COMMUN. MATH. SCI.
Vol. 17, No. 3, pp. 859–882

NONLOCAL APPROXIMATION OF ELLIPTIC OPERATORS WITH ANISOTROPIC COEFFICIENTS ON MANIFOLD*

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Abstract. In this paper, we give an integral approximation for the elliptic operators with anisotropic coefficients on smooth manifold. Using the integral approximation, the elliptic equation is transformed to an integral equation. The integral approximation preserves the symmetry and coercivity of the original elliptic operator. Based on these good properties, we prove the convergence between the solutions of the integral equation and the original elliptic equation.

Keywords. Nonlocal approximation; elliptic operator; anisotropic coefficients; Point Integral Method.

AMS subject classifications. 45P05; 45A05; 35A23.

1. Introduction

Recently, manifold model attracts more and more attention in many applications, including data analysis and image processing [3, 6, 7, 16, 18, 20, 21, 29, 33–35, 39]. In the manifold model, data or images are assumed to be distributed in a low dimensional manifold embedded in a high dimensional Euclidean space. Differential operators on the manifold, particularly the elliptic operators, encode lots of intrinsic information of the manifold.

Besides the data analysis and image processing, PDEs on manifolds also arise in many different applications, including material science [5, 13], fluid flow [15, 17], biology and biophysics [1, 2, 14, 31]. Many methods have been developed to solve PDEs on curved surfaces embedded in $\mathbb{R}^3$, such as surface finite element method [12], level set method [4, 40], grid-based particle method [22, 23] and closest point method [28, 36]. On the other hand, these methods do not apply in high dimensional problem directly.

In the past few years, many numerical methods to solve PDEs on manifold embedded in high dimensional space were developed. Liang et al. proposed to discretize the differential operators on point cloud by local least squares approximations of the manifold [26, 27]. Later, Lai et al. proposed local mesh method to approximate the differential operators on point cloud [19]. The main idea is to construct mesh locally around each point by using K nearest neighbors instead of constructing the global mesh. The other approach is so-called point integral method [24, 25, 37, 38]. In the point integral method, the differential operators are approximated by integral operators. Then it is easy to discretize the integral operators in manifold since there are not any differential operators inside. The convergence of the point integral method for elliptic operators with isotropic coefficients has been proved [24].

In this paper, we consider to solve general elliptic operators with anisotropic coefficients on manifold $\mathcal{M}$. We assume that $\mathcal{M}\in C^\infty$ is a compact d_0-dimensional manifold isometrically embedded in $\mathbb{R}^d$ with the standard Euclidean metric and $d_0\le d$. If $\mathcal{M}$ has a boundary, the boundary, $\partial\mathcal{M}$ is also a C^∞ smooth manifold.

Let $\Phi: V\subset\mathbb{R}^{d_0}\to\mathcal{M}\subset\mathbb{R}^d$ be a local parametrization of $\mathcal{M}$ and $\theta\in V$. For any

*Received: January 15, 2018; Accepted (in revised form): March 8, 2019. Communicated by Jianfeng Lu.

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differentiable function $f:\mathcal{M}\to\mathbb{R}$, let $F(\theta)=f(\Phi(\theta))$, define

$$D_k f(\Phi(\theta))=\sum_{i,j=1}^{d_0} g^{ij}(\theta)\frac{\partial\Phi^k}{\partial\theta_i}(\theta)\frac{\partial F}{\partial\theta_j}(\theta),\quad k=1,\cdots,d. \tag{1.1}$$

where $(g^{ij})_{i,j=1,\cdots,d_0}=G^{-1}$ and $G(\theta)=(g_{ij})_{i,j=1,\cdots,d_0}$ is the first fundamental form which is defined by

$$g_{ij}(\theta)=\sum_{k=1}^{d}\frac{\partial\Phi^k}{\partial\theta_i}(\theta)\frac{\partial\Phi^k}{\partial\theta_j}(\theta),\quad i,j=1,\cdots,d_0. \tag{1.2}$$

The general second order elliptic PDE on manifold $\mathcal{M}$ has following form,

$$-\sum_{i,j=1}^{d} D_i(a_{ij}(\mathbf{x})D_j u(\mathbf{x}))=f(\mathbf{x}),\quad \mathbf{x}\in\mathcal{M}. \tag{1.3}$$

The coefficients $a_{ij}(\mathbf{x})$ and source term $f(\mathbf{x})$ are smooth functions of spatial variables, i.e.

$$a_{ij},f\in C^1(\mathcal{M}),\quad i,j=1,\cdots,d.$$

The matrix $(a_{ij})_{i,j=1,\cdots,d}$ is symmetric and maps the tangent space $\mathcal{T}_{\mathbf{x}}$ into itself and satisfies following elliptic condition: there exist generic constants $0<a_0\le a_1<\infty$ independent of $\mathbf{x}$ such that for any $\xi=[\xi_1,\cdots,\xi_d]^T\in\mathcal{T}_{\mathbf{x}}$,

$$a_0\sum_{i=1}^{d}\xi_i^2\le\sum_{i,j=1}^{d}a_{ij}(\mathbf{x})\xi_i\xi_j\le a_1\sum_{i=1}^{d}\xi_i^2. \tag{1.4}$$

For any $\mathbf{x}\in\mathcal{M}$, the matrix $(a_{ij}(\mathbf{x}))$ gives a linear transform from $\mathbb{R}^d$ to $\mathbb{R}^d$, denoted as $A(\mathbf{x})$. The tangent space at $\mathbf{x}$, $\mathcal{T}_{\mathbf{x}}$, is an invariant subspace of $A(\mathbf{x})$. Confined on $\mathcal{T}_{\mathbf{x}}$, $A(\mathbf{x})$ introduces a linear transform from $\mathcal{T}_{\mathbf{x}}$ to $\mathcal{T}_{\mathbf{x}}$, denoted as $A_{\mathcal{T}}(\mathbf{x})$. In this paper, we consider the Neumann boundary condition, i.e.

$$\sum_{i,j=1}^{d}\mathbf{n}_i(\mathbf{x})a_{ij}(\mathbf{x})D_j u(\mathbf{x})=0,\quad \mathbf{x}\in\partial\mathcal{M}, \tag{1.5}$$

where $\mathbf{n}(\mathbf{x})$ is the outer normal of $\partial\mathcal{M}$ at $\mathbf{x}$, $\mathbf{n}_i$ denotes its i-th component.

In [24,25], the point integral method (PIM) was proposed for elliptic equations with isotropic coefficients, i.e.,

$$a_{ij}(\mathbf{x})=p^2(\mathbf{x})\delta_{ij}, \tag{1.6}$$

where $p(\mathbf{x})\ge C_0>0$ and

$$\delta_{ij}=\begin{cases}1,\ i=j,\\ 0,\ i\neq j.\end{cases}$$

The main ingredient of the point integral method is to approximate the elliptic equation by an integral equation:

$$\frac{1}{t}\int_{\mathcal{M}}R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))p(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}-2\int_{\partial\mathcal{M}}\frac{\partial u}{\partial\mathbf{n}}(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})p(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}$$

$$= \int_{\mathcal{M}} f(\mathbf{y}) \frac{\bar{R}_t(\mathbf{x},\mathbf{y})}{p(\mathbf{y})} d\mu_{\mathbf{y}}, \tag{1.7}$$

where $R_t(\mathbf{x},\mathbf{y})$ and $\bar{R}_t(\mathbf{x},\mathbf{y})$ are kernel functions given as following

$$R_t(\mathbf{x},\mathbf{y}) = C_t R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right), \quad \bar{R}_t(\mathbf{x},\mathbf{y}) = C_t \bar{R}\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) \tag{1.8}$$

where $C_t = (4t)^{-d_0/2}$ is the normalization factor with $d_0 = \dim(\mathcal{M})$. $R \in C^2(\mathbb{R}^+)$ be a positive function which is integrable over $[0,+\infty)$ and $\bar{R}(r) = \int_r^{+\infty} R(s)\mathrm{d}s$. The main advantage of the integral equation is that there are no differential operators in the equation. It is easy to be discretized from point clouds using numerical integration.

The main contribution of this paper is to generalize the point integral method to solve the general elliptic Equation (1.3). The key ingredient is to change the kernel function to

$$\bar{K}_t(\mathbf{x},\mathbf{y}) = \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}} \bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y}) \tag{1.9}$$

$$K_t(\mathbf{x},\mathbf{y}) = \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) + \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}} R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y}) \tag{1.10}$$

where $|A_{\mathcal{T}}(\mathbf{x})|$ is the determinant of $A_{\mathcal{T}}(\mathbf{x})$ and

$$R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) = R\left(\frac{(x_m - y_m)a^{mn}(\mathbf{x})(x_n - y_n)}{4t}\right), R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y}) = R\left(\frac{(x_m - y_m)a^{mn}(\mathbf{y})(x_n - y_n)}{4t}\right)$$

$$\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) = \bar{R}\left(\frac{(x_m - y_m)a^{mn}(\mathbf{x})(x_n - y_n)}{4t}\right), \bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y}) = \bar{R}\left(\frac{(x_m - y_m)a^{mn}(\mathbf{y})(x_n - y_n)}{4t}\right)$$

with matrix $(a^{ij}(\mathbf{x}))_{i,j=1,\cdots,d}$ being the inverse of the coefficient matrix $(a_{ij}(\mathbf{x}))_{i,j=1,\cdots,d}$ and $\bar{R}$ is the primitive function of R, i.e. $\bar{R}(r) = \int_r^{+\infty} R(s)\mathrm{d}s$.

Using above kernel function, we get an integral approximation to the original elliptic operator,

$$\begin{aligned} \int_{\mathcal{M}} \bar{K}_t(\mathbf{x},\mathbf{y}) \sum_{i,j=1}^d D_i(a_{ij}(\mathbf{y}) D_j u(\mathbf{y})) \mathrm{d}\mu_{\mathbf{y}} &\approx \frac{1}{t} \int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) \mathrm{d}\mu_{\mathbf{y}} \\ &- 2\int_{\partial\mathcal{M}} \sum_{i,j=1}^d \mathbf{n}_i(\mathbf{y}) a_{ij}(\mathbf{y}) D_j u(\mathbf{y}) \bar{K}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}}. \end{aligned} \tag{1.11}$$

Using above integral approximation, original Poisson Equation (1.3) can be approximately transferred to an integral Equation (2.1). The main result in this paper is to prove that the solution of (2.1) converges to the solution of the anisotropic elliptic equation.

