_{H^m}^2+\epsilon'||\nabla^{k+1}\rho||^2+C(||\rho||_{H^{m+1}}^4\\
&\quad+||\theta||_{H^{m+1}}^2||\rho||_{H^{m+1}}^2)+10\epsilon_1||\nabla^k \mathrm{div}u||^2+\frac{1}{2}\frac{d}{dt}\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k\mathrm{div}u|^2dx\\
&\quad+C(||u||_{m+1}+||\rho||_{H^{m+1}}||u||_{H^{m+1}}+||\theta||_{H^{m+2}}+||u||_{H^{m+1}}||\theta||_{H^{m+1}}+||u||_{H^m}^2)||\nabla^k\mathrm{div}u||^2\\
&\quad+C_{\epsilon_1}\Big[(||\theta||_{H^{m+1}}^2+||\rho||_{H^{m+1}}^2)^2(||u||_{H^{m+1}}^2+||\rho||_{H^{m+1}}^2+||\theta||_{H^{m+1}}^2+||u||_{H^{m+1}}^4+||f||_{H^{m+1}}^2\\
&\quad+||\rho||_{H^{m+1}}^4+||\theta||_{H^m}^2||\rho||_{H^{m+1}}^2+\epsilon||u||_{H^{m+2}}^2)+(||\theta||_{H^{m+1}}^2+||\rho||_{H^{m+1}}^2)(||u||_{H^{m+1}}^2\\
&\quad+||\rho||_{H^2}^2||\theta||_{H^{m+2}}^2+||\rho||_{H^m}^2)\Big]+C\epsilon(||\theta||_{H^2}+||\rho||_{H^2})(1+||\rho||_{H^{m+1}}+||\theta||_{H^{m+1}})||u||_{H^{m+2}}^2\\
&\quad+C\epsilon||u||_{H^{m+1}}^2||\rho||_{H^{m+2}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\nabla^k\mathrm{curl}u||^2+\frac{1}{2}||\nabla^k\mathrm{curl}u||^2+\epsilon||\nabla^{k+1}\mathrm{curl}u||^2\\
&\le 10\epsilon_1||\nabla^k\mathrm{div}u||^2+C(||\theta||_{H^{m+1}}^2||\rho||_{H^{m+1}}^2+||\rho||_{H^{m+1}}^4)+\frac{1}{2}\frac{d}{dt}\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k\mathrm{curl}u|^2dx\\
&\quad+C(||u||_{m+1}+||\rho||_{H^{m+1}}||u||_{H^{m+1}}+||\theta||_{H^{m+2}}+||u||_{H^{m+1}}||\theta||_{H^{m+1}}+||u||_{H^m}^2)||\nabla^k\mathrm{curl}u||^2\\
&\quad+C_{\epsilon_1}\Big[(||\theta||_{H^{m+1}}^2+||\rho||_{H^{m+1}}^2)^2(||u||_{H^{m+1}}^2+||\rho||_{H^{m+1}}^2+||\theta||_{H^{m+1}}^2+||u||_{H^{m+1}}^4+||f||_{H^{m+1}}^2\\
&\quad+||\rho||_{H^{m+1}}^4+||\theta||_{H^m}^2||\rho||_{H^{m+1}}^2+\epsilon||u||_{H^{m+2}}^2)+(||\theta||_{H^{m+1}}^2+||\rho||_{H^{m+1}}^2)(||u||_{H^{m+1}}^2\\
&\quad+||\rho||_{H^2}^2||\theta||_{H^{m+2}}^2+||\rho||_{H^m}^2)\Big]+C\epsilon(||\theta||_{H^2}+||\rho||_{H^2})(1+||\rho||_{H^{m+1}}+||\theta||_{H^{m+1}})||u||_{H^{m+2}}^2\\
&\quad+C\epsilon||u||_{H^{m+1}}^2||\rho||_{H^{m+2}}+C||\nabla u||_{H^2}||u||_{H^{m+1}}^2+C||\nabla^k\mathrm{curl}f||^2.
\end{aligned}
$$

