COMMUN. MATH. SCI.
Vol. 17, No. 4, pp. 1061–1069

ON LARGE TIME BEHAVIOR FOR THE CYLINDRICALLY SYMMETRIC VLASOV-POISSON SYSTEM*

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Abstract. A collisionless plasma is modeled by the Vlasov-Poisson system. Solutions in three space dimensions that have smooth, compactly supported initial data with cylindrical symmetry are considered. Using an identity of Rein and Illner (alt. Perthame) it is shown that almost every characteristic of the Vlasov equation (i.e. almost every particle) "escapes" to infinity for large time.

Keywords. collisionless plasma; Vlasov-Poisson; cylindrical symmetry.

AMS subject classifications. 35L60; 35Q83; 82C22; 82D10.

1. Introduction

Consider the Vlasov-Poisson system:

$$\begin{cases} \partial_t f_\alpha + m_\alpha^{-1} v\cdot\nabla_x f_\alpha + e_\alpha E\cdot\nabla_v f_\alpha = 0 \quad \alpha = 1,\ \ldots, N \\ \rho(t,x) = \sum_\alpha e_\alpha \int f_\alpha(t,x,v)dv \\ E = \nabla_x U \to 0 \quad \text{as } |x|\to\infty \\ \nabla\cdot E = \rho \end{cases}$$

where $t\geq 0$ is time, $x\in\mathbb{R}^3$ is position, and $v\in\mathbb{R}^3$ is momentum. f_α is the number density in phase space of particles of the α^{th} species which have mass $m_\alpha > 0$ and charge e_α. Collisional effects are neglected. The initial condition

$$f_\alpha(0,x,v) = f_{\alpha 0}(x,v) \geq 0,$$

for $(x,v)\in\mathbb{R}^6$ is given for each α where it is assumed that $f_{\alpha 0}\in C_0^1(\mathbb{R}^6)$ is nonnegative and compactly supported. It is known that solutions remain smooth for all time [15,18]. We will be interested in the case of cylindrical symmetry and in this case the existence and uniqueness of smooth solutions was established earlier in [10].

By a cylindrically symmetric solution we mean a solution for which

$$f_\alpha(t,Ox,Ov) = f_\alpha(t,x,v)$$

for every O of the form

$$O = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Such a solution is a function of t, r, u, a, x_3, v_3 where

$$r = \sqrt{x_1^2 + x_2^2},$$

*Received: January 3, 2019; Accepted (in revised form): March 23, 2019. Communicated by Pierre-Emmanuel Jabin.

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$$u = \sqrt{v_1^2 + v_2^2},$$

and $a \in [0,\pi]$ is defined by

$$ru\cos a = x_1 v_1 + x_2 v_2.$$

Define the characteristics of the Vlasov equation, $(X_\alpha(s,t,x,v), V_\alpha(s,t,x,v))$, by

$$\begin{cases} \frac{dX_\alpha}{ds} = m_\alpha^{-1} V_\alpha & X_\alpha(t,t,x,v) = x \\ \frac{dV_\alpha}{ds} = e_\alpha E(s,X_\alpha) & V_\alpha(t,t,x,v) = v. \end{cases}$$

Then

$$f_\alpha(s, X_\alpha(s,t,x,v), V_\alpha(s,t,x,v)) = f_\alpha(t,x,v).$$

Further introduce the notation

$$R_\alpha = \sqrt{X_{\alpha 1}^2 + X_{\alpha 2}^2},$$

$$U_\alpha = \sqrt{V_{\alpha 1}^2 + V_{\alpha 2}^2}$$

and $A_\alpha \in [0,\pi]$ defined by

$$R_\alpha U_\alpha \cos A_\alpha = X_{\alpha 1} V_{\alpha 1} + X_{\alpha 2} V_{\alpha 2}.$$