The rest of this paper is organized as follows. In Section 2, we show the integral equation along with its wellposedness. The convergence is analyzed in Section 3. In Sections 4 and 5, two theorems used to prove the convergence are proved. The conclusions and discussion of future work are provided in Section 6.

2. Integral equation and wellposedness

Using the integral approximation (1.11), it is easy to derive an integral equation to approximate the original elliptic equation with homogeneous Neumann boundary condition.

$$\frac{1}{t}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} = \int_{\mathcal{M}} \bar{K}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} - \bar{f}_t, \tag{2.1}$$

with $\bar{f}_t = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}}\int_{\mathcal{M}} \bar{K}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}}$. The necessary condition (also sufficient) that integral Equation (2.1) has a solution is that average of the right-hand side must be zero. So $\bar{f}_t$ is subtracted to make sure this condition is satisfied.

In the rest of this paper, we will focus on above integral equation. Before going to the details of the analysis, we give the assumptions used in this paper.

Assumption 2.1.
- Smoothness of the manifold: $\mathcal{M},\partial\mathcal{M}$ are both compact and C^∞ smooth d_0-dimensional submanifolds isometrically embedded in a Euclidean space $\mathbb{R}^d$.
- Assumptions on the kernel function $R(r)$:
  - (a) Smoothness: $R\in C^2(\mathbb{R}^+)$;
  - (b) Nonnegativity: $R(r)\geq 0$ for any $r\geq 0$.
  - (c) Compact support: $R(r)=0$ for $\forall r>1$;
  - (d) Nondegeneracy: $\exists\delta_0>0$ so that $R(r)\geq\delta_0$ for $0\leq r\leq\frac{1}{2}$.

Remark 2.1. The assumption on the kernel function is very mild. The compact support assumption can be relaxed to exponential decay, like Gaussian kernel. In the nondegeneracy assumption, $1/2$ may be replaced by a positive number θ_0 with $0<\theta_0<1$. Similar assumptions on the kernel function are also used in analysis of the nonlocal diffusion problem [11].

Based on above assumptions, with fixed $t>0$, wellposedness of (2.1) in $L^2(\mathcal{M})$ is straightforward from the well-known Fredholm theory.

Moreover, we can prove a stronger result based on following inequalities. The proofs are deferred to Appendix A. Similar inequalities were also obtained in the study of nonlocal equations [30].

Lemma 2.1. *There exists a constant $C>0$ independent of t so that for any function $u\in L_2(\mathcal{M})$ with $\int_{\mathcal{M}} u=0$ and for any sufficiently small t*

$$\frac{1}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}} \geq C\|u\|^2_{L_2(\mathcal{M})}. \tag{2.2}$$

Lemma 2.2. *For any function $u\in L^2(\mathcal{M})$, there exists a constant $C>0$ independent of t and u, such that*

$$\frac{1}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}} \geq C\int_{\mathcal{M}} |Dv|^2\mathrm{d}\mu_{\mathbf{x}} \tag{2.3}$$

where $D=(D_1,\cdots,D_d)^T$ is the gradient operator on manifold,

$$v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \tag{2.4}$$

and $w_t(\mathbf{x}) = \int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}$.

With above two lemmas, it is easy to get wellposedness of the integral Equation (2.1) in $H^1(\mathcal{M})$.

THEOREM 2.1. *Consider the integral equation*

$$\frac{1}{t}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} = r(\mathbf{x})$$

with $r\in H^1(\mathcal{M})$ *and* $\int_{\mathcal{M}} r(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}=0$.

(1) There exists a unique solution $u\in H^1(\mathcal{M})$ *with* $\int_{\mathcal{M}} u(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}=0$.

(2) There exist constants $C>0, T_0>0$ *independent of* t*, such that*

$$\|u\|_{L^2(\mathcal{M})}\le C\|r\|_{L^2(\mathcal{M})}, \quad \|u\|_{H^1(\mathcal{M})}\le C\left(\|r\|_{L^2(\mathcal{M})}+t\|Dr\|_{L^2(\mathcal{M})}\right)$$

as long as $t\le T_0$.

Proof. In $L^2(\mathcal{M})$, existence and uniqueness of the solution is a direct implication of the well-known Fredholm theory. Moreover, notice that the solution u has following expression,

$$u(\mathbf{x})=v(\mathbf{x})+t\left(\frac{r(\mathbf{x})}{w_t(\mathbf{x})}\right), \tag{2.5}$$

where

$$v(\mathbf{x})=\frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}, \quad w_t(\mathbf{x})=\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.$$

From the assumptions on the kernel function, we have $u\in H^1(\mathcal{M})$.

Upper bound of u in L^2 is given by Lemma 2.1 and following equality,

$$\int_{\mathcal{M}}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}} = 2\int_{\mathcal{M}} u(\mathbf{x})\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{x}}.$$

Now we turn to estimate $\|Du\|_{L^2(\mathcal{M})}$. Using (2.5) and Lemma 2.2, we have

$$\begin{aligned}
\|Du\|^2_{L^2(\mathcal{M})} &\le 2\|Dv\|^2_{L^2(\mathcal{M})}+2t^2\left\|D\left(\frac{r(\mathbf{x})}{w_t(\mathbf{x})}\right)\right\|^2_{L^2(\mathcal{M})}\\
&\le C\langle u, L_t u\rangle + Ct\|r\|^2_{L^2(\mathcal{M})}+Ct^2\|Dr\|^2_{L^2(\mathcal{M})}\\
&\le C\|u\|_{L^2(\mathcal{M})}\|r\|_{L^2(\mathcal{M})}+Ct\|r\|^2_{L^2(\mathcal{M})}+Ct^2\|Dr\|^2_{L^2(\mathcal{M})}\\
&\le C\|r\|^2_{L^2(\mathcal{M})}+Ct^2\|Dr\|^2_{L^2(\mathcal{M})}\\
&\le C\left(\|r\|_{L^2(\mathcal{M})}+t\|Dr\|_{L^2(\mathcal{M})}\right)^2.
\end{aligned}$$

This completes the proof. □

3. Convergence

In this section, we will prove that the solution of the integral Equation (2.1) converges to the solution of the elliptic Equation (1.3). First, we need the error estimate of the integral approximation (1.11) which is given in the theorem as follows:

THEOREM 3.1. *Under the assumptions in Assumption 2.1, let* $u(\mathbf{x})$ *be the solution of the problem* (1.3) *and* $u_t(\mathbf{x})$ *be the solution of the corresponding integral Equation* (2.1)*. Denote* $I_{in}=L_t(u-u_t)-I_{bd}$ *with*

$$L_t u(\mathbf{x})=\frac{1}{t}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}}. \tag{3.1}$$

and

$$\begin{aligned} I_{bd} = & \sum_{i,m,n=1}^{d} \int_{\partial\mathcal{M}} \mathbf{n}_i(\mathbf{y}) a_{im}(\mathbf{x}) \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) D_m D_n u(\mathbf{y})(x_n - y_n) \mathrm{d}\tau_{\mathbf{y}} \\ & -2 \sum_{i,j,k=1}^{d} \int_{\partial\mathcal{M}} \mathbf{n}_i(\mathbf{y})(y_k - x_k) D_k a_{ij}(\mathbf{x}) D_j u(\mathbf{y}) \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \\ & + \sum_{i,m,n=1}^{d} \int_{\partial\mathcal{M}} \mathbf{n}_i(\mathbf{y}) a_{im}(\mathbf{x}) \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) D_m D_n u(\mathbf{y})(x_n - y_n) \mathrm{d}\tau_{\mathbf{y}} \\ & + \sum_{i,j,k,m,n=1}^{d} \int_{\partial\mathcal{M}} \mathbf{n}_k(\mathbf{y}) a_{ij}(\mathbf{y}) D_j u(\mathbf{y}) \frac{D_i a^{mn}(\mathbf{y})(x_n - y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} a_{km}(\mathbf{x}) \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}}. \end{aligned} \tag{3.2}$$

If $u \in C^3(\mathcal{M})$, then there exist constants C, T_0 depending only on $\mathcal{M}$, $\partial\mathcal{M}$, so that for any $t \le T_0$,

$$\|I_{in}\|_{L^2(\mathcal{M})} \le Ct^{1/2}\|u\|_{C^3(\mathcal{M})} + Ct^{1/2}\|f\|_\infty, \tag{3.3}$$

$$\|D(I_{in})\|_{L^2(\mathcal{M})} \le C\|u\|_{C^3(\mathcal{M})}. \tag{3.4}$$

Furthermore, using the formula of I_{bd} (3.2), I_{bd} can be rewritten as

$$I_{bd} = \sum_{i=1}^{d} \int_{\partial\mathcal{M}} b_i(\mathbf{y})(x_i - y_i) \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \tag{3.5}$$

and it is easy to check that $\|b_i\|_{L^\infty(\partial\mathcal{M})} \le C\|u\|_{C^2(\mathcal{M})}$.

$$\begin{aligned} \|I_{bd}\|_{L^2}^2 \le & C \max_i \|b_i\|_{L^\infty(\partial\mathcal{M})}^2 \int_{\mathcal{M}} \left(\int_{\partial\mathcal{M}} |\mathbf{x}-\mathbf{y}| \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right)^2 \mathrm{d}\mu_{\mathbf{x}} \\ \le & C\|u\|_{C^2(\mathcal{M})}^2 \int_{\mathcal{M}} \left(\int_{\partial\mathcal{M}} |\mathbf{x}-\mathbf{y}|^2 \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \int_{\partial\mathcal{M}} \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \mathrm{d}\tau_{\mathbf{y}} \right) \mathrm{d}\mu_{\mathbf{x}} \\ \le & Ct^{-1/2}\|u\|_{C^2(\mathcal{M})}^2 \int_{\partial\mathcal{M}} \left(\int_{\mathcal{M}} |\mathbf{x}-\mathbf{y}|^2 \bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y}) \mathrm{d}\mu_{\mathbf{x}} \right) \mathrm{d}\tau_{\mathbf{y}} \\ \le & Ct^{1/2}\|u\|_{C^2(\mathcal{M})}^2. \end{aligned} \tag{3.6}$$

Considering $D(I_{bd})$, the derivative will apply on $x_i - y_i$ or $\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})$. In both cases, the derivative will generate a $t^{-1/2}$ factor. So we have

$$\|D(I_{bd})\|_{L^2(\mathcal{M})} \le Ct^{-1/4}\|u\|_{C^2(\mathcal{M})}. \tag{3.7}$$

Convergence immediately follows from Theorem 2.1, Theorem 3.1 and (3.6), (3.7).

$$\|u - u_t\|_{H^1(\mathcal{M})} \le Ct^{1/4}\|u\|_{C^3(\mathcal{M})}. \tag{3.8}$$

In this result, the convergence rate is relatively low, only $t^{1/4}$. The low convergence rate is due to the boundary term I_{bd}. The boundary term has a special structure as shown in (3.2). The convergence rate can be improved by exploiting the special structure of the boundary term. More specifically, we have a special stability result for the boundary term.

