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AN APPLICATION OF FDM BY GIBOU TO A NUMERICAL BLOW-UP FOR NONLINEAR EVOLUTION EQUATIONS ON A DOMAIN $\Omega\subset\mathbb{R}^N$*

SOON-YEONG CHUNG† AND JEA-HYUN PARK‡

Abstract. In this paper, using a finite difference method introduced by Gibou et al., we show the blow-up phenomenon of solutions to nonlinear evolution equations with Dirichlet boundary condition on an N-dimensional smooth bounded domain $\Omega\subset\mathbb{R}^N$.

We first present bounds of the discrete smallest eigenvalue and the corresponding eigenfunction to the discrete Dirichlet eigenvalue problem with the discrete Laplacian which is obtained by Gibou's method. We also show that the numerical solution is second order accurate to the theoretical solution and there exists a blow-up time of the numerical solution by finding upper and lower bounds of the blow-up time. Finally, using the above results, we prove that the theoretical solution has a blow-up time and we also give upper and lower bounds for the blow-up time of the theoretical solution.

Keywords. Blow-up; Nonlinear evolution equations; Semidiscretization; Convergence analysis.

AMS subject classifications. 35K61; 65M06.

1. Introduction

In recent decades, there has been an explosion of interest in blow-up of solutions to various linear or non-linear evolution equations representing many physical situations, such as flows in porous media, heat transfer, or biochemical kinetics (see [13, 15, 17–20] and references therein). Among methods to research long time behaviors of solutions, numerical approximation methods have been known to be an efficient tool to analyze qualitative properties of solutions, such as the blow-up, extinction, global existence, and so on (see [1–5, 11, 12]).

The purpose of this paper is to apply the finite difference method (FDM) introduced by Gibou et al. in [14] to show the blow-up phenomenon and to give upper and lower bounds for a blow-up time of a solution $u\in C_1^4(\Omega\times[0,T])$ to nonlinear evolution equations on a multi-dimensional smooth bounded domain $\Omega\subset\mathbb{R}^N$ $(N\geq 1)$

$$\begin{cases} u_t(x,t)=\Delta u(x,t)+f(u(x,t)), & x\in\Omega,\ t>0\\ u(x,t)=0, & x\in\partial\Omega,\ t\geq 0\\ u(x,0)=u_0(x) & x\in\Omega. \end{cases} \tag{1.1}$$

where $C_1^4(\Omega\times[0,T]):=\{u:\Omega\times[0,T]\to\mathbb{R}\mid u,\ \partial_x^k u,\ u_t\in C(\Omega\times[0,T]),k=1,2,3,4\}$, the initial condition u_0 is sufficiently large and the non-homogeneous term f is convex and satisfies $\int_\epsilon^\infty \frac{dz}{f(z)}<\infty$ for some $\epsilon>0$.

In general, if the given domain is regular, such as finite intervals, squares, rectangles, cubes, and so on, then the FDM by Shortley-Weller has been mainly used, which is a basic finite difference method mainly for solving elliptic partial differential equations on regular domains. But it has been known that the method constitutes a non-symmetric linear system for irregular domains [21]. For this reason, Gibou et al. [14] introduced a simple method, as a modified version of the Shortley-Weller method, in which they

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†Department of Mathematics and Program of Integrated Biotechnology, Sogang University, Seoul 121-742, Republic of Korea (sychung@sogang.ac.kr) .

‡ Corresponding author. Department of Mathematics, Kunsan National University, Kunsan 573-701, Republic of Korea (parkjhm@kunsan.ac.kr).

gave a symmetric linear system for N-dimensional irregular domains so that numerical solutions to Poisson equations turn out to be convergent to theoretical solutions.

The novelty of this paper is to present bounds of the smallest eigenvalue and its eigenfunction of the discrete Laplacian obtained by the FDM of Gibou et al. which are key tools to show the blow-up phenomenon of the theoretical solution to (1.1) on a multi-dimensional (irregular) domain $\Omega \subset \mathbb{R}^N$ ($N \geq 1$). More precisely, in this paper, we give bounds of the blow-up time of theoretical solution which are obtained by bounds of the blow-up time of the numerical solution to the discrete equation obtained by the FDM of Gibou et al. Specially, the upper bound of the blow-up time of the numerical solution depends on the smallest eigenvalue and its eigenfunction of the discrete Laplacian. Hence the bounds of the smallest eigenvalue and its eigenfunction of the discrete Laplacian is used when we discuss boundedness of the blow-up time of the numerical solution.

Actually, in the one-dimensional case, it is well-known that the smallest eigenvalue and its eigenfunction of the discrete Laplacian converge to the smallest eigenvalue and its eigenfunction of the Laplacian. These convergences were used as key tools in [1] when proving that the blow-up time of the semidiscrete problems converges to the theoretical one. On the other hand, in the multi-dimensional case, there is no result about the convergence of the smallest eigenvalue and its eigenfunction of the discrete Laplacian obtained by the FDM of Gibou et al., and some bounds of only eigenvalues of the discrete Laplacian were presented in [21]. We note that the bound of the smallest eigenvalue of the discrete Laplacian in this paper is slightly improved than the bound in [21], and as far as the authors know, this paper is the first to present the bounds of the eigenfunction corresponding to the smallest eigenvalue.

Moreover, we also expect that using the FDM by Gibou et al., various results for the behaviors of particles in networks can be applied to study properties of solutions to evolution-type equations. In fact, we analyze the behaviors of theoretical solutions to evolution-type equations in Remark 4.2 of this paper, as applying the FDM by Gibou et al. to the results for networks ([9, 16]).

This paper is organized as follows: we first consider the case $N = 2$. For this case, in Section 3, we present some bounds of the smallest discrete eigenvalue and the corresponding eigenfunction of discrete Dirichlet eigenvalue problems with the discrete Laplacian obtained by the method in [14]. In Section 4, we still consider the case $N = 2$, and we show that the numerical solution is second order accurate to the theoretical solution. To prove this result, we use some known results by Yoon and Min in [21]. Their results are very usefully employed to discuss results for the N-dimensional case in this paper. So, their results used in this paper are included in the Appendix of this paper for self-containedness. And then, we also show the existence of a finite blow-up time of the numerical solution by finding upper and lower bounds of the blow-up time. We also prove that the theoretical solution has a blow-up time and we present upper and lower bounds for the blow-up time of the theoretical solution. The case $N \geq 3$ is dealt with in Section 5. Actually, in this case, by similar procedure as in Section 3 and Section 4, we can obtain exactly the same results as Section 3 and Section 4 except only one result in Section 3. So, we are going to simply introduce the changed result.