For cylindrically symmetric solutions

$$|X_{\alpha 1} V_{\alpha 2} - X_{\alpha 2} V_{\alpha 1}| = R_\alpha U_\alpha \sin A_\alpha$$

is constant in s (conservation of angular momentum). This follows by differentiating $X_{\alpha 1} V_{\alpha 2} - X_{\alpha 2} V_{\alpha 1}$ with respect to s and using the fact that E is the gradient of a function of t, r, x_3. Define

$$\ell(x,v) = |x_1 v_2 - x_2 v_1| = ru\sin a.$$

Define

$$P(t) = \sup\Big\{ \sqrt{v_1^2 + v_2^2} : \exists s \in [0,t], x \in \mathbb{R}^3, v_3 \in \mathbb{R}, \alpha \text{ with } f_\alpha(s,x,v) \neq 0 \Big\}.$$

Then we may state:

LEMMA 1.1. *For C^1, compactly supported initial data that is cylindrically symmetric there is a constant $C > 0$ such that*

$$|E(t,x)| \leq C r^{-1/2} \min{}^{1/2}(P(t),\ r^{-1}) t^{-1/2} \log^{1/2}((1+r)(2+t)^4).$$

Our main goal is to understand the large-time behavior of solutions. One situation where this is known is the small data problem. It is shown in [1] (see [22] also) that for sufficiently small initial data

$$|E(t,x)| \leq C(1+t)^{-2}$$

hence $\lim_{s\to+\infty} V_\alpha(s,t,x,v)$ *exists with*

$$|V_\alpha(s,t,x,v) - \lim_{u\to+\infty} V_\alpha(u,t,x,v)| \le C(1+s)^{-1}$$

for every characteristic. So for small initial data the nonlinearity becomes negligible for large time and particles stream freely. We ask if this happens for arbitrary data.

DEFINITION 1.1. *We will say a characteristic,* $(X(s,t,x,v),V(s,t,x,v))$*, escapes if*

$$\lim_{s\to+\infty} V(s,t,x,v)$$

exists and is not zero.

Clearly $|X| \to +\infty$ linearly if (X,V) escapes. We have

THEOREM 1.1. *Let the initial data be* C^1*, compactly supported, and cylindrically symmetric. Then for each* α *the set*

$$\{(x,v) : f_{\alpha_0}(x,v) \ne 0 \text{ and } (X(s,0,x,v),V(s,0,x,v)) \text{ does not escape}\}$$

has Lebesgue measure zero.

Note that, in contrast to the small data result, this theorem provides no rate of decay. The two main ingredients for this theorem are the conservation of angular momentum along characteristics and a time decay identity from [13] (also [17]). According to [13] we have

$$\begin{aligned}&\tfrac{d}{dt}\left(t\sum_\alpha \iint m_\alpha f_\alpha |m_\alpha^{-1}v - t^{-1}x|^2 dv dx + t\int |E|^2 dx\right)\\ &= -\sum_\alpha \iint m_\alpha f_\alpha |m_\alpha^{-1}v - t^{-1}x|^2 dv dx,\end{aligned}$$

and hence there is a constant, $C>0$, dependent on initial data, such that

$$\sum_\alpha \int_1^\infty \iint f_\alpha |m_\alpha^{-1}v - t^{-1}x|^2 dv dx \le C \tag{1.1}$$

and for $t \ge 1$

$$\sum_\alpha \iint f_\alpha |m_\alpha^{-1}v - t^{-1}x|^2 dv dx \le Ct^{-1} \tag{1.2}$$

Some results on large-time behavior are obtained in the case of a single species of charge in [2, 3, 11], and [20]. Time asymptotics for low dimensional, multiple species problems are presented in [7, 8], and [9]. The asymptotic behavior of bounds on the v support of solutions started with [18] and has developed from there ([4,5,12,14,16]). For general references on mathematical results for collisionless plasma we mention [6] and [19]. Finally, as it contains techniques for handling cylindrically symmetric solutions, we cite [10].

