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SHARP ENERGY CRITERIA AND SINGULARITY OF BLOW-UP SOLUTIONS FOR THE DAVEY-STEWARTSON SYSTEM*

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Abstract. By analyzing the geometric characteristic of related algebra equations, we first find the sharp energy criteria of singular solutions and global solutions for the Davey-Stewartson systems. Then, we study the limiting behavior of singular solutions, and obtain the rate of convergence of singular solutions, rate of concentration of singular solutions for Davey-Stewartson systems with the L^2 super-critical nonlinearity.

Keywords. Davey-Stewartson system; Singular solution; Sharp energy criteria; Concentration.

AMS subject classifications. 35Q55; 45G05.

1. Introduction

The Davey-Stewartson systems were firstly proposed by Davey and Stewartson in 1974 (see [5, 28]). The original Davey-Stewartson system is the two coupled nonlinear differential equations, which describe the evolution of a three dimensional wave packet on water of finite depth. In terms of dimensionless variables, the two coupled nonlinear differential equations have the following form.

$$\begin{cases} i\psi_t + \psi_{x_1x_1} + \mu\psi_{x_2x_2} = a|\psi|^2\psi + b\psi\phi_{x_1}, \\ \nu\phi_{x_1x_1} + \phi_{x_2x_2} = -c(|\psi|^2)_{x_1}, \end{cases} \tag{1.1}$$

where $\psi(t,x_1,x_2)$ is the (complex) amplitude and $\phi(t,x_1,x_2)$ is the (real) mean velocity potential. The parameters $\mu,\nu>0$, a,b,c are real constants. In particular, as a mathematical model for the evolution of shallow-water waves, the Davey-Stewartson system is usually classified as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic according to the signs of (μ,ν): $(+,+)$, $(+,-)$, $(-,+)$, $(-,-)$. But the last case $(-,-)$ does not occur in the context of water waves. For the derivation and overview of the physical models for the Davey-Stewartson systems, one can refer to Chapter 11 and 12 in [28]. Usually, by using the Fourier transform of the second equation in Equation (1.1), the coupled system (1.1) is changed to the single equation (see [4, 8, 33]), and the generalized Davey-Stewartson system has the following form:

$$i\psi_t + \triangle\psi + a|\psi|^{p-1}\psi + bE(|\psi|^2)\psi = 0, \tag{1.2}$$

$$\psi(0,x) = \psi_0, \tag{1.3}$$

where $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$ is the Laplace operator on $\mathbb{R}^N$; E is the singular integral operator with symbol $\sigma_1(\xi) = \frac{{\xi_1}^2}{|\xi|^2}$, $\xi \in \mathbb{R}^N$, $E(|\psi|^2) = \mathcal{F}^{-1}[\frac{{\xi_1}^2}{|\xi|^2}\mathcal{F}[|\psi|^2]]$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and Fourier inverse transform on $\mathbb{R}^N$, respectively.

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As the operator E is nonlocal in physical space and singular in frequency space, studying the Davey-Stewartson system and its extensions is interesting and challenging. From the view-point of physics, the following problems are very basic and important. How large will the initial data be, whose corresponding wave will become unstable to the singular solution (i.e. there exists a finite time $0<T<+\infty$ such that $\lim_{t\to T^-}\|\psi(t)\|_{H^1}=+\infty$, which is also called the collapse solution or the blow-up solution)? How small will the initial data be, whose corresponding wave will be stable for all time (the global solution)? Then, the sharp thresholds of singular solutions and global solutions for Equation (1.2) are pursued strongly, as well as the structure of singular solutions, including the rate of convergence of singularity, the limiting behavior of singular solutions etc.

Plenty of mathematicians have devoted contributions on the global solutions and singular solutions for the Davey-Stewartson system. Here, we just list some works but this list is not exhaustive. Ghidaglia and Saut [8], Hayashi and Saut [13], and Linares and Ponce [20] studied global well-posedness for some generalized Davey-Stewartson systems in different spaces. Wang and Guo [29] investigated the initial value problem and scattering of global solutions to the generalized Davey-Stewartson systems. Cipolatti [3], Ohta [22,23] studied the existence and stability of the standing waves for the Davey-Stewartson system in the original and generalized forms. Gan and Zhang [12] studied the sharp threshold of blow-up and global existence, and proved that the standing waves are strongly unstable in the general form by the cross-constraint variational method. Ozawa in [24], Brown and Perry in [1] studied the exact singular solutions of the hyperbolic-elliptic Davey-Stewartson system, which is a very different Davey-Stewartson system in dimension two, namely the focusing DS II system. Papanicolaou, Sulem, and Wang [25] studied the rate of convergence of singular solutions for Equation (1.2) by numerical observations.

Motivated by these problems, we study the singular solutions of Equation (1.2) in the focusing case: $a=b=1$. First, we find the sharp energy criteria of singular solutions and global solutions for Equation (1.2) in Section 3 by using three sharp Gagliardo-Nirenberg-type inequalities.

By comparing with the results in [11,12,27], the sharp energy criteria obtained in this paper are precisely in the sense that they were expressed by the L^2-norms of the corresponding ground state solutions. These results also improved our previous results in [19,32] by extending the range of p to $1<p\leq 3$, and here these sharp energy criteria of singular solutions and global solutions may have a potential application for the physical scientists in the experimental study of the Davey-Stewartson systems.

For the rest of this paper, we are focusing on the limiting behavior of singular solutions for Equation (1.2). For the L^2 critical case: $p=3$ and $N=2$, Li, Zhang, Lai and Wu [18] studied the rate of convergence of singular, limiting behavior and concentration of singular solutions in H^1. Feng, Ren and Wang [7] studied the blow-up solutions with several blow-up points. Richards [26] studied the concentration of singular solutions in L^2, Zhu [33,34] proved the limiting profile and concentration of singular solutions in L^2 by the nonlinear profile decomposition argument.

However, when $N=3$ and $p\neq 3$, Equation (1.2) is not L^2 critical and loses the scaling invariance. The well-known arguments in [17,21] to study the dynamics of singular solutions for the L^2 critical nonlinear Schrödinger equations fail. We first inject the sharp Gagliardo-Nirenberg-type inequalities into the Strichartz estimates. And then we can obtain the following rate of convergence of singular solutions for Equation (1.2) with $p\neq 3$.