THEOREM 3.2. *Assume both the submanifolds $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞ smooth. Let*

$$r(\mathbf{x})=\sum_{i=1}^{d}\int_{\partial\mathcal{M}} b_i(\mathbf{y})(x_i-y_i)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}$$

where $b_i(\mathbf{y})\in L^\infty(\partial\mathcal{M})$ for any $1\le i\le d$. Assume $u(\mathbf{x})$ solves the following equation

$$L_t u = r-\bar{r},$$

where $\bar{r}=\frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} r(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}$. Then, there exist constants $C>0, T_0>0$ independent of t, such that

$$\|u\|_{H^1(\mathcal{M})}\le C\sqrt{t}\max_{1\le i\le d}(\|b_i\|_\infty).$$

as long as $t\le T_0$.

The proof can be found in Section 5.

Applying Theorem 2.1, Theorem 3.2 on I_{in}, I_{bd} respectively, we can get the convergence result with higher rate.

THEOREM 3.3. *Let u be the solution to Problem (1.3) with $f\in C^1(\mathcal{M})$ and u_t be the solution to the problem (2.1). Then there exist constants C and T_0 only depending on $\mathcal{M}$, such that for any $t\le T_0$*

$$\|u-u_t\|_{H^1(\mathcal{M})}\le Ct^{1/2}\|u\|_{C^3(\mathcal{M})}+Ct^{1/2}\|f\|_\infty.$$

REMARK 3.1. For the elliptic equation, it is well known that $\|u\|_{C^3(\mathcal{M})}\le C\|f\|_{C^1(\mathcal{M})}$, then we have

$$\|u-u_t\|_{H^1(\mathcal{M})}\le Ct^{1/2}\|f\|_{C^1(\mathcal{M})}.$$

Using more delicate analysis as that in [24, 37], the result can be further improved to

$$\|u-u_t\|_{H^1(\mathcal{M})}\le Ct^{1/2}\|f\|_{H^1(\mathcal{M})}.$$

4. Truncation errore estimate (Proof of Theorem 3.1)

First, we give some notations. Some of them have been introduced in the previous sections. For the convenience of the proof, we also list them here.

- $\Phi: V\subset\mathbb{R}^{d_0}\to U\subset\mathcal{M}$ is the local parametrization of the manifold. $\Phi^k, k=1,\cdots,d$ is its k-th component. For $\mathbf{x}\in\mathcal{M}$, $\theta(\mathbf{x})=\Phi^{-1}(\mathbf{x})$ is the local coordinate of $\mathbf{x}$, $\theta_i(\mathbf{x}), i=1,\cdots,d_0$ denotes i-th component of the coordinate.
- $\partial_k=\frac{\partial}{\partial\theta_k}, k=1,\cdots,d_0$ denotes the derivative in the parametrization space. $D_k=\sum_{i,j=1}^{d_0} g^{ij}\frac{\partial\Phi^k}{\partial\theta_i}\frac{\partial}{\partial\theta_j}, k=1,\cdots,d$ is the derivative on the manfiold.

We abuse the constant C. It may be different constant in different places. In the proof, without explicit notation, all functions are evaluated at $\mathbf{y}$.

We also need a proposition regarding the integral of the kernel functions over the manifold.

PROPOSITION 4.1. *Let* $\bar{w}_t(\mathbf{x})=\int_{\mathcal{M}}\bar{K}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}$ *and* $w_t(\mathbf{x})=\int_{\mathcal{M}}K_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}$.

$$\mathcal{M}_t=\{\mathbf{x}\in\mathcal{M}: dist(\mathbf{x},\partial\mathcal{M})\geq 2\sqrt{t}\}.$$

Then, we have for $\mathbf{x}\in\mathcal{M}_t$,

$$w_t(\mathbf{x})=2\int_{\mathbb{R}^{d_0}}R\left(\|\mathbf{z}\|^2\right)\mathrm{d}\mathbf{z}+O(\sqrt{t}),\quad \bar{w}_t(\mathbf{x})=2\int_{\mathbb{R}^{d_0}}\bar{R}\left(\|\mathbf{z}\|^2\right)\mathrm{d}\mathbf{z}+O(\sqrt{t})$$

Moreover, for $\mathbf{x}\in\mathcal{M}$,

$$\frac{1}{2}\int_{\mathbb{R}^{d_0}}R\left(\|\mathbf{z}\|^2\right)\mathrm{d}\mathbf{z}\leq w_t(\mathbf{x})\leq\frac{5}{2}\int_{\mathbb{R}^{d_0}}R\left(\|\mathbf{z}\|^2\right)\mathrm{d}\mathbf{z},$$
$$\frac{1}{2}\int_{\mathbb{R}^{d_0}}\bar{R}\left(\|\mathbf{z}\|^2\right)\mathrm{d}\mathbf{z}\leq \bar{w}_t(\mathbf{x})\leq\frac{5}{2}\int_{\mathbb{R}^{d_0}}\bar{R}\left(\|\mathbf{z}\|^2\right)\mathrm{d}\mathbf{z}$$

From Proposition 4.1, it is easy to get that $\bar{f}_t=O(\sqrt{t})\|f\|_{C^1}$. Then we focus on the integral approximation of the elliptic operator.

Using the Gauss formula, we have

$$\begin{aligned}&\int_{\mathcal{M}}\sum_{i,j=1}^{d}D_i(a_{ij}(\mathbf{y})D_ju(\mathbf{y}))\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\ &=-\sum_{i,j=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})D_i^{\mathbf{y}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}+\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}}\mathbf{n}_i(\mathbf{y})a_{ij}(\mathbf{y})D_ju(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}},\end{aligned}\tag{4.1}$$

$D_i^{\mathbf{y}}$ denotes D_i with respect to $\mathbf{y}$.

Substituting above expansion in the first term of (4.1), we get

$$\begin{aligned}&-\sum_{i,j=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})D_i^{\mathbf{y}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\ &=-\frac{1}{2t}\sum_{i,j,m,n=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})\sum_{l',k',i',j'=1}^{d_0}(\partial_{l'}\Phi^jg^{l'k'}\partial_{k'}u(\mathbf{y}))\partial_{i'}\Phi^ig^{i'j'}\partial_{j'}\Phi^na^{mn}(\mathbf{x})(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.\end{aligned}\tag{4.2}$$

The coefficients $a_{ij}(\mathbf{y})$ map the tangent space $\mathcal{T}_{\mathbf{y}}$ into itself which means that there exists $c_{l'l}(\mathbf{y})$ such that

$$\sum_{j=1}^{d}a_{ij}(\mathbf{y})\partial_{l'}\Phi^j(\theta(\mathbf{y}))=\sum_{l=1}^{d_0}c_{l'l}(\mathbf{y})\partial_l\Phi^i(\theta(\mathbf{y})),$$

where $\theta(\mathbf{y})=\Phi^{-1}(\mathbf{y})$ is the coordinate of $\mathbf{y}$ in the parametric space. Then

$$\begin{aligned}&-\sum_{i,j=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})D_i^{\mathbf{y}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\ &=-\frac{1}{2t}\sum_{i,m,n=1}^{d}\int_{\mathcal{M}}\sum_{l,l',k',i',j'=1}^{d_0}c_{l'l}(\mathbf{y})\partial_l\Phi^i\partial_{i'}\Phi^ig^{l'k'}g^{i'j'}\partial_{j'}\Phi^na^{mn}(\mathbf{x})(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\partial_{k'}u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\ &=-\frac{1}{2t}\sum_{m,n=1}^{d}\int_{\mathcal{M}}\sum_{l',k',j'=1}^{d_0}c_{l'j'}(\mathbf{y})\partial_{j'}\Phi^ng^{l'k'}a^{mn}(\mathbf{x})(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\partial_{k'}u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2t}\sum_{l,m,n=1}^{d}\int_{\mathcal{M}}\sum_{l',k'=1}^{d_0} a_{nl}(\mathbf{y})\partial_{l'}\Phi^l g^{l'k'}a^{mn}(\mathbf{x})(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\partial_{k'}u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&= -\frac{1}{2t}\sum_{l,m,n=1}^{d}\int_{\mathcal{M}} a_{nl}(\mathbf{y})a^{mn}(\mathbf{x})(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_l u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&= -\frac{1}{2t}\sum_{l=1}^{d}\int_{\mathcal{M}}(x_l-y_l)D_l u(\mathbf{y})R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&\quad -\frac{1}{2t}\sum_{l,m,n=1}^{d}\int_{\mathcal{M}}(a_{nl}(\mathbf{y})-a_{nl}(\mathbf{x}))a^{mn}(\mathbf{x})(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_l u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.
\end{aligned}
\tag{4.3}
$$

Notice that

$$
\begin{aligned}
&D_n^{\mathbf{y}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\\
=&\frac{1}{2t}\sum_{l,m=1}^{d}\sum_{i',j'=1}^{d_0}\partial_{i'}\Phi^n g^{i'j'}\partial_{j'}\Phi^l a^{ml}(\mathbf{x})(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\\
=&\frac{1}{2t}\sum_{l,m=1}^{d}\sum_{i',j',m'=1}^{d_0}\partial_{i'}\Phi^n g^{i'j'}\partial_{j'}\Phi^l a^{ml}(\mathbf{y})\partial_{m'}\Phi^m(\theta_{m'}(\mathbf{x})-\theta_{m'}(\mathbf{y}))R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{O(t)}{t}
\end{aligned}
\tag{4.4}
$$

where $\theta_{m'}(\mathbf{x})=(\Phi^{-1}(\mathbf{x}))_{m'}$ denotes the m'-th coordinate of $\mathbf{x}$ in the parameter space. To get the second equation in (4.4), we use the Taylor expansion of the coordinate function Φ,

$$
x_m-y_m=\Phi^m(\theta(\mathbf{x}))-\Phi^m(\theta(\mathbf{y}))=\sum_{m'=1}^{d_0}\partial_{m'}\Phi^m(\theta(\mathbf{y}))(\theta_{m'}(\mathbf{x})-\theta_{m'}(\mathbf{y}))+O(\|\mathbf{x}-\mathbf{y}\|).
$$