2. Preliminaries

We start this section with the discretization of (1.1) obtained by the method of Gibou et al. in the 2-dimensional case.

Let us consider a smooth bounded domain $\Omega \subset \mathbb{R}^2$ and a uniform grid $h\mathbb{Z}^2$ with step size $h > 0$ and denote by Ω^h the set of nodes of the grid belonging to Ω and by $\partial\Omega^h$ the set of intersection points between the boundary $\partial\Omega$ of Ω and grid lines as Figure 2.1. For convenience, by $\overline{\Omega^h}$, we denote $\Omega^h \cup \partial\Omega^h$. For $x,y \in \overline{\Omega^h}$, by $y \sim x$, we denote that the grid node y is a neighboring node of the grid node x. By $d(x,y)$, we denote the distance from a node $x \in \overline{\Omega^h}$ to its neighbor $y \in \overline{\Omega^h}$. We note that (x_i,y_i) is generally used as a notation of a grid node in numerical analysis. But in this paper, we use x instead of (x_i,y_i) for notational simplicity.

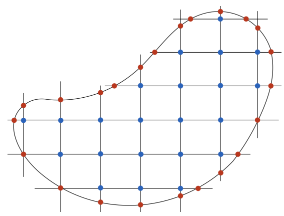

FIG. 2.1. *Blue vertices are grid nodes in* Ω^h *and red vertices are intersection points between* $\partial\Omega$ *and grid lines.*

We now consider the following discrete equation which is the discretization of (1.1) obtained by the method of Gibou et al.:

$$\begin{cases} U_t^h(x,t) = \Delta^h U^h(x,t) + f(U^h(x,t)), & x \in \Omega^h,\ t > 0, \\ U^h(x,t) = 0, & x \in \partial\Omega^h,\ t \geq 0, \\ U^h(x,0) = U_0^h(x) & x \in \Omega^h. \end{cases} \tag{2.1}$$

where $U_0^h(x) := u_0(x)$, $x \in \Omega^h$ and the discrete Laplacian operator Δ^h is defined by for each $x \in \Omega^h$,

$$\Delta^h U^h(x) := \sum_{\substack{y \in \overline{\Omega^h} \\ y \sim x}} \left[U^h(y) - U^h(x) \right] \frac{1}{hd(x,y)}.$$

We now recall the eigenvalue problem for the discrete Laplacian Δ^h which is defined by

$$\begin{cases} -\Delta^h \phi(x) = \lambda \phi(x), & x \in \Omega^h, \\ \phi(x) = 0, & x \in \partial\Omega^h. \end{cases} \tag{2.2}$$

Then it is well-known that the discrete smallest eigenvalue λ_0^h is strictly positive and there exists an eigenfunction ϕ_0^h corresponding to λ_0^h such that

(i) $\phi_0^h(x) > 0,\ x \in \Omega^h$,

(ii) $h^2 \sum_{x \in \Omega^h} \left| \phi_i^h(x) \right| = 1, \quad i = 0,1,\ldots,|\Omega^h| - 1.$

Moreover, the discrete smallest eigenvalue λ_0^h can be variationally characterized by the Rayleigh quotient as follows:

$$\lambda_0^h = \inf_{\substack{\phi \not\equiv 0 \\ \phi|_{\partial\Omega^h} \equiv 0}} \frac{\sum_{\substack{x,y \in \overline{\Omega^h} \\ y \sim x}} [\phi(x) - \phi(y)]^2 \frac{1}{hd(x,y)}}{\sum_{x \in \overline{\Omega^h}} \phi^2(x)},$$

(see [6,7] for more details). In addition, by μ_0 and Φ_0, we denote the smallest eigenvalue and its corresponding eigenfunction for the eigenvalue problem for the Laplacian Δ on the given domain Ω as follows:

$$\begin{cases} -\Delta \Phi(x) = \mu \Phi(x), & x \in \Omega, \\ \Phi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

and we assume that $\Phi_0>0$ in Ω and $\int_\Omega \Phi_0(x)dx=1$ throughout this paper. (see [10], for more properties of eigenvalues and eigenfunctions).

Finally, we introduce the comparison principle for the discrete Laplacian, which plays an important role throughout this paper. The proof of it can be done in a similar way as in [8], with some modification.

THEOREM 2.1 (Comparison Principle). *Suppose that a function f is locally Lipschitz continuous on $\mathbb{R}$. If two functions U^h and V^h on $\overline{\Omega^h}\times[0,\infty)$ satisfy*

$$\begin{cases} U_t^h-\Delta^h U^h-f(U^h)\geq V_t^h-\Delta^h V^h-f(V^h), & \text{in } \Omega^h\times(0,\infty),\\ U^h\geq V^h, & \text{on } \partial\Omega^h\times[0,\infty),\\ U^h(\cdot,0)\geq V^h(\cdot,0), & \text{in } \Omega^h, \end{cases} \tag{2.4}$$

then $U^h\geq V^h$ on $\Omega^h\times[0,\infty)$.

REMARK 2.1 (Uniqueness). *If a function f is locally Lipschitz continuous and $f(0)=0$, then by the above theorem, it is easy to see that the solution U^h to (2.1) is unique.*

3. Bounds for the discrete smallest eigenvalue and the corresponding eigenfunction

In this section, we give upper bounds for the smallest eigenvalue λ_0^h and the corresponding eigenfunction ϕ_0^h, which are very useful in the next section and seem to be interesting itself.

In the following theorems, the eigenvalue problems (2.2) and (2.3) are considered in $\Omega\subset\mathbb{R}^2$. Instead, the N dimensional case with $N\geq 3$ is going to be stated in Section 5.

THEOREM 3.1. *The discrete smallest eigenvalue λ_0^h to (2.2) satisfies that*

$$\lim_{h\to 0}\lambda_0^h\leq \mu_0, \tag{3.1}$$

where μ_0 is the smallest eigenvalue to (2.3).