THEOREM 1.1. *Let $N=3$ and $\psi_0\in H^1$. Assume that $\psi(t,x)$ is the corresponding singular solution of the Cauchy problem (1.2)-(1.3), where $0<T<+\infty$ is the singular time. Then, we have the following estimates.*

(i) For $2\le p<3$, there exists $C>0$ such that

$$\|\nabla\psi(t)\|_2\ge\frac{C}{(T-t)^{\frac{1}{6}}}\quad\text{for}\quad 0\le t<T<+\infty. \tag{1.4}$$

(ii) For $q>4$ and $2\le p<3$, there exists $C>0$ such that

$$\|\psi(t)\|_q\ge\frac{C}{(T-t)^{\frac{q-2}{6q}}}\quad\text{for}\quad 0\le t<T<+\infty. \tag{1.5}$$

(iii) For $3<q\le 4$ and $2\le p<\frac{7}{3}$, there exists $C>0$ such that

$$\|\psi(t)\|_q\ge\frac{C}{(T-t)^{\frac{1}{12}}}\quad\text{for}\quad 0\le t<T<+\infty. \tag{1.6}$$

In the proof of the above theorem, we adapt the ideas and techniques from [2]. But, when $p\neq 3$, Equation (1.2) does not have the scaling invariance, and the main difficulty is that there is no comparison of $\|\psi\|_{p_1}$ and $\|\psi\|_{p_2}$ when $p_1\neq p_2$. And we need some new ideas to obtain a uniform estimate for

$$\|\nabla(E(|\psi|^2)\psi)\|_{L_t^{\frac{8}{5}}((t,\tau);L_x^{\frac{4}{3}})}\quad\text{and}\quad\|\nabla(|\psi|^{p-1}\psi)\|_{L_t^{\frac{8}{5}}((t,\tau);L_x^{\frac{4}{3}})}.$$

And then we can obtain the rate of convergence of singular solutions for Equation (1.2) in Theorem 1.1.

Moreover, as a direct application of the rate of convergence of singular solutions, we can obtain the rate of $\dot{H}^{\frac{1}{2}}$-norm concentration of singular solutions for Equation (1.2). Let Q be the solution of

$$-\triangle\tilde{Q}+(-\triangle)^{\frac{1}{2}}\tilde{Q}-E(|\tilde{Q}|^2)\tilde{Q}=0, \tag{1.7}$$

where the pseudo-differential operator $(-\triangle)^s$ is defined by $\mathcal{F}[(-\triangle)^s f](\xi)\equiv|\xi|^{2s}\mathcal{F}[f](\xi)$ for $s\in\mathbb{R}$, which in turn defines the homogeneous Sobolev space $\dot{H}^s=\dot{H}^s(\mathbb{R}^N)\equiv\{f\in\mathcal{S}'(\mathbb{R}^N):\int|\xi|^{2s}|\mathcal{F}[f](\xi)|^2d\xi<\infty\}$ with its norm defined by $\|f\|_{\dot{H}^s}=\|(-\triangle)^{\frac{s}{2}}f\|_2$, where $\mathcal{S}'(\mathbb{R}^N)$ denotes the space of tempered distributions. The existence of nontrivial solutions of (1.7) has been given in [6]. Now, we obtain the rate of $\dot{H}^{\frac{1}{2}}$-norm concentration of singular solutions for Equation (1.2), which improves Feng and Cai's results in [6], as follows.

THEOREM 1.2. *Let $N=3$, $2\le p<3$ and $\psi_0\in H^1$. Assume that $\psi(t,x)$ is the corresponding blow-up solution of the Cauchy problem (1.2)-(1.3) satisfying $\sup_{t\in[0,T)}\|\psi(t)\|_{\dot{H}^{\frac{1}{2}}}<+\infty$, where $0<T<+\infty$ is the blow-up time. Then, there exists $y(t)\in\mathbb{R}^3$ such that*

$$\liminf_{t\in[0,T)}\|\psi(t,x)\|_{\dot{H}^{\frac{1}{2}}(|x-y(t)|\le(T-t)^{\frac{1}{6}-})}\ge\|\tilde{Q}\|^2_{\dot{H}^{\frac{1}{2}}}, \tag{1.8}$$

where $\tilde{Q}$ is the solution of (1.7) and $\frac{1}{6}-$ is denoted as the real number $\frac{1}{6}-\epsilon$ for any sufficiently small $\epsilon>0$.

2. Notations and preliminaries

In this paper, we use the notation $L^q := L^q(\mathbb{R}^3)$, $\|\cdot\|_{L^q(\mathbb{R}^3)} := \|\cdot\|_q$, $H^s := H^s(\mathbb{R}^3)$ and C standing for variant absolute constants. First, we recall some known facts of the singular integral operator E.

LEMMA 2.1 ([3, 9, 10]). *Let E be the singular integral operator defined in Fourier variables by*

$$\mathcal{F}[E(\psi)](\xi) = \sigma_1(\xi)\mathcal{F}[\psi](\xi),$$

where $\sigma_1(\xi) = \frac{{\xi_1}^2}{|\xi|^2}$, $\xi \in \mathbb{R}^3$ and $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^3$. Assume $1 < p < +\infty$.

(i) $E \in \mathcal{L}(L^p, L^p)$, where $\mathcal{L}(L^p, L^p)$ denotes the space of bounded linear operators from L^p to L^p.

(ii) If $\psi \in H^s$, then $E(\psi) \in H^s$, $s \in \mathbb{R}$.

(iii) If $\psi \in W^{m,p}$, then $E(\psi) \in W^{m,p}$ and $\partial_k E(\psi) = E(\partial_k \psi), k = 1, 2$.

(iv) E preserves the following operations:
- *translation: $E(\psi(\cdot + y))(x) = E(\psi)(x+y), y \in \mathbb{R}^3$;*
- *dilatation: $E(\psi(\lambda\cdot))(x) = E(\psi)(\lambda x), \lambda > 0$;*
- *conjugation: $\overline{E(\psi)} = E(\overline{\psi})$, where $\overline{\psi}$ is the complex conjugate of ψ.*

Guo and Wang [10] studied the local and global well-posedness for the Cauchy-problem (1.2)-(1.3). In particular, the local well-posedness is in the following.

PROPOSITION 2.1 ([10]). *Let $N = 3$ and $\psi_0 \in H^1$. There exists a unique solution $\psi(t,x)$ of the Cauchy problem (1.2)-(1.3) on the maximal time interval $[0,T)$ such that $\psi(t,x) \in C([0,T); H^1)$ and either $T = +\infty$ (global existence), or else $0 < T < +\infty$ and $\lim_{t\to T} \|\psi(t,x)\|_{H^1} = +\infty$ (blow-up). Furthermore, for all $t \in [0,T)$, $\psi(t,x)$ satisfies the conservation laws:*

(i) Conservation of mass:

$$M[\psi(t)] := \int |\psi(t,x)|^2 dx = M[\psi_0].$$

(ii) Conservation of energy:

$$H[\psi(t)] := \int (|\nabla\psi(t)|^2 - \frac{2}{p+1}|\psi(t)|^{p+1} - \frac{1}{2}E(|\psi(t)|^2)|\psi(t)|^2)dx = H[\psi_0].$$

Furthermore, in order to study the singular solution for Equation (1.2), in terms of Weinstein's arguments in [30], we need the following proposition, which can be obtained by some basic calculus(see also [12, 22, 23]).

LEMMA 2.2 ([12, 22, 23]). *Assume that $N = 3$, $\psi_0 \in H^1$, $|x|\psi_0 \in L^2$ and the corresponding solution $\psi(t,x)$ of the Cauchy-problem (1.2)-(1.3) on the interval $[0,T)$. Then, for all $t \in [0,T)$ we have $|x|\psi(t,x) \in L^2$. Moreover, let $J(t) := \int |x|^2|\psi(t,x)|^2 dx$. We have $J'(t) = -4\Im \int x\psi\nabla\overline{\psi}dx$, and*

$$J''(t) = 8\int |\nabla\psi|^2 dx - 12\frac{p-1}{p+1}\int |\psi|^{p+1}dx - 6\int E(|\psi|^2)|\psi|^2 dx. \tag{2.1}$$

Finally, we collect three sharp Gagliardo-Nirenberg-type inequalities.