Since $a^{ml}(\mathbf{y})$ also maps the tangent space $T_{\mathbf{y}}\mathcal{M}$ into itself, there exists $d_{l'l}(\mathbf{y})$ such that

$$
\sum_{m=1}^{d}a^{ml}(\mathbf{y})\partial_{m'}\Phi^m=\sum_{l'=1}^{d_0}d_{m'l'}(\mathbf{y})\partial_{l'}\Phi^l.
$$

It follows that

$$
\begin{aligned}
&D_n\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\\
=&\frac{1}{2t}\sum_{l=1}^{d}\sum_{i',j',l',m'=1}^{d_0}d_{m'l'}(\mathbf{y})\partial_{l'}\Phi^l\partial_{j'}\Phi^l g^{i'j'}\partial_{i'}\Phi^n(\theta_{m'}(\mathbf{x})-\theta_{m'}(\mathbf{y}))R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{O(t)}{t}\\
=&\frac{1}{2t}\sum_{i',m'=1}^{d_0}d_{m'i'}(\mathbf{y})\partial_{i'}\Phi^n(\theta_{m'}(\mathbf{x})-\theta_{m'}(\mathbf{y}))R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{O(t)}{t}\\
=&\frac{1}{2t}\sum_{m=1}^{d}\sum_{m'=1}^{d_0}a^{mn}(\mathbf{y})\partial_{m'}\Phi^m(\theta_{m'}(\mathbf{x})-\theta_{m'}(\mathbf{y}))R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{O(t)}{t}\\
=&\frac{1}{2t}\sum_{m=1}^{d}a^{mn}(\mathbf{y})(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{O(t)}{t}
\end{aligned}
$$

$$=\frac{1}{2t}\sum_{m=1}^{d}a^{mn}(\mathbf{x})(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{O(t)}{t}. \tag{4.5}$$

The last term of (4.3) becomes

$$\begin{aligned}
&\frac{1}{2t}\sum_{l,m,n=1}^{d}\int_{\mathcal{M}}(a_{nl}(\mathbf{y})-a_{nl}(\mathbf{x}))a^{mn}(\mathbf{x})(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_l u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
=&\sum_{l,n=1}^{d}\int_{\mathcal{M}}(a_{nl}(\mathbf{y})-a_{nl}(\mathbf{x}))D_n^{\mathbf{y}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_l u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^1}\\
=&-\sum_{l,n=1}^{d}\int_{\mathcal{M}}D_n a_{nl}(\mathbf{y})D_l u(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&+\sum_{l,n=1}^{d}\int_{\partial\mathcal{M}}\mathbf{n}_n(\mathbf{y})(a_{nl}(\mathbf{y})-a_{nl}(\mathbf{x}))\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_l u(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^1}.
\end{aligned}\tag{4.6}$$

Now, we turn to estimate the first term of (4.3). In this step, we need the help of Taylor's expansion of $u(\mathbf{x})$ at $\mathbf{y}$,

$$u(\mathbf{x})-u(\mathbf{y})=\sum_{j=1}^{d}(x_j-y_j)D_j u(\mathbf{y})+\frac{1}{2}\sum_{m,n=1}^{d}D_mD_n u(\mathbf{y})(x_m-y_m)(x_n-y_n)+O(\|\mathbf{x}-\mathbf{y}\|^3). \tag{4.7}$$

This expansion gives immediately

$$\begin{aligned}
&-\frac{1}{2t}\sum_{j=1}^{d}\int_{\mathcal{M}}(x_j-y_j)D_j u(\mathbf{y})R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
=&-\frac{1}{2t}\int_{\mathcal{M}}R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}}\\
&+\frac{1}{4t}\sum_{m,n=1}^{d}\int_{\mathcal{M}}R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_n u(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^3}.
\end{aligned}\tag{4.8}$$

Next, we focus on the second term of (4.8). It follows from (4.5) that

$$\begin{aligned}
\sum_{i=1}^{d}a_{im}(\mathbf{x})D_i\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})=&\frac{1}{2t}\sum_{i,n=1}^{d}a_{mi}(\mathbf{x})a^{ni}(\mathbf{x})(x_n-y_n)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{O(t)}{t}\\
=&\frac{1}{2t}(x_m-y_m)R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{O(t)}{t}.
\end{aligned}\tag{4.9}$$

The second term of (4.8) is calculated as

$$\begin{aligned}
&\frac{1}{4t}\sum_{m,n=1}^{d}\int_{\mathcal{M}}R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_n u(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}\\
=&\frac{1}{2}\sum_{i,m,n=1}^{d}\int_{\mathcal{M}}a_{im}(\mathbf{x})D_i\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_n u(\mathbf{y})(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2}\sum_{i,m,n=1}^{d}\int_{\mathcal{M}} a_{im}(\mathbf{x})\sum_{i',j'=1}^{d_0}(\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n)D_mD_nu(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&+\frac{1}{2}\sum_{i,m,n=1}^{d}\int_{\partial\mathcal{M}}\mathbf{n}_i(\mathbf{y})a_{im}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^3}.
\end{aligned}
\tag{4.10}
$$

Notice that in the support of $\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})$,

$$
\begin{aligned}
&\sum_{i,m=1}^{d}\sum_{i',j'=1}^{d_0} a_{im}(\mathbf{x})(\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n)D_m\\
=&\sum_{i,m=1}^{d}\sum_{i',j',i'',j''=1}^{d_0} a_{im}(\mathbf{y})(\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n)(\partial_{i''}\Phi^m g^{i''j''}\partial_{j''})+O(\sqrt{t})\\
=&\sum_{i=1}^{d}\sum_{i',j',l',i'',j''=1}^{d_0} c_{i''l'}\partial_{l'}\Phi^i\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n g^{i''j''}\partial_{j''}+O(\sqrt{t})\\
=&\sum_{l',i'',j''=1}^{d_0} c_{i''l'}\partial_{l'}\Phi^n g^{i''j''}\partial_{j''}+O(\sqrt{t})\\
=&\sum_{m=1}^{d}\sum_{i'',j''=1}^{d_0} a_{mn}(\mathbf{y})\partial_{i''}\Phi^m g^{i''j''}\partial_{j''}+O(\sqrt{t})\\
=&\sum_{m=1}^{d} a_{mn}(\mathbf{y})D_m+O(\sqrt{t}).
\end{aligned}
$$

From (4.10), we obtain

$$
\begin{aligned}
&\frac{1}{4t}\sum_{m,n=1}^{d}\int_{\mathcal{M}} R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}\\
=&\frac{1}{2}\sum_{m,n=1}^{d}\int_{\mathcal{M}} a_{mn}(\mathbf{y})D_mD_nu(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&+\frac{1}{2}\sum_{i,m,n=1}^{d}\int_{\partial\mathcal{M}}\mathbf{n}_i(\mathbf{y})a_{im}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_n-y_n)\mathrm{d}\tau_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^3}.
\end{aligned}
\tag{4.11}
$$

Now, using (4.1), (4.3), (4.8) and (4.11), we get

$$
\begin{aligned}
&\sum_{i,j=1}^{d}\int_{\mathcal{M}} D_i(a_{ij}(\mathbf{y})D_ju(\mathbf{y}))\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
=&-\frac{1}{2t}\int_{\mathcal{M}} R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}}+\frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathcal{M}} a_{ij}(\mathbf{y})D_iD_ju(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&+\sum_{i,j=1}^{d}\int_{\mathcal{M}} D_ia_{ij}(\mathbf{y})D_ju(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}+\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}}\mathbf{n}_i(\mathbf{y})a_{ij}(\mathbf{y})D_ju(\mathbf{y})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}
\end{aligned}
$$

$$+B.T.1+O(\sqrt{t})\|u\|_{C^3} \tag{4.12}$$

where

$$\begin{aligned} B.T.1=&\frac{1}{2}\sum_{i,m,n=1}^{d}\int_{\partial\mathcal{M}}\mathbf{n}_i(\mathbf{y})a_{im}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_n-y_n)\mathrm{d}\tau_{\mathbf{y}}\\ &-\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}}\mathbf{n}_i(\mathbf{y})(a_{ij}(\mathbf{y})-a_{ij}(\mathbf{x}))\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_ju(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}. \end{aligned} \tag{4.13}$$

Now, we change the kernel function to $\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})$ and get

$$\begin{aligned} &\int_{\mathcal{M}}\sum_{i,j=1}^{d}D_i(a_{ij}(\mathbf{y})D_ju(\mathbf{y}))\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\ =&-\sum_{i,j=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})D_i^{\mathbf{y}}\left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\right)\mathrm{d}\mu_{\mathbf{y}}\\ &+\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}}\mathbf{n}_i(\mathbf{y})a_{ij}(\mathbf{y})D_ju(\mathbf{y})\frac{\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\mathrm{d}\tau_{\mathbf{y}}. \end{aligned} \tag{4.14}$$

Direct calculation gives that the first term of (4.14) becomes

$$-\sum_{i,j=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})D_i^{\mathbf{y}}\left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\right)\mathrm{d}\mu_{\mathbf{y}} \tag{4.15}$$

$$\begin{aligned} =&-\frac{1}{2t}\sum_{i,j,m,n=1}^{d}\int_{\mathcal{M}}\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})\sum_{i',j'=1}^{d_0}\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n a^{mn}(\mathbf{y})(x_m-y_m)R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\ &+\frac{1}{4t}\sum_{i,j,m,n=1}^{d}\int_{\mathcal{M}}\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}a_{ij}(\mathbf{y})\partial_ju(\mathbf{y})D_ia^{mn}(\mathbf{y})(x_m-y_m)(x_n-y_n)R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\ &-\sum_{i,j=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})D_i\left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}. \end{aligned}$$

