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OPTIMAL DECAY RATES OF THE COMPRESSIBLE MAGNETO–MICROPOLAR FLUIDS SYSTEM IN $\mathbb{R}^3$*

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Abstract. In this paper, we consider the Cauchy problem of the compressible magneto–micropolar fluids system in $\mathbb{R}^3$ with initial data close to some constant steady state. Based on the spectral analysis on the semigroup generated by the linearized equations and the nonlinear energy estimates, we show that the solution of the magneto–micropolar fluids system converges to its constant equilibrium state at the exact same L^2–decay rate as the linearized equations, which shows that the convergence rate is optimal.

Keywords. Lower convergence rates; Upper decay rates; Spectral analysis; Energy method.

AMS subject classifications. 35Q35; 35B40; 35D35; 76N10.

1. Introduction

The magneto–micropolar fluid is a field of studying flow problems of microstructure related moving, conducting fluids under the action of magnetic fields. The fluid with rigid spherical particles suspended may rotate independently of the rotation and movement of the fluid, for example, the polymeric suspensions and liquid crystals. When a magnetic fluid includes relatively large particles, one can assume that the magnetic moment of a particle changes its orientation only for rotation of the particle itself. Then, the presence of an external magnetic field results in preventing the rotation of the particle and making an appearance of the mechanism of rotational viscosity of the fluid. This kind of fluid with internal rotation can be called magneto–micropolar fluid [2]. The equation of the motion can be expressed as follows:

$$\begin{cases} \tilde\rho_t+\operatorname{div}(\tilde\rho\tilde u)=0,\\ (\tilde\rho\tilde u)_t+\operatorname{div}(\tilde\rho\tilde u\otimes\tilde u)+\nabla P(\tilde\rho)-(\mu+\alpha)\Delta\tilde u-2\alpha\nabla\times\tilde w\\ =(\mu+\lambda-\alpha)\nabla\operatorname{div}\tilde u+\operatorname{curl}\tilde B\times\tilde B,\\ (\tilde\rho\tilde w)_t+\operatorname{div}(\tilde\rho\tilde u\otimes\tilde w)+4\alpha\tilde w-\mu'\Delta\tilde w-(\mu'+\lambda')\nabla\operatorname{div}\tilde w-2\alpha\nabla\times\tilde u=0,\\ \tilde B_t-\operatorname{curl}\left(\tilde u\times\tilde B\right)-\nu\Delta\tilde B=0,\ \operatorname{div}\tilde B=0,\\ (\tilde\rho,\tilde u,\tilde w,\tilde B)^T|_{t=0}=(\tilde\rho_0,\tilde u_0,\tilde w_0,\tilde B_0)^T, \end{cases} \tag{1.1}$$

for $(x,t)\in\mathbb{R}^3\times[0,\infty)$. Here the unknowns $\tilde\rho=\tilde\rho(x,t)$, $\tilde u=\tilde u(x,t)$, $\tilde w=\tilde w(x,t)$, $\tilde B=\tilde B(x,t)$ denote the density, velocity, the micro–rotational velocity, and the magnetic field, respectively. The pressure $P(\tilde\rho)$ is a C^1–function satisfying $P'(\rho_\infty)>0$ with some constant $\rho_\infty>0$. The constant $\nu>0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. The constant parameter $\alpha>0$ means dynamic microrotation viscosity. The constant coefficients μ, λ are the shear and bulk viscosity coefficients of the flow and satisfy the physical restrictions

$$\mu>0,\ 2\mu+3\lambda-4\alpha\geq 0.$$

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μ' and λ' are the angular viscosity coefficients satisfying

$$\mu' > 0,\ 2\mu' + 3\lambda' \geq 0.$$

When $\tilde{B} = 0$, the model becomes a micropolar fluid. There are some results about the incompressible viscous micropolar fluids. One can see [17, 33] for the weak solutions, [3, 4, 28] for strong solutions, [13, 19] and the reference therein for the regularity criteria. The problems about the compressible micropolar fluids have received considerable attention in the last few years. The initial boundary value problems of this model in the one–dimensional case were described by [7, 21–23, 35]. One can refer to [21–23] for the local or global existence of the strong or generalized solutions and [7, 35] for the asymptotic behavior. For the $3D$ case, this model with heat conduction was analyzed in relation to existence and stabilization of the spherically symmetric solutions to initial–boundary value problems. The global existence of the spherically symmetric generalized solution was investigated in [10] by the combination of the local existence [9], the uniqueness theorem [24] and the extension principle. The large-time behavior of the global unique spherically symmetric generalized solution can be seen in [14, 15]. One can refer to [5, 6] for the blowup criterion and [31] for the limit problem. For the Cauchy problem, results are much less. We can refer to [16, 26] and the references therein for the well–posedness in $1D$. Liu and Zhang [20] gave the global existence and decay rate of the classical solutions when the initial perturbation belonged to $H^N \cap L^1$ $(N \geq 4)$.

Taking the magnetic field into consideration, the magneto–micropolar fluid equations have attracted much attention due to their important physical background and mathematical complexity. Local strong solution was established by Rojas–Medar [27] and global strong solution by Ortega–Torres and Rojas–Medar [25]. Compared with the incompressible models, the results of the compressible equations of magneto–micropolar fluids are much less because of the strong nonlinearities and interactions among the physical quantities. For multi–dimensional compressible magneto–micropolar equations, Amirat and Hamdache [1] proved the global existence of weak solutions with finite energy. Recently, Wei–Guo–Li [34] considered the global existence and the time decay rate of the smooth solutions by combining the $L^p - L^q$ estimates for the linearized equations and the Fourier splitting method.

However, there is no result showing that the solution for the compressible magneto–micropolar fluids equations has an exact same decay rate as the linearized problem. Motivated by the work of [18, 29, 30], in this work, we use a similar method as [18] to derive the large-time behavior of the global classical solution of the compressible magneto–micropolar fluids equations in $\mathbb{R}^3$, provided the prescribed initial data is close to the constant steady state $(\rho_\infty, 0, 0, 0)^T$ with $\rho_\infty > 0$. We will show that the behavior of the perturbation is asymptotically equivalent to that of the linearized problem. To establish this result, much effort will be spent on the spectral analysis of the linearized system. To this end, we now linearize (1.1) near the constant state $(\rho_\infty, 0, 0, 0)^T$. Without loss of generality, the constant ρ_∞ is taken to be 1. We define

$$\tilde{m} = \tilde{\rho}\tilde{u},\ \tilde{W} = \tilde{\rho}\tilde{w},$$

and want to rewrite problem (1.1) in a symmetric form. By changing the unknown functions and denoting the perturbations by

$$\rho = \tilde{\rho} - 1,\ m = \frac{1}{\sqrt{P'(1)}}(\tilde{m} - 0),\ W = \frac{1}{\sqrt{P'(1)}}\left(\tilde{W} - 0\right),\ B = \frac{1}{\sqrt{P'(1)}}(\tilde{B} - 0),$$

problem (1.1) is reduced to

$$\begin{cases} \partial_t\rho+\sqrt{P'(1)}\mathrm{div}m=0, \\ \partial_t m+\sqrt{P'(1)}\nabla\rho-(\mu+\alpha)\Delta m-(\mu+\lambda-\alpha)\nabla\mathrm{div}m-2\alpha\nabla\times W=\mathbb{N}_1, \\ \partial_t W+4\alpha W-\mu'\Delta W-(\mu'+\lambda')\nabla\mathrm{div}W-2\alpha\nabla\times m=\mathbb{N}_2, \\ \partial_t B-\nu\Delta B=\mathbb{N}_3, \end{cases} \tag{1.2}$$

here

$$\begin{aligned} \mathbb{N}_1=&-\sqrt{P'(1)}\mathrm{div}\left(\frac{m\otimes m}{1+\rho}\right)-\frac{1}{\sqrt{P'(1)}}\nabla\left[P(1+\rho)-P(1)-P'(1)\rho\right] \\ &-(\mu+\alpha)\Delta\left(\frac{\rho m}{1+\rho}\right)-(\mu+\lambda-\alpha)\nabla\mathrm{div}\left(\frac{\rho m}{1+\rho}\right)-2\alpha\nabla\times\left(\frac{\rho W}{1+\rho}\right) \\ &+\sqrt{P'(1)}B\cdot\nabla B-\frac{\sqrt{P'(1)}}{2}\nabla\left(|B|^2\right), \\ \mathbb{N}_2=&-\sqrt{P'(1)}\mathrm{div}\left(\frac{m\otimes W}{1+\rho}\right)+4\alpha\frac{\rho W}{1+\rho}-\mu'\Delta\left(\frac{\rho W}{1+\rho}\right)-(\mu'+\lambda')\nabla\mathrm{div}\left(\frac{\rho W}{1+\rho}\right) \\ &-2\alpha\nabla\times\left(\frac{\rho m}{1+\rho}\right), \\ \mathbb{N}_3=&\sqrt{P'(1)}\mathrm{curl}\left(\frac{m\times B}{1+\rho}\right). \end{aligned}$$

Initial data of the system (1.2) is expressed as

$$\begin{aligned} (\rho,m,W,B)^T|_{t=0}&=\left(\tilde\rho_0-1,\frac{1}{\sqrt{P'(1)}}(\tilde m_0-0),\frac{1}{\sqrt{P'(1)}}\left(\tilde W_0-0\right),\frac{1}{\sqrt{P'(1)}}\left(\tilde B_0-0\right)\right)^T \\ &=(\rho_0,m_0,W_0,B_0)^T. \end{aligned} \tag{1.3}$$

Unfortunately, the method in [18] doesn't work directly in the magneto–micropolar fluids system because of its complex linearized system. Here this problem is overcome by introducing the decomposition as in [8] such that the solution of (1.2)–(1.3) can be decomposed into two parts in the form of

$$\begin{pmatrix}\rho\\ m\\ W\\ B\end{pmatrix}=\begin{pmatrix}\rho\\ m_{\shortparallel}\\ W_{\shortparallel}\\ B\end{pmatrix}+\begin{pmatrix}0\\ m_{\perp}\\ W_{\perp}\\ 0\end{pmatrix}, \tag{1.4}$$

where

$$m_{\shortparallel}=\Delta^{-1}\nabla\mathrm{div}m,\ m_\perp=-\Delta^{-1}\nabla\times(\nabla\times m),$$

and likewise for $W_{\shortparallel}$, $W_\perp$. For brevity, the first part on the right–hand side of (1.4) is called the fluid part and the second part is called the electromagnetic part, and we also denote

$$U_{\shortparallel}=(\rho,m_{\shortparallel},W_{\shortparallel},B)^T,\ U_\perp=(m_\perp,W_\perp)^T.$$

In the aid of the above decomposition, we give explicit representations of the solutions for the two eigenvalue problems. We now derive the equations of $U_{\shortparallel}$ and $U_\perp$ respectively.

Taking the divergence of the last two equations of (1.2)$_1$–(1.2)$_3$, and then applying $\Delta^{-1}\nabla$, it follows that

$$\begin{cases} \partial_t\rho+\sqrt{P'(1)}\mathrm{div}m_{\shortparallel}=0, \\ \partial_t m_{\shortparallel}+\sqrt{P'(1)}\nabla\rho-(2\mu+\lambda)\Delta m_{\shortparallel}=\Delta^{-1}\nabla\mathrm{div}\mathbb{N}_1=F_1, \\ \partial_t W_{\shortparallel}+4\alpha W_{\shortparallel}-(2u'+\lambda')\Delta W_{\shortparallel}=\Delta^{-1}\nabla\mathrm{div}\mathbb{N}_2=F_2, \\ (\rho,m_{\shortparallel},W_{\shortparallel})^T|_{t=0}=(\rho_0,m_{\shortparallel 0},W_{\shortparallel 0})^T, \end{cases} \tag{1.5}$$

with

$$\begin{aligned} F_1\sim&-\sqrt{P'(1)}\mathrm{div}\left(\frac{m\otimes m}{1+\rho}\right)-\frac{1}{\sqrt{P'(1)}}\nabla\left[P(1+\rho)-P(1)-P'(1)\rho\right] \\ &-(\mu+\alpha)\Delta\left(\frac{\rho m}{1+\rho}\right)-(\mu+\lambda-\alpha)\nabla\mathrm{div}\left(\frac{\rho m}{1+\rho}\right) \\ &+\sqrt{P'(1)}B\cdot\nabla B-\frac{\sqrt{P'(1)}}{2}\nabla\left(|B|^2\right), \end{aligned}$$

and

$$F_2\sim-\sqrt{P'(1)}\mathrm{div}\left(\frac{m\otimes W}{1+\rho}\right)+4\alpha\frac{\rho W}{1+\rho}-\mu'\Delta\left(\frac{\rho W}{1+\rho}\right)-(\mu'+\lambda')\nabla\mathrm{div}\left(\frac{\rho W}{1+\rho}\right).$$

Taking the curl of the last two equations of (1.2)$_1$–(1.2)$_3$ and then applying Δ^{-1}curl, one deduces that

$$\begin{cases} \partial_t m_\perp-(\mu+\alpha)\Delta m_\perp-2\alpha\nabla\times W_\perp=N_1, \\ \partial_t W_\perp+4\alpha W_\perp-\mu'\Delta W_\perp-2\alpha\nabla\times m_\perp=N_2, \\ (m_\perp,W_\perp)^T|_{t=0}=(m_{\perp 0},W_{\perp 0})^T, \end{cases} \tag{1.6}$$

with

$$N_1\sim-(\mu+\alpha)\Delta\left(\frac{\rho m}{1+\rho}\right)-2\alpha\nabla\times\left(\frac{\rho W}{1+\rho}\right),$$

and

$$N_2\sim 4\alpha\frac{\rho W}{1+\rho}-\mu'\Delta\left(\frac{\rho W}{1+\rho}\right)-2\alpha\nabla\times\left(\frac{\rho m}{1+\rho}\right),$$

where we have replaced $-\nabla\times\nabla\times W$ by $\Delta W-\nabla\mathrm{div}W$, likewise for $-\nabla\times\nabla\times m$.