LEMMA 2.3 ([32]). *Let $N=3$ and Q be the ground state solution of*

$$-\frac{3}{2}\triangle Q+\frac{1}{2}Q-|Q|^2Q-E(|Q|^2)Q=0, \quad Q\in H^1. \tag{2.2}$$

Then, we have the following sharp Gagliardo-Nirenberg inequality,

$$\int |u|^4+E(|u|^2)|u|^2dx\leq \frac{2}{\|Q\|_2^2}\|\nabla u\|_2^3\|u\|_2, \quad u\in H^1. \tag{2.3}$$

This inequality is sharp in the sense that the equality can be obtained by taking $u=Q$.

LEMMA 2.4 ([32]). *Let $N=3$ and R be the ground state solution of*

$$-\frac{3}{2}\triangle R+\frac{1}{2}R-E(|R|^2)R=0, \quad R\in H^1. \tag{2.4}$$

Then, we have the following sharp Gagliardo-Nirenberg inequality,

$$\int E(|u|^2)|u|^2dx\leq \frac{2}{\|R\|_2^2}\|\nabla u\|_2^3\|u\|_2, \quad u\in H^1. \tag{2.5}$$

This inequality is sharp in the sense that the equality can be obtained by taking $u=R$.

LEMMA 2.5 ([30]). *Let $N=3$ and P be the ground state solution of*

$$-\frac{3(p-1)}{4}\triangle P+\frac{5-p}{4}P-|P|^{p-1}=0, \quad P\in H^1. \tag{2.6}$$

Then, we have the following sharp Gagliardo-Nirenberg inequality,

$$\|u\|^{p+1}\leq \frac{p+1}{2\|P\|_2^{p-1}}\|\nabla u\|_2^{\frac{3(p-1)}{2}}\|u\|_2^{\frac{5-p}{2}}, \quad u\in H^1. \tag{2.7}$$

This inequality is sharp in the sense that the equality can be obtained by taking $u=P$.

3. Sharp energy criteria

In this section, we will answer the first question mentioned in the introduction: How to distinguish between the domains of initial datum for singular solutions and global solutions? The known results on this topic rely heavily on the scaling invariant of the evolution equation (see [14,15,31]). But the scaling invariant of Equation (1.2) fails when $p\neq 3$. Our main argument is choosing various ground state solutions corresponding to Equation (1.2) and constructing six pairs of invariant sets. Here, we will give the detailed proof for the case $1<p<1+\frac{2}{3}$, and for the other cases, we will just give the mainline.

Case 3.1: $1<p<1+\frac{2}{3}$

Applying the sharp Gagliardo-Nirenberg inequalities (2.5) and (2.7), we deduce that for a solution $\psi(t)$ of Equation (1.2), for all $t\in I$(the maximal existence interval)

$$\begin{aligned} H[\psi(t)]+M[\psi(t)]&\geq \|\nabla\psi(t)\|_2^2+\|\psi(t)\|_2^2-\frac{\|\psi(t)\|_2^p}{\|P\|_2^{p-1}}(\|\nabla\psi(t)\|_2^2+\|\psi(t)\|_2^2)\\ &\quad -\frac{1}{\|R\|_2^2}(\|\nabla\psi(t)\|_2^2+\|\psi(t)\|_2^2)^2\\ &=(1-\frac{\|\psi_0\|_2^p}{\|P\|_2^{p-1}})\|\psi(t)\|_{H^1}^2-\frac{1}{\|R\|_2^2}\|\psi(t)\|_{H^1}^4. \end{aligned} \tag{3.1}$$

Define $f_1(y)=(1-\frac{\|\psi_0\|_2^p}{\|P\|_2^{p-1}})y-\frac{1}{\|R\|_2^2}y^2,\quad y\geq 0$. We see that (3.1) is equivalent to

$$H[\psi(t)]+M[\psi(t)]\geq f_1(\|\psi(t)\|_{H^1}^2)$$

for all $t\in I$. Next, we will analyze the geometric characteristic of the algebra equation $f_(y)=0$ to construct the invariant flows. Obviously, $f_1(y)$ reaches the maximum at the point $y_1^*=\frac{\|R\|_2^2(\|P\|_2^{p-1}-\|\psi_0\|_2^p)}{2\|P\|_2^{p-1}}$ provided $\|P\|_2^{p-1}>\|\psi_0\|_2^p$. More precisely, $f_{1max}=f_1(y_1^*)=\frac{\|R\|_2^2(\|P\|_2^{p-1}-\|\psi_0\|_2^p)^2}{4\|P\|_2^{2(p-1)}}$. Now, we define

$$G_1:=\left\{u\in H^1: f_1(\|u\|_{H^1}^2)<H[u]+M[u]<\frac{\|R\|_2^2(\|P\|_2^{p-1}-\|\psi_0\|_2^p)^2}{4\|P\|_2^{2(p-1)}},\right.$$
$$\left.\|u\|_{H^1}^2<\frac{\|R\|_2^2(\|P\|_2^{p-1}-\|\psi_0\|_2^p)}{2\|P\|_2^{p-1}},\quad \|u\|_2^p<\|P\|_2^{p-1}\right\},$$

$$G_2:=\left\{u\in H^1: f_1(\|u\|_{H^1}^2)<H[u]+M[u]<K,\quad \|u\|_2^p<\tfrac{1}{3}\|P\|_2^{p-1},\right.$$
$$\left.\|u\|_{H^1}^2>\frac{\|R\|_2^2(4\|P\|_2^{p-1}-3(p-1)\|\psi_0\|_2^p)}{3\|P\|_2^{p-1}}\right\},$$

where $K=\frac{\|R\|_2^2(4\|P\|_2^{p-1}-3(p-1)\|\psi_0\|_2^p)(2\|P\|_2^{p-1}+3(p-3)\|\psi_0\|_2^p)}{36\|P\|_2^{2(p-1)}}$. We claim that G_1 and G_2 are two invariant evolution flows generated by the Cauchy problem (1.2)-(1.3). Indeed, let $\psi(t)$ be the solution of the Cauchy problem (1.2)-(1.3). It follows from the conservation of mass and energy that

$$f_1(\|\psi(t)\|_{H^1}^2)<H[\psi(t)]+M[\psi(t)]<\frac{\|R\|_2^2(\|P\|_2^{p-1}-\|\psi_0\|_2^p)^2}{4\|P\|_2^{2(p-1)}}:=f_{1max}$$

and $\|\psi(t)\|_2^p<\|P\|_2^{p-1}$ are true for all $t\in I$. If $\|\psi(t)\|_{H^1}^2<\frac{\|R\|_2^2(\|P\|_2^{p-1}-\|\psi_0\|_2^p)}{2\|P\|_2^{p-1}}$ is not true for all $t\in I$, then by the continuity of the solution $\psi(t,x)$ with respect to t, there exists a $t_0\in I$ such that $\|\psi(t_0)\|_{H^1}^2=\frac{\|R\|_2^2(\|P\|_2^{p-1}-\|\psi_0\|_2^p)}{2\|P\|_2^{p-1}}$. And then $f(\|\psi(t_0)\|_{H^1}^2)=f_1(y_1^*)=f_{1max}$, which is a contradiction with the fact that for all $t\in I$, $f(\|\psi(t)\|_{H^1}^2)<f_{1max}$. This completes the proof of the invariance of G_1. Furthermore, we can prove the invariance of G_2 by the same argument.