Next, we will estimate the three terms in (4.15) one by one.

$$\begin{aligned} &-\frac{1}{2t}\sum_{i,j,m,n=1}^{d}\int_{\mathcal{M}}\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})\sum_{i',j'=1}^{d_0}\partial_{i'}\Phi^i g^{i'j'}\partial_{j'}\Phi^n a^{mn}(\mathbf{y})(x_m-y_m)R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\ =&-\frac{1}{2t}\sum_{j=1}^{d}\int_{\mathcal{M}}(x_j-y_j)D_ju(\mathbf{y})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\ =&-\frac{1}{2t}\int_{\mathcal{M}}\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}}\\ &+\frac{1}{4t}\sum_{m,n=1}^{d}\int_{\mathcal{M}}\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^3}. \end{aligned} \tag{4.16}$$

The first equality is from (4.5). To get the second equality, we use the Taylor's expansion (4.7). The second term of (4.16) can be further calculated as

$$\frac{1}{4t}\sum_{m,n=1}^{d}\int_{\mathcal{M}}\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}$$

$$
\begin{aligned}
&= \frac{1}{4t}\sum_{m,n=1}^{d}\int_{\mathcal{M}}\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_m-y_m)(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^2}\\
&= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{x})D_iD_ju(\mathbf{y})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&\quad+\frac{1}{2}\sum_{i,m,n=1}^{d}\int_{\partial\mathcal{M}}\mathbf{n}_i(\mathbf{y})a_{im}(\mathbf{x})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^3}\\
&= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_iD_ju(\mathbf{y})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&\quad+\frac{1}{2}\sum_{i,m,n=1}^{d}\int_{\partial\mathcal{M}}\mathbf{n}_i(\mathbf{y})a_{im}(\mathbf{x})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_n-y_n)\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^3}.
\end{aligned}
\tag{4.17}
$$

To get the second equality, we use the same calculation as that in (4.11).

The second term of (4.15) is calculated as

$$
\begin{aligned}
&\frac{1}{4t}\sum_{i,j,m,n=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})\frac{D_ia^{mn}(\mathbf{y})(x_m-y_m)(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&= \frac{1}{4t}\sum_{i,j,m,n=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})\frac{D_ia^{mn}(\mathbf{y})(x_m-y_m)(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^1}\\
&= \frac{1}{2}\sum_{i,j,k,m,n=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})\frac{D_ia^{mn}(\mathbf{y})(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}a_{km}(\mathbf{x})D_k^{\mathbf{y}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^1}\\
&= \frac{1}{2}\sum_{i,j,k,m,n=1}^{d}\int_{\mathcal{M}}\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})D_ia^{mn}(\mathbf{y})a_{km}(\mathbf{y})\sum_{i',j'=1}^{d_0}(\partial_{i'}\Phi^k g^{i'j'}\partial_{j'}\Phi^n)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&\quad+\frac{1}{2}\sum_{i,j,k,m,n=1}^{d}\int_{\partial\mathcal{M}}\mathbf{n}_k(\mathbf{y})a_{ij}(\mathbf{y})D_ju(\mathbf{y})\frac{D_ia^{mn}(\mathbf{y})(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}a_{km}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^2}.
\end{aligned}
\tag{4.18}
$$

In addition, we have that

$$
\sum_{k,m,n=1}^{d}\sum_{i',j'=1}^{d_0}D_ia^{mn}(\mathbf{y})a_{km}(\mathbf{y})(\partial_{i'}\Phi^k g^{i'j'}\partial_{j'}\Phi^n)=-\frac{2D_i(\sqrt{|A_{\mathcal{T}}(\mathbf{y})|})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}. \tag{4.19}
$$

The derivation of this equation can be found in Appendix B.

Using above equation, we obtain

$$
\begin{aligned}
&\frac{1}{2}\sum_{i,j,k,m,n=1}^{d}\int_{\mathcal{M}}\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})D_ia^{mn}(\mathbf{y})a_{km}(\mathbf{y})\sum_{i',j'=1}^{d_0}(\partial_{i'}\Phi^k g^{i'j'}\partial_{j'}\Phi^n)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}\\
&=-\sum_{i,j=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})\frac{D_i\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}{|A_{\mathcal{T}}(\mathbf{y})|}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^1}\\
&=\sum_{i,j=1}^{d}\int_{\mathcal{M}}a_{ij}(\mathbf{y})D_ju(\mathbf{y})D_i\left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}+O(\sqrt{t})\|u\|_{C^1}.
\end{aligned}
\tag{4.20}
$$

Using (4.14)–(4.18) and (4.20),

$$
\int_{\mathcal{M}}\sum_{i,j=1}^{d}D_i(a_{ij}(\mathbf{y})D_ju(\mathbf{y}))\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}
$$

$$
\begin{aligned}
&= -\frac{1}{2t}\int_{\mathcal{M}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}} R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} \\
&\quad +\frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathcal{M}} a_{ij}(\mathbf{y})D_iD_ju(\mathbf{y})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \\
&\quad +\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}} \mathbf{n}_i(\mathbf{y})a_{ij}(\mathbf{y})D_ju(\mathbf{y})\frac{\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\mathrm{d}\tau_{\mathbf{y}} + B.T.2 + O(\sqrt{t})\|u\|_{C^3} \qquad (4.21)
\end{aligned}
$$

where

$$
\begin{aligned}
B.T.2 &= \frac{1}{2}\sum_{i,m,n=1}^{d}\int_{\partial\mathcal{M}} \mathbf{n}_i(\mathbf{y})a_{im}(\mathbf{x})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_n-y_n)\mathrm{d}\tau_{\mathbf{y}} \\
&\quad +\frac{1}{2}\sum_{i,j,k,m,n=1}^{d}\int_{\partial\mathcal{M}} \mathbf{n}_k(\mathbf{y})a_{ij}(\mathbf{y})D_ju(\mathbf{y})\frac{D_ia^{mn}(\mathbf{y})(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}a_{km}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}
\end{aligned} \qquad (4.22)
$$

Now, (4.12) and (4.21) imply that

$$
\begin{aligned}
&\int_{\mathcal{M}}\sum_{i,j=1}^{d} D_i(a_{ij}(\mathbf{y})D_ju(\mathbf{y}))\left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\right)\mathrm{d}\mu_{\mathbf{y}} \\
&= -\frac{1}{2t}\int_{\mathcal{M}}\left(\frac{R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}+\frac{R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} \\
&\quad +\frac{1}{2}\int_{\mathcal{M}}\sum_{i,j=1}^{d} D_i(a_{ij}(\mathbf{y})D_ju(\mathbf{y}))\left(\frac{\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}+\frac{\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)\mathrm{d}\mu_{\mathbf{y}} \\
&\quad +\int_{\partial\mathcal{M}}\sum_{i,j=1}^{d} \mathbf{n}_i(\mathbf{y})a_{ij}(\mathbf{y})D_ju(\mathbf{y})\left(\frac{\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}+\frac{\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)\mathrm{d}\tau_{\mathbf{y}} + I_{bd} + O(\sqrt{t})\|u\|_{C^3}
\end{aligned} \qquad (4.23)
$$

where

$$
\begin{aligned}
I_{bd} &= \frac{1}{2}\sum_{i,m,n=1}^{d}\int_{\partial\mathcal{M}} \mathbf{n}_i(\mathbf{y})a_{im}(\mathbf{x})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_n-y_n)\mathrm{d}\tau_{\mathbf{y}} \\
&\quad -\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}} \mathbf{n}_i(\mathbf{y})(a_{ij}(\mathbf{y})-a_{ij}(\mathbf{x}))\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_ju(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}} \\
&\quad +\frac{1}{2}\sum_{i,m,n=1}^{d}\int_{\partial\mathcal{M}} \mathbf{n}_i(\mathbf{y})a_{im}(\mathbf{x})\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})D_mD_nu(\mathbf{y})(x_n-y_n)\mathrm{d}\tau_{\mathbf{y}} \\
&\quad +\frac{1}{2}\sum_{i,j,k,m,n=1}^{d}\int_{\partial\mathcal{M}} \mathbf{n}_k(\mathbf{y})a_{ij}(\mathbf{y})D_ju(\mathbf{y})\frac{D_ia^{mn}(\mathbf{y})(x_n-y_n)}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}a_{km}(\mathbf{x})\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}.
\end{aligned} \qquad (4.24)
$$

Finally, it follows from (4.23) that

$$\int_{\mathcal{M}}\sum_{i,j=1}^{d} D_i(a_{ij}(\mathbf{y})D_ju(\mathbf{y}))\left(\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\right)\mathrm{d}\mu_{\mathbf{y}}$$
$$=-\frac{1}{t}\int_{\mathcal{M}}\left(\frac{R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}+\frac{R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}}$$
$$+2\int_{\partial\mathcal{M}}\sum_{i,j=1}^{d}\mathbf{n}_i(\mathbf{y})a_{ij}(\mathbf{y})D_ju(\mathbf{y})\left(\frac{\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}}+\frac{\bar{R}_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}}\right)\mathrm{d}\tau_{\mathbf{y}}+2B.T.+O(\sqrt{t})\|u\|_{C^3}.$$

Now, we go to estimate the gradient of the residual. Notice that the gradient is on $\mathbf{x}$. If the gradient is applied on $\mathbf{x}-\mathbf{y}$ or $R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})$, the error is amplified by $t^{-1/2}$. This is just what we want to get. The most difficult term is from the Taylor's expansion of $u(\mathbf{x})-u(\mathbf{y})$. Directly applying the gradient will give fourth order derivative of u which is not allowed. Here we use a trick in [37], using Newton-Leibniz formula to get an integral formula for the residual in Taylor's expansion.

$$u(\mathbf{x})-u(\mathbf{y})-(\mathbf{x}-\mathbf{y})\cdot Du(\mathbf{y})-\frac{1}{2}\sum_{i,j=1}^{d}(x_i-y_i)(x_j-y_j)D_iD_ju(\mathbf{y})$$
$$=\xi_i\xi_{i'}\int_0^1\int_0^1 s_1\left(\partial_i\Phi^j(\theta(\mathbf{y})+s_1\xi)\partial_{i'}\Phi^{j'}(\theta(\mathbf{y})+s_2s_1\xi)D_{j'}D_ju(\Phi(\theta(\mathbf{y})+s_2s_1\xi))\right)\mathrm{d}s_2\mathrm{d}s_1$$
$$-\xi_i\xi_{i'}\int_0^1\int_0^1 s_1\left(\partial_i\Phi^j(\theta(\mathbf{y}))\partial_{i'}\Phi^{j'}(\theta(\mathbf{y}))D_{j'}D_ju(\Phi(\theta(\mathbf{y})))\right)\mathrm{d}s_2\mathrm{d}s_1$$

with $\xi=\theta(\mathbf{x})-\theta(\mathbf{y})$ and ξ_i is its i-th component. Taking derivative with respect to $\mathbf{x}$, we can get the bound of the gradient.