Proof. By the same arguments with the proof of Theorem 3.2 in [21], for each $h>0$, we have

$$\lambda_0^h\leq \frac{\sum_{\substack{x,y\in\overline{\Omega^h}\\ y\sim x}}\left[\Phi_0^h(x)-\Phi_0^h(y)\right]^2\frac{1}{hd(x,y)}}{\sum_{x\in\overline{\Omega^h}}\left[\Phi_0^h(x)\right]^2},$$

where $\Phi_0^h(x):=\Phi_0(x)$, $x\in\overline{\Omega^h}$. Since $h\geq d(x,y)$, the right-hand side of the above inequality satisfies that for each $h>0$,

$$\frac{\sum_{\substack{x,y\in\overline{\Omega^h}\\ y\sim x}}\left[\Phi_0^h(x)-\Phi_0^h(y)\right]^2\frac{1}{hd(x,y)}}{\sum_{x\in\overline{\Omega^h}}\left[\Phi_0^h(x)\right]^2}\leq\frac{\sum_{\substack{x,y\in\overline{\Omega^h}\\ y\sim x}}\left[\frac{\Phi_0^h(x)-\Phi_0^h(y)}{d(x,y)}\right]^2hd(x,y)}{\sum\limits_{x\in\overline{\Omega^h}}\left[\Phi_0^h(x)\right]^2h\left[h/2+d(x,y)/2\right]}.$$

Hence we have

$$\lim_{h\to 0}\lambda_0^h\leq\lim_{h\to 0}\frac{\sum_{\substack{x,y\in\overline{\Omega^h}\\ x\sim y}}\left[\frac{\Phi_0^h(x)-\Phi_0^h(y)}{d(x,y)}\right]^2hd(x,y)}{\sum\limits_{x\in\overline{\Omega^h}}\left[\Phi_0^h(x)\right]^2h\left[h/2+d(x,y)/2\right]}=\frac{\int_\Omega|\nabla\Phi_0|^2}{\int_\Omega\Phi_0^2}=\mu_0.$$

∎

THEOREM 3.2. *The eigenfunction ϕ_0^h corresponding to the smallest eigenvalue λ_0^h satisfies the following:*

(i) $0 < \phi_0^h(x) \le \frac{C_\Omega \lambda_0^h}{h}, \quad x \in \Omega^h.$

(ii) $\sum_{\substack{x,y\in\overline{\Omega^h}\\ x\sim y}} \left[\phi_0^h(x) - \phi_0^h(y)\right]^2 < \frac{C_\Omega (\lambda_0^h)^2}{h},$

(iii) $\sum_{x\in\partial^\circ\Omega^h} \phi_0^h(x) \le \lambda_0^h,$

where $C_\Omega > 0$ is a constant depending on Ω and $\partial^\circ\Omega^h$ is defined by

$$\partial^\circ\Omega^h := \{x \in \Omega^h \mid y \sim x \text{ for some } y \in \partial\Omega^h\}.$$

Proof. (i) It follows from $h \ge d(x,y)$ that

$$\lambda_0^h \ge \frac{\sum_{\substack{x,y\in\overline{\Omega^h}\\ x\sim y}} \left[\phi_0^h(x) - \phi_0^h(y)\right]^2}{h^2 \sum_{x\in\overline{\Omega^h}} \left[\phi_0^h(x)\right]^2}.$$

Since $\phi_0^h(x) > 0$ for all $x \in \Omega^h$ and $h^2 \sum_{x\in\Omega^h} \phi_0^h(x) = 1$, we have

$$\sum_{\substack{x,y\in\overline{\Omega^h}\\ x\sim y}} \left[\phi_0^h(x) - \phi_0^h(y)\right]^2 \le \lambda_0^h \max_{x\in\overline{\Omega^h}} \phi_0^h(x). \tag{3.2}$$

Moreover, by the assumption $h^2 \sum_{x\in\Omega^h} \phi_0^h(x) = 1$,

$$\sum_{\substack{x,y\in\overline{\Omega^h}\\ x\sim y}} \left[\phi_0^h(x) - \phi_0^h(y)\right]^2 \le \frac{\lambda_0^h}{h^2}. \tag{3.3}$$

Let $x_0 \in \Omega^h$ satisfy $\phi_0^h(x_0) = \max_{x\in\overline{\Omega^h}} \phi_0^h(x)$. Then there exists the shortest path P_{x_0} from x_0 to the boundary $\partial\Omega^h$ where a path from $x \in \overline{\Omega^h}$ to $y \in \overline{\Omega^h}$ is a sequence of nodes $x = x_0, x_1, \ldots, x_n = y$ satisfying $x_{j-1} \sim x_j$ for $j = 1, \ldots, N$. Let $x_n \in \partial\Omega^h$ be the final node in the path P_{x_0}. Then since we choose the shortest path, it is clear that $x_j \in \Omega^h$, $j = 0, \ldots, n-1$.

Applying the triangle inequality and Hölder's inequality for the path P_{x_0}, we have

$$\phi_0^h(x_0) \le \sum_{i=1}^{n} |\phi_0^h(x_{i-1}) - \phi_0^h(x_i)| \le \sqrt{n} \left[\sum_{i=1}^{n} (\phi_0^h(x_{i-1}) - \phi_0^h(x_i))^2\right]^{\frac{1}{2}}. \tag{3.4}$$

Since the domain Ω is bounded, it is clear that there exists $C_\Omega > 0$ such that

$$n \le \frac{C_\Omega}{h}. \tag{3.5}$$

Thus it follows from (3.3), (3.4), and (3.5) that

$$\phi_0^h(x_0) \le \frac{C_\Omega^{\frac{1}{2}} (\lambda_0^h)^{\frac{1}{2}}}{h^{\frac{3}{2}}}.$$

We now apply the above inequality to (3.2). Then we obtain

$$\sum_{\substack{x,y\in\overline{\Omega^h}\\ x\sim y}} \left[\phi_0^h(x) - \phi_0^h(y)\right]^2 \le \frac{C_\Omega^{\frac{1}{2}} (\lambda_0^h)^{\frac{1}{2}}}{h^{\frac{3}{2}}}. \tag{3.6}$$

Thus it follows from (3.4), (3.5), and (3.6) that

$$\phi_0^h(x_0) \le \frac{C_\Omega^{\frac{3}{4}}(\lambda_0^h)^{\frac{3}{4}}}{h^{\frac{5}{4}}}.$$

Repeating the above argument infinitely many times, we eventually reach the desired result.

(ii) It is trivial in view of (i) and (3.2).

(iii) From the discrete eigenvalue problem (2.2), we have

$$\begin{aligned}\lambda_0^h \sum_{x\in\Omega^h} \phi_0^h(x) &= \sum_{x\in\Omega^h} \sum_{\substack{y\in\Omega^h\\ y\sim x}} [\phi_0^h(x)-\phi_0^h(y)]\frac{1}{hd(x,y)} + \sum_{x\in\partial^\circ\Omega^h} \sum_{\substack{y\in\partial\Omega^h\\ y\sim x}} [\phi_0^h(x)-\phi_0^h(y)]\frac{1}{hd(x,y)}\\ &\ge \frac{1}{h^2}\sum_{x\in\partial^\circ\Omega^h} \phi_0^h(x)\end{aligned}$$

where a set $\partial^\circ\Omega^h \subset \Omega^h$ is defined by $\partial^\circ\Omega^h := \left\{x\in\Omega^h \mid x\sim y \text{ for some y} \in \partial\Omega^h\right\}$. Thus the assumption $h^2\sum_{x\in\Omega^h}\phi_0^h(x)=1$ implies the desired result. ∎

4. Numerical blow-up for the solutions

In this section, we show that the blow-up phenomenon of the solution u to (1.1) occurs in finite time, with a lower bound and an upper bound of the blow-up time T_B of u. In order to do this, we need to first discuss the convergence property of U^h and its blow-up time.