Hence, we can obtain the following Theorem.

THEOREM 3.1. *Let $N=3$, $1<p<1+\frac{2}{3}$, R and P be the ground state solution of (2.4) and (2.6) respectively. Then, we have the following:*

(i) if $\psi_0\in G_1\cup\{0\}$, then $\psi(t,x)$ exists globally in H^1;

(ii) if $\psi_0\in G_2$ and $|x|\psi_0\in L^2$, then $\psi(t,x)$ is a singular solution in a finite time, where $\psi(t,x)$ is the solution of Equation (1.2) corresponding to ψ_0.

Proof.

(i) It follows from the invariance of G_1 that the corresponding solution $\psi(t,x)$ of Equation (1.2) with the initial data $\psi_0\in G_1$, must be bounded in H^1 for all $t\in I$, and then these solutions exist globally by Proposition 2.1.

(ii) Suppose $\psi_0 \in G_2$, then the solution $\psi(t) \in G_2$, that is, for all $t \in I$

$$H[\psi(t)] + M[\psi(t)] < K \quad \text{and} \quad \|\psi(t)\|_{H^1}^2 > \frac{\|R\|_2^2(4\|P\|_2^{p-1} - 3(p-1)\|\psi_0\|_2^p)}{3\|P\|_2^{p-1}}.$$

Injecting above estimates into Lemma 2.2, we see that

$$\begin{aligned} J''(t) &\leq 12K - 4(\|\nabla\psi(t)\|_2^2 + \|\psi(t)\|_2^2) + \frac{(18-6p)\|\psi_0\|_2^p}{\|P\|_2^{p-1}}\|\nabla\psi(t)\|_2\|\psi(t)\|_2 \\ &\leq 12K - \frac{4\|P\|_2^{p-1} - (18-6p)\|\psi_0\|_2^p}{\|P\|_2^{p-1}}\|\psi(t)\|_{H^1}^2 \\ &< -\frac{\|R\|_2^2(4\|P\|_2^{p-1} - 3(p-1)\|\psi_0\|_2^p)(2\|P\|_2^{p-1} + (3p-9)\|\psi_0\|_2^p)}{3\|P\|_2^{2(p-1)}} \\ &< 0. \end{aligned}$$

This implies that for sufficiently large $|t|$, $J(t)$ is negative, while $J(t) := \int |x|^2|\psi|^2 dx$ is non-negative, which means that both T_- and T_+ are finite. Specifically, the solution $\psi(t,x)$ is a singular solution of the Cauchy problem (1.2)-(1.3). □

Case 3.2: $p = 1 + \frac{2}{3}$

Applying the sharp inequalities (2.5) and (2.7), we have, for all $t \in I$

$$H[\psi(t)] + M[\psi(t)] \geq (1 - \frac{\|\psi_0\|_2^{\frac{2}{3}}}{\|P\|_2^{\frac{2}{3}}})\|\psi(t)\|_{H^1}^2 - \frac{1}{\|R\|_2^2}\|\psi(t)\|_{H^1}^4 = f_2(\|\psi(t)\|_{H^1}^2),$$

where $f_2(y) = (1 - \frac{\|\psi_0\|_2^{\frac{2}{3}}}{\|P\|_2^{\frac{2}{3}}})y - \frac{1}{\|R\|_2^2}y^2, \quad y \geq 0$. Obviously, $f_2(y)$ reaches the maximum at the point $y_2^* = \frac{\|R\|_2^2(\|P\|_2^{\frac{2}{3}} - \|\psi_0\|_2^{\frac{2}{3}})}{2\|P\|_2^{\frac{2}{3}}}$ provided $\|P\|_2 > \|\psi_0\|_2$. More precisely, $f_{2max} = f_2(y_2^*) = \frac{\|R\|_2^2(\|P\|_2^{\frac{2}{3}} - \|\psi_0\|_2^{\frac{2}{3}})^2}{4\|P\|_2^{\frac{2}{3}}}$. Define

$$G_3 := \Big\{ u \in H^1 : f_2(\|u\|_{H^1}^2) < H[u] + M[u] < \frac{\|R\|_2^2(\|P\|_2^{\frac{2}{3}} - \|\psi_0\|_2^{\frac{2}{3}})^2}{4\|P\|_2^{\frac{4}{3}}}, \\ \|u\|_{H^1}^2 < \frac{\|R\|_2^2(\|P\|_2^{\frac{2}{3}} - \|\psi_0\|_2^{\frac{2}{3}})}{2\|P\|_2^{\frac{2}{3}}}, \|u\|_2 < \|P\|_2 \Big\},$$

$$G_4 := \Big\{ u \in H^1 : f_2(\|u\|_{H^1}^2) < H[u] + M[u] < \frac{2\|R\|_2^2(\|P\|_2^{\frac{2}{3}} - \|\psi_0\|_2^{\frac{2}{3}})^2}{9\|P\|_2^{\frac{4}{3}}}, \\ \|u\|_{H^1}^2 > \frac{2\|R\|_2^2(\|P\|_2^{\frac{2}{3}} - \|\psi_0\|_2^{\frac{2}{3}})}{3\|P\|_2^{\frac{2}{3}}}, \|u\|_2 < \|P\|_2 \Big\}.$$

We can prove that G_3 and G_4 are two invariant evolution flows generated by the Cauchy problem (1.2)-(1.3). Then, we have the following theorem.

THEOREM 3.2. *Let $N = 3$, $p = 1 + \frac{2}{3}$, R and P be the ground state solution of (2.4) and (2.6), respectively. Then, we have the following:*

(i) if $\psi_0 \in G_3 \cup \{0\}$, then $\psi(t,x)$ exists globally in H^1;

(ii) if $\psi_0 \in G_4$ and $|x|\psi_0 \in L^2$, then $\psi(t,x)$ is a singular solution in a finite time, where $\psi(t,x)$ is the solution of Equation (1.2) corresponding to ψ_0.