Thus, there exists a large enough constant $M_3>0$ such that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}||\nabla^{k+1}\rho||^2+\frac{3}{4}||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\mathrm{div}u||^2+\frac{1}{2}||\nabla^k\rho||^2+\frac{1}{2}||\nabla^k\mathrm{curl}u||^2\\
&\quad-\frac{1}{M_3}\langle\nabla^k\mathrm{div}u,\nabla^k\rho\rangle\Big)+\frac{\epsilon}{4}(||\nabla^k\triangle\rho||^2+||\nabla^{k+1}\rho||^2)+\frac{1}{2M_3}(||\nabla^{k+1}\rho||^2+||\nabla^k\rho||^2)\\
&\quad+\frac{1}{2}||\nabla^{k+1}\theta||^2+\frac{1}{2}||\nabla^k\triangle\theta||^2+\frac{1}{2}(||\nabla^k\mathrm{div}u||^2+||\nabla^k\mathrm{curl}u||^2)+\frac{\epsilon}{2}(||\nabla^{k+1}\mathrm{div}u||^2\\
&\quad+||\nabla^{k+1}\mathrm{curl}u||^2)\\
&\le\frac{1}{2}\frac{d}{dt}\int\frac{\rho}{1+\tau\rho}|\nabla^{k+1}\rho|^2dx+C\epsilon||\rho||_{H^{m+1}}||\rho||_{H^{m+2}}^2+3\epsilon_1||u||_{H^{m+1}}^2+C||u||_{H^m}||\rho||_{H^{m+1}}^2\\
&\quad+C_{\epsilon_1}||\rho||_{H^{m+1}}^4+C_{\epsilon_1}||\rho||_{H^{m+1}}^6+C(||\theta||_{H^{m+1}}^2||u||_{H^2}^2+||\theta||_{H^2}^2||u||_{H^{m+1}}^2+||u||_{H^m}^4\\
&\quad+||\rho||_{H^m}^2||\theta||_{H^{m+2}}^2+||u||_{H^{m+1}}^3+||f||_{H^{m+1}}^2)+C_{\epsilon'}||\rho||_{H^m}^2||u||_{H^m}^2+\epsilon'||\nabla^{k+1}\rho||^2\\
&\quad+C(||\theta||_{H^{m+1}}^2||\rho||_{H^{m+1}}^2+||\rho||_{H^{m+1}}^4)+\frac{1}{2}\frac{d}{dt}\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k\mathrm{div}u|^2dx\\
&\quad+\frac{1}{2}\frac{d}{dt}\int\frac{\theta-\rho}{1+\rho-\tau\rho+\tau\theta}|\nabla^k\mathrm{div}u|^2dx+C||\theta||_{H^m}^2||\rho||_{H^{m+1}}^2||\rho||_{H^{m+1}}^2+10\epsilon_1||\nabla^k\mathrm{div}u||^2\\
&\quad+C(||u||_{m+1}+||\rho||_{H^{m+1}}||u||_{H^{m+1}}+||\theta||_{H^{m+2}}+||u||_{H^{m+1}}||\theta||_{H^{m+1}}+||u||_{H^m}^2)||u||_{H^{m+1}}^2\\
&\quad+C_{\epsilon_1}\Big[(||\theta||_{H^{m+1}}^2+||\rho||_{H^{m+1}}^2)^2(||u||_{H^{m+1}}^2+||\rho||_{H^{m+1}}^2+||\theta||_{H^{m+1}}^2+||u||_{H^{m+1}}^4+||f||_{H^{m+1}}^2\\
&\quad+||\rho||_{H^{m+1}}^4+||\theta||_{H^m}^2||\rho||_{H^{m+1}}^2+\epsilon||u||_{H^{m+2}}^2)+(||\theta||_{H^{m+1}}^2+||\rho||_{H^{m+1}}^2)(||u||_{H^{m+1}}^2\\
&\quad+||\rho||_{H^2}^2||\theta||_{H^{m+2}}^2+||\rho||_{H^m}^2)\Big]+C\epsilon(||\theta||_{H^2}+||\rho||_{H^2})(1+||\rho||_{H^{m+1}}+||\theta||_{H^{m+1}})||u||_{H^{m+2}}^2\\
&\quad+C\epsilon||u||_{H^{m+1}}^2||\rho||_{H^{m+2}}.
\end{aligned}
$$

It completes the proof by choosing $\epsilon' = \frac{1}{4M_3}, \epsilon_1 = \frac{1}{104}$. □

Now, we show the existence of time periodic solutions to the regularized system.

PROPOSITION 3.1. *Assume that the assumptions of the force f in Theorem 1.1 hold. Let $\tau \in [0,1]$, then regularized system (2.1)-(2.5) admits a time periodic solution $(\rho,u,\theta) \in X_\delta$.*

Proof. First, one can see that solving problem (2.1)-(2.5) is equivalent to solving the equation:

$$U - \mathcal{L}(U,1) = 0,\ U = (\rho,u,\theta) \in X.$$

In what follows, we use the topological degree theory to solve this problem. We have to show that there exists a small enough constant $\delta > 0$ such that

$$(1 - \mathcal{L}(\cdot,\tau))\partial B_\delta(0) \neq 0,\ \forall \tau \in [0,1], \tag{3.16}$$

where B_δ is the ball of radius δ centered at 0 in X.

Applying Lemma 3.4, integrating it over $[0,T]$ and by the periodicity of the solution, we have

$$\begin{aligned}
&\epsilon \int_0^T (||\rho||^2_{H^{m+2}} + ||u||^2_{H^{m+2}})dt + \int_0^T (||\rho||^2_{H^{m+1}} + ||u||^2_{H^{m+1}} + ||\theta||^2_{H^{m+2}})dt \\
\leq & C\epsilon\Big(\sup_{t\in[0,T]} ||\rho||_{H^{m+1}} \int_0^T ||\rho||^2_{H^{m+2}} dt + \sup_{t\in[0,T]} ||u||^2_{m+1} \int_0^T ||\rho||^2_{H^{m+2}} dt\Big) + \frac{1}{2}\int_0^T ||u||^2_{H^{m+1}} dt \\
&+ C\epsilon \sup_{t\in[0,T]} (||\rho||_{H^2} + ||\theta||_{H^2})(1 + ||\rho||_{H^{m+1}} + ||\theta||_{H^{m+1}}) \int_0^T ||u||^2_{H^{m+2}} dt \\
&+ C\epsilon \sup_{t\in[0,T]} (||\theta||^4_{H^{m+1}} + ||\rho||^4_{H^{m+1}}) \int_0^T ||u||^2_{H^{m+2}} dt + C\int_0^T ||f||^2_{H^{m+1}} dt \\
&+ C \sup_{t\in[0,T]} \Big[||u||_{H^m} ||\rho||^2_{H^{m+1}} + ||\rho||^4_{H^{m+1}} + ||\rho||^6_{H^{m+1}} + ||\theta||^2_{H^{m+1}} ||u||^2_{H^2} + ||\theta||^2_{H^2} ||u||^2_{H^{m+1}} \\
&+ ||u||^4_{H^m} + ||u||^3_{H^{m+1}} + ||\rho||^2_{H^m} ||u||^2_{H^m} + ||\theta||^2_{H^{m+1}} ||\rho||^2_{H^{m+1}} + ||\rho||^4_{H^{m+1}} \\
&+ (||u||_{m+1} + ||\rho||_{H^{m+1}} ||u||_{H^{m+1}} + ||u||_{H^{m+1}} ||\theta||_{H^{m+1}} + ||u||^2_{H^m}) ||u||^2_{H^{m+1}} \\
&+ (||\theta||^4_{H^{m+1}} + ||\rho||^4_{H^{m+1}})(||u||^2_{H^{m+1}} + ||\rho||^2_{H^{m+1}} + ||\theta||^2_{H^{m+1}} + ||u||^4_{H^{m+1}} + ||\rho||^4_{H^{m+1}} \\
&+ ||\theta||^2_{H^m} ||\rho||^2_{H^{m+1}}) + (||\theta||^2_{H^{m+1}} + ||\rho||^2_{H^{m+1}})(||u||^2_{H^{m+1}} + ||\rho||^2_{H^m}) + ||\theta||^2_{H^m} ||\rho||^4_{H^{m+1}} \Big] \\
&+ C \sup_{t\in[0,T]} \Big[||\rho||^2_{H^m} + ||\rho||^2_{H^2} (||\theta||^2_{H^{m+1}} + ||\rho||^2_{H^{m+1}}) \Big] \int_0^T ||\theta||^2_{H^{m+2}} dt \\
&+ C \sup_{t\in[0,T]} ||u||^2_{H^{m+1}} \int_0^T ||\theta||^2_{H^{m+2}} dt + C \sup_{t\in[0,T]} (||\theta||^4_{H^{m+1}} + ||\rho||^4_{H^{m+1}}) \int_0^T ||f||^2_{H^{m+1}} dt,
\end{aligned}$$