In fact, Gibou et al. [14] considered the convergence property of discrete solution of the Poisson Equation (6.1), by showing that the solution U^h to (6.1) approximates the solution u to (6.2) with an accuracy of the second order.

We now state and prove one of the main theorems in which the discrete solution U^h to (2.1) also approximates the theoretical solution u to (1.1) with an accuracy of the second order.

THEOREM 4.1. *For a time $T>0$, let u be a function in $C_1^4(\Omega\times[0,T])$ which satisfies (1.1) up to the interval $[0,T]$. If a nonlinear source f is locally Lipschitz continuous on $\mathbb{R}$, then the solution U^h to (2.1) is well-defined on $\overline{\Omega^h}\times[0,T]$ and satisfies that*

$$\max_{\substack{x\in\Omega^h\\ t\in[0,T]}} \left|U^h(x,t)-u^h(x,t)\right| \le C_T h^2, \tag{4.1}$$

for a sufficiently small $h>0$ and a constant $C_T>0$, being independent of h, where the function u^h is defined by $u^h(x,t):=u(x,t)$ for $(x,t)\in\overline{\Omega^h}\times[0,T]$.

Proof. We first choose a time T^h as

$$T^h := \begin{cases} T_B^h-\delta, & \text{if } U^h \text{ has a blow-up time } T_B^h,\\ \infty, & \text{if not, (that is, } U^h \text{ is global),}\end{cases}$$

where $\delta>0$ is sufficiently small. Then since $U_0^h(x)=u_0^h(x)$ for all $x\in\Omega^h$, there exists the greatest time T_0^h such that

$$T_0^h \le \min\{T,T^h\} \quad \text{and} \quad \left\|U^h(\cdot,t)-u^h(\cdot,t)\right\|_{l^\infty}<1, \quad t\in[0,T_0^h) \tag{4.2}$$

where the norm $\|\cdot\|_{l^\infty}$ is defined by $\|v\|_{l^\infty} := \max_{x\in\Omega^h}|v(x)|$ for a function $v:\Omega^h\to\mathbb{R}$.

For each $t\in[0,T_0^h]$, putting $e(x,t):=U^h(x,t)-u^h(x,t)$, $x\in\overline{\Omega^h}$, then it follows that for all $x\in\overline{\Omega^h}$ and $t\in[0,T_0^h]$,

$$\begin{aligned}|e_t(x,t)-\Delta^h e(x,t)| &\le |f(U^h(x,t))-f(u^h(x,t))|+|\Delta u(x,t)-\Delta^h u^h(x,t)|\\ &\le |f(U^h(x,t))-f(u^h(x,t))|\\ &\quad +C_1h^2\mathbb{I}_{\Omega^h\setminus\partial^\circ\Omega^h}(x)+(C_2+C_3h)\mathbb{I}_{\partial^\circ\Omega^h}(x),\end{aligned}$$

for positive constants C_1, C_2, and C_3, where the last part follows from Lemma 6.1 in the Appendix and the function $\mathbb{I}_T$ is defined by

$$\mathbb{I}_T(x)=\begin{cases}1, & x\in T,\\ 0, & x\in\overline{\Omega^h}\setminus T,\end{cases}$$

for a subset $T\subset\overline{\Omega^h}$.

On the other hand, by the (locally Lipchitz) continuity, there exists $R_T>0$ and $M_T>0$ such that

$$\max_{\substack{x\in\Omega\\ t\in[0,T]}}|u(x,t)|<R_T, \tag{4.3}$$

and $|f(s)-f(t)|\le M_T|s-t|$, $s,t\in[0,R_T+1]$, respectively. Hence it follows from (4.2) that

$$U^h(x,t)\le R_T+1,\quad x\in\overline{\Omega^h},\ t\in[0,T_0^h]$$

and so that

$$|f(U^h(x,t))-f(u^h(x,t))|\le M_T|U^h(x,t)-u^h(x,t)|$$

for all $x\in\overline{\Omega^h}$ and $t\in[0,T_0^h]$. Therefore, the function e satisfies

$$|e_t(x,t)-\Delta^h e(x,t)|\le M_T|e(x,t)|+C_1h^2\mathbb{I}_{\Omega^h\setminus\partial^\circ\Omega^h}(x)+(C_2+C_3h)\mathbb{I}_{\partial^\circ\Omega^h}(x)$$

for all $x\in\overline{\Omega^h}$ and $t\in[0,T_0^h]$.

Let us consider a function $V^h:\overline{\Omega^h}\times[0,T_0^h]\to\mathbb{R}$ defined by

$$V^h(x,t):=\exp(M_Tt)\left[C_1hv^h(x)+(C_2+C_3h)w^h(x)\right],\quad x\in\overline{\Omega^h},\ t\in[0,T_0^h]$$

where w^h and v^h come from Lemma 6.2 in the Appendix. Then it is easy to see that $V^h(x,t)=0$ for each $(x,t)\in\partial\Omega^h\times[0,T_0^h]$ and $V^h(x,0)>0$ for each $x\in\Omega^h$. Moreover, a simple calculation gives that

$$\frac{dV^h}{dt}(x,t)-\Delta^hV^h(x,t)\ge M_TV^h(x,t)+C_1h^2\mathbb{I}_{\Omega^h\setminus\partial^\circ\Omega^h}(x)+(C_2+C_3h)\mathbb{I}_{\partial^\circ\Omega^h}(x)$$

for each $x\in\Omega^h$ and $t\in[0,T_0^h]$. Hence it follows from the comparison principle that

$$|e(x,t)|\le|V^h(x,t)|\le\exp(M_Tt)\left[C_1hv^h(x)+(C_2+C_3h)w^h(x)\right]$$

for $x\in\Omega^h$ and $t\in[0,T_0^h]$. By Lemma 6.2 in the Appendix, we have

$$|e(x,t)|\le C_Th^2, x\in\Omega^h,\ t\in[0,T_0^h], \tag{4.4}$$

for some constant $C_T := Ce^{M_T T}$, being independent of h.