Proof. The global existence part is obvious (see the first part of the proof of Theorem 3.1). Now, we give the proof of (ii). It follows from the invariance of G_4 that for all $t \in I$(the maximal existence interval),

$$\begin{aligned} J''(t) &\le 12(H[\psi(t)]+M[\psi(t)]) - 4(\|\nabla\psi(t)\|_2^2 + \|\psi(t)\|_2^2) \\ &\quad + \frac{8\|\psi_0\|_2^{\frac{2}{3}}}{\|P\|_2^{\frac{2}{3}}}\|\nabla\psi(t)\|_2\|\psi(t)\|_2 \\ &\le 12(H[\psi(t)]+M[\psi(t)]) - 4\Big(1-\frac{\|\psi_0\|_2^{\frac{2}{3}}}{\|P\|_2^{\frac{2}{3}}}\Big)\|\psi(t)\|_{H^1}^2 \\ &< 0. \end{aligned}$$

∎

Case 3.3: $1+\frac{2}{3} < p < 1+\frac{4}{3}$

Applying the sharp Gagliardo-Nirenberg inequalities (2.5) and (2.7), we obtain that, for all $t \in I$(the maximal existence interval),

$$H[\psi(t)]+M[\psi(t)] \ge \|\psi(t)\|_{H^1}^2 - \frac{\|\psi_0\|_2^{\frac{5-p}{2}}}{\|P\|_2^{p-1}}\|\psi(t)\|_{H^1}^{\frac{3(p-1)}{2}} - \frac{\|\psi_0\|_2^2}{\|R\|_2^2}\|\psi(t)\|_{H^1}^3. \tag{3.2}$$

Denote $f_3(y) := y - \frac{\|\psi_0\|_2^{\frac{5-p}{2}}}{\|P\|_2^{p-1}} y^{\frac{3(p-1)}{4}} - \frac{\|\psi_0\|_2^2}{\|R\|_2^2} y^{\frac{3}{2}}$ $\;y \ge 0$. Then,

$$f_3'(y) = 1 - \frac{3(p-1)\|\psi_0\|_2^{\frac{5-p}{2}}}{4\|P\|_2^{p-1}} y^{\frac{3(p-1)}{4}-1} - \frac{3\|\psi_0\|_2^2}{2\|R\|_2^2} y^{\frac{1}{2}} = 1 - F(y). \tag{3.3}$$

where $F(y) = \frac{3(p-1)\|\psi_0\|_2^{\frac{5-p}{2}}}{4\|P\|_2^{p-1}} y^{\frac{3(p-1)}{4}-1} + \frac{3\|\psi_0\|_2^2}{2\|R\|_2^2} y^{\frac{1}{2}}$. Obviously, $F(y)$ has only one positive minimizer $y^* > 0$. Writing $F_{min} = F(y^*)$ and adding the condition of $F_{min} < 1$, we claim that there exists a unique positive solution y_0 for the equation $f_3'(y) = 0$. Indeed, we have $f_3'(y^*) > 0$, and $\lim_{y\to+\infty} f_3'(y) = -\infty$. Since $f_3'(y)$ is continuous on $[0,+\infty)$, there exists a unique positive $y_0 \in [y^*,+\infty)$ such that $f_3'(y_0) = 0$. Therefore, we can deduce that 0 and y_0 are two minimizers of $f_3'(y)$, and $f_3'(y)$ is increasing on the interval $[0,y_0)$ and decreasing on the interval $[y_0,+\infty)$. Note that

$$f_{3max} = f_3(y_0) = \frac{1}{3}y_0 + \frac{(p-3)\|\psi_0\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}} y_0^{\frac{3(p-1)}{4}}. \tag{3.4}$$

We define

$$G_5 := \left\{u \in H^1 : f_3(\|u\|_{H^1}^2) < H[u]+M[u] < f_3(y_0), \|u\|_{H^1}^2 < y_0, F_{min} < 1\right\},$$

$$G_6 := \left\{u \in H^1 : f_3(\|u\|_{H^1}^2) < H[u]+M[u] < f_3(y_0), \|u\|_{H^1}^2 > y_0, F_{min} < 1\right\}.$$

We can prove that G_5 and G_6 are two invariant evolution flows generated by the Cauchy problem (1.2)-(1.3). Then, we have the following theorem.

THEOREM 3.3. *Let $N=3$, $1+\frac{2}{3} < p < 1+\frac{4}{3}$, R and P be the ground state solution of (2.4) and (2.6) respectively. Then, we have the following:*

(i) if $\psi_0\in G_5$, then $\psi(t,x)$ exists globally in H^1;

(ii) if $\psi_0\in G_6$ and $|x|\psi_0\in L^2$, then $\psi(t,x)$ is a singular solution in a finite time, where $\psi(t,x)$ is the solution of Equation (1.2) corresponding to ψ_0.

Proof. Here, we just give the proof of blow-up case, as the global existence case is obvious. From the invariance of G_6 and Lemma 2.2, we obtain

$$\begin{aligned} J''(t) &= 12(H[\psi(t)]+M[\psi(t)]) - 4\|\nabla\psi(t)\|_2^2 - 12\|\psi(t)\|_2^2 + \frac{36-12p}{p+1}\|\psi(t)\|_{p+1}^{p+1} \\ &\le 12 f_3(y_0) - 4\|\psi(t)\|_{H^1}^2 + \frac{(18-6p)\|\psi(t)\|_2^{\frac{5-p}{2}}}{\|P\|_2^{p-1}}(\|\psi(t)\|_{H^1}^2)^{\frac{3(p-1)}{4}}. \end{aligned} \tag{3.5}$$

Let $g(y)=\frac{(18-6p)\|\psi_0\|_2^{\frac{5-p}{2}}}{\|P\|_2^{p-1}}y^{\frac{3(p-1)}{4}}-4y$. By some simple computations, we know that $g_{max}=g(y_m)$ as well as $g(y)$ is increasing on the interval $[0,y_m)$ and decreasing on the interval $[y_m,+\infty)$. Moreover, we find that $y_0\in(y_m,+\infty)$, and then when $y>y_0$,

$$g(y)\le g(y_0)=\frac{(18-6p)\|\psi_0\|_2^{\frac{5-p}{2}}}{\|P\|_2^{p-1}}y_0^{\frac{3(p-1)}{4}}-4y_0=-12f_3(y_0).$$

By injecting above fact into (3.5), we deduce that when $\|\psi(t)\|_{H^1}^2>y_0$,

$$J''(t)<12f_3(y_0)+g(y_0)\le 0.$$

□

Case 3.4: $p=1+\frac{4}{3}$

By (2.5) and (2.7), we deduce that for all $t\in I$

$$\begin{aligned} H[\psi(t)]+M[\psi(t)] &\ge \|\nabla\psi(t)\|_2^2+\|\psi(t)\|_2^2-\frac{\|\psi(t)\|_2^{\frac{4}{3}}}{\|P\|_2^{\frac{4}{3}}}(\|\nabla\psi(t)\|_2^2+\|\psi(t)\|_2^2) \\ &\quad -\frac{1}{\|R\|_2^2}(\|\nabla\psi(t)\|_2^2+\|\psi(t)\|_2^2)^2 \\ &= (1-\frac{\|\psi_0\|_2^{\frac{4}{3}}}{\|P\|_2^{\frac{4}{3}}})\|\psi(t)\|_{H^1}^2-\frac{1}{\|R\|_2^2}\|\psi(t)\|_{H^1}^4 := f_4(\|\psi(t)\|_{H^1}^2), \end{aligned}$$

where the function $f_4(y)$ is defined on $[0,\infty)$ by $f_4(y)=(1-\frac{\|\psi_0\|_2^{\frac{4}{3}}}{\|P\|_2^{\frac{4}{3}}})y-\frac{1}{\|R\|_2^2}y^2$. We easily observe that $f_4(y)$ has a similar structure as $f_2(y)$, by the same argument as in Case 3.2, we define

$$\begin{aligned} G_7 := \Big\{ u\in H^1 : f_4(\|u\|_{H^1}^2) < H[u]+M[u] < \frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}}-\|\psi_0\|_2^{\frac{4}{3}})^2}{4\|P\|_2^{\frac{8}{3}}}, \\ \|u\|_{H^1}^2 < \frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}}-\|\psi_0\|_2^{\frac{4}{3}})}{2\|P\|_2^{\frac{4}{3}}}, \|u\|_2<\|P\|_2 \Big\}, \end{aligned}$$

$$\begin{aligned} G_8 := \Big\{ u\in H^1 : f_4(\|u\|_{H^1}^2) < H[u]+M[u] < \frac{2\|R\|_2^2(\|P\|_2^{\frac{4}{3}}-\|\psi_0\|_2^{\frac{4}{3}})^2}{9\|P\|_2^{\frac{8}{3}}}, \\ \|u\|_{H^1}^2 > \frac{2\|R\|_2^2(\|P\|_2^{\frac{4}{3}}-\|\psi_0\|_2^{\frac{4}{3}})}{3\|P\|_2^{\frac{4}{3}}}, \|u\|_2<\|P\|_2 \Big\}. \end{aligned}$$

By the same argument as in the Case 3.1, we can prove that G_7 and G_8 are two invariant evolution flows generated by the Cauchy problem (1.2)-(1.3). Then, we have the following theorem.