We now record the following existence and uniqueness of the solution for (1.2)–(1.3). The proof has been described in [34], for convenience, we just state the result in the following theorem.

THEOREM 1.1. *Let $(\rho_0,u_0,w_0,B_0)^T\in H^3$. There exists some sufficiently small δ such that if $\|(\rho_0,u_0,w_0,B_0)\|_{H^3}\le\delta$, the Cauchy problem (1.2)–(1.3) admits a unique solution $(\rho,u,w,B)^T\in H^3$. Moreover, there exists a constant C such that*

$$\|(\rho,u,w,B)(t)\|_{H^3}^2+\int_0^t\left(\|\nabla\rho\|_{H^2}^2+\|\nabla(u,w,B)\|_{H^3}^2\right)(\tau)d\tau\le C\|(\rho_0,u_0,w_0,B_0)\|_{H^3}^2. \tag{1.7}$$

The main purpose of the paper is to prove the following theorem concerning the convergence rates of the solution of Equations (1.2)–(1.3).

THEOREM 1.2. *Under the assumptions of*

$$\|(\rho_0,m_0,W_0,B_0)\|_{H^3\cap L^1}\lesssim\delta; \tag{1.8}$$

$$\int_{\mathbb{R}^3}\rho_0(x)dx\neq 0;\ \int_{\mathbb{R}^3}m_{\perp 0}(x)dx\neq 0;\ \int_{\mathbb{R}^3}B_0(x)dx\neq 0, \tag{1.9}$$

and

$$x\rho_0(x),\ xm_{\shortparallel 0}(x),\ xm_{\perp 0}(x),\ xB_0(x)\in L^1, \tag{1.10}$$

then for $t>t_0$ with some t_0 suitably large, we have

$$(1+t)^{-\frac{3}{4}}\lesssim\|(\rho,m,B)\|_{L^2}\lesssim(1+t)^{-\frac{3}{4}}, \tag{1.11}$$

$$(1+t)^{-\frac{5}{4}}\lesssim\|W\|_{L^2}\lesssim(1+t)^{-\frac{5}{4}}, \tag{1.12}$$

and

$$(1+t)^{-\frac{5}{4}}\lesssim\|\nabla(\rho,m,B)\|_{L^2}\lesssim(1+t)^{-\frac{5}{4}}, \tag{1.13}$$

$$(1+t)^{-\frac{7}{4}}\lesssim\|\nabla W\|_{L^2}\lesssim(1+t)^{-\frac{7}{4}}. \tag{1.14}$$

In order to obtain the optimal decay rates (1.11)–(1.14), in addition to the method in [18], we also need to introduce the decomposition (1.4) to overcome the difficulty caused by the curl term. We consider the linearized systems (2.1) and (3.1) near a constant equilibrium state and investigate the spectrum of the semigroup in terms of the decomposition of wave modes at the lower frequency and high frequency, respectively. Under the conditions (1.8)–(1.10), we obtain the lower and upper decay rates of the linearized cases. Moreover, in Section 4, in virtue of the careful analysis on the semigroup, and the energy estimates, we show that the difference $(\rho_h,m_h,W_h,B_h)^T=(\rho-\bar{\rho},m-\bar{m},W-\bar{W},B-\bar{B})^T$ has a faster decay rate than the linearized one obtained in Proposition 3.1. This implies the solution $(\rho,m,W,B)^T$ has the same decay rate as $(\bar{\rho},\bar{m},\bar{W},\bar{B})^T$.

In the following, we present some remarks about Theorem 1.1 and Theorem 1.2.

REMARK 1.1. To avoid confusion, we should point out that, the notations u, w in Theorem 1.1 have the following relations with those used in our paper:

$$u=\tilde{u}-0=\frac{\widetilde{m}}{\widetilde{\rho}}=\frac{\sqrt{P'(1)}m}{1+\rho},\ w=\tilde{w}-0=\frac{\tilde{W}}{\widetilde{\rho}}=\frac{\sqrt{P'(1)}W}{1+\rho}.$$

REMARK 1.2. In this work, we investigate both lower and upper decay rates of the linearized equations and prove that the solution of the nonlinear system also has the same decay property, in this case, we give a more elaborate decay result to show that the convergence rate is optimal compared with [34].

Notation. Throughout the paper, we use $f \lesssim g$ to denote $f \le Cg$ and $f \gtrsim g$ to denote $f \ge Cg$, where $C>0$ is a generic constant. $f \sim g$ means $f \gtrsim g$ and $f \lesssim g$. For simplicity, the notation $\|(f,g)\|_X$ means $\|f\|_X + \|g\|_X$ with $f,g \in X$.

The rest of the paper is structured as follows. In Section 2, we will analyze the property of the solution semigroup $e^{t\mathbb{A}}$ and obtain the decay property of the solution to the linearized fluid part. In Section 3, we use a similar method as in Section 2 to explore the decay rate of the solution to the electromagnetic part of the system. In Section 4, by using the linear estimates and some energy methods, we will prove the difference $(\rho_h, m_{h_{\shortparallel}}, m_{h\perp}, W_{h_{\shortparallel}}, W_{h\perp}, B_h)^T$ has a faster decay rate than $(\bar{\rho}, \bar{m}_{\shortparallel}, \bar{m}_{\perp}, \bar{W}_{\shortparallel}, \bar{W}_{\perp}, \bar{B})^T$, which implies the optimal time decay rate of the solution to the magneto–micropolar fluid system (1.2)–(1.3) and give the proof of Theorem 1.2.

2. L^2 decay rates for the solution of the fluid part

In this section, we will give some analysis on the semigroup generated by the linearized fluid part and obtain the upper and lower bound decay rates for the solution of the linearized equations.

2.1. Spectral representation. The solution $\bar{U}_{\shortparallel} = (\bar{\rho}, \bar{m}_{\shortparallel}, \bar{W}_{\shortparallel}, \bar{B})^T$ of the linearized fluid part satisfies

$$\begin{cases} \partial_t \bar{\rho} + \sqrt{P'(1)} \operatorname{div} \bar{m}_{\shortparallel} = 0, \\ \partial_t \bar{m}_{\shortparallel} + \sqrt{P'(1)} \nabla \bar{\rho} - (2\mu + \lambda) \Delta \bar{m}_{\shortparallel} = 0, \\ \partial_t \bar{W}_{\shortparallel} + 4\alpha \bar{W}_{\shortparallel} - (2u' + \lambda') \Delta \bar{W}_{\shortparallel} = 0, \\ \partial_t \bar{B} - \nu \Delta \bar{B} = 0, \\ (\bar{\rho}, \bar{m}_{\shortparallel}, \bar{W}_{\shortparallel}, \bar{B})^T|_{t=0} = (\rho_0, m_{\shortparallel 0}, W_{\shortparallel 0}, B_0)^T. \end{cases} \tag{2.1}$$

Taking the Fourier transform to system (2.1) with respect to the space variable, by the semigroup theory for evolution equations, the solution $\bar{U}_{\shortparallel} = (\bar{\rho}, \bar{m}_{\shortparallel}, \bar{W}_{\shortparallel}, \bar{B})^T$ can be expressed as

$$\begin{cases} \hat{\bar{U}}_{1\shortparallel}(\xi) = e^{t\mathbb{A}(\xi)} \hat{\bar{U}}_{1\shortparallel 0}(\xi), \ \hat{\bar{U}}_{1\shortparallel 0} = (\hat{\rho}_0, \hat{m}_{\shortparallel 0})^T, \\ \hat{\bar{W}}_{\shortparallel}(\xi) = e^{-\left[4\alpha + (2\mu' + \lambda')|\xi|^2\right] t} \hat{W}_{\shortparallel 0}, \\ \hat{\bar{B}}(\xi) = e^{-\nu |\xi|^2 t} \hat{B}_0(\xi), \end{cases} \tag{2.2}$$

where we denote $\hat{\bar{U}}_{1\shortparallel}(\xi) = (\hat{\bar{\rho}}, \hat{\bar{m}}_{\shortparallel})^T$ and $\mathbb{A}(\xi)$ is defined as

$$\begin{pmatrix} 0 & -\sqrt{P'(1)} i \xi^T \\ -\sqrt{P'(1)} i \xi & -(2\mu + \lambda)|\xi|^2 \mathbb{I}_{3\times 3} \end{pmatrix}.$$

The characteristic polynomial of $\mathbb{A}(\xi)$ is

$$\det(\mathbb{A}(\xi) - \bar{\lambda}\mathbb{I}) = \left(\bar{\lambda} + (2\mu + \lambda)|\xi|^2\right)^2 \left(\bar{\lambda}^2 + (2\mu + \lambda)|\xi|^2 \bar{\lambda} + P'(1)|\xi|^2\right) = 0, \tag{2.3}$$

which implies the eigenvalues of (2.3)

$$\begin{aligned} \lambda_0 &= -(2\mu + \lambda)|\xi|^2 \ (\textit{double}), \\ \lambda_1 &= -(\mu + \lambda/2)|\xi|^2 + \frac{i}{2}\sqrt{4P'(1)|\xi|^2 - (2\mu + \lambda)^2 |\xi|^4}, \\ \lambda_2 &= -(\mu + \lambda/2)|\xi|^2 - \frac{i}{2}\sqrt{4P'(1)|\xi|^2 - (2\mu + \lambda)^2 |\xi|^4}. \end{aligned}$$

The exponential matrix $e^{t\mathbb{A}}$ has the spectral resolution

$$e^{t\mathbb{A}} = e^{\lambda_0 t}P_0 + e^{\lambda_1 t}P_1 + e^{\lambda_2 t}P_2,$$

where the project operators P_0, P_1 and P_2 can be computed as

$$P_i = \prod_{j\neq i} \frac{\mathbb{A}(\xi) - \lambda_j \mathbb{I}}{\lambda_i - \lambda_j}.$$

Note the fact that

$$\nabla \mathrm{div} m_{\shortparallel} = \nabla \mathrm{div} m = \Delta m_{\shortparallel},$$

by a direct computation, we can obtain the exact expression of the semigroup

$$e^{t\mathbb{A}} = \begin{pmatrix} \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} & -\frac{i\xi^T \sqrt{P'(1)}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)}{\lambda_1 - \lambda_2} \\ -\frac{i\xi \sqrt{P'(1)}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)}{\lambda_1 - \lambda_2} & \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \mathbb{I}_{3\times 3} \end{pmatrix}. \tag{2.4}$$

Now we turn to deal with the terms of the exponential matrix $e^{t\mathbb{A}}$. We need to verify the approximation of the semigroup $e^{t\mathbb{A}}$ for both low frequency and high frequency. In terms of the definition of the eigenvalues, we are able to obtain that

$$\frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} = \begin{cases} e^{-\left(\mu + \frac{\lambda}{2}\right)|\xi|^2 t} \left[\cos(bt) + \left(\mu + \frac{\lambda}{2}\right) \frac{\sin(bt)}{b} |\xi|^2\right], \ |\xi| \leq \eta, \\ e^{-R_0 t}, \ |\xi| \geq \eta, \end{cases} \tag{2.5}$$

$$\frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = \begin{cases} e^{-\left(\mu + \frac{\lambda}{2}\right)|\xi|^2 t} \left[\cos(bt) - \left(\mu + \frac{\lambda}{2}\right) \frac{\sin(bt)}{b} |\xi|^2\right], \ |\xi| \leq \eta, \\ e^{-R_0 t}, \ |\xi| \geq \eta, \end{cases} \tag{2.6}$$

$$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = \begin{cases} \frac{\sin(bt)}{b} e^{-\left(\mu + \frac{\lambda}{2}\right)|\xi|^2 t}, \ |\xi| \leq \eta, \\ e^{-R_0 t}, \ |\xi| \geq \eta, \end{cases} \tag{2.7}$$

where

$$b = \frac{1}{2}\sqrt{4P'(1)|\xi|^2 - (2\mu + \lambda)^2 |\xi|^4} \sim |\xi| + O(|\xi|^3), \ |\xi| \leq \eta, \tag{2.8}$$

and

$$\eta = \frac{2\sqrt{P'(1)}}{2\mu + \lambda}.$$

2.2. The L^2–decay rates of $(\bar{\rho}, \bar{m}_{\shortparallel}, \bar{W}_{\shortparallel}, \bar{B})^T$. In aid of the exponential matrix $e^{t\mathbb{A}}$ and the analysis of the low and high frequency (2.5)–(2.7), we can deduce the following upper bound decay rates of $(\bar{\rho}, \bar{m}_{\shortparallel}, \bar{B})^T$ and the decay rates of $\bar{W}_{\shortparallel}$.