We now show that the inequality (4.4) is still true up to the interval $[0,T]$. We first show that $T < T^h$. If U^h is global, it is clear. So let us assume that U^h has a blow-up time T_B^h. Then it suffices to show that $T < T_B^h$, since $T^h = T_B^h - \delta$ for sufficiently small $\delta > 0$. Let us suppose that $T_B^h \le T$, on the contrary. Then it follows from (4.2) and (4.4) that

$$1 = \left\| U^h(\cdot, T_0^h) - u^h(\cdot, T_0^h) \right\|_{l^\infty} \le C_T h^2,$$

which leads to a contradiction for a sufficiently small $h > 0$.

Then we see that $T_0^h \le T$, by the definition of T_0^h and it remains to show that $T_0^h = T$. But if $T_0^h < T$, then we can reach a contradiction by the same argument as the above, which is desired. □

REMARK 4.1.

(i) What is interesting in the above theorem is that the convergence property (4.1) works not only for blow-up solutions but also for global solutions, under only the condition of the local Lipchitz continuity of f.

(ii) If u is a blow-up solution with a blow-up time T_B, then the constant C_T goes to infinity as T goes to T_B. By this observation, even though U^h has a blow-up time T_B^h for all $h > 0$, we can not directly say that the blow-up time T_B^h converges to T_B, in general. This is a reason why we impose a convexity on nonlinear source f, instead of a local Lipschitz continuity in Theorem 4.4.

REMARK 4.2.

(i) In [9], authors discussed the network version for the discrete evolution equations with a nonlinear absorption term, that is, they consider $-f$ instead of f in (2.1). In particular, as applying the FDM by Gibou et al. to some results in [9], we can obtain the following result:
If the locally Lipschitz continuous function f satisfies

$$\int_0^1 \frac{1}{f(s)} ds < \infty,$$

then the solution U^h to

$$\begin{cases} U_t^h(x,t) = \Delta^h U^h(x,t) - f(U^h(x,t)), & x \in \Omega^h,\ t > 0, \\ U^h(x,t) = 0, & x \in \partial\Omega^h,\ t \ge 0, \\ U^h(x,0) = U_0^h(x) & x \in \Omega^h. \end{cases}$$

is global. Moreover, the solution U^h satisfies

$$0 \le U^h(x,t) \le F^{-1}\left(\left[F(\max_{\xi \in \Omega} u_0(\xi)) - t \right]_+ \right),\ x \in \Omega^h,\ t \ge 0,$$

where $F(s) := \int_0^s \frac{1}{f(s)} ds$, and $[s]_+ := \max\{s, 0\}$.
We note that the super-solution $F^{-1}\left([F(\max_{\xi \in \Omega} u_0(\xi)) - t]_+ \right)$ becomes extinct at $t = F(\max_{\xi \in \Omega} u_0(\xi))$. Hence the solution U^h is extinct in finite time.
Moreover, we can get a convergence estimate result for the case $-f$ as

$$\max_{\substack{x \in \Omega^h \\ t \in [0,T]}} \left| U^h(x,t) - u^h(x,t) \right| \le Ch^2, \tag{4.5}$$

using the same arguments with the proof of Theorem 4.1. In this case, since the discrete solution U^h is not blow-up, the constant C is independent of time t. Moreover, since the super-solution $F^{-1}\left([F(\max_{\xi\in\Omega} u_0(\xi))-t]_+\right)$ is independent of h, it follows from (4.5) that the continuous solution u satisfies

$$0\le u(x,t)\le F^{-1}\left(\left[F(\max_{\xi\in\Omega} u_0(\xi))-t\right]_+\right).$$

Thus the solution u is extinct in finite time (see [9], for more details).

(ii) Similar to Remark 4.2 (i), we can also consider the case $-f+g$ where f and g are locally Lipschitz continuous and strictly positive on $(0,\infty)$. In this case, since it is not guaranteed that $(f-g)(x)>0$, $x\in(0,\infty)$, it can not be discussed by the method of Remark 4.2 (i). We note that it is not easy to find sub-solution or super-solution for the case that the nonlinear source (or absorption) is not positive.

On the other hand, the special case $(f-g)(x)=\delta(x^3-x)$ was discussed in [16]. In the paper, they considered the Cucker-Smale-type (C-S-type) model with the Rayleigh friction

$$\begin{aligned}\frac{d\mathbf{x}_i}{dt}&=\mathbf{v}_i,\\ \frac{d\mathbf{v}_i}{dt}&=\frac{\lambda}{N}\sum_{i=1}^{N}\psi(\|\mathbf{x}_j-\mathbf{x}_i\|)(\mathbf{v}_j-\mathbf{v}_i)+\delta\mathbf{v}_i(1-\|\mathbf{v}_i\|^2),\end{aligned}$$

and presented flocking estimates for the C-S-type model where λ and ψ are a nonnegative coupling strength and a communication weight and $\|\cdot\|$ is the standard l^2-norm in $\mathbb{R}^d$. In particular, they assumed that $\psi(s)\ge\psi_*>0$ for some positive ψ_*. By this assumption, their arguments of some results can be applied to our case even if the C-S-type model has no boundary and the weight ψ depends on time. We note that, by the same proof of Theorem 3.1 in [16], it is proved that the solution U^h to

$$\begin{cases} U_t^h(x,t)=\Delta^h U^h(x,t)+\delta U^h(x,t)\left(1-(U^h(x,t))^2\right), & x\in\Omega^h,\ t>0,\\ U^h(x,t)=0, & x\in\partial\Omega^h,\ t\ge 0,\\ U^h(x,0)=U_0^h(x) & x\in\Omega^h.\end{cases}$$

satisfies

$$0\le U^h(x,t)\le\left((\max_{x\in\Omega} u_0(x))^{-1}e^{-\delta t}+(1-e^{-\delta t})\right)^{-\frac{1}{2}},\ x\in S,\ t>0.$$

Hence the discrete solution U^h is global. Therefore by the same arguments as in Remark 4.2 (i), the theoretical solution u for the case $(f-g)(x)=\delta(x^3-x)$ is global.

THEOREM 4.2. *Let the function f be locally Lipschitz continuous on $\mathbb{R}$ and satisfy $f(s)>0$ for all $s>0$. If the solution U^h to (2.1) has a blow-up time T_B^h, then the blow-up time T_B^h satisfies*

$$\int_{\|U_0^h\|_{l^\infty}}^{\infty}\frac{dz}{f(z)}\le T_B^h.$$

Proof. Let V be a solution to

$$\begin{cases} \frac{dV}{dt}(t) = f(V(t)), \ t>0, \\ V(0) = \left\|U_0^h\right\|_{l^\infty}. \end{cases} \tag{4.6}$$

Then it is easy to see that V is strictly increasing on $[0,\infty)$ and $V(t) > \left\|U_0^h\right\|_{l^\infty}, \quad t\in (0,\infty)$, since $V(0)>0$.