THEOREM 3.4. *Let $N=3$, $p=1+\frac{4}{3}$, R and P be the ground state solution of the nonlinear elliptic Equation (2.4) and (2.6) respectively. Then, we have the following:*

(i) if $\psi_0\in G_7\cup\{0\}$, then $\psi(t,x)$ exists globally in H^1;

(ii) if $\psi_0\in G_8$ and $|x|\psi_0\in L^2$, then $\psi(t,x)$ is a singular solution in a finite time, where $\psi(t,x)$ is the solution of Equation (1.2) corresponding to ψ_0.

Proof. The proof of Theorem 3.4 is similar to that of Theorem 3.2. □

Case 3.5: $1+\frac{4}{3}<p<3$

Let us recall the estimate (3.2). Here, we use the same algebra function

$$f_5(y):=y-\frac{\|\psi_0\|_2^{\frac{5-p}{2}}}{\|P\|_2^{p-1}}y^{\frac{3(p-1)}{4}}-\frac{\|\psi_0\|_2^2}{\|R\|_2^2}y^{\frac{3}{2}},$$

but the parameter p and the method are different from the case $1+\frac{2}{3}<p<1+\frac{4}{3}$. We have $f_5'(y)=1-\frac{3(p-1)\|\psi_0\|_2^{\frac{5-p}{2}}}{4\|P\|_2^{p-1}}y^{\frac{3(p-1)}{4}-1}-\frac{3\|\psi_0\|_2^2}{2\|R\|_2^2}y^{\frac{1}{2}}$. We claim that there exists a unique positive solution y_0 for the equation $f_5'(y)=0$. Indeed, by some computations, we have for $y>0$

$$f_5''(y)=-\frac{3(p-1)(3p-7)\|\psi_0\|_2^{\frac{5-p}{2}}}{16\|P\|_2^{p-1}}y^{\frac{3(p-1)}{4}-1}-\frac{3\|\psi_0\|_2^2}{4\|R\|_2^2}y^{-\frac{1}{2}}<0, \tag{3.6}$$

which implies that $f_5'(y)$ is decreasing on $[0,+\infty)$. Notice that $f_5'(0)=1$ and $f_5'((\frac{2\|P\|_2}{3\|\psi_0\|_2})^{\frac{1}{2}})<0$. Since $f_5'(y)$ is continuous on $[0,+\infty)$, there exists a unique positive y_0 such that $f_5'(y_0)=0$.

$$f_{5max}=f_5(y_0)=\frac{3p-7}{3(p-1)}y_0+\frac{(3-p)\|\psi_0\|_2^2}{(p-1)\|R\|_2^2}y_0^{\frac{3}{2}}>\frac{3p-7}{3(p-1)}y_0. \tag{3.7}$$

We define

$$G_9:=\left\{u\in H^1: f_5(\|u\|_{H^1}^2)<H[u]+M[u]<\frac{3p-7}{3(p-1)}y_0, \|u\|_{H^1}^2<y_0\right\},$$

$$G_{10}:=\left\{u\in H^1: f_5(\|u\|_{H^1}^2)<H[u]+M[u]<\frac{3p-7}{3(p-1)}y_0, \|u\|_{H^1}^2>y_0\right\}.$$

By the same argument as in the Case 3.1, we can prove that G_9 and G_{10} are two invariant evolution flows generated by the Cauchy problem (1.2)-(1.3). Then, we have the following theorem.

THEOREM 3.5. *Let $N=3$, $1+\frac{4}{3}<p<3$, R and P be the ground state solution of the nonlinear elliptic Equation (2.4) and (2.6) respectively. Then, we have the following:*

(i) if $\psi_0\in G_9\cup\{0\}$, then $\psi(t,x)$ exists globally in H^1;

(ii) if $\psi_0\in G_{10}$ and $|x|\psi_0\in L^2$, then $\psi(t,x)$ is a singular solution in a finite time,

where $\psi(t,x)$ is the solution of Equation (1.2) corresponding to ψ_0.

Proof. The proof of Theorem 3.5 is similar to that of Theorem 3.3. ◻

Case 3.6: $p=3$

Applying the sharp Gagliardo-Nirenberg inequalities (2.5) and (2.7), we see that for all $t\in I$(the maximal existence interval),

$$H[\psi(t)]+M[\psi(t)]\geq \|\psi(t)\|_{H^1}^2-\frac{1}{\|Q\|_2^2}\|\psi(t)\|_{H^1}^4:=f_6(\|\psi(t)\|_{H^1}^2), \tag{3.8}$$

where the function $f_6(y)$ is defined on $[0,+\infty)$ by $f_6(y)=y-\frac{1}{\|Q\|_2^2}y^2$. Obviously, $f_6(y)$ reaches the maximum at the point $y_6^*=\frac{\|Q\|_2^2}{2}$. More precisely, $f_{6max}=f_6(y_6^*)=\frac{\|Q\|_2^2}{4}$. We define

$$G_{11}:=\left\{u\in H^1: f_6(\|u\|_{H^1}^2)<H[u]+M[u]<\frac{\|Q\|_2^2}{4}, \|u\|_{H^1}^2<\frac{\|Q\|_2^2}{2}\right\},$$

$$G_{12}:=\left\{u\in H^1: f_6(\|u\|_{H^1}^2)<H[u]+M[u]<\frac{2\|Q\|_2^2}{9}, \|u\|_{H^1}^2>\frac{2\|Q\|_2^2}{3}\right\}.$$

By the same argument as in the Case 3.1, we can prove that G_{11} and G_{12} are two invariant evolution flows generated by the Cauchy problem (1.2)-(1.3). Then, we have the following theorem.

THEOREM 3.6. *Let $N=3$, $p=3$ and Q be the ground state solution of (2.2). Then we have the following:*

(i) if $\psi_0\in G_{11}\cup\{0\}$, then $\psi(t,x)$ exists globally in H^1;

(ii) if $\psi_0\in G_{12}$ and $|x|\psi_0\in L^2$, then $\psi(t,x)$ is a singular solution in a finite time, where $\psi(t,x)$ is the solution of Equation (1.2) corresponding to ψ_0.