where C is independent of ϵ.

Since $\delta < 1$ is appropriately small so that $C\delta < \frac{1}{2}$, there holds

$$\begin{aligned}
&\frac{\epsilon}{2} \int_0^T (||\rho||^2_{H^{m+2}} + ||u||^2_{H^{m+2}})dt + \int_0^T (||\rho||^2_{H^{m+1}} + \frac{1}{2}||u||^2_{H^{m+1}} + ||\theta||^2_{H^{m+2}})dt \\
\leq & C(\delta^3 + \delta^4 + \delta^6 + \delta^8 + (1 + \delta^4)\eta^2).
\end{aligned}$$

Hence, there exists $t^*\in(0,T)$ such that

$$\begin{aligned}&T(||\rho(T^*)||^2_{H^{m+1}}+||u(T^*)||^2_{H^{m+1}}+||\theta(T^*)||^2_{H^{m+2}})\\ \le& C(\delta^3+\delta^4+\delta^6+\delta^8+(1+\delta^4)\eta^2).\end{aligned}$$

Recalling Lemma 3.4, integrating from t^* to t for any $t^*\le t\le t^*+T$ and by the periodicity of the solution, we have

$$\begin{aligned}&\sup_{t\in[0,T]}(||\rho||^2_{H^{m+1}}+||u||^2_{H^{m+1}}+||\theta||^2_{H^{m+1}})+\int_0^T||\theta||^2_{H^{m+2}}dt\\ \le& C(||\rho(T^*)||^2_{H^{m+1}}+||u(T^*)||^2_{H^{m+1}}+||\theta(T^*)||^2_{H^{m+1}})+C(\delta^3+\delta^4+\delta^6+\delta^8+(1+\delta^4)\eta^2)\\ \le& C(\delta^3+\delta^4+\delta^6+\delta^8+(1+\delta^4)\eta^2).\end{aligned}$$

When $\delta=\eta^{\frac14}$ and η is suitably small, the above inequality leads to

$$\sup_{t\in[0,T]}(||\rho||^2_{H^{m+1}}+||u||^2_{H^{m+1}}+||\theta||^2_{H^{m+1}})+\int_0^T||\theta||^2_{H^{m+2}}dt\le\frac12\delta^4.$$

Therefore, (3.15) holds.

Now, note that $\mathcal{L}((\rho,u,\theta),0)\equiv0$. From the properties of Leray-Schauder degree, we have

$$\deg(I-\mathcal{L}(\cdot,1),B_\delta(0),0)=\deg(I-\mathcal{L}(\cdot,0),B_\delta(0),0)=\deg(I,B_\delta(0),0)=1.$$

Thus, we obtain that problem (2.1)-(2.5) has a time periodic solution $(\rho,u,\theta)\in X_\delta$ which completes the proof. ∎

3.4. Limiting as $\epsilon\to0$. In this section, we devote to proving the existence of periodic solutions to Equations (1.6)-(1.10) by passing to the limit in the regularized equations.

Proof. **(Proof of the existence in Theorem 1.1.)**

Let $(\rho_\epsilon,u_\epsilon,\theta_\epsilon)$ be a time periodic solution of regularized Equations (2.1)-(2.5). From the proof of Proposition 3.1, we get

$$\sup_{t\in[0,T]}(||\rho_\epsilon||^2_{H^{m+1}}+||u_\epsilon||^2_{H^{m+1}}+||\theta_\epsilon||^2_{H^{m+1}})+\int_0^T||\theta_\epsilon||^2_{H^{m+2}}dt\le C\delta^2,$$

where C and δ are independent of ϵ.

Moreover, integrating (3.7) from t to $t+h$, and then integrating it from 0 to T, we have

$$\begin{aligned}&\int_0^T(||\rho_\epsilon(t+h)||^2_{H^{m+1}}+||u_\epsilon(t+h)||^2_{H^{m+1}}+||\theta_\epsilon(t+h)||^2_{H^{m+1}})-(||\rho_\epsilon(t)||^2_{H^{m+1}}\\ &+||u_\epsilon(t)||^2_{H^{m+1}}+||\theta_\epsilon(t)||^2_{H^{m+1}})dt\le C\delta,\end{aligned}$$

where C is independent of ϵ. Hence, there exists a subsequence $(\rho_\epsilon,u_\epsilon,\theta_\epsilon)$ which is denoted by itself such that

$$\begin{aligned}&(\rho_\epsilon,u_\epsilon,\theta_\epsilon)\rightharpoonup(\rho,u,\theta),\ \text{weakly}*\ \text{in}\ L^\infty(0,T,H^{m+1}),\\ &(\rho_\epsilon,u_\epsilon,\theta_\epsilon)\to(\rho,u,\theta),\ \text{strongly}\ \ \text{in}\ L^2(0,T,H^{m+1}).\end{aligned}$$

In what follows, we will show $\rho_\epsilon \in C^{\alpha,\beta}([0,T]\times\Omega)$, $\alpha,\beta\in(0,1)$. Using Sobolev's imbedding theorem, we have $\rho_\epsilon\in C^\alpha(\omega)$ for any t. So, we only need to prove

$$|\rho_\epsilon(x,t_1)-\rho_\epsilon(x,t_2)|\le C|t_1-t_2|^\beta,\ \forall t_1,t_2\in(0,T),x\in\Omega$$

for some $\beta\in(0,1)$.