Let us define $V^h(x,t) := V(t)$ for all $x \in \overline{\Omega^h}$ and $t \geq 0$. Then it is easy to see that $V^h(x,t) > 0 = U^h(x,t)$ for all $(x,t) \in \partial\Omega^h \times [0,\infty)$, and $V^h(x,0) \geq U_0^h(x)$ for all $x \in \Omega^h$. Moreover, it is clear that

$$V_t^h(x,t) - \Delta^h V^h(x,t) - f(V^h(x,t)) = 0 = U_t^h(x,t) - \Delta^h U^h(x,t) - f(U^h(x,t))$$

for all $(x,t) \in \Omega^h \times [0,\infty)$. Thus by the comparison principle,

$$V^h(x,t) \geq U^h(x,t), \quad x \in \Omega^h, \ t>0.$$

Thus the function V^h also has a blow-up time $t_B \leq T_B^h$. Finally, by a simple calculation, the Equation (4.6) implies

$$t_B = \int_{\left\|U_0^h\right\|_{l^\infty}}^{\infty} \frac{dz}{f(z)}.$$

∎

In the next theorem, we establish sufficient conditions to guarantee that blow-up phenomenon occurs. We also give an upper bound of the blow-up time T_B^h. The upper bound is composed of two functions $Q^h : [0,\infty) \to \mathbb{R}$ and $G : [Q^h(0),\infty) \to (0, \int_{Q^h(0)}^{\infty} \frac{dz}{f(z) - \lambda_0^h z}]$ which are defined by

$$Q^h(t) := \sum_{x \in \Omega^h} h^2 \phi_0^h(x) U^h(x,t), \quad t \geq 0,$$

and

$$G(s) := \int_s^\infty \frac{dz}{f(z) - \lambda_0^h z}, \quad s \geq Q^h(0).$$

In particular, assuming that the function f is convex and satisfies

$$\int_\epsilon^\infty \frac{dz}{f(z)} < \infty \tag{4.7}$$

for some $\epsilon > 0$, it is clear that for each $C > 0$, there exists the smallest value $z_C \geq 0$ such that

$$f(z) - Cz > 0, \quad z > z_C, \tag{4.8}$$

and the value z_C non-decreasing with respect to C. In particular, we denote by z_0^h the smallest one of z_0 satisfying

$$f(z) - \lambda_0^h z > 0, \quad z > z_0.$$

as in (4.8) with $C = \lambda_0^h$. Therefore, if the function Q^h satisfies

$$Q^h(0) > z_0^h,$$

then the function G is well-defined. It is also clear that the function G is strictly decreasing and has an inverse function G^{-1} which is also strictly decreasing.

THEOREM 4.3. *For each fixed $h>0$, suppose that a function f is convex and satisfies*

$$\int_{\epsilon}^{\infty} \frac{dz}{f(z)} < \infty \tag{4.9}$$

for some ϵ. If the function Q^h satisfies

$$Q^h(0) > z_0^h, \tag{4.10}$$

then Q^h has a blow-up time T_B^h. Moreover, the blow-up time T_B^h satisfies

$$T_B^h \leq \int_{Q^h(0)}^{\infty} \frac{dz}{f(z)-\lambda_0^h z} < \infty. \tag{4.11}$$

Proof. We first show that the function Q^h satisfies

$$\frac{dQ^h}{dt}(t) \geq f(Q^h(t)) - \lambda_0^h Q^h(t), \quad t \geq 0.$$

Since U^h is the solution to (2.1), we have

$$\frac{dQ^h}{dt}(t) = \sum_{x\in\Omega^h} h^2 \phi_0^h(x)\Delta^h U^h(x,t) + \sum_{x\in\Omega^h} h^2\phi_0^h(x) f(U^h(x,t))$$

for all $t \geq 0$. Since Δ^h is symmetric and f is convex, Q^h satisfies

$$\begin{aligned}\frac{dQ^h}{dt}(t) &\geq \sum_{x\in\Omega^h} h^2 U^h(x,t)\Delta^h\phi_0^h(x) + f\left(\sum_{x\in\Omega^h} h^2\phi_0^h(x)U^h(x,t)\right)\\ &= -\lambda_0^h Q^h(t) + f(Q^h(t)), \quad t\geq 0.\end{aligned} \tag{4.12}$$

Secondly, we show that

$$Q^h(t) > Q^h(0), \quad t > 0. \tag{4.13}$$

The inequality (4.10) implies $f(Q^h(0)) - \lambda_0^h Q^h(0) > 0$. Thus there exists τ_0 such that $Q^h(t) > Q^h(0)$ for all $t \in (0,\tau_0)$. Suppose that there exists $t_0 > \tau_0$ such that $Q^h(t_0) = Q^h(0)$ and $Q^h(t) > Q^h(0)$ for all $t\in(0,t_0)$. Then at the time t_0, the function Q^h satisfies

$$0 \geq \frac{dQ^h}{dt}(t_0).$$

On the other hand, by (4.12), we have

$$\frac{dQ^h}{dt}(t_0) \geq f(Q^h(t_0)) - \lambda_0^h Q^h(t_0) = f(Q^h(0)) - \lambda_0^h Q^h(0) > 0,$$

which is a contradiction. Hence $Q^h(t) > Q^h(0)$ for all $t > 0$.

We now show that

$$\int_{Q^h(0)}^{\infty} \frac{dz}{f(z)-\lambda_0^h z} < \infty.$$

It follows from (4.10) and (4.13) that

$$f(z)-\lambda_0^h z>0, \quad z\geq Q^h(0). \tag{4.14}$$

Since the function f is convex and satisfies (4.9) and (4.14), these facts imply that there exists $z_1\geq Q^h(0)>z_0^h$ such that $f(z)-\lambda_0^h z$ is strictly increasing on $[z_1,\infty)$. Hence there exists $\epsilon_0>0$ such that

$$\frac{f(z)}{z}\geq \lambda_0^h+\epsilon_0, \quad z\geq Q^h(0).$$

Hence by a simple calculation, we have

$$\int_{Q^h(0)}^{\infty}\frac{dz}{f(z)-\lambda_0^h z}\leq \frac{\lambda_0^h+\epsilon_0}{\epsilon_0}\int_{Q^h(0)}^{\infty}\frac{dz}{f(z)}.$$