Proof. The proof of Theorem 3.6 is similar to that of Theorem 3.2. ◻

4. Rate of convergence of singular solutions and concentration

In this section, we study the rate and concentration of singular solutions for Equation (1.2). Here, our main tools are the Strichartz estimates(see [2, 16]). As $p\neq 3$, Equation (1.2) does not have the scaling invariance, and the singular integral operator E also has a bad effect on the study of the rate of convergence of singular solutions. We inject the Gagliardo-Nirenberg inequality and conservation of mass into the Strichartz estimates, and we give the proof of Theorem 1.1.

Proof. (**Proof of Theorem 1.1**) First of all, we recall the Strichartz admissible pair and Strichartz estimates for the linear and nonlinear Schrödinger equations.

A pair is called L^2 Strichartz admissible, iff $2\leq q,r\leq+\infty$, $(q,r)\neq(2,\infty)$ and $\frac{2}{q}=N(\frac{1}{2}-\frac{1}{r})$, where N is the space dimension. It is easy to check that for $N=3$, $(q,r)=(\infty,2)$ and $(q,r)=(\frac{8}{3},4)$ are L^2 Strichartz admissible pairs.

Denote the Schrödinger semigroup by $e^{it\triangle}\psi(x)=\mathcal{F}^{-1}[e^{-it|\xi|^2}\mathcal{F}[\psi]]$ for any tempered distribution ψ, and $f(\psi)=E(|\psi|^2)\psi+|\psi|^{p-1}\psi$. Then, the Cauchy problem (1.2)-(1.3) is equivalent to

$$\psi(t,x)=e^{it\triangle}\psi_0-i\int_0^t e^{i(t-\tau)\triangle}f(\psi(\tau,x))d\tau. \tag{4.1}$$

For any L^2 Strichartz admissible pair (q,r), we have the following Strichartz estimates:

$$\|e^{it\triangle}\psi\|_{L_t^q((t,\tau);L_x^r)} \leq C\|\psi\|_{L_x^2}, \tag{4.2}$$

$$\|\int_0^t e^{i(t-\tau)\triangle} f(\psi)d\tau\|_{L_t^q((t,\tau);L_x^r)} \leq C\|f(\psi)\|_{L_t^{q'}((t,\tau);L_x^{r'})}, \tag{4.3}$$

where $\frac{1}{q}+\frac{1}{q'}=1$, $\frac{1}{r}+\frac{1}{r'}=1$ and $\|\psi\|_{L_t^q((t,\tau);L_x^r)} := (\int_t^\tau (\int_{\mathbb{R}^2} |\psi(t,x)|^r dx)^{\frac{q}{r}} dt)^{\frac{1}{q}}$.

Secondly, by injecting the fact that $(\infty,2)$ and $(q,r)=(\frac{8}{3},4)$ are L^2 Strichartz admissible pairs, and the corresponding $(q',r')=(\frac{8}{5},\frac{4}{3})$, into the Strichartz estimate (4.2)-(4.3), we obtain

$$\begin{aligned}&\|\nabla\psi\|_{L_t^\infty((t,\tau);L_x^2)} + \|\nabla\psi\|_{L_t^{\frac{8}{3}}((t,\tau);L_x^4)} \\ &\leq C\|\nabla\psi_0\|_{L_x^2} + C\|\nabla(E(|\psi|^2)\psi)\|_{L_t^{\frac{8}{5}}((t,\tau);L_x^{\frac{4}{3}})} + C\|\nabla(|\psi|^{p-1}\psi)\|_{L_t^{\frac{8}{5}}((t,\tau);L_x^{\frac{4}{3}})}.\end{aligned} \tag{4.4}$$

Indeed, for the term $\|\nabla(E(|\psi|^2)\psi)\|_{L_t^{\frac{8}{5}}((t,\tau);L_x^{\frac{4}{3}})}$, by the interpolation estimate and the Gagliardo-Nirenberg inequality: $\|\psi\|_4^4 \leq C\|\psi\|_2\|\nabla\psi\|_2^3$, we deduce that

$$\|\nabla(E(|\psi|^2)\psi)\|_{L^{\frac{4}{3}}} \leq C\|\psi\|_{L_x^4}^2\|\nabla\psi\|_{L_x^4} \leq C\|\nabla\psi\|_{L_x^2}^{\frac{3}{2}}\|\nabla\psi\|_{L_x^4}.$$

Then, for any $0<t<\tau<T<+\infty$, we deduce that

$$\begin{aligned}&\|\nabla(E(|\psi|^2)\psi)\|_{L_t^{\frac{8}{5}}((t,\tau);L_x^{\frac{4}{3}})} \\ \leq & C(1+\|\nabla\psi\|_{L_t^\infty((t,\tau);L_x^2)})^{\frac{3}{2}}(\int_t^\tau 1\cdot\|\nabla\psi\|_{L_x^4}^{\frac{8}{5}}d\tau)^{\frac{5}{8}} \\ \leq & C(1+\|\nabla\psi\|_{L_t^\infty((t,\tau);L_x^2)})^{\frac{3}{2}}\ (\tau-t)^{\frac{1}{4}}\ \ \|\nabla\psi\|_{L_t^{\frac{8}{3}}((t,\tau);L_x^4)} \\ \leq & C(\tau-t)^{\frac{1}{4}}\ (1+\|\nabla\psi\|_{L_t^\infty((t,\tau);L_x^2)} + \|\nabla\psi\|_{L_t^{\frac{8}{3}}((t,\tau);L_x^4)})^{\frac{5}{2}}.\end{aligned} \tag{4.5}$$

For the term $\|\nabla(|\psi|^{p-1}\psi)\|_{L_t^{\frac{8}{5}}((t,\tau);L_x^{\frac{4}{3}})}$, we see that

$$\|\nabla(|\psi|^{p-1}\psi)\|_{L_x^{\frac{4}{3}}} \leq C\|\psi\|_{L_x^{2(p-1)}}^{p-1}\|\nabla\psi\|_{L_x^4} \leq C\|\psi\|_{L_x^2}^{\frac{4-p}{2}}\|\nabla\psi\|_{L_x^2}^{\frac{3(p-2)}{2}}\|\nabla\psi\|_{L_x^4},$$

where in the last step, we use $0\leq\frac{3(p-2)}{2}<\frac{3}{2}$ for $2\leq p<3$, and the Gagliardo-Nirenberg inequality: $\|\psi\|_{L_x^{2(p-1)}}^{2(p-1)} \leq C\|\psi\|_{L_x^2}^{4-p}\|\nabla\psi\|_{L_x^2}^{3(p-2)}$. Thus, by the same argument in (4.5), we have

$$\|\nabla(|\psi|^{p-1}\psi)\|_{L_t^{\frac{8}{5}}((t,\tau);L_x^{\frac{4}{3}})} \leq C(\tau-t)^{\frac{1}{4}}\ (1+\|\nabla\psi\|_{L_t^\infty((t,\tau);L_x^2)} + \|\nabla\psi\|_{L_t^{\frac{8}{3}}((t,\tau);L_x^4)})^{\frac{5}{2}}, \tag{4.6}$$

for any $0<t<\tau<T<+\infty$. This completes the proof of Claim (4.4).