With a similar argument in Lemma 3.2, we have

$$\int_0^T\int|\frac{\partial\rho}{\partial t}|^2dxdt\le C\sup_{t\in[0,T]}(||\mathrm{div}u||^2+||\mathrm{div}(\rho u)||^2)\le C,$$

where $C>0$ is a constant independent of ϵ. Taking a ball B_r of radius r centered at x with $r=|t_1-t_2|^\zeta$, $\zeta=\frac{1}{3+2\alpha}$, we obtain that

$$\begin{aligned}&\int_{B_r}|\rho_\epsilon(y,t_1)-\rho_\epsilon(y,t_2)|dy=\int_{B_r}\Big|\int_{t_2}^{t_1}\frac{\partial\rho_\epsilon(y,t)}{\partial t}dt\Big|dy\\ \le& C\Big(\int_{B_r}\int_{t_2}^{t_1}\Big|\frac{\partial\rho_\epsilon(y,t)}{\partial t}\Big|^2dtdy\Big)^{\frac12}|t_1-t_2|^{\frac12}r^{\frac32}\le C|t_1-t_2|^{\frac12}r^{\frac32}.\end{aligned}$$

By the mean value theorem, there exists $x^*\in B_r$ such that

$$|\rho_\epsilon(x^*,t_1)-\rho_\epsilon(x^*,t_2)|\le C|t_1-t_2|^{\frac12}r^{-\frac32}\le C|t_1-t_2|^{\frac{1-3\zeta}{2}}.$$

Then, we have

$$\begin{aligned}&|\rho_\epsilon(x,t_1)-\rho_\epsilon(x,t_2)|\le|\rho_\epsilon(x,t_1)-\rho_\epsilon(x^*,t_1)|+|\rho_\epsilon(x^*,t_1)-\rho_\epsilon(x^*,t_2)|+|\rho_\epsilon(x^*,t_2)-\rho_\epsilon(x,t_2)|\\ &\le C(|x-x^*|^\alpha+|t_1-t_2|^{\frac{1-3\zeta}{2}})\le C(|t_1-t_2|^{\alpha\zeta}+|t_1-t_2|^{\frac{1-3\zeta}{2}})\le C|t_1-t_2|^{\frac{\alpha}{3+2\alpha}}.\end{aligned}$$

Repeating the above process with u,θ, it concludes that

$$|u_\epsilon(x_1,t_1)-u_\epsilon(x_2,t_2)|\le C(|x_1-x_2|^\alpha+|t_1-t_2|^\beta),$$

$$|\theta_\epsilon(x_1,t_1)-\theta_\epsilon(x_2,t_2)|\le C(|x_1-x_2|^\alpha+|t_1-t_2|^\beta)$$

for some constants $\alpha,\beta\in(0,1)$.

From Arzelà-Ascoli theorem, we get

$$(\rho_\epsilon,u_\epsilon,\theta_\epsilon)\to(\rho,u,\theta)\text{ uniformly.}$$

As $\epsilon\to0$, one deduces that the limit function $(\rho,u,\theta)\in X_\delta$ is the desired time periodic solution in Theorem 1.1 which completes the proof. □

4. Uniqueness

In this part, we show the uniqueness of time periodic solutions to system (1.6)-(1.10).

Proof. **(Proof of the uniqueness in Theorem 1.1.)** We denote two time periodic solutions by (ρ_1,u_1,θ_1) and (ρ_2,u_2,θ_2). Let $\rho=\rho_1-\rho_2,u=u_1-u_2,\theta=\theta_1-\theta_2$. Then, (ρ,u,θ) is a time periodic solution of the following system:

$$\rho_t+\mathrm{div}u=-\mathrm{div}(\rho u_1)-\mathrm{div}(\rho_2u),\tag{4.1}$$

$$u_t+u-E+\nabla\theta+\nabla\rho+u\cdot\nabla u_1+u_2\cdot\nabla u=\frac{\rho-\theta-\theta\rho_2+\theta_2\rho}{(1+\rho_1)(1+\rho_2)}\nabla\rho_1+\frac{\rho_2-\theta_2}{1+\rho_2}\nabla\rho, \tag{4.2}$$

$$\begin{aligned}\frac{3}{2}\theta_t+\theta+\mathrm{div}u-\triangle\theta=&-\frac{3}{2}u\nabla\theta_1-\frac{3}{2}u_2\nabla\theta-\theta\mathrm{div}u_1-\theta_2\mathrm{div}u+\kappa(|u_1|^2-|u_2|^2)\\&-\frac{\rho}{(1+\rho_1)(1+\rho_2)}\triangle\theta_1+\frac{\rho_2}{1+\rho_2}\triangle\theta,\end{aligned} \tag{4.3}$$

$$E=\nabla\phi, \tag{4.4}$$

$$\triangle\phi=\rho. \tag{4.5}$$

Take the L^2 inner product of Equations (4.1)-(4.3) with ρ,u,θ, respectively. There holds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\rho||^2+\langle\mathrm{div}u,\rho\rangle=\langle-\mathrm{div}(\rho u_1)-\mathrm{div}(\rho_2 u),\rho\rangle,\\&\frac{1}{2}\frac{d}{dt}||u||^2+||u||-\langle E,u\rangle+\langle\nabla\theta,u\rangle+\langle\nabla\rho,u\rangle+\langle u\cdot\nabla_1,u\rangle+\langle u_2\cdot\nabla u,u\rangle\\=&\langle\frac{\rho-\theta-\theta\rho_2+\theta_2\rho}{(1+\rho_1)(1+\rho_2)}\nabla\rho_1,u\rangle+\langle\frac{\rho_2-\theta_2}{1+\rho_2}\nabla\rho,u\rangle,\\&\frac{3}{4}\frac{d}{dt}||\theta||^2+||\theta||^2+\langle\mathrm{div}u,\theta\rangle+||\nabla\theta||^2=\langle-\frac{3}{2}u\nabla\theta_1-\frac{3}{2}u_2\nabla\theta-\theta\mathrm{div}u_1\\&-\theta_2\mathrm{div}u+\kappa(|u_1|^2-|u_2|^2)-\frac{\rho}{(1+\rho_1)(1+\rho_2)}\triangle\theta_1+\frac{\rho_2}{1+\rho_2}\triangle\theta,\theta\rangle.\end{aligned}$$