Lemma 2.1. *Under the assumption that $(\rho_0, m_0, W_0, B_0)^T \in H^3 \cap L^1$, we can deduce for $k = 0,1,2,3$*

$$\left\|\nabla^k \left(\bar{\rho}, \bar{m}_{\shortparallel}, \bar{B}\right)\right\|_{L^2} \lesssim (1+t)^{-\frac{3}{4} - \frac{k}{2}} \left(\|(\rho_0, m_{\shortparallel 0}, B_0)\|_{L^1} + \left\|\nabla^k (\rho_0, m_{\shortparallel 0}, B_0)\right\|_{L^2}\right) \tag{2.9}$$

and

$$\left\|\nabla^k \bar{W}_{\shortparallel}\right\|_{L^2} \sim e^{-ct}\|W_{\shortparallel 0}\|_{H^k}. \tag{2.10}$$

Proof. Observe that the lower and high frequency analysis on the semigroup implies

$$\hat{\bar{\rho}}(\xi,t),\ \hat{\bar{m}}_{\shortparallel}(\xi,t) \lesssim \begin{cases} e^{-(\mu+\frac{\lambda}{2})|\xi|^2 t}(|\hat{\rho}_0|+|\hat{m}_{\shortparallel 0}|), & |\xi|\le\eta, \\ e^{-R_0 t}(|\hat{\rho}_0|+|\hat{m}_{\shortparallel 0}|), & |\xi|\ge\eta. \end{cases} \tag{2.11}$$

By Plancherel's Theorem and (2.11), it implies

$$\begin{aligned}
\|(\nabla^k\bar{\rho},\nabla^k\bar{m}_{\shortparallel})\|_{L^2}^2 &= \int_{|\xi|\le\eta}|\xi|^{2k}\left(\left|\hat{\bar{\rho}}\right|^2+\left|\hat{\bar{m}}_{\shortparallel}\right|^2\right)d\xi+\int_{|\xi|\ge\eta}|\xi|^{2k}\left(\left|\hat{\bar{\rho}}\right|^2+\left|\hat{\bar{m}}_{\shortparallel}\right|^2\right)d\xi \\
&\lesssim \int_{|\xi|\le\eta}e^{-2(\mu+\frac{\lambda}{2})|\xi|^2 t}|\xi|^{2k}\left(|\hat{\rho}_0|^2+|\hat{m}_{\shortparallel 0}|^2\right)d\xi+\int_{|\xi|\ge\eta}e^{-2R_0 t}|\xi|^{2k}\left(|\hat{\rho}_0|^2+|\hat{m}_{\shortparallel 0}|^2\right)d\xi \\
&\lesssim \|(\hat{\rho}_0,\hat{m}_{\shortparallel 0})\|_{L^\infty}^2\int_{|\xi|\le\eta}|\xi|^{2k}e^{-2(\mu+\frac{\lambda}{2})|\xi|^2 t}d\xi+e^{-2R_0 t}\int_{|\xi|\ge\eta}|\xi|^{2k}\left(|\hat{\rho}_0|^2+|\hat{m}_{\shortparallel 0}|^2\right)d\xi \\
&\lesssim (1+t)^{-\frac{3}{2}-k}\left(\|(\rho_0,m_{\shortparallel 0})\|_{L^1}^2+\left\|\nabla^k(\rho_0,m_{\shortparallel 0})\right\|_{L^2}^2\right).
\end{aligned}$$

For the term $\bar{B}$, from $(2.2)_3$, we can calculate directly that

$$\begin{aligned}
\|\nabla^k\bar{B}\|_{L^2}^2 = \left\|\widehat{\nabla^k\bar{B}}\right\|_{L^2}^2 &= \int_{|\xi|\le\eta}|\xi|^{2k}e^{-2\nu|\xi|^2 t}|\hat{B}_0|^2 d\xi+\int_{|\xi|\ge\eta}|\xi|^{2k}e^{-2\nu|\xi|^2 t}|\hat{B}_0|^2 d\xi \\
&\lesssim \|B_0\|_{L^1}^2\int_{|\xi|\le\eta}|\xi|^{2k}e^{-2\nu|\xi|^2 t}d\xi+e^{-2R_0 t}\int_{|\xi|\ge\eta}|\xi|^{2k}|\hat{B}_0|^2 d\xi \\
&\lesssim (1+t)^{-\frac{3}{2}-k}\left(\|B_0\|_{L^1}^2+\left\|\nabla^k B_0\right\|_{L^2}^2\right).
\end{aligned}$$

To this end, we have proved (2.9).

Note that

$$\hat{\bar{W}}_{\shortparallel} = e^{-\left[4\alpha+(2\mu'+\lambda')|\xi|^2\right]t}\hat{W}_{\shortparallel 0} = \begin{cases} O(1)e^{-ct}\hat{W}_{\shortparallel 0}, & |\xi|\le\eta, \\ O(1)e^{-R_0 t}\hat{W}_{\shortparallel 0}, & |\xi|\ge\eta, \end{cases}$$

with the constants $R_0\ge c>0$, it follows that

$$\begin{aligned}
\left\|\widehat{\nabla^k\bar{W}_{\shortparallel}}\right\|_{L^2}^2 &= \int_{|\xi|\le\eta}|\xi|^{2k}\left|\hat{\bar{W}}_{\shortparallel}\right|^2 d\xi+\int_{|\xi|\ge\eta}|\xi|^{2k}\left|\hat{\bar{W}}_{\shortparallel}\right|^2 d\xi \\
&\sim e^{-2ct}\left\|\nabla^k W_{\shortparallel 0}\right\|_{L^2}^2+e^{-2R_0 t}\left\|\nabla^k W_{\shortparallel 0}\right\|_{L^2}^2 \\
&\sim e^{-2ct}\left\|\nabla^k W_{\shortparallel 0}\right\|_{L^2}^2,
\end{aligned}$$

therefore, we prove (2.10). Hence, we complete the proof of Lemma 2.1. ∎

Before obtaining the lower decay rates for the solution $(\bar{\rho},\bar{m}_{\shortparallel},\bar{B})^T$, let's present some properties of $(\hat{\rho}_0(\xi),\hat{m}_{\shortparallel 0}(\xi),\hat{B}_0(\xi))^T$ as well as $\hat{m}_{\perp 0}(\xi)$. For simplicity, we let $Y=\hat{\rho}_0(0)$, $\hat{m}_{\perp 0}(0)$ or $\hat{B}_0(0)$, $X=\hat{\rho}_0(\xi)$, $\hat{m}_{\shortparallel 0}(\xi)$, $\hat{m}_{\perp 0}(\xi)$ or $\hat{B}_0(\xi)$.

LEMMA 2.2. *Let $|\xi|\in[0,\eta]$, under the conditions of (1.8)–(1.10), we know that*

$$Y\neq 0, \tag{2.12}$$

and there exist some $\bar{\xi}_i\in(0,\xi)$, with $i=1,2,3,4$, such that

$$X(\xi)=X(0)+\frac{\partial X(\bar{\xi}_i)}{\partial\xi}\xi. \tag{2.13}$$

Proof. We just prove the case $Y=\hat{\rho}_0(0)$ and $X=\hat{\rho}_0(\xi)$, the other terms can be proved in the same way. From (1.9), we can easily obtain

$$\hat{\rho}_0(0)=\int_{\mathbb{R}^3}e^{-i\xi x}\rho_0(x)dx|_{\xi=0}=\int_{\mathbb{R}^3}\rho_0(x)dx\neq 0. \tag{2.14}$$

Since $\rho_0\in L^1$, we can derive

$$\left|\int_{\mathbb{R}^3}e^{-i\xi x}\rho_0(x)dx\right|\lesssim\int_{\mathbb{R}^3}|\rho_0(x)|dx<\infty. \tag{2.15}$$

The condition (1.10) implies

$$\left|\int_{\mathbb{R}^3}\frac{\partial\left(e^{-i\xi x}\rho_0(x)\right)}{\partial\xi}dx\right|=\left|\int_{\mathbb{R}^3}ixe^{-i\xi x}\rho_0(x)dx\right|\lesssim\int_{\mathbb{R}^3}|x\rho_0(x)|dx<\infty. \tag{2.16}$$

It concludes from (2.15)–(2.16) that $\hat{\rho}_0(\xi)$ is continuous in $[-\eta,\eta]$, and has the derivative of order one. Thus, there exists $\bar{\xi}_1\in(0,\xi)$ such that

$$\hat{\rho}_0(\xi)=\hat{\rho}_0(0)+\frac{\partial\hat{\rho}_0(\bar{\xi}_1)}{\partial\xi}\xi.$$

□

Having completed this preparatory work, we now go to the proof of the lower bound time decay rates of $\bar{\rho}$, $\bar{m}_{\shortparallel}$ and $\bar{B}$.

LEMMA 2.3. *Under the conditions of (1.8)–(1.10), we obtain*

$$\left\|(\bar{\rho},\bar{m}_{\shortparallel},\bar{B})\right\|_{L^2}\geq C_0(1+t)^{-\frac{3}{4}}, \tag{2.17}$$

and

$$\left\|\nabla(\bar{\rho},\bar{m}_{\shortparallel},\bar{B})\right\|_{L^2}\geq C_0(1+t)^{-\frac{5}{4}}. \tag{2.18}$$

Proof. In terms of the formula (2.2) and the frequency analysis (2.5)–(2.8), we can see when $|\xi|\leq\eta$,

$$\begin{aligned}\hat{\bar{\rho}}&=e^{-(\mu+\frac{\lambda}{2})|\xi|^2t}\left[\cos(bt)+\left(\mu+\frac{\lambda}{2}\right)\frac{\sin(bt)}{b}|\xi|^2\right]\hat{\rho}_0-i\xi^T\sqrt{P'(1)}\frac{\sin(bt)}{b}e^{-(\mu+\frac{\lambda}{2})|\xi|^2t}\hat{m}_{\shortparallel 0}\\&=e^{-(\mu+\frac{\lambda}{2})|\xi|^2t}\cos(bt)\hat{\rho}_0+\left(\mu+\frac{\lambda}{2}\right)e^{-(\mu+\frac{\lambda}{2})|\xi|^2t}\frac{\sin(bt)}{b}|\xi|^2\hat{\rho}_0\\&\quad-i\xi^T\sqrt{P'(1)}\frac{\sin(bt)}{b}e^{-(\mu+\frac{\lambda}{2})|\xi|^2t}\hat{m}_{\shortparallel 0}.\end{aligned}$$

By Plancherel's Theorem, it is easy to verify that

$$
\begin{aligned}
\|\bar{\rho}\|_{L^2}^2 =& \int_{|\xi|\leq\eta} \left|\hat{\bar{\rho}}\right|^2 d\xi + \int_{|\xi|\geq\eta} \left|\hat{\bar{\rho}}\right|^2 d\xi \\
\gtrsim& \int_{|\xi|\leq\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t} \left[\cos(bt)\hat{\rho}_0 - \frac{i\xi^T\sqrt{P'(1)}\sin(bt)}{b}\hat{m}_{\shortparallel 0}\right]^2 d\xi \\
&- \int_{|\xi|\leq\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t} \frac{\sin^2(bt)}{|b|^2}|\xi|^4|\hat{\rho}_0|^2 d\xi - \int_{|\xi|\geq\eta} e^{-2R_0 t}\left(|\hat{\rho}_0|^2 + |\hat{m}_{\shortparallel 0}|^2\right) d\xi \\
\gtrsim& \int_{|\xi|\leq\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t} \left[\cos(|\xi|t)\hat{\rho}_0(0) - \sin(|\xi|t)\hat{m}_{\shortparallel 0}(0)\right]^2 d\xi \\
&- \int_{|\xi|\leq\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}|\xi|^6 t^2 \left(|\hat{\rho}_0(0)|^2 + |\hat{m}_{\shortparallel 0}(0)|^2\right) d\xi \\
&- \int_{|\xi|\leq\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}|\xi|^2 \left(\left|\frac{\partial\hat{\rho}_0(\bar{\xi}_1)}{\partial\xi}\right|^2 + \left|\frac{\partial\hat{m}_{\shortparallel 0}(\bar{\xi}_2)}{\partial\xi}\right|^2\right) d\xi \\
&- \int_{|\xi|\leq\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t} \frac{\sin^2(bt)}{|b|^2}|\xi|^4|\hat{\rho}_0|^2 d\xi - \int_{|\xi|\geq\eta} e^{-2R_0 t}\left(|\hat{\rho}_0|^2 + |\hat{m}_{\shortparallel 0}|^2\right) d\xi \\
=& J_1 + J_2 + J_3 + J_4 + J_5,
\end{aligned} \tag{2.19}
$$

where we have used (2.13) and the fact that

$$
\left|\cos\left(|\xi|t + |\xi|^3 t\right) - \sin\left(|\xi|t + |\xi|^3 t\right)\right| \gtrsim |\cos(|\xi|t) - \sin(|\xi|t)| - |\xi|^3 t.
$$

J_5 is estimated by

$$
|J_5| = e^{-2R_0 t}\int_{|\xi|\geq\eta} \left(|\hat{\rho}_0|^2 + |\hat{m}_{\shortparallel 0}|^2\right) d\xi \leq Ce^{-2R_0 t}\|(\rho_0, m_{\shortparallel 0})\|_{L^2}^2. \tag{2.20}
$$

In terms of the Taylor expansion (2.8), we bound

$$
\begin{aligned}
|J_4| \lesssim& \int_{|\xi|\leq\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}|\xi|^2|\hat{\rho}_0|^2 d\xi \\
\lesssim& \|\hat{\rho}_0\|_{L^\infty}^2 \int_{|\xi|\leq\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}|\xi|^2 d\xi \\
\lesssim& (1+t)^{-\frac{5}{2}}\|\rho_0\|_{L^1}^2.
\end{aligned} \tag{2.21}
$$

Like (2.21), we deduce

$$
|J_2| + |J_3| \lesssim (1+t)^{-\frac{5}{2}}. \tag{2.22}
$$

To estimate the term J_1 on the right-hand side of (2.19), we make the change of variables $y = |\xi|\sqrt{t}$, to deduce for sufficiently large time t

$$
\begin{aligned}
J_1 \gtrsim& t^{-\frac{3}{2}}\int_{y\leq\eta\sqrt{t}} e^{-2\left(\mu+\frac{\lambda}{2}\right)y^2}\left[\cos\left(y\sqrt{t}\right)\hat{\rho}_0(0) - \sin\left(y\sqrt{t}\right)\hat{m}_{\shortparallel 0}(0)\right]^2 dy \\
\geq& C_0 t^{-\frac{3}{2}}\sum_{k=0}^{[\eta t/\pi]-1}\int_{\frac{k\pi}{\sqrt{t}}}^{\frac{k\pi+\frac{\pi}{12}}{\sqrt{t}}} e^{-2\left(\mu+\frac{\lambda}{2}\right)y^2}\left[\cos\left(y\sqrt{t}\right) - \sin\left(y\sqrt{t}\right)\right]^2 dy
\end{aligned}
$$

$$
\begin{aligned}
&\geq C_0 t^{-\frac{3}{2}} \sum_{k=0}^{[\eta t/\pi]-1} \int_{\frac{k\pi}{\sqrt{t}}}^{\frac{k\pi+\frac{\pi}{12}}{\sqrt{t}}} e^{-2(\mu+\frac{\lambda}{2})y^2} \cos^2\left(y\sqrt{t}+\pi/4\right) dy \\
&\geq C_0 t^{-\frac{3}{2}} \sum_{k=0}^{[\eta t/\pi]-1} \int_{\frac{k\pi}{\sqrt{t}}}^{\frac{k\pi+\frac{\pi}{12}}{\sqrt{t}}} e^{-2(\mu+\frac{\lambda}{2})y^2} dy \\
&\geq C_0(1+t)^{-\frac{3}{2}},
\end{aligned} \tag{2.23}
$$

with a positive constant C_0 depending on $\hat{\rho}_0(0)$ and $\hat{m}_{_{\shortparallel}0}(0)$.