The proof of the lemma is Section 2, the proof of the theorem is Section 3. The letter, C, denotes a generic positive constant which may change from line to line but is determined by the initial conditions.

2. The Proof of the Lemma

By an elementary calculation we have

$$\begin{aligned}|E(t,x)| &= \left|\frac{1}{4\pi}\int \rho(t,y)\frac{x-y}{|x-y|^3}dy\right| \\ &\leq \frac{1}{4\pi}\int \frac{|\rho(t,y)|}{|x-y|^2}dy \\ &= \frac{1}{4\pi}\int_0^\infty \int |\rho(t,r',z')| \int_0^{2\pi} \frac{r'd\theta dz'dr'}{r^2+(r')^2-2rr'\cos\theta+(x_3-z')^2} \\ &= \frac{1}{2}\int_0^\infty \int \frac{|\rho(t,r',z')|}{D_- D_+} r'dr'dz' \qquad (2.1)\end{aligned}$$

where

$$D_\pm = \sqrt{(r\pm r')^2+(x_3-z')^2}.$$

Also define

$$\kappa(t,x) = \sum_\alpha \int f_\alpha(t,x,v)|m_\alpha^{-1}v - t^{-1}x|^2 dv$$

and

$$M(t,r) = \min(P(t), r^{-1}).$$

Because $\ell = ru\sin a$ is conserved on characteristics, it is bounded uniformly on the support of f_α and

$$f_\alpha(t,x,v) \neq 0 \Rightarrow u\sin a \leq CM(t,r).$$

Let $q>0$ and

$$\mathcal{S} = \{v : u\sin a \leq CM \text{ and } |m_\alpha^{-1}v - t^{-1}x| \leq q\}.$$

Then on $\mathcal{S}$ the component of (v_1,v_2) that is perpendicular to (x_1,x_2) is bounded by $u\sin a \leq CM$ and all components of $v - m_\alpha t^{-1}x$ are bounded by Cq. Hence $\mathcal{S}$ is included within a rectangular box with side lengths CM by Cq by Cq. Hence,

$$\int f_\alpha dv \leq \int_{\mathcal{S}} C dv + q^{-2}\kappa \leq Cq^2M + q^{-2}\kappa.$$

Taking

$$q^2 = \sqrt{M^{-1}\kappa}$$

yields

$$\int f_\alpha dv \leq C\sqrt{M\kappa}. \qquad (2.2)$$

Note that

$$\frac{M(t,r')}{D_+}r' \leq M(t,r)$$

so, for any $B \geq b > 0$, (2.2) yields

$$\begin{aligned}
&\iint\limits_{0<r',b<D_-<B} \frac{|\rho(t,r',z')|}{D_-D_+} r'dz'dr' \\
&\leq C \iint\limits_{0<r',b<D_-<B} \frac{\sqrt{M(t,r')\kappa}}{D_-D_+} r'dz'dr' \\
&= C \iint\limits_{0<r',b<D_-<B} \frac{\sqrt{\kappa r'}}{D_-} \sqrt{\frac{M(t,r')r'}{D_+}} \frac{1}{\sqrt{D_+}} dz'dr' \\
&\leq C\sqrt{M} \frac{1}{\sqrt{r}} \iint\limits_{0<r',b<D_-<B} \frac{\sqrt{\kappa r'}}{D_-} dz'dr' \\
&\leq C\sqrt{\frac{M}{r}} \sqrt{\int_0^\infty \int \kappa r' dz'dr'} \sqrt{\int\limits_{0<r',b<D_-<B} \int D_-^{-2} dz'dr'}.
\end{aligned} \tag{2.3}$$

But

$$\iint\limits_{b<D_-<B} D_-^{-2} dr'dz' = \int_b^B \frac{2\pi\lambda d\lambda}{\lambda^2} = 2\pi \log(\frac{B}{b}),$$

so using (1.2) in (2.3) yields

$$\iint\limits_{0<r',b<D_-<B} \frac{|\rho(t,r',z')|}{D_-D_+} r'dz'dr' \leq Cr^{-1/2} M^{1/2} t^{-1/2} \log^{1/2} \frac{B}{b}. \tag{2.4}$$