By using Hölder's, Sobolev's and Young's inequalities, we obtain that

$$\frac{1}{2}\frac{d}{dt}||\rho||^2+\langle\mathrm{div}u,\rho\rangle\le C||\rho||(||\rho||_{H^1}||u_1||_{H^2}+||\rho_2||_{H^2}||u||_{H^1}), \tag{4.6}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u||^2+\frac{1}{2}||u||^2+\langle\nabla\theta,u\rangle+\langle\nabla\rho,u\rangle\le& C\big(||\rho||^2+||u||_{H^1}^2(||u_1||_{H^2}^2+||u_2||_{H^2}^2)\\&+||\rho_1||_{H^3}^2(||\rho||^2+||\theta||^2)+||\rho||_{H^1}^2(||\rho_2||_{H^2}^2+||\theta_2||_{H^2}^2)\big),\end{aligned} \tag{4.7}$$

$$\begin{aligned}&\frac{3}{4}\frac{d}{dt}||\theta||^2+\frac{1}{2}||\theta||^2+\langle\mathrm{div}u,\theta\rangle+||\nabla\theta||^2\le C\big(||u||^2||\theta_1||_{H^3}^2+||u_2||_{H^2}^2||\theta||_{H^1}^2+||\theta||^2||u_1||_{H^3}^2\\&+||\theta_2||_{H^2}^2||u||_{H^1}^2+(||u_1||_{H^2}^2+||u_2||_{H^2}^2)||u||^2+||\rho||_{H^1}^2||\theta_1||_{H^3}^2+||\rho_2||_{H^2}^2||\theta||_{H^2}^2\big).\end{aligned} \tag{4.8}$$

On the other hand, multiplying Equation (4.2) by $-\nabla\mathrm{div}u$ and integrating over Ω by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\mathrm{div}u||^2+||\mathrm{div}u||^2+\langle E,\nabla\mathrm{div}u\rangle-\langle\nabla\theta,\nabla\mathrm{div}u\rangle-\langle\nabla\rho,\nabla\mathrm{div}u\rangle=\langle u\cdot\nabla u_1,\nabla\mathrm{div}u\rangle\\&+\langle u_2\cdot\nabla u,\nabla\mathrm{div}u\rangle+\langle\frac{\rho-\theta-\theta\rho_2+\theta_2\rho}{(1+\rho_1)(1+\rho_2)}\nabla\rho,-\nabla\mathrm{div}u\rangle+\langle\frac{\rho_2-\theta_2}{1+\rho_2}\nabla\rho,-\nabla\mathrm{div}u\rangle.\end{aligned}$$

Hence, there holds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||\mathrm{div}u||^2+||\mathrm{div}u||^2-\langle\rho,\mathrm{div}u\rangle-\langle\nabla\theta,\nabla\mathrm{div}u\rangle-\langle\nabla\rho,\nabla\mathrm{div}u\rangle\\\le& C\big(||\nabla u_1||_\infty||\nabla u||^2+||\triangle u_1||_4||u||_4||\mathrm{div}u||+||\nabla u_2||_\infty||\nabla u||^2+||\mathrm{div}u||||\nabla\rho||_2(||\rho||_{H^3}\\&+||\theta||_{H^3}+||\rho_2||_{H^3}||\theta||_{H^3}+||\theta_2||_{H^3}||\rho||_{H^3})+||\mathrm{div}u||||\triangle\rho||_4(||\rho||_4+||\theta||_4)\end{aligned}$$

$$+||\text{div}u||||\nabla\rho||(||\rho_2||_{H^3}+||\theta_2||_{H^3}))+|\langle\frac{\rho_2-\theta_2}{1+\rho_2}\triangle\rho,\text{div}u\rangle|.$$

By using the representation of $\nabla\rho$ from (4.2), we infer that

$$\langle\frac{\rho_2-\theta_2}{1+\rho_2}\triangle\rho,\text{div}u\rangle=\langle\frac{\rho_2-\theta_2}{1+\rho_2}\text{div}[\frac{1+\rho_2}{1+\theta_2}(-u_t-u+E-\nabla\theta-u\cdot\nabla u_1-u_2\cdot\nabla u$$
$$+\frac{\rho-\theta-\theta\rho_2+\theta_2\rho}{(1+\rho_1)(1+\rho_2)}\nabla\rho_1)],\text{div}u\rangle.$$