In terms of (2.20)–(2.23), we know that

$$
\|\bar{\rho}\|_{L^2}^2 \geq C_0(1+t)^{-\frac{3}{2}} - C(1+t)^{-\frac{5}{2}} \geq C_0(1+t)^{-\frac{3}{2}}, \tag{2.24}
$$

for t large enough. Likewise, one can obtain

$$
\|\bar{m}_{_{\shortparallel}}\|_{L^2}^2 \geq C_0(1+t)^{-\frac{3}{2}}. \tag{2.25}
$$

Now we deal with the decay rate for the first–order derivative of $(\bar{\rho}, \bar{m}_{_{\shortparallel}})^T$. By employing the same method as proving (2.24), it implies

$$
\begin{aligned}
\|\nabla\bar{\rho}\|_{L^2}^2 &= \int_{|\xi|\leq\eta} |\xi|^2 \left|\hat{\bar{\rho}}\right|^2 d\xi + \int_{|\xi|\geq\eta} |\xi|^2 \left|\hat{\bar{\rho}}\right|^2 d\xi \\
&\gtrsim \int_{|\xi|\leq\eta} |\xi|^2 e^{-2(\mu+\frac{\lambda}{2})|\xi|^2 t} \left[\cos(bt)\hat{\rho}_0 - \frac{i\xi^T\sqrt{P'(1)}\sin(bt)}{b}\hat{m}_{_{\shortparallel}0}\right]^2 d\xi \\
&\quad - \int_{|\xi|\leq\eta} e^{-2(\mu+\frac{\lambda}{2})|\xi|^2 t} \frac{\sin^2(bt)}{|b|^2} |\xi|^6 |\hat{\rho}_0|^2 d\xi - \int_{|\xi|\geq\eta} e^{-2R_0 t}|\xi|^2 \left(|\hat{\rho}_0|^2 + |\hat{m}_{_{\shortparallel}0}|^2\right) d\xi \\
&\geq C_0(1+t)^{-\frac{5}{2}} - C(1+t)^{-\frac{7}{2}} \geq C_0(1+t)^{-\frac{5}{2}}.
\end{aligned}
$$

Similarly,

$$
\|\nabla\bar{m}_{_{\shortparallel}}\|_{L^2}^2 \geq C_0(1+t)^{-\frac{5}{2}}.
$$

On the other hand, it derives from $(2.2)_3$ and (2.13) for $l=0,1$,

$$
\begin{aligned}
\left\|\nabla^l\bar{B}\right\|_{L^2}^2 &= \int_{|\xi|\leq\eta} |\xi|^{2l} e^{-2\nu|\xi|^2 t} |\hat{B}_0|^2 d\xi + \int_{|\xi|\geq\eta} |\xi|^{2l} e^{-2\nu|\xi|^2 t} |\hat{B}_0|^2 d\xi \\
&\gtrsim \int_{|\xi|\leq\eta} |\xi|^{2l} e^{-2\nu|\xi|^2 t} |\hat{B}_0(0)|^2 d\xi - \int_{|\xi|\leq\eta} |\xi|^{2l+2} e^{-2\nu|\xi|^2 t} \left|\frac{\partial\hat{B}_0(\bar{\xi}_4)}{\partial\xi}\right|^2 d\xi \\
&\quad - \int_{|\xi|\geq\eta} |\xi|^{2l} e^{-2R_0 t} |\hat{B}_0|^2 d\xi \\
&\geq C_0(1+t)^{-\frac{3}{2}-l} - C(1+t)^{-\frac{5}{2}-l} - e^{-2R_0 t}\left\|\nabla^l B_0\right\|_{L^2}^2 \\
&\geq C_0(1+t)^{-\frac{3}{2}-l}.
\end{aligned}
$$

To this end, we have proved Lemma 2.3. □

3. The L^2 decay rates for the solution of electromagnetic part

In this section, we should deal with the electromagnetic part. First of all, we give the spectral analysis on the semigroup generated by the linearized electromagnetic part.

3.1. Spectral representation for electromagnetic part. Recall that the electromagnetic part $\bar{U}_\perp = (\bar{m}_\perp, \bar{W}_\perp)^T$ satisfies the following equations

$$\begin{cases} \partial_t \bar{m}_\perp - (\mu+\alpha)\Delta \bar{m}_\perp - 2\alpha \nabla \times \bar{W}_\perp = 0, \\ \partial_t \bar{W}_\perp + 4\alpha \bar{W}_\perp - \mu' \Delta \bar{W}_\perp - 2\alpha \nabla \times \bar{m}_\perp = 0, \\ (\bar{m}_\perp, \bar{W}_\perp)^T|_{t=0} = (m_{\perp 0}, W_{\perp 0})^T. \end{cases} \tag{3.1}$$

From [20], we know that

$$\begin{pmatrix} \hat{\bar{m}}_\perp \\ \hat{\bar{W}}_\perp \end{pmatrix} = \hat{G} \begin{pmatrix} \hat{m}_{\perp 0} \\ \hat{W}_{\perp 0} \end{pmatrix}, \tag{3.2}$$

where

$$\hat{G} = \begin{pmatrix} \frac{k_1 e^{k_2 t} - k_2 e^{k_1 t}}{k_1 - k_2} - (\mu+\alpha)|\xi|^2 \frac{e^{k_1 t} - e^{k_2 t}}{k_1 - k_2} & \frac{e^{k_1 t} - e^{k_2 t}}{k_1 - k_2} 2\alpha i \xi \times \\ \frac{e^{k_1 t} - e^{k_2 t}}{k_1 - k_2} 2\alpha i \xi \times & \frac{k_1 e^{k_2 t} - k_2 e^{k_1 t}}{k_1 - k_2} - (\mu' |\xi|^2 + 4\alpha) \frac{e^{k_1 t} - e^{k_2 t}}{k_1 - k_2} \end{pmatrix},$$

with the real character roots

$$\begin{cases} k_1 = \frac{-[(\mu+\alpha+\mu')|\xi|^2 + 4\alpha] + \sqrt{(\mu+\alpha-\mu')^2 |\xi|^4 + 16\alpha^2 + 8\alpha(\mu'+\alpha-\mu)|\xi|^2}}{2}, \\ k_2 = \frac{-[(\mu+\alpha+\mu')|\xi|^2 + 4\alpha] - \sqrt{(\mu+\alpha-\mu')^2 |\xi|^4 + 16\alpha^2 + 8\alpha(\mu'+\alpha-\mu)|\xi|^2}}{2}. \end{cases} \tag{3.3}$$

Here we use the notations $\hat{G}_{11}$, $\hat{G}_{12}$, $\hat{G}_{21}$ and $\hat{G}_{22}$ to denote the four elements of $\hat{G}$ for simplicity. Due to the definition of $k_{1,2}$ in (3.3), in terms of Taylor expansion, we can easily find that for $|\xi| \ll 1$,

$$\hat{G}_{11} \sim |\xi|^2 e^{-ct} + e^{-\mu|\xi|^2 t}, \ \hat{G}_{12} = \hat{G}_{21} \sim |\xi| e^{-\mu|\xi|^2 t}, \ \hat{G}_{22} \sim e^{-ct} + |\xi|^2 e^{-\mu|\xi|^2 t}. \tag{3.4}$$

When $|\xi| \gg 1$, from [20], we know

$$|\hat{G}_{11}| + |\hat{G}_{12}| + |\hat{G}_{21}| + |\hat{G}_{22}| \le C e^{-R_0 t}. \tag{3.5}$$

Now we estimate the decay rates of $(\hat{\bar{m}}_\perp, \hat{\bar{W}}_\perp)^T$ according to the analysis on the terms of $\hat{G}$.

3.2. The L^2 decay rates of $(\bar{m}_\perp, \bar{W}_\perp)^T$. With the help of the analysis on the terms of $\hat{G}$, we show the following results.

LEMMA 3.1. *It holds that for $k = 0,1,2,3$,*

$$\left\| \nabla^k \bar{m}_\perp \right\|_{L^2}^2 \lesssim (1+t)^{-\frac{3}{2}-k} \left(\| (m_{\perp 0}, W_{\perp 0}) \|_{L^1}^2 + \left\| \nabla^k (m_{\perp 0}, W_{\perp 0}) \right\|_{L^2}^2 \right), \tag{3.6}$$

$$\left\| \nabla^k \bar{W}_\perp \right\|_{L^2}^2 \lesssim (1+t)^{-\frac{5}{2}-k} \left(\| (m_{\perp 0}, W_{\perp 0}) \|_{L^1}^2 + \left\| \nabla^k (m_{\perp 0}, W_{\perp 0}) \right\|_{L^2}^2 \right), \tag{3.7}$$

and for $l = 0,1$,

$$\left\| \nabla^l \bar{m}_\perp \right\|_{L^2}^2 \ge C_0 (1+t)^{-\frac{3}{2}-l}, \tag{3.8}$$

$$\left\| \nabla^l \bar{W}_\perp \right\|_{L^2}^2 \ge C_0 (1+t)^{-\frac{5}{2}-l}. \tag{3.9}$$

Proof. Since

$$\hat{\bar{m}}_\perp = \hat{G}_{11}\hat{m}_{\perp 0} + \hat{G}_{12}\hat{W}_{\perp 0}, \tag{3.10}$$

by Plancherel's Theorem and (3.4)–(3.5), one can deduce

$$\begin{aligned}
\left\|\nabla^k \bar{m}_\perp\right\|_{L^2}^2 = \left\|\widehat{\nabla^k \bar{m}_\perp}\right\|_{L^2}^2 &\lesssim \int_{|\xi|\le\eta} |\xi|^{2k}\left|\hat{G}_{11}\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k}\left|\hat{G}_{12}\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad + \int_{|\xi|\ge\eta} |\xi|^{2k}\left|\hat{G}_{11}\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k}\left|\hat{G}_{12}\hat{W}_{\perp 0}\right|^2 d\xi \\
&\lesssim \int_{|\xi|\le\eta} |\xi|^{2k} e^{-2\mu|\xi|^2 t}\left|\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k+2} e^{-2\mu|\xi|^2 t}\left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad + \int_{|\xi|\le\eta} |\xi|^{2k+4} e^{-2ct}\left|\hat{m}_{\perp 0}\right|^2 d\xi \\
&\quad + \int_{|\xi|\ge\eta} |\xi|^{2k} e^{-2R_0 t}\left|\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k} e^{-2R_0 t}\left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\lesssim \left\|\left(\hat{m}_{\perp 0}, \hat{W}_{\perp 0}\right)\right\|_{L^\infty}^2 \int_{|\xi|\le\eta} |\xi|^{2k} e^{-2\mu|\xi|^2 t} d\xi + e^{-2ct}\int_{|\xi|\le\eta} |\xi|^{2k}\left|\hat{m}_{\perp 0}\right|^2 d\xi \\
&\quad + e^{-2R_0 t}\int_{|\xi|\ge\eta} |\xi|^{2k}\left(\left|\hat{m}_{\perp 0}\right|^2 + \left|\hat{W}_{\perp 0}\right|^2\right) d\xi \\
&\lesssim (1+t)^{-\frac{3}{2}-k}\left(\left\|\left(m_{\perp 0}, W_{\perp 0}\right)\right\|_{L^1}^2 + \left\|\nabla^k\left(m_{\perp 0}, W_{\perp 0}\right)\right\|_{L^2}^2\right).
\end{aligned} \tag{3.11}$$

Furthermore, we just estimate like (3.11) to derive

$$\begin{aligned}
\left\|\nabla^k \bar{W}_\perp\right\|_{L^2}^2 = \left\|\widehat{\nabla^k \bar{W}_\perp}\right\|_{L^2}^2 &\lesssim \int_{|\xi|\le\eta} |\xi|^{2k}\left|\hat{G}_{21}\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k}\left|\hat{G}_{22}\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad + \int_{|\xi|\ge\eta} |\xi|^{2k}\left|\hat{G}_{21}\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k}\left|\hat{G}_{22}\hat{W}_{\perp 0}\right|^2 d\xi \\
&\lesssim \int_{|\xi|\le\eta} |\xi|^{2k+2} e^{-2\mu|\xi|^2 t}\left|\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k+4} e^{-2\mu|\xi|^2 t}\left|\hat{W}_{\perp 0}\right|^2 + |\xi|^{2k} e^{-2ct}\left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad + \int_{|\xi|\ge\eta} |\xi|^{2k} e^{-2R_0 t}\left|\hat{m}_{\perp 0}\right|^2 + |\xi|^{2k} e^{-2R_0 t}\left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\lesssim \left\|\left(\hat{m}_{\perp 0}, \hat{W}_{\perp 0}\right)\right\|_{L^\infty}^2 \int_{|\xi|\le\eta} |\xi|^{2k+2} e^{-2\mu|\xi|^2 t} d\xi + e^{-2ct}\int_{|\xi|\le\eta} |\xi|^{2k}\left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad + e^{-2R_0 t}\int_{|\xi|\ge\eta} |\xi|^{2k}\left(\left|\hat{m}_{\perp 0}\right|^2 + \left|\hat{W}_{\perp 0}\right|^2\right) d\xi \\
&\lesssim (1+t)^{-\frac{5}{2}-k}\left(\left\|\left(m_{\perp 0}, W_{\perp 0}\right)\right\|_{L^1}^2 + \left\|\nabla^k\left(m_{\perp 0}, W_{\perp 0}\right)\right\|_{L^2}^2\right).
\end{aligned}$$