Using mass conservation we have

$$\iint\limits_{0<r',B<D_-} \frac{|\rho(t,r',z')|}{D_-D_+} r'dz'dr' \leq \frac{1}{rB} \int_0^\infty \int |\rho| r'dz'dr' \leq Cr^{-1}B^{-1}. \tag{2.5}$$

Letting

$$Q(t) = \sup\{|v| : \exists s \in [0,t], x \in \mathbb{R}^3, \alpha \text{ with } f_\alpha(s,x,v) \neq 0\}$$

we also have

$$\begin{aligned}
&\iint\limits_{0<r',D_-<b} \frac{|\rho(t,r',z')|}{D_-D_+} r'dz'dr' \\
&\leq CQ^3(t) \iint\limits_{0<r',D_-<b} \frac{r'}{D_-D_+} dz'dr' \\
&\leq CQ^3(t) \iint\limits_{0<r',D_-<b} \frac{1}{D_-} dz'dr' \\
&\leq CQ^3(t) \int_0^b \frac{1}{\lambda} 2\pi\lambda d\lambda \leq CbQ^3(t).
\end{aligned} \tag{2.6}$$

Using (2.4), (2.5), and (2.6) in (2.1) yields

$$|E(t,x)| \le C(r^{-1/2}M^{1/2}t^{-1/2}\log^{1/2}\frac{B}{b} + r^{-1}B^{-1} + Q^3(t)b).$$

Taking

$$b = r^{-1/2}M^{1/2}t^{-1/2}Q^{-3}(t)$$

and

$$B = \max\ (b, r^{-1/2}M^{-1/2}t^{1/2})$$

yields

$$|E| \le Cr^{-1/2}M^{1/2}t^{-1/2}(1+\log^{1/2}(1+M^{-1}tQ^3(t))). \tag{2.7}$$

From Theorem 1.1 of [21]

$$Q(t) \le C(t+2)^{11/15}\log^{4/15}(t+2)$$

and so the lemma follows from (2.7).

3. The Proof of the Theorem

Proof. From (1.1) it follows that

$$\begin{aligned} C &\ge \int_1^\infty \iint f_\alpha(t,x,v)|m_\alpha^{-1}v - t^{-1}x|^2 dv dx dt \\ &= \iint f_{\alpha 0}(x,v) \int_1^\infty \eta_\alpha^2(t,x,v) dt dv dx \end{aligned}$$

where we define

$$\eta_\alpha(t,x,v) = |m_\alpha^{-1}V_\alpha(t,0,x,v) - t^{-1}X_\alpha(t,0,x,v)|.$$

Hence

$$\left\{(x,v) \in S_{\alpha 0} : \int_1^\infty \eta_\alpha^2(t,x,v)dt = \infty\right\}$$

has measure zero, where we define

$$S_{\alpha 0} = \{(x,v) : f_{\alpha 0}(x,v) \ne 0\}.$$

Also

$$\{(x,v) \in S_{\alpha 0} : \ell(x,v) = 0\}$$

has measure zero. Consider any $(x,v) \in S_{\alpha 0}$ with

$$\textstyle\int_1^\infty \eta_\alpha^2(t,x,v)dt \text{ finite} \tag{3.1}$$

and

$$\ell(X_\alpha(t,0,x,v), V_\alpha(t,0,x,v)) \ne 0.$$

We will suppress the dependence on (x,v) and write, for example,

$$\eta_\alpha(t)=\eta_\alpha(t,x,v).$$

Note that

$$\begin{aligned}\eta_\alpha^2(t)=&|m_\alpha^{-1}V_\alpha-t^{-1}X_\alpha|^2\\ =&(m_\alpha^{-1}U_\alpha\cos A_\alpha-t^{-1}R_\alpha)^2\\ &+(m_\alpha^{-1}R_\alpha^{-1}\ell)^2+(m_\alpha^{-1}V_{\alpha 3}-t^{-1}X_{\alpha 3})^2.\end{aligned}\tag{3.2}$$