Note that

$$\langle\frac{\rho_2-\theta_2}{1+\rho_2}\frac{1+\rho_2}{1+\theta_2}(-\text{div}u_t),\text{div}u\rangle=\langle\frac{\rho_2-\theta_2}{1+\theta_2},-\frac{1}{2}\frac{d}{dt}|\text{div}u|^2\rangle$$
$$=-\frac{1}{2}\frac{d}{dt}\langle\frac{\rho_2-\theta_2}{1+\theta_2},|\text{div}u|^2\rangle+\frac{1}{2}\langle\frac{d}{dt}(\frac{\rho_2-\theta_2}{1+\theta_2}),|\text{div}u|^2\rangle,$$
$$\langle\frac{d}{dt}(\frac{\rho_2-\theta_2}{1+\theta_2}),|\text{div}u|^2\rangle\leq C(||\partial_t\rho_2||_\infty+||\partial_t\theta_2||_\infty)||\text{div}u||^2$$
$$\leq C(||\text{div}u_2+\text{div}(\rho_2u_2)||_\infty+||\theta_2+\text{div}u_2-\triangle\theta_2+\frac{3}{2}u_2\cdot\nabla\theta_2+\theta_2\text{div}u_2-\kappa|u_2|^2$$
$$+\frac{\rho_2}{1+\rho_2}\triangle\theta_2||_\infty)||\text{div}u||^2$$
$$\leq C(||u_2||_{H^3}+||\rho_2||_{H^3}||u_2||_{H^3}+||\theta_2||_{H^4}+||u_2||^2_{H^2}+||u_2||_{H^3}||\theta_2||_{H^3})||\text{div}u||^2,$$

and

$$\langle\frac{\rho_2-\theta_2}{1+\rho_2}\text{div}[\frac{1+\rho_2}{1+\theta_2}(-u+E-\nabla\theta-u\cdot\nabla u_1-u_2\cdot\nabla u+\frac{\rho-\theta-\theta\rho_2+\theta_2\rho}{(1+\rho_1)(1+\rho_2)}\nabla\rho_1)],\text{div}u\rangle$$
$$\leq C||\text{div}u||(||\rho_2||_\infty+||\theta_2||_\infty)\big[(||\nabla\rho_2||_\infty+||\nabla\theta_2||_\infty)(||u||+||E||+||\nabla\theta||+||\nabla u_1||_\infty||u||$$
$$+||u_2||_\infty||\nabla u||+||\nabla\rho_1||_\infty(||\rho||+||\theta||))+(||\rho_2||_\infty+||\theta_2||_\infty)(||\text{div}u||+||\text{div}E||+||\triangle\theta||$$
$$+||u_1||_{H^3}||u||_4+||\nabla u_1||_\infty||\nabla u||+||\nabla u_2||_\infty||\nabla u||+(||\rho||_4+||\theta||_4)||\triangle\rho_1||_4$$
$$+(||\nabla\rho||+||\nabla\theta||)||\nabla\rho_1||_\infty)\big].$$

It concludes that

$$\frac{1}{2}\frac{d}{dt}||\text{div}u||^2+||\text{div}u||^2-\langle\rho,\text{div}u\rangle-\langle\nabla\theta,\nabla\text{div}u\rangle-\langle\nabla\rho,\nabla\text{div}u\rangle$$
$$\leq C(||u_1||_{H^3}||u||^2_{H^1}+||u_2||_{H^3}||u||^2_{H^1}+||u||_{H^1}||\rho||_{H^1}(||\rho||_{H^3}+||\theta||_{H^3}+||\rho_2||_{H^3}||\theta||_{H^3}$$
$$+||\theta_2||_{H^3}||\rho||_{H^3})+||u||_{H^1}||\rho||_{H^3}(||\rho||_{H^1}+||\theta||_{H^1})+||u||_{H^1}||\rho||_{H^1}(||\rho_2||_{H^3}+||\theta_2||_{H^3}))$$
$$+C(||u_2||_{H^3}+||\rho_2||_{H^3}||u_2||_{H^3}+||\theta_2||_{H^4}+||u_2||^2_{H^2}+||u_2||_{H^3}||\theta_2||_{H^3})||u||^2_{H^1}$$
$$+C||u||_{H^1}(||\rho_2||_{H^3}+||\theta_2||_{H^3})[||u||_{H^1}+||\rho||_{H^1}+||\theta||_{H^2}+(||u_1||_{H^3}+||u_2||)||u||_{H^1}$$
$$+||\rho_1||_{H^3}(||\rho||_{H^1}+||\theta||_{H^1})+||\rho||_{H^1}(||\rho_2||_{H^2}+||\theta_2||_{H^2})]+\frac{1}{2}\frac{d}{dt}\langle\frac{\rho_2-\theta_2}{1+\theta_2},|\text{div}u|^2\rangle. \tag{4.9}$$

Similarly, we get

$$\frac{1}{2}\frac{d}{dt}||\text{curl}u||^2+||\text{curl}u||^2$$
$$\leq C(||u_1||_{H^3}||u||^2_{H^1}+||u_2||_{H^3}||u||^2_{H^1}+||u||_{H^1}||\rho||_{H^1}(||\rho||_{H^3}+||\theta||_{H^3}+||\rho_2||_{H^3}||\theta||_{H^3}$$

$$+||\theta_2||_{H^3}||\rho||_{H^3})+||u||_{H^1}||\rho||_{H^3}(||\rho||_{H^1}+||\theta||_{H^1})+||u||_{H^1}||\rho||_{H^1}(||\rho_2||_{H^3}+||\theta_2||_{H^3}))$$
$$+C(||u_2||_{H^3}+||\rho_2||_{H^3}||u_2||_{H^3}+||\theta_2||_{H^4}+||u_2||^2_{H^2}+||u_2||_{H^3}||\theta_2||_{H^3})||u||^2_{H^1}$$
$$+C||u||_{H^1}(||\rho_2||_{H^3}+||\theta_2||_{H^3})[||u||_{H^1}+||\rho||_{H^1}+||\theta||_{H^2}+(||u_1||_{H^3}+||u_2||)||u||_{H^1}$$
$$+||\rho_1||_{H^3}(||\rho||_{H^1}+||\theta||_{H^1})+||\rho||_{H^1}(||\rho_2||_{H^2}+||\theta_2||_{H^2})]+\frac{1}{2}\frac{d}{dt}\langle\frac{\rho_2-\theta_2}{1+\theta_2},|\mathrm{div}u|^2\rangle. \quad (4.10)$$