On the other hand, together with (2.13), it stems from (3.4)–(3.5) again that

$$\begin{aligned}
\left\|\nabla^l \hat{\bar{m}}_\perp\right\|_{L^2}^2 &= \int_{|\xi|\le\eta} |\xi|^{2l}\left|\hat{G}_{11}\hat{m}_{\perp 0} + \hat{G}_{12}\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad + \int_{|\xi|\ge\eta} |\xi|^{2l}\left|\hat{G}_{11}\hat{m}_{\perp 0} + \hat{G}_{12}\hat{W}_{\perp 0}\right|^2 d\xi
\end{aligned}$$

$$
\begin{aligned}
&\gtrsim \int_{|\xi|\le\eta} |\xi|^{2l} e^{-2\mu|\xi|^2 t} |\hat{m}_{\perp 0}|^2 \, d\xi - \int_{|\xi|\le\eta} |\xi|^{2l+4} e^{-2ct} |\hat{m}_{\perp 0}|^2 \, d\xi \\
&\quad - \int_{|\xi|\le\eta} |\xi|^{2l+2} e^{-2\mu|\xi|^2 t} \left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad - \int_{|\xi|\ge\eta} e^{-2R_0 t} |\xi|^{2l} \left(|\hat{m}_{\perp 0}|^2 + \left|\hat{W}_{\perp 0}\right|^2 \right) d\xi \\
&\gtrsim \int_{|\xi|\le\eta} |\xi|^{2l} e^{-2\mu|\xi|^2 t} d\xi \, |\hat{m}_{\perp 0}(0)|^2 - \int_{|\xi|\le\eta} |\xi|^{2l+2} e^{-2\mu|\xi|^2 t} \left| \frac{\partial \hat{m}_{\perp 0}(\bar{\xi}_3)}{\partial \xi} \right|^2 d\xi \\
&\quad - e^{-2ct} \int_{|\xi|\le\eta} |\xi|^{2l+4} |\hat{m}_{\perp 0}|^2 \, d\xi - \int_{|\xi|\le\eta} |\xi|^{2l+2} e^{-2\mu|\xi|^2 t} \left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad - e^{-2R_0 t} \int_{|\xi|\ge\eta} |\xi|^{2l} \left(|\hat{m}_{\perp 0}|^2 + |\hat{W}_{\perp 0}|^2 \right) d\xi \\
&\ge C_0 (1+t)^{-\frac{3}{2}-l},
\end{aligned}
$$

with a positive constant C_0 depending on $\hat{m}_{\perp 0}(0)$. With this method once again, it gives

$$
\begin{aligned}
\left\| \nabla^l \hat{\bar{W}}_\perp \right\|_{L^2}^2 &= \int_{|\xi|\le\eta} |\xi|^{2l} \left| \hat{G}_{21} \hat{m}_{\perp 0} + \hat{G}_{22} \hat{W}_{\perp 0} \right|^2 d\xi + \int_{|\xi|\ge\eta} |\xi|^{2l} \left| \hat{G}_{21} \hat{m}_{\perp 0} + \hat{G}_{22} \hat{W}_{\perp 0} \right|^2 d\xi \\
&\gtrsim \int_{|\xi|\le\eta} |\xi|^{2l+2} e^{-2\mu|\xi|^2 t} |\hat{m}_{\perp 0}|^2 \, d\xi - \int_{|\xi|\le\eta} |\xi|^{2l+4} e^{-2\mu|\xi|^2 t} \left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad - \int_{|\xi|\le\eta} e^{-2ct} |\xi|^{2l} \left|\hat{W}_{\perp 0}\right|^2 d\xi - \int_{|\xi|\ge\eta} e^{-2R_0 t} |\xi|^{2l} \left(|\hat{m}_{\perp 0}|^2 + \left|\hat{W}_{\perp 0}\right|^2 \right) d\xi \\
&\gtrsim \int_{|\xi|\le\eta} |\xi|^{2l+2} e^{-2\mu|\xi|^2 t} d\xi \, |\hat{m}_{\perp 0}(0)|^2 - \int_{|\xi|\le\eta} |\xi|^{2l+4} e^{-2\mu|\xi|^2 t} \left| \frac{\partial \hat{m}_{\perp 0}(\bar{\xi}_3)}{\partial \xi} \right|^2 d\xi \\
&\quad - \int_{|\xi|\le\eta} |\xi|^{2l+4} e^{-2\mu|\xi|^2 t} \left|\hat{W}_{\perp 0}\right|^2 d\xi - e^{-2ct} \int_{|\xi|\le\eta} |\xi|^{2l} \left|\hat{W}_{\perp 0}\right|^2 d\xi \\
&\quad - e^{-2R_0 t} \int_{|\xi|\ge\eta} |\xi|^{2l} \left(|\hat{m}_{\perp 0}|^2 + |\hat{W}_{\perp 0}|^2 \right) d\xi \\
&\ge C_0 (1+t)^{-\frac{5}{2}-l}.
\end{aligned}
$$

This completes the proof of Lemma 3.1. □

It should be noted that the solution $(\bar{\rho}, \bar{m}, \bar{B}, \bar{W})$ of the linearized magneto–micropolar fluids system has an optimal decay rate. In fact, from Lemma 2.1, Lemma 2.3 and Lemma 3.1, we can present the following important proposition.

PROPOSITION 3.1. *Under the conditions of* (1.8)–(1.10), *it holds for* $l = 0, 1$

$$
C_0 (1+t)^{-\frac{3}{4}-\frac{l}{2}} \le \left\| \nabla^l (\bar{\rho}, \bar{m}, \bar{B}) \right\|_{L^2} \le C\delta (1+t)^{-\frac{3}{4}-\frac{l}{2}}, \tag{3.12}
$$

and

$$
C_0 (1+t)^{-\frac{5}{4}-\frac{l}{2}} \le \left\| \nabla^l \bar{W} \right\|_{L^2} \le C\delta (1+t)^{-\frac{5}{4}-\frac{l}{2}}, \tag{3.13}
$$

with $C_0 > 0$ *depending on* $\hat{\rho}_0(0)$, $\hat{m}_{\shortparallel 0}(0)$, $\hat{m}_{\perp 0}(0)$, $\hat{B}_0(0)$ *suitably smaller than* δ.

Proof. The upper decay rates hold clearly. For the lower decay rates, in view of (3.9) and (2.10), we can obtain

$$\begin{aligned}\left\|\nabla^l\bar{W}\right\|_{L^2}&=\left\|\nabla^l\bar{W}_{\shortparallel}+\nabla^l\bar{W}_{\perp}\right\|_{L^2}\geq\left\|\nabla^l\bar{W}_{\perp}\right\|_{L^2}-\left\|\nabla^l\bar{W}_{\shortparallel}\right\|_{L^2}\geq C_0(1+t)^{-\frac{5}{4}-\frac{l}{2}}-Ce^{-ct}\\&\geq C_0(1+t)^{-\frac{5}{4}-\frac{l}{2}}.\end{aligned}$$

For the term $\bar{m}$, using the same method as proving (2.17)–(2.18), we get

$$\left\|\nabla^l\bar{m}\right\|_{L^2}^2=\left\|\nabla^l\bar{m}_{\shortparallel}+\nabla^l\bar{m}_{\perp}\right\|_{L^2}^2\geq C_0(1+t)^{-\frac{3}{2}-l}.$$

□

4. Estimates on the nonlinear equations

In this section, we want to deduce the L^2 decay rate for the nonlinear system. Define

$$\begin{aligned}&\rho_h=\rho-\bar{\rho},\ m_{h\shortparallel}=m_{\shortparallel}-\bar{m}_{\shortparallel},\ m_{h\perp}=m_{\perp}-\bar{m}_{\perp},\\&B_h=B-\bar{B},\ W_{h\shortparallel}=W_{\shortparallel}-\bar{W}_{\shortparallel},\ W_{h\perp}=W_{\perp}-\bar{W}_{\perp}.\end{aligned}$$

The nonlinear system (1.2)–(1.3) is reformulated as follows:

$$\begin{cases}\partial_t\rho_h+\sqrt{P'(1)}\mathrm{div}m_{h\shortparallel}=0,\\\partial_t m_{h\shortparallel}+\sqrt{P'(1)}\nabla\rho_h-(2\mu+\lambda)\Delta m_{h\shortparallel}=\Delta^{-1}\nabla\mathrm{div}\mathbb{N}_1=F_1,\\\partial_t W_{h\shortparallel}+4\alpha W_{h\shortparallel}-(2u'+\lambda')\Delta W_{h\shortparallel}=\Delta^{-1}\nabla\mathrm{div}\mathbb{N}_2=F_2,\\\partial_t B_h-\nu\Delta B_h=\mathbb{N}_3,\\(\rho_h,m_{h\shortparallel},W_{h\shortparallel},B_h)^T|_{t=0}=(\rho_{h0},m_{h\shortparallel 0},W_{h\shortparallel 0},B_{h0})^T=(0,0,0,0)^T,\end{cases}\tag{4.1}$$

and

$$\begin{cases}\partial_t m_{h\perp}-(\mu+\alpha)\Delta m_{h\perp}-2\alpha\nabla\times W_{h\perp}=N_1,\\\partial_t W_{h\perp}+4\alpha W_{h\perp}-\mu'\Delta W_{h\perp}-2\alpha\nabla\times m_{h\perp}=N_2,\\(m_{h\perp},W_{h\perp})^T|_{t=0}=(m_{h\perp 0},W_{h\perp 0})^T=(0,0)^T.\end{cases}\tag{4.2}$$

For the sake of convenience, we denote

$$F_1=\nabla f_1+\mathrm{div}f_2,\quad N_1=\nabla f,\ \mathbb{N}_3=\mathrm{curl}f_3,$$

with

$$f_1\sim\rho^2+\mathrm{div}\left(\frac{\rho m}{1+\rho}\right)+|B|^2,\ f_2\sim\frac{m\otimes m}{1+\rho}+\nabla\left(\frac{\rho m}{1+\rho}\right)+B_iB,$$

$$f\sim\nabla\left(\frac{\rho m}{1+\rho}\right)+\frac{\rho W}{1+\rho},\ f_3\sim\frac{m\times B}{1+\rho},$$

here we have used the fact that

$$B\cdot\nabla B=B_j\partial_jB_i=\partial_j(B_jB_i).$$

In order to deduce the L^2 decay rate for the nonlinear system, we just need to prove that the solutions $(\rho_h,m_{h\shortparallel},W_{h\shortparallel},B_h)^T$ and $(m_{h\perp},W_{h\perp})^T$ to (4.1) and (4.2) have faster decay rates than $(\bar{\rho},\bar{m}_{\shortparallel},\bar{W}_{\shortparallel},\bar{B})^T$ and $(\bar{m}_{\perp},\bar{W}_{\perp})^T$.

We can represent the solution in terms of the Duhamel's principle

$$\begin{cases} \hat{U}_{1h_{\shortparallel}} = (\hat{\rho}_h, \hat{m}_{h_{\shortparallel}})^T = \int_0^t e^{(t-s)\mathbb{A}}(0,\hat{F}_1)^T(s)ds, \\ \hat{W}_{h_{\shortparallel}} = \int_0^t e^{-\left[4\alpha+(2\mu'+\lambda')|\xi|^2\right](t-s)}\hat{F}_2(s)ds, \\ \hat{U}_{h\perp} = (\hat{m}_{h\perp},\hat{W}_{h\perp})^T = \int_0^t \hat{G}(\xi,t-s)(\hat{N}_1,\hat{N}_2)^T(s)ds, \\ \hat{B}_h = \int_0^t e^{-\nu|\xi|^2(t-s)}\hat{\mathbb{N}}_3(s)ds. \end{cases} \tag{4.3}$$

In the following, we should assume

$$N(t) = \sup_{0\le\tau\le t}\left\{ \|(\rho_h, m_{h_{\shortparallel}}, W_{h_{\shortparallel}}, m_{h\perp}, W_{h\perp}, B_h)\|_{H^2}(1+\tau)^{\frac{3}{4}+\frac{\epsilon}{2}} \right. \\ \left. + \left\|\nabla^3(\rho_h, m_{h_{\shortparallel}}, W_{h_{\shortparallel}}, m_{h\perp}, W_{h\perp}, B_h)\right\|_{L^2}\right\}. \tag{4.4}$$

We claim that for $0<\epsilon\le 1$,

$$N(t)\le C\delta, \tag{4.5}$$

with δ defined in Theorem 1.1.