We note that there exists $x\in\Omega^h$ such that $U_0^h(x)>0$ for sufficiently small $h>0$, since $u_0\not\equiv 0$ and $u_0\geq 0$. Thus $Q^h(0)>0$ and it follows from (4.9) that

$$\int_{Q^h(0)}^{\infty}\frac{dz}{f(z)-\lambda_0^h z}<\infty.$$

Finally, we show that Q^h has a blow-up time. It follows from (4.14) that

$$1\leq \frac{\frac{dQ^h}{dt}(t)}{f(Q^h(t))-\lambda_0^h Q^h(t)},$$

which implies

$$Q^h(t)\geq G^{-1}\left(G(Q^h(0))-t\right), \quad t\geq 0.$$

Hence $Q^h(t)\to\infty$ as $t\to G(Q^h(0))$. Therefore we have

$$T_B^h\leq G(Q^h(0))=\int_{Q^h(0)}^{\infty}\frac{dz}{f(z)-\lambda_0^h z}.$$

∎

Now we are in a stage to state and prove the main theorem as a final result. In fact, we prove that the solution u to (1.1) blows up in a finite time and its blow-up time T_B can be obtained by the limit of the blow-up time T_B^h as h goes to zero.

For notational simplicity, we denote by z_μ the smallest one of z_0 satisfying

$$f(z)-(\mu_0+\delta_0)z>0, \quad z>z_0$$

as in (4.8) with $C=\mu_0+\delta_0$. Here $\delta_0>0$ is a fixed number.

THEOREM 4.4. *Suppose f is convex and satisfies*

$$\int_{\epsilon}^{\infty}\frac{dz}{f(z)}<\infty \tag{4.15}$$

for some $\epsilon>0$. If the initial condition u_0 is sufficiently large in a sense that

$$u_0(x)\geq v_0(x), \quad x\in\Omega, \tag{4.16}$$

where v_0 is a function on Ω satisfying

$$\begin{cases} -\Delta v_0 \geq \mu_0 z_\mu + \delta_0, & \textit{in } \Omega, \\ v_0 = 0, & \textit{on } \partial\Omega, \end{cases} \tag{4.17}$$

for a fixed number $\delta_0 > 0$, then there exists a blow-up time T_B of the solution u to (1.1) and the blow-up time T_B has the following upper bound:

$$\int_{\|u_0\|_\infty}^{\infty} \frac{dz}{f(z)} \leq T_B \leq \int_{z_\mu}^{\infty} \frac{dz}{f(z) - (\mu_0 + \delta_0) z}. \tag{4.18}$$

Proof.

(i) We first show that $\lim_{h\to 0} Q^h(0) > 0$. Let us define $V_0^h(x) := v_0(x)$ for all $x \in \overline{\Omega^h}$. Since $u_0(x) \geq v_0(x)$ for all $x \in \Omega$, it is easy to see that the function V_0^h satisfies

$$U_0^h(x) \geq V_0^h(x), \quad x \in \overline{\Omega^h}. \tag{4.19}$$

Hence it follows from (4.17), (4.19), and Lemma 6.1 that

$$Q^h(0) \geq \frac{h^2}{\lambda_0^h} \left[\sum_{x \in \Omega^h \setminus \partial^\circ \Omega^h} \phi_0^h(x)(\mu_0 z_\mu + \delta_0 - C_1 h^2) - \sum_{x \in \partial^\circ \Omega^h} \phi_0^h(x)(C_2 + C_3 h) \right].$$

Recalling $\lim_{h\to 0} \lambda_0^h \leq \mu_0$ in Theorem 3.1 and the inequality (iii) in Theorem 3.2, we obtain

$$0 \leq \lim_{h\to 0} h^2 \sum_{x \in \partial^\circ \Omega^h} \phi_0^h(x) \leq \lim_{h\to 0} h^2 \lambda_0^h = 0,$$

so that we have

$$\lim_{h\to 0} \frac{h^2}{\lambda_0^h} \sum_{x \in \partial^\circ \Omega^h} \phi_0^h(x)(C_2 + C_3 h) = 0.$$

On the other hand, since $h^2 \sum_{x \in \overline{\Omega^h}} \phi_0^h(x) = 1$, $h > 0$, we have

$$\lim_{h\to 0} \frac{h^2}{\lambda_0^h} \sum_{x \in \Omega^h \setminus \partial^\circ \Omega^h} \phi_0^h(x)(\mu_0 z_\mu + \delta_0 - C_1 h^2) \geq z_\mu + \frac{\delta_0}{\mu_0}.$$

Thus it follows that

$$\lim_{h\to 0} Q^h(0) \geq z_\mu + \frac{\delta_0}{\mu_0} > 0. \tag{4.20}$$

(ii) Secondly, we show that the solution U^h to (6.1) has a blow-up time T_B^h. Since f is convex and satisfies (4.15), there exists z_0^h such that

$$f(z) - \lambda_0^h z > 0, \quad z > z_0^h$$

for each $h > 0$. Moreover, we note that Theorem 3.1 implies

$$z_0^h < z_\mu + \frac{\delta_0}{\mu_0} < Q^h(0) \leq \|u_0\|_{L^\infty(\Omega)}$$

for sufficiently small $h>0$. Thus it follows from Theorem 4.2 and Theorem 4.3 that there exists a blow-up time T_B^h satisfying

$$\int_{\|u_0\|_{L^\infty(\Omega)}}^\infty \frac{dz}{f(z)} < \int_{\|U_0^h\|_{l^\infty}}^\infty \frac{dz}{f(z)} \le T_B^h < \int_{z_\mu}^\infty \frac{dz}{f(z)-(\mu_0+\delta_0)z}. \tag{4.21}$$

(iii) We now show that there exists a blow-up time T_B of the solution u. If u has no blow-up time T_B, that is, u is global, then it follows from Theorem 4.1 that U^h has no blow-up time. It is a contradiction that U^h has a blow-up time T_B^h by (4.21). Thus u has a blow-up time T_B in a sense that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty, \quad t\to T_B.$$

Moreover, if we suppose that

$$T_B > \liminf_{h\to 0} T_B^h,$$

then it is a contradiction to Theorem 4.1. Hence we have

$$T_B < \int_{z_\mu}^\infty \frac{dz}{f(z)-(\mu_0+\delta_0)z}.$$

(iv) We now show that $\lim_{h\to 0}\left\|U^h(\cdot,T_B)\right\|_{l^\infty} = \infty$. Suppose that there exists $M>0$ and a sequence $\{h_j\}$ such that $h_j\to 0$ as $j\to\infty$ and

$$\left\|U^{h_j}(\cdot,T_B)\right\|_{l^\infty} < M, \quad j=1,2,\dots.$$

Then we can take a sufficiently small $\epsilon_0>0$ such that

$$\left\|u^{h_j}(\cdot,T_B-\epsilon_0)\right\|_{l^\infty} - \left\|U^{h_j}(\cdot,T_B-\epsilon_0)\right\|_{l^\infty} \ge 1, \quad j=1,2,\dots,$$

which is a contradiction to Theorem 4.1.