5. Stability (Proof of Theorem 3.2)

The key point is to show that

$$\left|\int_{\mathcal{M}}u(\mathbf{x})(r(\mathbf{x})-\bar{r})\mathrm{d}\mu_{\mathbf{x}}\right|\le C\sqrt{t}\max_{1\le i\le d}(\|b_i\|_\infty)\|u\|_{H^1(\mathcal{M})}. \tag{5.1}$$

Notice that

$$|\bar{r}|=\frac{1}{|\mathcal{M}|}\left|\sum_{i=1}^{d}\int_{\mathcal{M}}\int_{\partial\mathcal{M}}b_i(\mathbf{y})(x_i-y_i)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mathbf{x}\right|\le C\sqrt{t}\max_{1\le i\le d}(\|b_i\|_\infty).$$

Then it is sufficient to show that

$$\left|\int_{\mathcal{M}}u(\mathbf{x})\left(\int_{\partial\mathcal{M}}\sum_{i=1}^{d}b_i(\mathbf{y})(x_i-y_i)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)\mathrm{d}\mu_{\mathbf{x}}\right|\le C\sqrt{t}\max_{1\le i\le d}(\|b_i\|_\infty)\|u\|_{H^1(\mathcal{M})}. \tag{5.2}$$

Notice that

$$(x_i-y_i)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})=2t\sum_{j=1}^{d}a_{ij}(\mathbf{x})D_j^{\mathbf{y}}\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})$$

$$=-2t\sum_{j=1}^{d}a_{ij}(\mathbf{x})\left(D_j^{\mathbf{x}}\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})+\frac{1}{4t}\sum_{m,n=1}^{d}D_j^{\mathbf{x}}a^{mn}(\mathbf{x})(x_m-y_m)(x_n-y_n)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\right) \quad (5.3)$$

where $\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})=C_t\bar{\bar{R}}\left(\frac{1}{4t}\sum_{m,n=1}^{d}(x_m-y_m)a^{mn}(\mathbf{x})(x_n-y_n)\right)$ and $\bar{\bar{R}}(r)=\int_r^{\infty}\bar{R}(s)\mathrm{d}s$.

By integration by parts, we have

$$\begin{aligned}&\sum_{i,j=1}^{d}\int_{\mathcal{M}}u(\mathbf{x})\int_{\partial\mathcal{M}}b_i(\mathbf{y})a_{ij}(\mathbf{x})D_j^{\mathbf{x}}\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\mathrm{d}\mathbf{x}\\ =&\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}}\int_{\partial\mathcal{M}}\mathbf{n}_j(\mathbf{x})a_{ij}(\mathbf{x})b_i(\mathbf{y})u(\mathbf{x})\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}}\\ &-\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}}\int_{\mathcal{M}}b_i(\mathbf{y})D_j^{\mathbf{x}}[u(\mathbf{x})a_{ij}(\mathbf{x})]\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\tau_{\mathbf{y}}. \end{aligned} \quad (5.4)$$

For the boundary term,

$$\begin{aligned}&\left|\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}}\int_{\partial\mathcal{M}}\mathbf{n}_j(\mathbf{x})a_{ij}(\mathbf{x})b_i(\mathbf{y})u(\mathbf{x})\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}}\right|\\ \le& C\max_{1\le i\le d}(\|b_i\|_\infty)\int_{\partial\mathcal{M}}\int_{\partial\mathcal{M}}|u(\mathbf{x})|\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}}\\ \le& C\max_{1\le i\le d}(\|b_i\|_\infty)\left(\int_{\partial\mathcal{M}}\left(\int_{\partial\mathcal{M}}|u(\mathbf{x})|\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\right)^2\mathrm{d}\tau_{\mathbf{y}}\right)^{1/2}\\ \le& C\max_{1\le i\le d}(\|b_i\|_\infty)\left(\int_{\partial\mathcal{M}}\left(\int_{\partial\mathcal{M}}\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\right)\left(\int_{\partial\mathcal{M}}|u(\mathbf{x})|^2\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{x}}\right)\mathrm{d}\tau_{\mathbf{y}}\right)^{1/2}\\ \le& Ct^{-1/2}\max_{1\le i\le d}(\|b_i\|_\infty)\|u\|_{L^2(\partial\mathcal{M})}\le Ct^{-1/2}\max_{1\le i\le d}(\|b_i\|_\infty)\|u\|_{H^1(\mathcal{M})}. \end{aligned} \quad (5.5)$$

The bound of the second term of (5.4) is straightforward. By using the assumption that the coefficients $a_{ij}(\mathbf{x})$ are smooth functions, we have

$$\begin{aligned}\left|\sum_{i,j=1}^{d}b_i(\mathbf{y})D_j^{\mathbf{x}}[u(\mathbf{x})a_{ij}(\mathbf{x})]\right|\le&\sum_{i,j=1}^{d}|D_j^{\mathbf{x}}u(\mathbf{x})||b_i(\mathbf{y})a_{ij}(\mathbf{x})|+\sum_{i,j=1}^{d}|u(\mathbf{x})||b_i(\mathbf{y})D_j^{\mathbf{x}}a_{ij}(\mathbf{x})|\\ \le& C\max_{1\le i\le d}(\|b_i\|_\infty)(|Du(\mathbf{x})|+|u(\mathbf{x})|)\end{aligned}$$

where the constant C depends on the curvature of the manifold $\mathcal{M}$.

Then, we have

$$\begin{aligned}&\left|\sum_{i,j=1}^{d}\int_{\partial\mathcal{M}}\int_{\mathcal{M}}b_i(\mathbf{y})D_j^{\mathbf{x}}[u(\mathbf{x})a_{ij}(\mathbf{x})]\bar{\bar{R}}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\tau_{\mathbf{y}}\right|\\ \le& C\max_{1\le i\le d}(\|b_i\|_\infty)\int_{\partial\mathcal{M}}\int_{\mathcal{M}}(|Du(\mathbf{x})|+|u(\mathbf{x})|)\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\tau_{\mathbf{y}}\\ \le& C\max_{1\le i\le d}(\|b_i\|_\infty)\left(\int_{\mathcal{M}}(|Du(\mathbf{x})|^2+|u(\mathbf{x})|^2)\left(\int_{\partial\mathcal{M}}\bar{\bar{R}}_t(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)\mathrm{d}\mu_{\mathbf{x}}\right)^{1/2}\end{aligned}$$

$$\leq Ct^{-1/4}\max_{1\leq i\leq d}(\|b_i\|_\infty)\|u\|_{H^1(\mathcal{M})}. \tag{5.6}$$

and

$$\left|\int_{\mathcal{M}} u(\mathbf{x})\left(\int_{\partial\mathcal{M}}\sum_{i,j,m,n=1}^{d} b_i(\mathbf{y})a_{ij}(\mathbf{x})D_j^{\mathbf{x}}a^{mn}(\mathbf{x})(x_m-y_m)(x_n-y_n)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)\mathrm{d}\mathbf{x}\right|$$
$$\leq Ct\int_{\mathcal{M}}|u(\mathbf{x})|\left(\int_{\partial\mathcal{M}}\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}\right)\mathrm{d}\mathbf{x}\leq Ct^{3/4}\|u\|_{L^2}. \tag{5.7}$$

Then, the inequality (5.2) is obtained from (5.3)–(5.7). Now, using Lemma 2.1, we have

$$\|u\|_{L^2(\mathcal{M})}^2\leq C\langle u,L_tu\rangle\leq C\sqrt{t}\max_i(\|b_i\|_\infty)\|u\|_{H^1(\mathcal{M})}. \tag{5.8}$$

Note $r(\mathbf{x})=\sum_{i=1}^d\int_{\partial\mathcal{M}}b_i(\mathbf{y})(x_i-y_i)\bar{R}_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}$. A direct calculation gives us that

$$\|r(\mathbf{x})\|_{L^2(\mathcal{M})}\leq Ct^{1/4}\max_{1\leq i\leq d}(\|b_i\|_\infty),\text{ and} \tag{5.9}$$
$$\|Dr(\mathbf{x})\|_{L^2(\mathcal{M})}\leq Ct^{-1/4}\max_{1\leq i\leq d}(\|b_i\|_\infty). \tag{5.10}$$

The integral equation $L_tu=r-\bar{r}$ gives that

$$u(\mathbf{x})=v(\mathbf{x})+\frac{t}{w_t(\mathbf{x})}(r(\mathbf{x})-\bar{r}) \tag{5.11}$$

where

$$v(\mathbf{x})=\frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}}K_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}},\quad w_t(\mathbf{x})=\int_{\mathcal{M}}K_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}. \tag{5.12}$$

By Lemma 2.2, we have

$$\begin{aligned}
&\|Du\|_{L^2(\mathcal{M})}^2\\
&\leq 2\|Dv\|_{L^2(\mathcal{M})}^2+2t^2\left\|D\left(\frac{r(\mathbf{x})-\bar{r}}{w_t(\mathbf{x})}\right)\right\|_{L^2(\mathcal{M})}^2\\
&\leq C\langle u,L_tu\rangle+Ct\|r\|_{L^2(\mathcal{M})}^2+Ct^2\|Dr\|_{L^2(\mathcal{M})}^2\\
&\leq C\sqrt{t}\max_{1\leq i\leq d}(\|b_i\|_\infty)\|u\|_{H^1(\mathcal{M})}+Ct\|r\|_{L^2(\mathcal{M})}^2+Ct^2\|Dr\|_{L^2(\mathcal{M})}^2\\
&\leq C\max_{1\leq i\leq d}(\|b_i\|_\infty)\left(\sqrt{t}\|u\|_{H^1(\mathcal{M})}+Ct^{3/2}\right).
\end{aligned} \tag{5.13}$$

Using (5.8) and (5.13), we have

$$\|u\|_{H^1(\mathcal{M})}^2\leq C\max_{1\leq i\leq d}(\|b_i\|_\infty)\left(\sqrt{t}\|u\|_{H^1(\mathcal{M})}+Ct^{3/2}\right), \tag{5.14}$$

which proves the theorem.