Thirdly, let $F_t(\tau) := 1+\|\nabla\psi\|_{L_t^\infty((t,\tau);L_x^2)} + \|\nabla\psi\|_{L_t^{\frac{8}{3}}((t,\tau);L_x^4)}$. Then, it follows from (4.4), (4.5) and (4.6) that there exists $C_0>0$ such that

$$F_t(\tau) \leq C_0(1+\|\nabla\psi(t)\|_{L_x^2}) + C_0(\tau-t)^{\frac{1}{4}}\ \ F_t^{\frac{5}{2}}(\tau), \tag{4.7}$$

for any $0<t<\tau<T<+\infty$. Now, we investigate the properties of $F_t(\tau)$. For any $t\in(0,T)$, if $T<+\infty$, then by Proposition 2.1 the solution $\psi(t,x)$ is the singular solution of the Cauchy problem (1.2)-(1.3), and $\lim_{\tau\to T}F_t(\tau)=+\infty$. And $F_t(\tau)$ is continuous and nondecreasing on (t,T), and for $\tau>t$, we see that $F_t(\tau)\to 1+\|\nabla\psi(t)\|_{L^2_x}$ as $\tau\to t$. It follows from the continuity of $F_t(\tau)$ with respect to τ that there exists $\tau_0\in(t,T)$ such that

$$F_t(\tau_0)=(C_0+1)(1+\|\nabla\psi(t)\|_{L^2_x}), \tag{4.8}$$

where $C_0>0$ is the constant in (4.7). Taking $\tau=\tau_0$ in (4.7), yields

$$\begin{aligned}1+\|\nabla\psi(t)\|_{L^2_x}&=F_t(\tau_0)-C_0(1+\|\nabla\psi(t)\|_{L^2_x})\\&\le C_0(\tau_0-t)^{\frac14}\ (C_0+1)^{\frac52}\ (1+\|\nabla\psi(t)\|_{L^2_x})^{\frac52}\\&\le (C_0+1)^{\frac72}(T-t)^{\frac14}\ \ (1+\|\nabla\psi(t)\|_{L^2_x})^{\frac52},\end{aligned}$$

and so

$$1+\|\nabla\psi(t)\|_{L^2_x}\ge\frac{1}{(C_0+1)^{\frac73}\ (T-t)^{\frac16}}\quad\text{for}\quad 0<t<T<+\infty.$$

This completes the proof of (1.4).

Finally, we prove L^q-norm of blow-up solutions for Equation (1.2). By the assumption $3<q\le 4$, we use the Hölder interpolation estimate to obtain

$$\int E(|\psi(t)|^2)||\psi(t)|^2dx\le C\|\psi(t)\|_4^4\le C\|\nabla\psi(t)\|_2^{\frac{12-3q}{6-q}}\ \|\psi(t)\|_q^{\frac{2q}{6-q}} \tag{4.9}$$

for all $0\le t<T<+\infty$. It follows from $2\le p<\frac73$ and the Gagliardo-Nirenberg inequality that there exists $0<C_0<\frac{p+1}{4}$ and $C_1>0$ such that

$$\|\psi(t)\|_{p+1}^{p+1}\le C\ \|\psi(t)\|_2^{\frac{5-p}{2}}\|\nabla\psi(t)\|_2^{\frac{3(p-1)}{2}}\le C_0\|\nabla\psi(t)\|_2^2+C_1 \tag{4.10}$$

for all $0\le t<T<+\infty$. Inject (4.9) and (4.10) into the energy. For all $0\le t<T<+\infty$, we have

$$(1-\frac{2C_0}{p+1})\|\nabla\psi(t)\|_2^2\le 2H[\psi_0]+\frac{2C_1}{p+1}+C\|\nabla\psi(t)\|_2^{\frac{12-3q}{6-q}}\ \|\psi(t)\|_q^{\frac{2q}{6-q}},$$

which implies that

$$\|\psi(t)\|_q^{\frac{2q}{6-q}}\ge C\|\nabla\psi(t)\|_2^{\frac{q}{6-q}}$$

and then (1.6) follows from (1.4).

For (1.5), it follows from the assumption $2\le p<3$ and the Hölder interpolation estimate that for all $0\le t<T<+\infty$,

$$\|\psi(t)\|_{p+1}^{p+1}\le C\|\psi(t)\|_2^{\theta(p+1)}\ \|\psi(t)\|_4^{(1-\theta)(p+1)}\le C_2+\|\psi(t)\|_4^4, \tag{4.11}$$

where $\theta=\frac{3-p}{p+1}\in(0,1)$ and $C_2>0$. We remark that when $2<4<q$, from the Hölder interpolation estimate, we deduce that

$$\|\psi(t)\|_4^4\le C\|\psi(t)\|_2^{\frac{2(q-4)}{q-2}}\ \|\psi(t)\|_q^{\frac{2q}{q-2}} \tag{4.12}$$

for all $0 \leq t < T < +\infty$. Now, inject (4.11) and (4.12) into the energy. For all $0 \leq t < T < +\infty$

$$\|\nabla\psi(t)\|_2^2 \leq 2H[\psi_0] + \frac{2C_2}{p+1} + C\|\psi(t)\|_2^{\frac{2(q-4)}{q-2}} \|\psi(t)\|_q^{\frac{2q}{q-2}},$$

which implies that

$$\|\psi(t)\|_q \geq C\|\nabla\psi(t)\|_2^{\frac{q-2}{q}}$$

and then (1.5) follows from (1.4). This completes the proof of Theorem 1.1. □

At the end of this section, we obtain the rate of $\dot{H}^{\frac{1}{2}}$-norm concentration of singular solutions for Equation (1.2) and give the proof of Theorem 1.2.

Proof. (**Proof of Theorem 1.2**) Denote $\lambda(t) > 0$ such that

$$\lambda(t)\|\nabla\psi(t)\|_2^2 \to +\infty, \quad \text{as} \quad t \to T, \tag{4.13}$$

where $\tilde{Q}$ is the solution of (1.7). Then, it follows from the results in [6] that if $\psi(t,x)$ is the singular solution of the Cauchy problem (1.2)-(1.3) satisfying (4.13) and $\sup_{t\in[0,T)} \|\psi(t)\|_{\dot{H}^{\frac{1}{2}}} < +\infty$, then there exists $y(t) \in \mathbb{R}^3$ such that

$$\liminf_{t\in[0,T)} \int_{|x-y(t)|\leq\lambda(t)} |(-\triangle)^{\frac{1}{4}}\psi(t,x)|^2 dx \geq \|\tilde{Q}\|_{\dot{H}^{\frac{1}{2}}}^2. \tag{4.14}$$

From Theorem 1.1, we can obtain (1.8) by taking $\lambda(t) = (T-t)^{\frac{1}{6}-}$ in (4.14). Indeed, it follows from (1.4) that $\|\nabla\psi(t)\|_2 \geq C(\frac{1}{T-t})^{\frac{1}{6}}$. And then we see that

$$\lambda(t)\|\nabla\psi(t)\|_2^2 = (T-t)^{\frac{1}{6}-} \|\nabla\psi(t)\|_2^2 \to +\infty \quad \text{as} \quad t \to T.$$

Therefore, $\lambda(t) = (T-t)^{\frac{1}{6}-}$ satisfies (4.13), which implies that (4.14) is true, so is (1.8). □

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