(v) We now show that the blow-up time T_B satisfies

$$\int_{\|u_0\|_{L^\infty(\Omega)}}^\infty \frac{dz}{f(z)} \le T_B.$$

Putting $U^h(x^h,T_B) := \left\|U^{h_j}(\cdot,T_B)\right\|_{l^\infty}$, then we have

$$U_t^h(x^h,t) = \Delta^h U^h(x^h,t) + f(U^h(x^h,t)) \le f(U^h(x^h,t)), \quad t\in[0,T^h)$$

which implies

$$\int_{U^h(x^h,0)}^{U^h(x^h,T_B)} \frac{dz}{f(z)} \le T_B$$

for sufficiently small $h>0$. By (iv), we have

$$\lim_{h\to 0}\int_{U^h(x^h,0)}^{U^h(x^h,T_B)} \frac{dz}{f(z)} = \int_{\|u_0\|_{L^\infty(\Omega)}}^\infty \frac{dz}{f(z)}.$$

□

REMARK 4.3.

(i) In the above, it seems that the condition (4.16) and (4.17) for the initial value u_0 is not only a little bit technical but also original. In general, in order to get a blow-up solution, it is necessary for the initial condition u_0 to be sufficiently large. A good example of the initial value u_0 seems to be $u_0(x)=v(x)+w(x)$, where w is a nonnegative function in Ω with $w|_{\partial\Omega}=0$ and v is a (super-)solution to

$$\begin{cases} -\Delta v=\mu_0 z_\mu+\delta_0, & \text{in } \Omega,\\ v=0, & \text{on } \partial\Omega. \end{cases}$$

In fact, it is well-known that the solution v is smooth and positive in Ω.

(ii) In the above theorem, it is not certain whether $\lim_{h\to 0} T_B^h = T_B$. But if we knew the convergence

$$\lim_{h\to 0}\max_{x\in\Omega^h}|\Phi_0(x)-\phi_0^h(x)|=0, \tag{4.22}$$

then we would prove that $\lim_{h\to 0} T_B^h = T_B$, following a similar procedure in [1], in which they dealt with the Equation (1.1) in $\mathbb{R}^1$. In fact, the convergence (4.22) is true for a domain of special type, such as intervals in $\mathbb{R}^1$ or rectangular domains in $\mathbb{R}^N$. As a matter of fact, as far as the authors know, the convergence (4.22) is still open for the domain in $\mathbb{R}^N$.

5. Multi-dimensional space

We now discuss $\Omega\subset\mathbb{R}^N$ where $N\geq 3$. In this case, the discrete eigenfunction ϕ_0^h is assumed to be

$$\sum_{x\in\Omega^h} h^N\phi_0^h(x)=1 \tag{5.1}$$

and the function Q^h is given by

$$Q^h(x,t):=\sum_{x\in\Omega^h} h^N\phi_0^h(x)U^h(x,t),\quad t\geq 0.$$

Then Theorem 3.2 can be stated as follows:

THEOREM 5.1. *For each $h>0$, the eigenfunction ϕ_0^h corresponding to the smallest eigenvalue λ_0^h satisfies the following:*

(i) $0<\phi_0^h(x)\leq \frac{R_2\lambda_0^h}{h^{N-1}},\quad x\in\Omega^h.$

(ii) $\sum_{\substack{x,y\in\overline{\Omega^h}\\ x\sim y}}\left[\phi_0^h(x)-\phi_0^h(y)\right]^2<\frac{R_2(\lambda_0^h)^2}{h^{2N-3}},$

(iii) $\sum_{x\in\partial^\circ\Omega^h}\phi_0^h(x)\leq\frac{\lambda_0^h}{h}.$

Proof. It is proved by the same argument as in the proof of Theorem 3.2. ∎

Then the same procedure as in the previous Section 3 and Section 4, with a little changes in calculation and notation, can be applied to the case in $\mathbb{R}^N$ so that we can obtain exactly the same result as Theorem 4.4.

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Appendix. In this appendix, we introduce theoretical analysis of the consistency and convergence accuracy of the method by Gibou et al. which had been discussed in [21] by Yoon and Min. They showed that the discrete solution U^h to the discrete Poisson equation

$$\begin{cases} -\Delta^h U^h(x) = f(x), & x \in \Omega^h, \\ U^h = 0, & x \in \partial\Omega^h, \end{cases} \tag{6.1}$$

is second order accurate to the theoretical solution u to the Poisson equation

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{6.2}$$

Here, we only introduce some results related to the consistency and convergence accuracy which are only needed to prove results in Section 3 and Section 4 (see [21], for other results and their proofs).

LEMMA 6.1 (Consistency error). *For a function $u \in C^4(\Omega)$,*

$$\left|\Delta u(x) - \Delta^h u^h(x)\right| \le \begin{cases} C_1 h^2, & x \in \Omega^h \setminus \partial^\circ\Omega^h, \\ C_2 + C_3 h, & x \in \partial^\circ\Omega^h, \end{cases} \tag{6.3}$$

where the function u^h is defined by $u^h(x) := u(x)$, $x \in \overline{\Omega^h}$, C_1, C_2, and C_3 are constants independent of h.

LEMMA 6.2.

(i) *Let w^h be a solution to*

$$-\Delta^h w^h(x) = \begin{cases} 0, & x \in \Omega^h \setminus \partial^\circ\Omega^h, \\ 1, & x \in \partial^\circ\Omega^h, \end{cases}$$

and $w^h(x) = 0$ for all $x \in \partial\Omega^h$. Then $0 < w^h(x) \le h^2$ for all $x \in S^h$.

(ii) *Let v^h be a solution to*

$$-\Delta^h v^h(x) = \begin{cases} 1, & x \in \Omega^h \setminus \partial^\circ\Omega^h, \\ 0, & x \in \partial^\circ\Omega^h, \end{cases}$$

and $v^h(x) = 0$ for all $x \in \partial\Omega^h$. Then $0 < v^h(x) \le C_{v_0}$ for all $x \in S^h$, where v_0 is an analytic solution to

$$-\Delta v_0(x) = \begin{cases} 2, & x \in \Omega, \\ 0, & x \in \partial\Omega, \end{cases}$$

and $C_{v_0} := \max_{x\in\Omega} v_0(x) + 1$.

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