Multiplying Equation (4.3) by $-\triangle\theta$ and integrating over Ω by parts, we have

$$\frac{3}{4}\frac{d}{dt}||\nabla\theta||^2+||\nabla\theta||^2+\langle\nabla\mathrm{div}u,\nabla\theta\rangle+\frac{1}{2}||\triangle\theta||^2$$
$$\le C\big(||u||^2_{H^1}(||\theta_1||^2_{H^3}+||\theta_2||^2_{H^2}+||u_1||^2_{H^2}+||u_2||^2_{H^2})+||\theta||^2_{H^2}(||u_1||^2_{H^3}+||u_2||^2_{H^2}+||\rho_2||^2_{H^2})$$
$$+||\theta_1||^2_{H^3}||\rho||^2_{H^1}\big). \quad (4.11)$$

Multiplying Equation (4.1) by $-\triangle\rho$ and integrating over Ω by parts yields

$$\frac{1}{2}\frac{d}{dt}||\nabla\rho||^2+\langle\nabla\mathrm{div}u,\nabla\rho\rangle=-\langle\nabla(\rho\mathrm{div}u_1+u_1\nabla\rho+\rho_2\mathrm{div}u+u\nabla\rho_2),\nabla\rho\rangle.$$

Note that

$$-\langle u_1\nabla^2\rho,\nabla\rho\rangle=\frac{1}{2}\int\mathrm{div}u_1|\nabla\rho|^2dx\le C||\nabla u_1||_\infty||\nabla\rho||^2.$$

We have

$$\frac{1}{2}\frac{d}{dt}||\nabla\rho||^2+\langle\nabla\mathrm{div}u,\nabla\rho\rangle\le C||\nabla\rho||(||\nabla\rho||||u_1||_{H^3}+||\rho||_4||u_1||_{H^3}+||u_1||_{H^3}||\nabla\rho||$$
$$+||\rho_2||_{H^3}||\mathrm{div}u||+||\nabla u||||\rho_2||_{H^3}+||u||_4||\rho_2||_{H^3})-\langle\rho_2\nabla\mathrm{div}u,\nabla\rho\rangle.$$

By using the representation of div$u$ from (4.1), there holds

$$-\langle\rho_2\nabla\mathrm{div}u,\nabla\rho\rangle=\langle\rho_2\nabla(\frac{\rho_t+u_1\cdot\nabla\rho+\rho\mathrm{div}u_1+u\cdot\nabla\rho_2}{1+\rho_2}),\nabla\rho\rangle.$$

Note that

$$\langle\frac{\rho_2}{1+\rho_2}\nabla\rho_t,\nabla\rho\rangle=\int\frac{\rho_2}{1+\rho_2}\frac{1}{2}\frac{d}{dt}|\nabla\rho|^2dx$$
$$=\frac{1}{2}\frac{d}{dt}\int\frac{\rho_2}{1+\rho_2}|\nabla\rho|^2dx+\frac{1}{2}\int|\nabla\rho|^2\frac{1}{(1+\rho_2)^2}(\mathrm{div}u_2+\mathrm{div}(\rho_2u_2))dx$$
$$\le\frac{1}{2}\frac{d}{dt}\int\frac{\rho_2}{1+\rho_2}|\nabla\rho|^2dx+C||\nabla\rho||^2(||u_2||_{H^3}+||\rho_2||_{H^3}||u_2||_{H^3}),$$

and

$$\langle\frac{\rho_2}{1+\rho_2}u_1\nabla^2\rho,\nabla\rho\rangle=-\frac{1}{2}\int\mathrm{div}(\frac{\rho_2}{1+\rho_2}u_1)|\nabla\rho|^2dx\le C||\nabla\rho||^2||u_1||_{H^3}||\rho_2||_{H^3}.$$

It concludes that

$$\frac{1}{2}\frac{d}{dt}||\nabla\rho||^2+\langle\nabla\mathrm{div}u,\nabla\rho\rangle$$

$$\leq C||\rho||_{H^1}(||\rho||_{H^1}||u_1||_{H^3}+||u||_{H^1}||\rho_2||_{H^3})+\frac{1}{2}\frac{d}{dt}\int\frac{\rho_2}{1+\rho_2}|\nabla\rho|^2dx+C||\rho||_{H^1}^2(||u_2||_{H^3}$$
$$+||\rho_2||_{H^3}||u_2||_{H^3})+C||\rho||_{H^1}^2||u||_{H^3}||\rho_2||_{H^3}+C||\rho_2||_{H^3}||\rho||_{H^1}(||u_1||_{H^3}||\rho||_{H^1}$$
$$+||u||_{H^1}||\rho||_{H^3}). \tag{4.12}$$

Now, we give the estimate of $||\rho||_{H^1}$. Multiplying Equation (4.2) by $\nabla\rho$ and integrating it over Ω by parts, there holds

$$-\langle \text{div}\partial_t u,\rho\rangle+||\nabla\rho||^2+\langle\nabla\theta,\nabla\rho\rangle-\langle\text{div}u,\rho\rangle-\langle E,\nabla\rho\rangle+\langle u\cdot\nabla u_1,\nabla\rho\rangle+\langle u_2\cdot\nabla u,\nabla\rho\rangle$$
$$=\int\frac{\rho-\theta-\theta\rho_2+\theta_2\rho}{(1+\rho_1)(1+\rho_2)}\nabla\rho_1\nabla\rho dx+\int\frac{\rho_2-\theta_2}{1+\rho_2}|\nabla\rho|^2dx.$$

Note that

$$-\langle\text{div}\partial_t u,\rho\rangle=-\frac{d}{dt}\langle\text{div}u,\rho\rangle+\langle\text{div}u,-\text{div}u-\text{div}(\rho u_1)-\text{div}(\rho_2 u)\rangle.$$