6. Conclusion

In this paper, we give an integral approximation for the elliptic operators with anisotropic coefficients on smooth manifold. The integral approximation is proved to preserve the symmetry and coercivity of the original elliptic operator. Using the integral approximation, we get an integral equation which approximates the original elliptic equation. Moreover, we prove the convergence between the solutions of the integral equation and the original elliptic equation.

The integral approximation of the differential operators was also studied in nonlocal diffusion and peridynamics [8–11, 30, 32, 41]. In Euclidean spaces, the non-divergence-type anisotropic elliptic equation was studied using the integral approximation [32]. In this paper, we extend the integral approximation to smooth manifolds.

The integral approximation can be used to deal with Dirichlet boundary condition also. One simple way is to approximate Dirichlet boundary condition by Robin boundary condition [25, 38].

$$u(\mathbf{x})+\beta\sum_{i,j=1}^{d}\mathbf{n}_i(\mathbf{x})a_{ij}(\mathbf{x})D_ju(\mathbf{x})=b(\mathbf{x}),\quad \mathbf{x}\in\partial\mathcal{M},$$

with $0<\beta\ll 1$. Then the integral approximation leads to an integral equation

$$\begin{aligned}&\frac{1}{t}\int_{\mathcal{M}}K_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}}-\frac{2}{\beta}\int_{\partial\mathcal{M}}\bar{K}_t(\mathbf{x},\mathbf{y})(b(\mathbf{y})-u(\mathbf{y}))\mathrm{d}\tau_{\mathbf{y}}\\ =&\frac{1}{t}\int_{\mathcal{M}}\bar{K}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mu_{\mathbf{y}}.\end{aligned}$$

This method is very simple. However, the boundary term introduces many difficulties in both theoretical analysis and numerical computation. Weighted nonlocal Laplacian (WNLL) gives another option to deal with Dirichlet boundary condition. WNLL has good theoretical properties. The symmetry and maximum principle are both preserved while the accuracy is only first order. To improve the accuracy, we are working on high order approximation of the Dirichlet boundary condition.

One advantage of the integral equation is that there are not any differential operators inside. It is easy to develop the numerical scheme in high dimensional point cloud. One simpliest discretization is to replace the integral by a weighted summation over the point cloud. Suppose P is a point cloud sample manifold $\mathcal{M}$ and $S\subset P$ is a sample of the boundary $\partial\mathcal{M}$. V_j is the volume weight of $\mathcal{M}$ at point $\mathbf{x}_j\in P$, A_l is the area weight of $\partial\mathcal{M}$ at point $\mathbf{x}_l\in S$. Then the simpliest discretization is

$$\sum_{\mathbf{x}_j\in P}K_t(\mathbf{x}_i,\mathbf{x}_j)(u_i-u_j)V_j-2\sum_{\mathbf{x}_l\in S}\bar{K}_t(\mathbf{x}_i,\mathbf{x}_l)g(\mathbf{x}_l)A_l=\sum_{\mathbf{x}_j\in P}\bar{K}_t(\mathbf{x}_i,\mathbf{x}_j)f(\mathbf{x}_j),\quad \mathbf{x}_i\in P.$$

with Neumann boundary condition, $\frac{\partial u}{\partial\mathbf{n}}(\mathbf{x})=g(\mathbf{x}),\mathbf{x}\in\partial\mathcal{M}$. This discretization is easy to implement and has good stability. Following the methods in [24, 37], we can prove the convergence of above scheme. In high dimensional cases, this is almost the only practical discretization. However, the accuracy of above discretization is relatively low. For the applications with high accuracy requirement, local mesh [19] or moving least squares [26, 27] seem to be good way to discretize the integral equation on point cloud. But the convergence is much more difficult to analyze. Finite element method (FEM) gives another numerical method with good theoretical property to solve the integral equation. The convergence can be proved even if the mesh is not having a regular shape. However, a global mesh is required in FEM, which is not practical especially in high dimensional problems.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (NSFC-11671005).

Appendix A. Proof of Lemma 2.1 and 2.2. Lemma 2.1 is a direct consequence of following two lemmas which have been proved in [37].

LEMMA A.1. *If t is small enough, then for any function $u\in L^2(\mathcal{M})$, there exists a constant $C>0$ independent of t and u, such that*

$$\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{32t}\right)(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}} \le C\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}}.$$

LEMMA A.2. *Assume both $\mathcal{M}$ and $\partial\mathcal{M}$ are C^∞. There exists a constant $C>0$ independent of t so that for any function $u\in L_2(\mathcal{M})$ with $\int_{\mathcal{M}} u(\mathbf{x})\mathrm{d}\mu_{\mathbf{x}}=0$ and for any sufficiently small t*

$$\frac{C_t}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right)(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}} \ge C\|u\|^2_{L_2(\mathcal{M})} \tag{A.1}$$

with $C_t=(4t)^{-d_0/2}$ be the normalization factor.

Notice that

$$K_t(\mathbf{x},\mathbf{y}) \le \frac{CC_t}{\delta_0} R\left(\frac{\lambda\|\mathbf{x}-\mathbf{y}\|^2}{8t}\right)$$

where δ_0 is the nondeneneracy constant in Assumption 2.1 and λ is the lower bound of the coefficient matrix in (1.4). Using this fact, Lemma 2.1 is an easy corollary of Lemma A.1 and A.2.

Now, we turn to prove Lemma 2.2. We start with the calculation of Dv,

$$\begin{aligned}
D_i v(\mathbf{x}) =& \frac{1}{w_t^2(\mathbf{x})}\int_{\mathcal{M}}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y}')D_i^{\mathbf{x}}K_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}} \\
&-\frac{1}{w_t^2(\mathbf{x})}\int_{\mathcal{M}}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})D_i^{\mathbf{x}}K(\mathbf{x},\mathbf{y}',t)u(\mathbf{y})\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}} \\
=& \frac{1}{w_t^2(\mathbf{x})}\int_{\mathcal{M}}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y}')D_i^{\mathbf{x}}K_t(\mathbf{x},\mathbf{y})(u(\mathbf{y})-u(\mathbf{y}'))\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}} \\
=& \frac{1}{w_t^2(\mathbf{x})}\int_{\mathcal{M}}\int_{\mathcal{M}} \mathcal{Q}(\mathbf{x},\mathbf{y},\mathbf{y}',t)(u(\mathbf{y})-u(\mathbf{y}'))\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}}
\end{aligned}$$

where $\mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}',t)=K_t(\mathbf{x},\mathbf{y}')D_i^{\mathbf{x}}K_t(\mathbf{x},\mathbf{y})$, $D_i^{\mathbf{x}}$ denotes D_i with respect to $\mathbf{x}$.

Notice that $\mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}',t)=0$ when $|\mathbf{x}-\mathbf{y}|^2\ge 4t/\lambda$ or $|\mathbf{x}-\mathbf{y}'|^2\ge 4t/\lambda$. This implies that $\mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}',t)=0$ when $|\mathbf{y}-\mathbf{y}'|^2\ge 16t/\lambda$ or $|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2\ge 4t/\lambda$. Thus from the assumption on R, we have

$$\mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}';t)^2 \le \frac{1}{\delta_0^2}\mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}';t)^2 R\left(\frac{\lambda\|\mathbf{y}-\mathbf{y}'\|^2}{32t}\right)R\left(\frac{\lambda\|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}\|^2}{8t}\right).$$

We can upper bound the norm of Dv as follows:

$$|Dv(\mathbf{x})|^2 = \frac{1}{w_t^4(\mathbf{x})}\sum_{i=1}^d\left(\int_{\mathcal{M}}\int_{\mathcal{M}} \mathcal{Q}_i(\mathbf{x},\mathbf{y},\mathbf{y}';t)(u(\mathbf{y})-u(\mathbf{y}'))\mathrm{d}\mathbf{y}'\mathrm{d}\mathbf{y}\right)^2$$

$$\le \frac{1}{w_t^4(\mathbf{x})}\sum_{i=1}^d \int_{\mathcal{M}}\int_{\mathcal{M}} \mathcal{Q}_i^2(\mathbf{x},\mathbf{y},\mathbf{y}';t)\left(R\left(\frac{\lambda\|\mathbf{y}-\mathbf{y}'\|^2}{32t}\right)R\left(\frac{\lambda\|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}\|^2}{8t}\right)\right)^{-1}\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}}$$
$$\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{\lambda\|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}\|^2}{8t}\right)R\left(\frac{\lambda\|\mathbf{y}-\mathbf{y}'\|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}}$$
$$=\frac{C}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} t\sum_{i=1}^d \mathcal{Q}_i^2(\mathbf{x},\mathbf{y},\mathbf{y}';t)\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}}$$
$$\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{\lambda|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}|^2}{8t}\right)R\left(\frac{\lambda|\mathbf{y}-\mathbf{y}'|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}}.$$

By direct calculation, it is easy to check that

$$\int_{\mathcal{M}}\int_{\mathcal{M}} t\sum_{i=1}^d \mathcal{Q}_i^2(\mathbf{x},\mathbf{y},\mathbf{y}';t)\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}} \le CC_t^2$$

where $C>0$ is a generic constant.