We claim that there exists $T\geq 1$ such that

$$\frac{R_\alpha(T)}{T}>T^{-1/2}\sqrt{\int_T^\infty \eta_\alpha^2(s)ds}.\tag{3.3}$$

Suppose this is not the case, then

$$\frac{R_\alpha(t)}{t}\leq t^{-1/2}\sqrt{\int_t^\infty \eta_\alpha^2(s)ds}$$

for all $t\geq 1$. Then it follows (using (3.2)) that

$$\begin{aligned}\int_1^\infty \eta_\alpha^2(t)dt\geq&\int_1^\infty (m_\alpha^{-1}R_\alpha^{-1}(t)\ell)^2dt\\ \geq&(m_\alpha^{-1}\ell)^2\int_1^\infty\left(t^{1/2}\sqrt{\int_t^\infty \eta_\alpha^2(s)ds}\right)^{-2}dt\\ =&(m_\alpha^{-1}\ell)^2\int_1^\infty t^{-1}\left(\int_t^\infty \eta_\alpha^2(s)ds\right)^{-1}dt\\ \geq&(m_\alpha^{-1}\ell)^2\int_1^\infty t^{-1}\left(\int_1^\infty \eta_\alpha^2(s)ds\right)^{-1}dt=\infty.\end{aligned}$$

This contradicts (3.1) and hence (3.3) is established.

Now by (3.2)

$$\begin{aligned}\frac{d}{ds}(s^{-1}R_\alpha)=&s^{-1}\left(\frac{dR_\alpha}{ds}-s^{-1}R_\alpha\right)\\ =&s^{-1}\left(m_\alpha^{-1}U_\alpha\cos A_\alpha-s^{-1}R_\alpha\right)\geq -s^{-1}\eta_\alpha(s)\end{aligned}$$

so for $t\geq T$ we have

$$\begin{aligned}t^{-1}R_\alpha(t)\geq&T^{-1}R_\alpha(T)-\int_T^t s^{-1}\eta_\alpha(s)ds\\ \geq&T^{-1}R_\alpha(T)-\left(\int_T^\infty s^{-2}ds\right)^{1/2}\left(\int_T^\infty \eta_\alpha^2ds\right)^{1/2}\\ =&T^{-1}R_\alpha(T)-T^{-1/2}\left(\int_T^\infty \eta_\alpha^2ds\right)^{1/2}.\end{aligned}\tag{3.4}$$

Letting

$$D = T^{-1}R_\alpha(T) - T^{-1/2}\left(\int_T^\infty \eta_\alpha^2 ds\right)^{1/2},$$

(3.3), (3.4), and the lemma yield

$$\begin{aligned}
|E(t,X_\alpha(t))| &\le C R_\alpha^{-1/2}\min{}^{1/2}(P,R_\alpha^{-1})t^{-1/2}\log^{1/2}((1+R_\alpha)(2+t)^4)\\
&\le C R_\alpha^{-1}t^{-1/2}(\log(1+R_\alpha)+\log(2+t))^{1/2}\\
&\le C t^{-1/2}(R_\alpha^{-9/10}+R_\alpha^{-1}\log^{1/2}(2+t))\\
&\le C t^{-1/2}((Dt)^{-9/10}+(Dt)^{-1}\log^{1/2}(2+t)).
\end{aligned}$$

Therefore

$$\int_T^\infty |E(t,X_\alpha(t))|dt$$

is finite and

$$\lim_{t\to+\infty} V_\alpha(t)$$

exists. It follows from (3.4) that this limit is not zero and therefore this characteristic escapes. ∎

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