LEMMA 4.1. *Under the assumption of (4.4), it holds*

$$\|(\rho_h, m_{h_{\shortparallel}}, W_{h_{\shortparallel}}, m_{h\perp}, W_{h\perp}, B_h)\|_{L^2} \lesssim (1+t)^{-\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}\left(\delta^2+N^2(t)\right). \tag{4.6}$$

Proof. From (4.3) and (2.4), we know

$$\hat{\rho}_h(\xi,t) = -\int_0^t \frac{i\xi^T\sqrt{P'(1)}\left(e^{\lambda_1(t-s)}-e^{\lambda_2(t-s)}\right)}{\lambda_1-\lambda_2}\hat{F}_1(s)ds,$$

and

$$\hat{m}_{h_{\shortparallel}}(\xi,t) = \int_0^t \frac{\lambda_1 e^{\lambda_1(t-s)}-\lambda_2 e^{\lambda_2(t-s)}}{\lambda_1-\lambda_2}\hat{F}_1(s)ds.$$

It derives from (2.5)–(2.7) that for some constant $C' = \min\{2\mu+\lambda, 2\nu\}$,

$$\begin{aligned} &\left\|\frac{i\xi^T\left(e^{\lambda_1 t}-e^{\lambda_2 t}\right)}{\lambda_1-\lambda_2}\hat{F}_1\right\|_{L^2}^2 + \left\|\frac{\lambda_1 e^{\lambda_1 t}-\lambda_2 e^{\lambda_2 t}}{\lambda_1-\lambda_2}\hat{F}_1\right\|_{L^2}^2 + \left\|e^{-\nu|\xi|^2 t}\hat{\mathbb{N}}_3\right\|_{L^2}^2 \\ \le &\int_{|\xi|\le\eta} e^{-2\left(\mu+\frac{\lambda}{2}\right)|\xi|^2 t}\left|\hat{F}_1\right|^2 d\xi + \int_{|\xi|\le\eta} e^{-2\nu|\xi|^2 t}\left|\hat{\mathbb{N}}_3\right|^2 d\xi \\ &\quad + \int_{|\xi|\ge\eta} e^{-2R_0 t}\left(\left|\hat{F}_1\right|^2+\left|\hat{\mathbb{N}}_3\right|^2\right)d\xi \\ \lesssim &\int_{|\xi|\le\eta} e^{-C'|\xi|^2 t}|\xi|^2\left|\left(\hat{f}_1,\hat{f}_2,\hat{f}_3\right)\right|^2 d\xi + e^{-2R_0 t}\int_{|\xi|\ge\eta}\left(\left|\hat{F}_1\right|^2+\left|\hat{\mathbb{N}}_3\right|^2\right)d\xi \\ \le &C(1+t)^{-\frac{5}{2}}\left(\|(f_1,f_2,f_3)\|_{L^1}^2 + \|(F_1,\mathbb{N}_3)\|_{L^2}^2\right). \end{aligned}$$

Thus, one has

$$\|(\rho_h,m_{h_{\shortparallel}},B_h)\|_{L^2}\leq C\int_0^t(1+t-s)^{-\frac{5}{4}}(\|(f_1,f_2,f_3)(s)\|_{L^1}+\|(F_1,\mathbb{N}_3)(s)\|_{L^2})ds, \tag{4.7}$$

and

$$\|W_{h_{\shortparallel}}\|_{L^2}\leq C\int_0^t e^{-c(t-s)}\|F_2(s)\|_{L^2}ds. \tag{4.8}$$

In light of (3.4)–(3.5), it can be computed directly

$$\begin{aligned}\left\|\hat{G}_{11}\hat{N}_1(s)\right\|_{L^2}^2\lesssim&\int_{|\xi|\leq\eta}|\xi|^4e^{-2ct}\left|\hat{N}_1\right|^2+e^{-2\mu|\xi|^2t}\left|\hat{N}_1\right|^2d\xi+\int_{|\xi|\geq\eta}e^{-2R_0t}\left|\hat{N}_1\right|^2d\xi\\ \lesssim&\int_{|\xi|\leq\eta}|\xi|^4e^{-2ct}\left|\hat{N}_1\right|^2d\xi+\int_{|\xi|\leq\eta}e^{-2\mu|\xi|^2t}|\xi|^2\left|\hat{f}\right|^2d\xi+\int_{|\xi|\geq\eta}e^{-2R_0t}\left|\hat{N}_1\right|^2d\xi\\ \lesssim&(1+t)^{-\frac{5}{2}}\left(\|f\|_{L^1}^2+\|N_1\|_{L^2}^2\right),\end{aligned}$$

and

$$\begin{aligned}\left\|\hat{G}_{12}\hat{N}_2(s)\right\|_{L^2}^2\lesssim&\int_{|\xi|\leq\eta}|\xi|^2e^{-2\mu|\xi|^2t}\left|\hat{N}_2\right|^2d\xi+\int_{|\xi|\geq\eta}e^{-2R_0t}\left|\hat{N}_2\right|^2d\xi\\ \lesssim&(1+t)^{-\frac{5}{2}}\left(\|N_2\|_{L^1}^2+\|N_2\|_{L^2}^2\right).\end{aligned}$$

Likely, we have

$$\begin{aligned}\left\|\hat{G}_{21}\hat{N}_1(s)\right\|_{L^2}^2\lesssim&\int_{|\xi|\leq\eta}|\xi|^2e^{-2\mu|\xi|^2t}\left|\hat{N}_1\right|^2d\xi+\int_{|\xi|\geq\eta}e^{-2R_0t}\left|\hat{N}_1\right|^2d\xi\\ \lesssim&\int_{|\xi|\leq\eta}e^{-2\mu|\xi|^2t}|\xi|^4\left|\hat{f}\right|^2d\xi+\int_{|\xi|\geq\eta}e^{-2R_0t}\left|\hat{N}_1\right|^2d\xi\\ \lesssim&(1+t)^{-\frac{7}{2}}\left(\|f\|_{L^1}^2+\|N_1\|_{L^2}^2\right),\end{aligned}$$

and

$$\begin{aligned}\left\|\hat{G}_{22}\hat{N}_2(s)\right\|_{L^2}^2\lesssim&\int_{|\xi|\leq\eta}e^{-2ct}\left|\hat{N}_2\right|^2d\xi+\int_{|\xi|\leq\eta}|\xi|^4e^{-2\mu|\xi|^2t}\left|\hat{N}_2\right|^2d\xi+\int_{|\xi|\geq\eta}e^{-2R_0t}\left|\hat{N}_2\right|^2d\xi\\ \lesssim&e^{-2ct}\|N_2\|_{L^2}^2+(1+t)^{-\frac{7}{2}}\left(\|N_2\|_{L^1}^2+\|N_2\|_{L^2}^2\right)\\ \lesssim&(1+t)^{-\frac{7}{2}}\left(\|N_2\|_{L^1}^2+\|N_2\|_{L^2}^2\right).\end{aligned}$$

Therefore, we end up with

$$\|m_{h\perp}\|_{L^2}\leq\int_0^t(1+t-s)^{-\frac{5}{4}}(\|f(s)\|_{L^1}+\|(N_1,N_2)(s)\|_{L^1\cap L^2})ds, \tag{4.9}$$

and for $k=0,1$

$$\left\|\nabla^kW_{h\perp}\right\|_{L^2}\leq\int_0^t(1+t-s)^{-\frac{7}{4}-\frac{k}{2}}(\|(f,N_1)(s)\|_{L^1}+\|(N_1,N_2)(s)\|_{L^2})ds. \tag{4.10}$$

Now, it turns to estimate the nonlinear terms f_1, f_2, f_3, f, F_1, F_2, N_1, N_2, $\mathbb{N}_3$. Note that we denote

$$m_h = m - \bar{m} = m_{\shortparallel} + m_{\perp} - \bar{m}_{\shortparallel} - \bar{m}_{\perp} = m_{h\shortparallel} + m_{h\perp},$$
$$W_h = W - \bar{W} = W_{\shortparallel} + W_{\perp} - \bar{W}_{\shortparallel} - \bar{W}_{\perp} = W_{h\shortparallel} + W_{h\perp}.$$

It is easy to verify from (4.4), Lemma 2.1, Lemma 3.1 and Lemma 5.2 that

$$\left\| \frac{m\otimes m}{1+\rho} \right\|_{L^1} \lesssim \|\bar{m}\|_{L^2}^2 + \|\bar{m}\|_{L^2}\|m_h\|_{L^2} + \|m_h\|_{L^2}^2 \lesssim (1+t)^{-\frac{3}{2}}(\delta^2+N^2(t)), \tag{4.11}$$

likewise, one has

$$\|\nabla\rho\cdot\nabla m\|_{L^1} + \left\|\rho\nabla^2 m\right\|_{L^1} + \|\rho\nabla m\|_{L^1} \lesssim (1+t)^{-\left(\frac{3}{2}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)). \tag{4.12}$$

We can estimate the remaining terms as (4.11) and (4.12), hence

$$\|(f_1,f_2,f_3,f,N_1,N_2)\|_{L^1} \lesssim (1+t)^{-\frac{3}{2}}\left(\delta^2+N^2(t)\right). \tag{4.13}$$

It derives from the assumption (4.4), Lemma 2.1, Lemma 3.1 and Lemma 5.2 once again that

$$\begin{aligned}\|m\cdot\nabla m\|_{L^2} = {} & \|\bar{m}\|_{L^\infty}\|\nabla\bar{m}\|_{L^2} + \|\bar{m}\|_{L^\infty}\|\nabla m_h\|_{L^2} \\ & + \|\nabla\bar{m}\|_{L^2}\|m_h\|_{L^\infty} + \|m_h\|_{L^\infty}\|\nabla m_h\|_{L^2} \\ \lesssim {} & (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)),\end{aligned} \tag{4.14}$$

and likely

$$\|\rho W\|_{L^2} + \left\|\rho\nabla^2 m\right\|_{L^2} + \|\nabla m\cdot\nabla\rho\|_{L^2} \lesssim (1+t)^{-\left(\frac{3}{2}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)). \tag{4.15}$$

We can estimate like (4.14)–(4.15) to imply

$$\|(F_1,F_2,N_1,N_2,\mathbb{N}_3)\|_{L^2} \lesssim (1+t)^{-\left(\frac{3}{2}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)). \tag{4.16}$$

In light of (4.7)–(4.10), (4.13) and (4.16) together with (5.1) of Lemma 5.1, we have

$$\begin{aligned}\|(\rho_h, m_{h\shortparallel}, m_{h\perp}, B_h)\|_{L^2} & \le C\int_0^t (1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{3}{2}}ds(\delta^2+N^2(t)) \\ & \le C(1+t)^{-\frac{5}{4}}(\delta^2+N^2) \\ & \le C(1+t)^{-\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t));\end{aligned}$$

$$\|W_{h\shortparallel}\|_{L^2} \lesssim \int_0^t e^{-c(t-s)}(1+s)^{-\frac{3}{2}}ds(\delta^2+N^2(t)) \lesssim (1+t)^{-\frac{3}{2}}(\delta^2+N^2(t)), \tag{4.17}$$

and

$$\|W_{h\perp}\|_{L^2} \le C\int_0^t (1+t-s)^{-\frac{7}{4}}(1+s)^{-\frac{3}{2}}ds(\delta^2+N^2(t)) \le C(1+t)^{-\frac{3}{2}}(\delta^2+N^2(t)), \tag{4.18}$$

which proves (4.6). □

In the following lemma, we employ the energy method to obtain the decay rates of the derivatives of $(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp}, B_h)^T$.

Lemma 4.2. *Under the same conditions of Lemma 4.1, it holds that*

$$\|\nabla(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, m_{h\perp}, W_{h\perp}, B_h)\|_{H^1} \le C(1+t)^{-\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)). \tag{4.19}$$

Proof. Applying ∇^k with $k=0,1,2$ to (4.1) and multiplying by $\nabla^k\rho_h$, $\nabla^k m_{h\shortparallel}$, $\nabla^k W_{h\shortparallel}$, $\nabla^k B_h$, respectively, we have

$$\begin{aligned}&\frac{d}{dt}\left\|\nabla^k(\rho_h, m_{h\shortparallel}, W_{h\shortparallel}, B_h)\right\|_{L^2}^2 + \left\|\nabla^{k+1}(m_{h\shortparallel}, W_{h\shortparallel}, B_h)\right\|_{L^2}^2 + \left\|\nabla^k W_{h\shortparallel}\right\|_{L^2}^2 \\ &\lesssim \int_{\mathbb{R}^3}\nabla^k F_1\cdot\nabla^k m_{h\shortparallel}dx + \int_{\mathbb{R}^3}\nabla^k F_2\cdot\nabla^k W_{h\shortparallel}dx + \int_{\mathbb{R}^3}\nabla^k \mathbb{N}_3\cdot\nabla^k B_h dx.\end{aligned} \tag{4.20}$$

A similar method as (4.20) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla^k(m_{h\perp}, W_{h\perp})\right\|_{L^2}^2 + (\mu+\alpha)\left\|\nabla^{k+1}m_{h\perp}\right\|_{L^2}^2 + \mu'\left\|\nabla^{k+1}W_{h\perp}\right\|_{L^2}^2 + 4\alpha\left\|\nabla^k W_{h\perp}\right\|_{L^2}^2 \\ &\le \int_{\mathbb{R}^3}\nabla^k N_1\cdot\nabla^k m_{h\perp}dx + \int_{\mathbb{R}^3}\nabla^k N_2\cdot\nabla^k W_{h\perp}dx + 4\alpha\int_{\mathbb{R}^3}\nabla^k W_{h\perp}\cdot\mathrm{curl}\nabla^k m_{h\perp}dx \\ &\le \int_{\mathbb{R}^3}\nabla^k N_1\cdot\nabla^k m_{h\perp}dx + \int_{\mathbb{R}^3}\nabla^k N_2\cdot\nabla^k W_{h\perp}dx + 4\alpha\left\|\nabla^k W_{h\perp}\right\|_{L^2}^2 + \alpha\left\|\nabla^{k+1}m_{h\perp}\right\|_{L^2}^2,\end{aligned}$$

further implies,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\nabla^k(m_{h\perp}, W_{h\perp})\right\|_{L^2}^2 + \mu\left\|\nabla^{k+1}m_{h\perp}\right\|_{L^2}^2 + \mu'\left\|\nabla^{k+1}W_{h\perp}\right\|_{L^2}^2 \\ &\le \int_{\mathbb{R}^3}\nabla^k N_1\cdot\nabla^k m_{h\perp}dx + \int_{\mathbb{R}^3}\nabla^k N_2\cdot\nabla^k W_{h\perp}dx.\end{aligned} \tag{4.21}$$

We now estimate the right-hand side of (4.20)–(4.21). For $k=0$, by Hölder's inequality, we deduce

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{\rho W}{1+\rho}W_{h\shortparallel}dx &\le \|\rho\|_{L^3}\|W\|_{L^2}\|W_{h\shortparallel}\|_{L^6} \\ &\lesssim (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2 + \epsilon_1\|\nabla W_{h\shortparallel}\|_{L^2}^2.\end{aligned} \tag{4.22}$$

By the integration by parts, we know

$$\begin{aligned}\int_{\mathbb{R}^3}\mathrm{div}\left(\frac{m\otimes m}{1+\rho}\right)\cdot m_{h\shortparallel}dx &= -\int_{\mathbb{R}^3}\frac{m\otimes m}{1+\rho}\cdot\nabla m_{h\shortparallel}dx \le \|m\|_{L^\infty}\|m\|_{L^2}\|\nabla m_{h\shortparallel}\|_{L^2} \\ &\lesssim (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2 + \epsilon_1\|\nabla m_{h\shortparallel}\|_{L^2}^2,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3}\Delta\left(\frac{\rho m}{1+\rho}\right)\cdot m_{h\shortparallel}dx &\le \left\|\nabla\left(\frac{\rho m}{1+\rho}\right)\right\|_{L^2}\|\nabla m_{h\shortparallel}\|_{L^2} \\ &\le \|m\nabla\rho\|_{L^2}\|\nabla m_{h\shortparallel}\|_{L^2} + \|\rho\nabla m\|_{L^2}\|\nabla m_{h\shortparallel}\|_{L^2} \\ &\lesssim (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2 + \epsilon_1\|\nabla m_{h\shortparallel}\|_{L^2}^2.\end{aligned} \tag{4.23}$$