Finally, we have

$$\int_{\mathcal{M}} |Dv(\mathbf{x})|^2\mathrm{d}\mu_{\mathbf{x}}$$
$$\le \frac{CC_t^2}{t}\int_{\mathcal{M}}\left(\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{\lambda\|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}\|^2}{8t}\right)R\left(\frac{\lambda\|\mathbf{y}-\mathbf{y}'\|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}}\right)\mathrm{d}\mu_{\mathbf{x}}$$
$$=\frac{CC_t^2}{t}\int_{\mathcal{M}}\int_{\mathcal{M}}\left(\int_{\mathcal{M}} R\left(\frac{\lambda\|\mathbf{x}-\frac{\mathbf{y}+\mathbf{y}'}{2}\|^2}{8t}\right)\mathrm{d}\mu_{\mathbf{x}}\right)R\left(\frac{\lambda\|\mathbf{y}-\mathbf{y}'\|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}}$$
$$\le \frac{CC_t}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{\lambda\|\mathbf{y}-\mathbf{y}'\|^2}{32t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}}.$$

Using Lemma A.1,

$$\int_{\mathcal{M}} |Dv(\mathbf{x})|^2\mathrm{d}\mu_{\mathbf{x}}$$
$$\le \frac{CC_t}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} R\left(\frac{\Lambda\|\mathbf{y}-\mathbf{y}'\|^2}{2t}\right)(u(\mathbf{y})-u(\mathbf{y}'))^2\mathrm{d}\mu'_{\mathbf{y}}\mathrm{d}\mu_{\mathbf{y}}$$
$$\le C\int_{\mathcal{M}}\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mu_{\mathbf{x}}\mathrm{d}\mu_{\mathbf{y}}.$$

Appendix B. Proof of Proposition 4.1. We need a proposition on the local distortion between the manifold $\mathcal{M}$ around $\mathbf{x}\in\mathcal{M}$ and its tangent space $\mathcal{T}_{\mathbf{x}}$.

PROPOSITION B.1 (Propsition 1 in [37]). *Assume both $\mathcal{M}$ and $\partial\mathcal{M}$ are compact and C^2 smooth. σ is the minimum of the reaches of $\mathcal{M}$ and $\partial\mathcal{M}$. For any point $\mathbf{x}\in\mathcal{M}$, there is a neighborhood $U\subset\mathcal{M}$ of $\mathbf{x}$, so that there is a parametrization $\Phi:\Omega\subset\mathbb{R}^k\to U$ satisfying the following conditions. For any $\rho\le 0.1$,*

(i) Ω is convex and contains at least half of the ball $B_{\Phi^{-1}(\mathbf{x})}(\frac{\rho}{5}\sigma)$, i.e., vol$(\Omega\cap B_{\Phi^{-1}(\mathbf{x})}(\frac{\rho}{5}\sigma))>\frac{1}{2}(\frac{\rho}{5}\sigma)^k w_k$ where w_k is the volume of unit ball in $\mathbb{R}^k$;

(ii) $B_{\mathbf{x}}(\frac{\rho}{10}\sigma)\cap\mathcal{M}\subset U$.

(iii) *The determinant of the Jacobian of* Φ *is bounded:* $(1-2\rho)^k \le |D\Phi| \le (1+2\rho)^k$ *over* Ω.

(iv) *For any points* $\mathbf{y},\mathbf{z}\in U$, $1-2\rho \le \frac{|\mathbf{y}-\mathbf{z}|}{|\Phi^{-1}(\mathbf{y})-\Phi^{-1}(\mathbf{z})|} \le 1+3\rho$.

This proposition basically says that there exists a local parametrization of small distortion if $(\mathcal{M},\partial\mathcal{M})$ satisfies certain smoothness, and moreover, the parameter domain is convex and big enough. From this proposition, we can easily get that

$$\int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} = \int_{\mathcal{T}_{\mathbf{x}}} K_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} + O(\sqrt{t}), \quad \mathbf{x}\in\mathcal{M}_t,$$
$$\frac{1}{2}\int_{\mathcal{T}_{\mathbf{x}}} K_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} - O(\sqrt{t}) \le \int_{\mathcal{M}} K_t(\mathbf{x},\mathbf{y})\mathrm{d}\mu_{\mathbf{y}} \le \int_{\mathcal{T}_{\mathbf{x}}} K_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} + O(\sqrt{t}), \quad \mathbf{x}\in\partial\mathcal{M}_t,$$

where $\mathcal{T}_{\mathbf{x}}$ denotes the tangent space of $\mathcal{M}$ at $\mathbf{x}$,

$$\mathcal{M}_t = \{\mathbf{x}\in\mathcal{M} : \mathrm{dist}(\mathbf{x},\partial\mathcal{M}) \ge 2\sqrt{t}\}, \quad \partial\mathcal{M}_t = \mathcal{M}\backslash\mathcal{M}_t.$$

Then, we only need to calculate the integral over the tangent space. First, we have

$$\begin{aligned}
&\int_{\mathcal{T}_{\mathbf{x}}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} \\
=&C_t \int_{\mathcal{T}_{\mathbf{x}}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} R\left(\frac{(\mathbf{x}-\mathbf{y})^T A_{\mathcal{T}}^{-1}(\mathbf{x})(\mathbf{x}-\mathbf{y})}{4t}\right)\mathrm{d}\mathbf{y} \\
=&C_t(4t)^{d_0/2}\int_{\mathcal{T}_{\mathbf{x}}} R\left(\|\mathbf{z}\|^2\right)\mathrm{d}\mathbf{z}, \qquad \text{with } \mathbf{z} = \frac{1}{2\sqrt{t}}\left(A_{\mathcal{T}}^{-1}(\mathbf{x})\right)^{1/2}(\mathbf{x}-\mathbf{y}) \\
=&\int_{\mathbb{R}^{d_0}} R\left(\|\mathbf{z}\|^2\right)\mathrm{d}\mathbf{z}.
\end{aligned} \tag{B.1}$$

Moreover,

$$\begin{aligned}
\int_{\mathcal{T}_{\mathbf{x}}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{y})|}} R_t^{\mathbf{y}}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} &= \int_{\mathcal{T}_{\mathbf{x}}} \frac{1}{\sqrt{|A_{\mathcal{T}}(\mathbf{x})|}} R_t^{\mathbf{x}}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} + O(\sqrt{t}) \\
&= \int_{\mathbb{R}^{d_0}} R\left(\|\mathbf{z}\|^2\right)\mathrm{d}\mathbf{z} + O(\sqrt{t}).
\end{aligned} \tag{B.2}$$

Putting above two equations together, we obtain

$$\int_{\mathcal{T}_{\mathbf{x}}} K_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} = 2\int_{\mathbb{R}^{d_0}} R\left(\|\mathbf{z}\|^2\right)\mathrm{d}\mathbf{z} + O(\sqrt{t}). \tag{B.3}$$

Appendix C. Derivation of Equation (4.19). Denote $\mathbf{A}(\mathbf{x}) = (a_{ij}(\mathbf{x})) \in \mathbb{R}^{d\times d}$. Let $\mathbf{X} = [\partial_1\Phi, \partial_2\Phi, \cdots, \partial_{d_0}\Phi]$ be an orthonormal basis of the tangent space $\mathcal{T}_{\mathbf{x}}(\mathcal{M})$ at $\mathbf{x}$ and $\mathbf{Y}$ be the orthogonal completion of $\mathbf{X}$ in $\mathbb{R}^d$. Then we have $\mathbf{AX} = \mathbf{XC}$, $\mathbf{AY} = \mathbf{YD}$, since the tangent space $\mathcal{T}_{\mathbf{x}}(\mathcal{M})$ is an invariant subspace of $\mathbf{A}(\mathbf{x})$. This gives a decomposition of $\mathbf{A}$

$$\mathbf{A} = \mathbf{P}\begin{bmatrix}\mathbf{C} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}\end{bmatrix}\mathbf{P}^{-1}, \qquad \mathbf{P} = [\mathbf{X},\mathbf{Y}], \quad \mathbf{P}^{-1} = \begin{bmatrix}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathbf{T}}\\ (\mathbf{Y}^{\mathbf{T}}\mathbf{Y})^{-1}\mathbf{Y}^{\mathbf{T}}\end{bmatrix}. \tag{C.1}$$

Using these notations, we have

$$D_i a^{mn}(\mathbf{y}) a_{km}(\mathbf{y}) (\partial_{i'}\Phi^k g^{i'j'} \partial_{j'}\Phi^n)$$

$$
\begin{aligned}
&=\mathrm{trace}(D_i(\mathbf{A}^{-1})\mathbf{A}\mathbf{X}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}})\\
&=\mathrm{trace}(D_i(\mathbf{A}^{-1})\mathbf{X}\mathbf{C}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}})\\
&=\mathrm{trace}\left(\mathbf{P}D_i\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)\mathbf{P}^{-1}\mathbf{X}\mathbf{C}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathbf{T}}\right)\\
&\quad+\mathrm{trace}\left(D_i(\mathbf{P})\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)\mathbf{P}^{-1}\mathbf{X}\mathbf{C}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathbf{T}}\right)\\
&\quad+\mathrm{trace}\left(\mathbf{P}\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)D_i(\mathbf{P}^{-1})\mathbf{X}\mathbf{C}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathbf{T}}\right).
\end{aligned}
$$

Then, we calculate three terms one by one.

$$
\mathrm{trace}\left(\mathbf{P}D_i\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)\mathbf{P}^{-1}\mathbf{X}\mathbf{C}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}}\right)=\mathrm{trace}\left(D_i(\mathbf{C}^{-1})\mathbf{C}\right),
$$

$$
\mathrm{trace}\left(D_i(\mathbf{P})\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)\mathbf{P}^{-1}\mathbf{X}\mathbf{C}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}}\right)=\mathrm{trace}\left(D_i(\mathbf{X})(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}}\right),
$$

$$
\mathrm{trace}\left(\mathbf{P}\left(\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{-1}\end{bmatrix}\right)D_i(\mathbf{P}^{-1})\mathbf{X}\mathbf{C}(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}}\right)=\mathrm{trace}\left(\mathbf{X}D_i((\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}})\right).
$$

Also notice that

$$
\begin{aligned}
&\mathrm{trace}\left(D_i(\mathbf{X})(\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}}\right)+\mathrm{trace}\left(\mathbf{X}D_i((\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}})\right)\\
=&D_i\left(\mathrm{trace}\left((\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-\mathbf{1}}\mathbf{X}^{\mathbf{T}}\mathbf{X}\right)\right)=0.
\end{aligned}
$$

Combining all the calculations together, we get

$$
D_i a^{mn}(\mathbf{y})a_{km}(\mathbf{y})(\partial_{i'}\Phi^k g^{i'j'}\partial_{j'}\Phi^n)=\mathrm{trace}\left(D_i(\mathbf{C}^{-1})\mathbf{C}\right)=\frac{-2}{\sqrt{\det(\mathbf{C})}}D_i(\sqrt{\det(\mathbf{C})}).
$$

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