We have

$$\frac{1}{2}||\rho||^2-\frac{d}{dt}\langle\text{div}u,\rho\rangle$$
$$\leq C\big(||\nabla\theta||^2+||\text{div}u||^2+||\rho||_{H^1}||u||_{H^1}(||u_1||_{H^2}+||u_2||_{H^2})+||\rho||_{H^1}^2||u_1||_{H^2}^2$$
$$+||\rho_2||_{H^2}^2||u||_{H^1}^2+(||\rho_2||_{H^2}+||\theta_2||_{H^2})||\rho||_{H^1}^2+||\rho||_{H^1}||\rho_1||_{H^3}(||\rho||+||\theta||)\big). \tag{4.13}$$

Choosing $M_4>0$ appropriately large, summing up $\frac{1}{M_4}\times(4.13)+(4.9)+(4.10)+(4.11)+(4.12)+(4.6)+\frac{1}{8CM_4}\times(4.7)+(4.8)$, integrating from 0 to T and by the periodicity of solutions $(\rho_1,u_1,\theta_1),(\rho_2,u_2,\theta_2)\in X_\delta$, we have

$$\int_0^T(||\rho||_{H^1}^2+||u||_{H^1}^2+||\theta||_{H^2}^2)dt\leq C\delta\int_0^T(||\rho||_{H^1}^2+||u||_{H^1}^2+||\theta||_{H^2}^2)dt.$$

It implies that $\rho=u=\theta=0$ providing $\delta<\frac{1}{2C}$ small enough. The proof of uniqueness is complete. □

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REFERENCES

[1] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag Berlin, Heidelberg, 1990. 1

[2] M. Rudin, A. Gnudi, and W. Quade, *A generalized approach to the hydrodynamic model of semiconductor equations*, in G. Baccmani (ed.) Process and Device Modelling for Microelectronics, Elsevier Science, Amsterdam, 1993. 1

[3] S. Sze, *Physics of Semiconductor Devices*, Wiley, New York, 2006. 1

[4] T. Luo, R. Natalini, and Z. Xin, *Large time behavior of the solutions to a hydrodynamic model for semiconductors*, SIAM J. Appl. Math., 59:810–830, 1998. 1

[5] L. Hisao and S. Wang, *Asymptotic behavior of global smooth solutions to the full 1D hydrodynamic model for semiconductors*, Math. Models Meth. Appl. Sci., 12:777–796, 2002. 1

[6] B. Zhang, *Global existence and asymptotic stability to the full 1D hydrodynamic model for semiconductor devices*, Indiana Univ. Math. J., 44:971–1005, 1995. 1

[7] C. Zhu and H. Hattori, *Asymptotic behavior of the solution to a nonisentropic hydrodynamic model of semiconductors*, J. Diff. Eqs., 144:353–389, 1998. 1

[8] S. Nishibata and M. Suzuki, *Asymptotic stability of a stationary solution to a thermal hydrodynamic model for semiconductors*, Arch. Ration. Mech. Anal., 192:187–215, 2009. 1

[9] S. Nishibata and M. Suzuki, *Relaxation limit and initial layer to hydrodynamic models for semiconductors*, J. Diff. Eqs., 249:1385–1409, 2010. 1

[10] L. Hisao, S. Jiang, and P. Zhang, *Global existence and exponential stability of smooth solutions to a full hydrodynamic model to semiconductors*, Monatsh. Math., 136:269–285, 2002. 1

[11] J. Xu and W. Yong, *Relaxation–time limits of non–isentropic hydrodynamic models for semiconductors*, J. Diff. Eqs., 247:1777–1795, 2009. 1

[12] Y. Li, *Large–time behaviour of the solutions for a multidimensional non–isentropic hydrodynamic model for semiconductors*, Proc. Edinb. Math. Soc., 49:145–172, 2006. 1

[13] P. Wang and J. Xu, *Relaxation–time limit and initial layer in the isentropic hydrodynamic model for semiconductors*, J. Math. Anal. Appl., 414:402–413, 2014. 1

[14] E. Feireisl, P. B. Mucha, A. Novotný, and M. Pokorný, *Time–periodic solutions to the full Navier–Stokes–Fourier system*, Arch. Ration. Mech. Anal., 204:745–786, 2012. 1

[15] E. Feireisl, Š. Matušu–Nečasová, H. Petzeltová, and I. Straškraba, *On the motion of a viscous compressible fluid driven by a time–periodic external force*, Arch. Ration. Mech. Anal., 149:69–96, 1999. 1

[16] A. Valli, *Periodic and stationary solutions for compressible Navier–Stokes equations via a stability method*, Ann. Sc. Norm. Super. Pisa Cl. Sci., 10:607–647, 1983. 1

[17] J. Serrin, *A note on the exstencie of periodic solutions of the Navier–Stokes equations*, Arch. Ration. Mech. Anal., 3:120–122, 1959. 1

[18] C. Jin and T. Yang, *Time periodic solution for a 3D compressible Navier–Stokes system with an external force in* $\mathbb{R}^3$, J. Diff. Eqs., 259:2576–2601, 2015. 1

[19] C. Jin and T. Yang, *Time periodic solution to the compressible Navier–Stokes equations in a periodic domain*, Acta Math. Sci. Ser. A Chin. Ed., 36:1015–1029, 2016. 1

[20] Z. Tan, Q. Xu, and H. Qiao, *Time periodic solution to the compressible Euler equations with damping in a periodic domain*, Nonlinearity, 29:2024–2049, 2016. 1

[21] J. L. Massera, *The existence of periodic solutions of systems of differential equations*, Duke Math. J., 17:457–475, 1950. 1

[22] Y. Li , F. Z. Cong, Z. H. Lin, and W. B. Liu, *Periodic solutions for evolution equations*, Nonlinear Anal., 36:275–293, 1999. 1

[23] M. Cheng, *Time–periodic and stationary solutions to the compressible Hall-magnetohydrodynamic system*, Z. Angew. Math. Phys., 68:38, 2017. 1

[24] M. E. Taylor, *Partial Differential Equations III. Nonlinear Equations*, Second Edition, Appl. Math. Sci., Springer, New York, 117, 2011. 2

[25] A. Tychonoff, *Ein Fixpunktsatz*, Math. Ann., 111:767–776, 1935. 3.1