We can estimate the remaining terms like (4.22)–(4.23), and it implies

$$\int_{\mathbb{R}^3} F_1\cdot m_{h_{\shortparallel}}dx+\int_{\mathbb{R}^3} F_2\cdot W_{h_{\shortparallel}}dx+\int_{\mathbb{R}^3} \mathbb{N}_3\cdot B_h dx$$
$$\lesssim (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2+\epsilon_1\|(\nabla m_{h_{\shortparallel}},\nabla W_{h_{\shortparallel}},\nabla B_h)\|_{L^2}^2, \tag{4.24}$$

and

$$\int_{\mathbb{R}^3} N_1\cdot m_{h\perp}dx+\int_{\mathbb{R}^3} N_2\cdot W_{h\perp}dx\lesssim (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2$$
$$+\epsilon_1\|(\nabla m_{h\perp},\nabla W_{h\perp})\|_{L^2}^2. \tag{4.25}$$

Combining the assumption (4.4), Lemma 2.1, Lemma 3.1 and Lemma 5.2, we estimate

$$\begin{aligned}\left\|\nabla m\cdot\nabla^2\rho\right\|_{L^2}\lesssim& \|\nabla m_h\|_{L^\infty}\left\|\nabla^2\rho_h\right\|_{L^2}+\|\nabla\bar{m}\|_{L^\infty}\left\|\nabla^2\rho_h\right\|_{L^2}+\|\nabla m_h\|_{L^6}\left\|\nabla^2\bar{\rho}\right\|_{L^3}\\&+\|\nabla\bar{m}\|_{L^\infty}\left\|\nabla^2\bar{\rho}\right\|_{L^2}\\ \lesssim& (1+t)^{-\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)),\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}\left\|m\cdot\nabla^3\rho\right\|_{L^2}\lesssim& \|m_h\|_{L^\infty}\left\|\nabla^3\rho_h\right\|_{L^2}+\|\bar{m}\|_{L^\infty}\left\|\nabla^3\rho_h\right\|_{L^2}+\|m_h\|_{L^\infty}\left\|\nabla^3\bar{\rho}\right\|_{L^2}\\&+\|\bar{m}\|_{L^\infty}\left\|\nabla^3\bar{\rho}\right\|_{L^2}\\ \lesssim& (1+t)^{-\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)).\end{aligned} \tag{4.27}$$

In light of (4.14)–(4.15) and (4.26)–(4.27), we end up with

$$\|\nabla(F_1,F_2,N_1,N_2,\mathbb{N}_3)\|_{L^2}\lesssim (1+t)^{-\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t)). \tag{4.28}$$

The intergration by parts and (4.28) imply that

$$\begin{aligned}&\int_{\mathbb{R}^3}\nabla F_1\cdot\nabla m_{h_{\shortparallel}}dx+\int_{\mathbb{R}^3}\nabla^2 F_1\cdot\nabla^2 m_{h_{\shortparallel}}dx+\int_{\mathbb{R}^3}\nabla F_2\cdot\nabla W_{h_{\shortparallel}}dx+\int_{\mathbb{R}^3}\nabla^2 F_2\cdot\nabla^2 W_{h_{\shortparallel}}dx\\&\quad+\int_{\mathbb{R}^3}\nabla\mathbb{N}_3\cdot\nabla B_h dx+\int_{\mathbb{R}^3}\nabla^2\mathbb{N}_3\cdot\nabla^2 B_h dx\\&\lesssim\|\nabla(F_1,F_2,\mathbb{N}_3)\|_{L^2}^2+\epsilon_1\left\|\left(\nabla m_{h_{\shortparallel}},\nabla^3 m_{h_{\shortparallel}},\nabla B_h,\nabla^3 B_h,\nabla W_{h_{\shortparallel}},\nabla^3 W_{h_{\shortparallel}}\right)\right\|_{L^2}^2\\&\lesssim\epsilon_1\left\|\left(\nabla m_{h_{\shortparallel}},\nabla^3 m_{h_{\shortparallel}},\nabla B_h,\nabla^3 B_h,\nabla W_{h_{\shortparallel}},\nabla^3 W_{h_{\shortparallel}}\right)\right\|_{L^2}^2\\&\qquad+(1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^3}\nabla N_1\cdot\nabla m_{h\perp}dx+\int_{\mathbb{R}^3}\nabla^2 N_1\cdot\nabla^2 m_{h\perp}dx+\int_{\mathbb{R}^3}\nabla N_2\cdot\nabla W_{h\perp}dx+\int_{\mathbb{R}^3}\nabla^2 N_2\cdot\nabla^2 W_{h\perp}dx\\&\lesssim\epsilon_1\left\|\left(\nabla m_{h\perp},\nabla^3 m_{h\perp},\nabla W_{h\perp},\nabla^3 W_{h\perp}\right)\right\|_{L^2}^2+(1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2.\end{aligned} \tag{4.30}$$

Plugging (4.24)–(4.25) and (4.29)–(4.30) into (4.20)–(4.21), it implies

$$\frac{d}{dt}\|(\rho_h,m_{h_{\shortparallel}},W_{h_{\shortparallel}},m_{h\perp},W_{h\perp},B_h)\|_{H^2}^2+\|\nabla(m_{h_{\shortparallel}},m_{h\perp},W_{h\perp},B_h)\|_{H^2}^2+\|W_{h_{\shortparallel}}\|_{H^3}^2$$

$$\lesssim (1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2. \tag{4.31}$$

Applying ∇^l with $l=0,1$ to $(4.1)_2$ and multiplying by $\nabla^l\nabla\rho_h$, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\nabla^l m_{h_{\shortparallel}}\cdot\nabla^{l+1}\rho_h dx+\left\|\nabla^{l+1}\rho_h\right\|_{L^2}^2\\ \lesssim&\int_{\mathbb{R}^3}\nabla^l F_1\cdot\nabla^{l+1}\rho_h dx+\left\|\nabla^{l+1}m_{h_{\shortparallel}}\right\|_{L^2}^2+\left\|\nabla^{l+2}m_{h_{\shortparallel}}\right\|_{L^2}^2.\end{aligned} \tag{4.32}$$

In view of (4.16) and (4.28), one can deduce

$$\begin{aligned}&\int_{\mathbb{R}^3}F_1\cdot\nabla\rho_h dx+\int_{\mathbb{R}^3}\nabla F_1\cdot\nabla^2\rho_h dx\\ \lesssim&\|F_1\|_{L^2}^2+\|\nabla F_1\|_{L^2}^2+\epsilon_1\left\|\left(\nabla\rho_h,\nabla^2\rho_h\right)\right\|_{L^2}^2\\ \lesssim&(1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2+\epsilon_1\left\|\left(\nabla\rho_h,\nabla^2\rho_h\right)\right\|_{L^2}^2.\end{aligned} \tag{4.33}$$

Plugging (4.33) into (4.32), it gives rise to

$$\frac{d}{dt}\sum_{l=0}^{1}\int_{\mathbb{R}^3}\nabla^l m_{h_{\shortparallel}}\cdot\nabla^{l+1}\rho_h dx+\|\nabla\rho_h\|_{H^1}^2\lesssim(1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2+\|\nabla m_{h_{\shortparallel}}\|_{H^2}^2. \tag{4.34}$$

Multiplying (4.34) by η_1 for suitably small η_1 and adding it to (4.31), then there exists some positive constant a, such that

$$\frac{d}{dt}M(t)+aM(t)\lesssim\|(\rho_h,m_{h_{\shortparallel}},m_{h\perp},W_{h\perp},B_h)\|_{L^2}^2+(1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2,$$

where

$$\begin{aligned}M=&\|(\rho_h,m_{h_{\shortparallel}},W_{h_{\shortparallel}},m_{h\perp},W_{h\perp},B_h)\|_{H^2}^2+\sum_{l=0}^{1}\int_{\mathbb{R}^3}\nabla^l m_{h_{\shortparallel}}\cdot\nabla^{l+1}\rho_h dx\\ \sim&\|(\rho_h,m_{h_{\shortparallel}},W_{h_{\shortparallel}},m_{h\perp},W_{h\perp},B_h)\|_{H^2}^2.\end{aligned}$$

By Grönwall's inequality, and (4.6) of Lemma 4.1, we obtain

$$M\lesssim\int_0^t e^{-a(t-s)}(1+s)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}ds(\delta^2+N^2)^2\lesssim(1+t)^{-2\left(\frac{3}{4}+\frac{\epsilon}{2}\right)}(\delta^2+N^2(t))^2.$$

Therefore, we have proved (4.19). □

Now, we need to prove that

$$\left\|\nabla^3(\rho_h,m_h,W_h,B_h)\right\|_{L^2}\le C.$$

In fact, note that

$$(\tilde\rho,\tilde u,\tilde w,\tilde B)=\left(1+\rho,\frac{\sqrt{P'(1)}m}{1+\rho},\frac{\sqrt{P'(1)}W}{1+\rho},\sqrt{P'(1)}B\right)=(1+\rho,u,w,\sqrt{P'(1)}B).$$

By the existence Theorem 1.1, we obtain

$$\|(\rho,m,W,B)\|_{H^3}^2\lesssim\|(\rho,u,w,B)\|_{H^3}^2\lesssim\|(\rho_0,u_0,w_0,B_0)\|_{H^3}^2\lesssim\|(\rho_0,m_0,W_0,B_0)\|_{H^3}^2\lesssim\delta^2.$$

Thus

$$\|(\rho_h,m_h,W_h,B_h)\|_{H^3}\lesssim\|(\rho,m,W,B)\|_{H^3}+\left\|(\bar\rho,\bar m,\bar W,\bar B)\right\|_{H^3}$$
$$\lesssim\|(\rho_0,u_0,w_0,B_0)\|_{H^3}\lesssim\|(\rho_0,m_0,W_0,B_0)\|_{H^3}. \tag{4.35}$$

The combination of (4.6), (4.19) and (4.35) leads to

$$N(t)\lesssim\delta^2+N^2(t)+\delta,$$

which together with the smallness of $\delta>0$ leads to the estimate (4.5). From (4.17)–(4.18) and (4.5), we know

$$\|W_{h_{\shortparallel}}\|_{L^2}\lesssim\int_0^t e^{-c(t-s)}(1+s)^{-\frac32}ds(\delta^2+N^2(t))\lesssim(1+t)^{-\frac32},$$

and

$$\|W_{h\perp}\|_{L^2}\lesssim\int_0^t(1+t-s)^{-\frac74}(1+s)^{-\frac32}ds(\delta^2+N^2(t))\lesssim(1+t)^{-\frac32}.$$

Hence, we obtain the optimal decay rate for $(\rho,m,W,B)^T$ and prove (1.11)–(1.12) of Theorem 1.2.

In the following, we want to prove (1.13)–(1.14) of Theorem 1.2. Assume

$$N_1(t)=\sup_{0\le s\le t}\Big\{\|\nabla(\rho_h,m_{h_{\shortparallel}},W_{h_{\shortparallel}},m_{h\perp},W_{h\perp},B_h)\|_{H^1}(1+s)^{\frac54+\epsilon'}$$
$$+\left\|\nabla^3(\rho_h,m_{h_{\shortparallel}},W_{h_{\shortparallel}},m_{h\perp},W_{h\perp},B_h)\right\|_{L^2}\Big\}. \tag{4.36}$$

We claim that for $0<\epsilon'\le1/4$,

$$N_1(t)\le\delta. \tag{4.37}$$

By the assumption (4.36), Lemma 2.1 and Lemma 3.1, estimating in the same way as proving (4.16) in Lemma 4.1, we obtain

$$\|(F_1,F_2,N_1,N_2,\mathbb{N}_3)\|_{L^2}+\|\nabla(F_1,F_2,N_1,N_2,\mathbb{N}_3)\|_{L^2}\lesssim(1+t)^{-(\frac54+\epsilon')}(\delta^2+N_1^2(t)). \tag{4.38}$$

Like (4.7)–(4.10), we know

$$\begin{aligned}&\|\nabla(\rho_h,m_{h_{\shortparallel}},W_{h_{\shortparallel}},m_{h\perp},W_{h\perp},B_h)\|_{L^2}\\
\le&C\int_0^t(1+t-s)^{-\frac74}(\|(f_1,f_2,f_3,f,N_1,N_2)(s)\|_{L^1}+\|\nabla(F_1,F_2,N_1,N_2,\mathbb{N}_3)(s)\|_{L^2})ds\\
\le&C\int_0^t(1+t-s)^{-\frac74}(1+s)^{-(\frac54+\epsilon_1)}ds(\delta^2+N_1^2)\\
\le&C(1+t)^{-(\frac54+\epsilon_1)}(\delta^2+N_1^2).\end{aligned} \tag{4.39}$$

Taking (4.20) and (4.21) with $k=1,2$ and (4.32) with $l=1$, we obtain

$$\frac{d}{dt}\left(\|\nabla(\rho_h,m_{h_{\shortparallel}},W_{h_{\shortparallel}},m_{h\perp},W_{h\perp},B_h)\|_{H^1}^2+\epsilon_1\int_{\mathbb{R}^3}\nabla m_{h_{\shortparallel}}\cdot\nabla^2\rho_h dx\right)$$

$$
\begin{aligned}
&+\left\|\nabla^2(m_{h\shortparallel},m_{h\perp},W_{h\shortparallel},W_{h\perp},B_h)\right\|_{H^1}^2+\left\|\nabla^2\rho_h\right\|_{L^2}^2\\
\lesssim&\int_{\mathbb{R}^3}F_1\cdot\nabla^2 m_{h\shortparallel}dx+\int_{\mathbb{R}^3}\nabla F_1\cdot\nabla^3 m_{h\shortparallel}dx+\int_{\mathbb{R}^3}N_1\cdot\nabla^2 m_{h\perp}dx\\
&+\int_{\mathbb{R}^3}\nabla N_1\cdot\nabla^3 m_{h\perp}dx+\int_{\mathbb{R}^3}N_2\cdot\nabla^2 W_{h\perp}dx+\int_{\mathbb{R}^3}\nabla N_2\cdot\nabla^3 W_{h\perp}dx\\
&+\int_{\mathbb{R}^3}\mathbb{N}_3\cdot\nabla^2 B_h dx+\int_{\mathbb{R}^3}\nabla\mathbb{N}_3\cdot\nabla^3 B_h dx+\int_{\mathbb{R}^3}F_2\cdot\nabla^2 W_{h\shortparallel}dx\\
&+\int_{\mathbb{R}^3}\nabla F_2\cdot\nabla^3 W_{h\shortparallel}dx+\epsilon_1\int_{\mathbb{R}^3}\nabla F_1\cdot\nabla^2\rho_h dx\\
\lesssim&\,\epsilon\left\|(\nabla^2\rho_h,\nabla^2 m_{h\shortparallel},\nabla^3 m_{h\shortparallel},\nabla^2 m_{h\perp},\nabla^3 m_{h\perp},\nabla^2 W_{h\shortparallel},\nabla^3 W_{h\shortparallel},\nabla^2 W_{h\perp},\nabla^3 W_{h\perp})\right\|_{L^2}^2\\
&+\|(F_1,F_2,N_1,N_2,\mathbb{N}_3)\|_{L^2}^2+\|\nabla(F_1,F_2,N_1,N_2,\mathbb{N}_3)\|_{L^2}^2. \qquad (4.40)
\end{aligned}
$$

From (4.40) and (4.38), we have

$$
\begin{aligned}
\frac{d}{dt}M_1(t)+bM_1(t)&\le C(1+t)^{-2\left(\frac{5}{4}+\epsilon_1\right)}(\delta^2+N_1^2)^2+\|\nabla(\rho_h,m_{h\shortparallel},W_{h\shortparallel},m_{h\perp},W_{h\perp},B_h)\|_{L^2}^2\\
&\le C(1+t)^{-2\left(\frac{5}{4}+\epsilon_1\right)}(\delta^2+N_1^2)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
M_1(t)=&\left(\|\nabla(\rho_h,m_{h\shortparallel},W_{h\shortparallel},m_{h\perp},W_{h\perp},B_h)\|_{H^1}^2+\epsilon_1\int_{\mathbb{R}^3}\nabla m_{h\shortparallel}\cdot\nabla^2\rho_h dx\right)\\
&\sim\|\nabla(\rho_h,m_{h\shortparallel},W_{h\shortparallel},m_{h\perp},W_{h\perp},B_h)\|_{H^1}^2.
\end{aligned}
$$

By Grönwall's inequality, we have

$$
M_1(t)\le C\int_0^t e^{-b(t-s)}(1+s)^{-2\left(\frac{5}{4}+\epsilon_1\right)}(\delta^2+N_1^2)^2ds\le C(1+t)^{-2\left(\frac{5}{4}+\epsilon_1\right)}(\delta^2+N_1^2)^2.
$$

That is,

$$
\|\nabla(\rho_h,m_{h\shortparallel},W_{h\shortparallel},m_{h\perp},W_{h\perp},B_h)\|_{H^1}\le C(1+t)^{-\left(\frac{5}{4}+\epsilon_1\right)}(\delta^2+N_1^2). \qquad (4.41)
$$

It holds from (4.35) and (4.41) that

$$
N_1(t)\lesssim\delta^2+N_1^2(t)+\delta,
$$

which together with the smallness of $\delta>0$ leads to the estimates (4.37). This in turn gives

$$
\begin{aligned}
\|\nabla W_{h\perp}\|_{L^2}\le&\, C\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{9}{4}}\left(\|(f,N_2)(s)\|_{L^1}+\|\nabla(N_1,N_2)(s)\|_{L^2}\right)ds\\
&+C\int_{\frac{t}{2}}^t(1+t-s)^{-\frac{9}{4}}\left(\|(f,N_2)(s)\|_{L^1}+\|\nabla(N_1,N_2)(s)\|_{L^2}\right)ds\\
\le&\, C\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{9}{4}}(1+s)^{-\left(\frac{5}{4}+\epsilon_1\right)}ds\\
&+C\int_{\frac{t}{2}}^t(1+t-s)^{-\frac{9}{4}}\left(\|(f,N_2)(s)\|_{L^1}+\|\nabla(N_1,N_2)(s)\|_{L^2}\right)ds
\end{aligned}
$$

$$
\begin{aligned}
&\le C(1+t)^{-\frac{7}{4}-\epsilon_1}\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{2}{4}+\epsilon_1}(1+s)^{-\left(\frac{5}{4}+\epsilon_1\right)}ds\\
&\qquad +C\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{9}{4}}(\|(f,N_2)(s)\|_{L^1}+\|\nabla(N_1,N_2)(s)\|_{L^2})ds\\
&\le C(1+t)^{-\frac{7}{4}-\epsilon_1}\int_0^{\frac{t}{2}}(1+s)^{-\frac{2}{4}+\epsilon_1}(1+s)^{-\left(\frac{5}{4}+\epsilon_1\right)}ds\\
&\qquad +C\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{9}{4}}(\|(f,N_2)(s)\|_{L^1}+\|\nabla(N_1,N_2)(s)\|_{L^2})ds\\
&\le C(1+t)^{-\left(\frac{7}{4}+\epsilon_1\right)} \qquad (4.42)
\end{aligned}
$$

and

$$\|\nabla W_{h_{\text{II}}}\|_{L^2}\le C(1+t)^{-\left(\frac{7}{4}+\epsilon'\right)}, \qquad (4.43)$$

for some constant $\epsilon'>0$. Therefore, we can obtain the optimal time decay rate for $(\nabla\rho,\nabla m,\nabla B,\nabla W)^T$ and prove (1.13)–(1.14) of Theorem 1.2.

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Appendix. Analytic tools.

LEMMA 5.1. *Let $r_1>1$, $0\le r_2\le r_1$, then it holds that*

$$\int_0^t(1+t-s)^{-r_1}(1+s)^{-r_2}ds\le C(r_1,r_2)(1+t)^{-r_2} \qquad (5.1)$$

where $C(r_1,r_2)$ is defined as

$$C(r_1,r_2)=\frac{2^{r_2+1}}{r_1-1}.$$

Proof. The proof can be seen in [12]. ∎

LEMMA 5.2. *Let $l\ge 0$ be an integer, it holds*

$$\left\|\nabla^l(gh)\right\|_{L^{p_0}}\lesssim\|g\|_{L^{p_1}}\left\|\nabla^l h\right\|_{L^{p_2}}+\left\|\nabla^l g\right\|_{L^{p_3}}\|h\|_{L^{p_4}}. \qquad (5.2)$$

In the above, $p_0,p_1,p_2,p_3,p_4\in[1,+\infty]$ such that

$$\frac{1}{p_0}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Proof. The proof can be seen in [11]. ∎

REFERENCES

[1] Y. Amirat and K. Hamdache, *Weak solutions to the equations of motion for compressible magnetic fluids*, J. Math. Pures Appl., 91:433–467, 2009. 1

[2] B. Berkovski and V. Bashtovoy, *Magnetic Fluids and Applications Handbook*, Begell House, New York, 1996. 1

[3] J.L. Boldrini, M. Durán, and M.A. Rojas–Medar, *Existence and uniqueness of strong solution for the incompressible micropolar fluid equations in domains of* $\mathbb{R}^3$, Ann. Univ. Ferrara Sez. VII Sci. Mat., 56(1):37–51, 2010. 1

[4] J.L. Boldrini, M.A. Rojas-Medar, and E. Fernández-Cara, *Semi–Galerkin approximation and strong solutions to the equations of the nonhomogeneous asymmetric fluids*, J. Math. Pures Appl., 82:1499–1525, 2003. 1

[5] M.T. Chen, *Blowup criterion for viscous, compressible micropolar fluids with vacuum*, Nonlinear Anal. Real World Appl., 13(2):850–859, 2012. 1

[6] M.T. Chen, B. Huang, and J.W. Zhang, *Blowup criterion for the three–dimensional equations of compressible viscous micropolar fluids with vacuum*, Nonlinear Anal., 79:1–11, 2013. 1

[7] H.B. Cui and H.Y. Yin, *Stationary solutions to the one–dimensional micropolar fluid model in a half line: existence, stability and convergence rate*, J. Math. Anal. Appl., 449(1):464–489, 2017. 1

[8] R.J. Duan, *Global smooth flows for the compressible Euler–Maxwell system. The relaxation case*, J. Hyperbolic Diff. Eqs., 8:375–413, 2011. 1

[9] I. Dražić and N. Mujaković, *3D flow of a compressible viscous micropolar fluid with spherical symmetry: a local existence theorem*, Bound. Value Probl., 2012:69, 2012. 1

[10] I. Dražić and N. Mujaković, *3D flow of a compressible viscous micropolar fluid with spherical symmetry: a global existence theorem*, Bound. Value Probl., 2015:98, 2015. 1

[11] R.J. Duan, L.Z. Ruan, and C.J. Zhu, *Optimal decay rates to conservation laws with diffusion–type terms of regularity–gain and regularity–loss*, Math. Models Meth. Appl. Sci., 22(7):1250012, 2012. 4

[12] R.J. Duan, S. Ukai, T. Yang, and H.J. Zhao, *Optimal convergence rates for the compressible Navier–Stokes equations with potential forces*, Math. Models Meth. Appl. Sci., 17(5):737–758, 2007. 4

[13] B.Q. Dong and Z.F. Zhang, *Global regularity of the 2D micropolar fluid flows with zero angular viscosity*, J. Diff. Eqs., 249(1):200–213, 2010. 1

[14] L. Huang and C.X. Kong, *Global behavior for compressible viscous micropolar fluid with spherical symmetry*, J. Math. Anal. Appl., 443(2):1158–1178, 2016. 1

[15] L. Huang and R.X. Lian, *Exponential stability of spherically symmetric solutions for compressible viscous micropolar fluid*, J. Math. Phys., 56(7):071503, 2015. 1

[16] J. Jin and R. Duan, *Stability of rarefaction waves for 1D compressible viscous micropolar fluid model*, J. Math. Anal. Appl., 450(2):1123–1143, 2017. 1

[17] G. Lukaszewicz, *On nonstationary flows of incompressible asymmetric fluids*, Math. Meth. Appl. Sci., 13(3):219–232, 1990. 1

[18] H.L. Li, A. Matsumura, and G.J. Zhang, *Optimal decay rate of the compressible Navier–Stokes–Poisson system in* $\mathbb{R}^3$, Arch. Ration. Mech. Anal., 196(2):681–713, 2010. 1, 1, 1

[19] M. Loayza and M.A. Rojas–Medar, *A weak* L^p *Prodi–Serrin type regularity criterion for the micropolar fluid equations*, J. Math. Phys., 57(2):021512, 2016. 1

[20] Q.Q. Liu and P.X. Zhang, *Optimal time decay of the compressible micropolar fluids*, J. Diff. Eqs., 260(10):7634–7661, 2016. 1, 3.1, 3.1

[21] N. Mujaković, *Global in time estimates for one–dimensional compressible viscous micropolar fluid model*, Glas. Mat. Ser. III, 40(60):103–120, 2005. 1

[22] N. Mujaković, *Nonhomogeneous boundary value problem for one–dimensional compressible viscous micropolar fluid model: a local existence theorem*, Ann. Univ. Ferrara Sez. VII Sci. Mat., 53(2):361–379, 2007. 1

[23] N. Mujaković, *The existence of a global solution for one dimensional compressible viscous micropolar fluid with nonhomogeneous boundary conditions for temperature*, Nonlinear Anal. Real World Appl., 19:19–30, 2014. 1

[24] N. Mujaković and I. Dražić, *3D flow of a compressible viscous micropolar fluid with spherical symmetry: uniqueness of a generalized solution*, Bound. Value Probl., 2014:226, 2014. 1

[25] E.E. Ortega–Torres and M.A. Rojas–Medar, *Magneto–micropolar fluid motion: global existence of strong solutions*, Abstr. Appl. Anal., 4:109–125, 1999. 1

[26] Y.M. Qin, T.G. Wang, and G.L. Hu, *The Cauchy problem for a 1D compressible viscous micropolar fluid model: analysis of the stabilization and the regularity*, Nonlinear Anal. Real World Appl., 13(3):1010–1029, 2012. 1

[27] M.A. Rojas–Medar, *Magneto–micropolar fluid motion: existence and uniqueness of strong solution*, Math. Nachr., 188:301–319, 1997. 1

[28] M.A. Rojas–Medar and E.E. Orteg-Torres, *The equations of a viscous asymmetric fluid: An interactive approach*, Z. Angew. Math. Mech., 85(7):471–489, 2005. 1

[29] M.E. Schonbek, *Large time behavior of solutions to the Navier–Stokes equations*, Comm. Part. Diff. Eqs., 11(7):733–763, 1986. 1

[30] M.E. Schonbek, *Lower bounds of rates of decay for solutions to the Navier–Stokes equations*, J. Amer. Math. Soc., 4(3):423–449, 1991. 1

[31] J.R. Su, *Incompressible limit of a compressible micropolar fluid model with general initial data*, Nonlinear Anal., 132:1–24, 2016. 1

[32] P.B. Silva and E.G. Santos, *Global weak solutions for asymmetric incompressible fluids with variable density*, C.R. Math. Acad. Sci. Paris, 346:575–578, 2008.

[33] P.B. Silva and E.G. Santos, *Global weak solutions for variable density asymmetric incompressible fluids*, J. Math. Anal. Appl., 387:953–969, 2012. 1

[34] R.Y. Wei, B.L. Guo, and Y. Li, *Global existence and optimal convergence rates of solutions for* 3*D compressible magneto–micropolar fluid equations*, J. Diff. Eqs., 263(5):2457–2480, 2017. 1, 1, 1.2

[35] H.Y. Yin, *Stability of stationary solutions for inflow problem on the micropolar fluid model*, Z. Angew. Math. Phys., 68(2):44, 2017. 1