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A NOTE ON THE SOLUTION TO THE MOVING CONTACT LINE PROBLEM WITH THE NO-SLIP BOUNDARY CONDITION*

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Abstract. When two immiscible fluids flow on a solid substrate, a moving contact line forms at the location where the fluid-fluid interface meets the solid surface. Under the no-slip boundary condition, the velocity field is necessarily multi-valued at the moving contact line. In this paper we show that the Stokes equation with the no-slip boundary condition does not admit such multi-valued solution when the fluid-fluid interface is assumed to be flat near the moving contact line.

Keywords. Stokes flow; no-slip boundary condition; the moving contact line problem.

AMS subject classifications. 76D05; 35Q30; 30E25.

1. Introduction

The moving contact line problem is a classical problem in fluid mechanics. It refers to the motion of a contact line, i.e. the intersection of a fluid-fluid interface with a solid substrate, relative to the solid wall, such as what occurs in the spreading dynamics of a droplet. Traditional hydrodynamics imposes the requirement that the velocity of the fluids in contact with the solid substrate must be equal to the velocity of the substrate, i.e., the no-slip condition. However, this condition is obviously violated at the moving contact line. In order to resolve this difficulty, much effort has been devoted to analyzing the contact line dynamics and to developing alternative models. For samples of these work, we refer to the research articles [3–6, 9–12, 16, 18–21, 24, 25] and references therein. We also refer to the monographs and review articles [2, 13, 14, 17, 22] as well as the collected volume edited by Velarde [23], for the latest development for this problem.

In one of the earlier works [11], Huh and Scriven proposed a solution to the steady Stokes flow for a moving contact line. It was assumed that the velocity field was governed by the Stokes equation with the no-slip boundary condition at the solid wall and the usual stress conditions at the fluid interface. Furthermore, the fluid interface was assumed to be planar. Under these conditions, the authors looked for a solution for the stream function of the form

$$\psi(r,\theta)=r\left(A\sin\theta+B\cos\theta+\theta\left(C\sin\theta+D\cos\theta\right)\right),$$

where r and θ are the polar coordinates, $r=0$ is at the location of the contact line, and $\theta=0,\pi$ correspond to the solid surface on the two sides of the contact line, respectively. The coefficients A, B, C and D were then determined by the conditions imposed on the fluid interface and the solid wall. The solution they found corresponds to a velocity field with multiple values at $r=0$, which leads to a singular viscous stress with a divergence rate of $1/r$ at the moving contact line.

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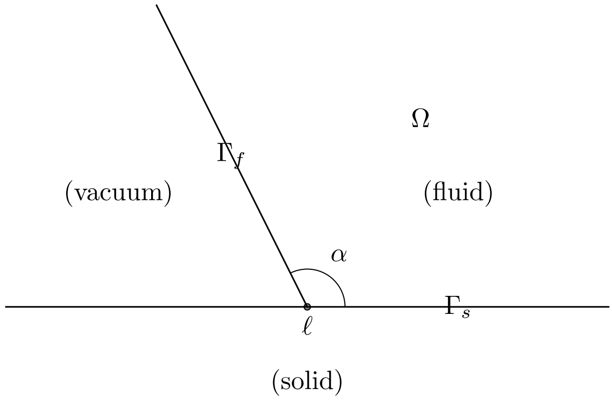

FIG. 2.1. *The geometry of the vacuum-fluid-solid system near the moving contact line.*

While the solution obtained in Ref. [11] satisfies all the conditions imposed on the fluid interface and the solid wall as $r \to \infty$, it fails to satisfy the balance of the stress condition at the fluid interface near the contact line. Therefore, what Huh and Scriven obtained is, in a rigorous sense, *not* a solution to their setup. In fact, even the existence of solutions to this problem remains unclear.

The purpose of this paper is to show that, under the setup in Ref. [11], there is no solution to the Stokes equation with the no-slip boundary condition and the usual conditions on the planar interface. This includes solutions which are possibly multi-valued at the contact line. The precise statement is given in Theorem 2.1. For simplicity, we will focus on a vacuum-fluid-solid system.

The paper is organized as follows. In Section 2 we describe the mathematical model and introduce the main result (see Theorem 2.1). In Section 3, we rewrite all the real-valued variables in the model into their respective complex form (see Lemma 4.1). The main result is proved in Section 4 using techniques from complex analysis. The paper is concluded in Section 5.

2. Mathematical model and the main result

Consider a vacuum-fluid-solid system in two dimensions. We focus on the velocity field near the moving contact line, where the vacuum-fluid interface is assumed to be flat. This geometry is the same as that in the pioneering work of Huh and Scriven [11]. The geometry is shown in Figure 2.1, where ℓ denotes the moving contact line, Γ_s denotes the fluid-solid interface with $\Gamma_s \cap \ell = \emptyset$, Γ_f denotes the vacuum-fluid interface with $\Gamma_f \cap \ell = \emptyset$, and Ω is the interior of the fluid with $\Omega \cap (\ell \cup \Gamma_s \cup \Gamma_f) = \emptyset$. In addition, $\alpha \in (0,\pi)$ is the contact angle formed between Γ_s and Γ_f.

We choose the Cartesian coordinate with the origin being fixed at the contact line ℓ and the x_1-axis coinciding with the solid surface Γ_s. We consider the steady-state problem where the solid substrate moves horizontally with a constant speed and the vacuum-fluid interface remains still in space. With these settings, the dimensionless Stokes equations in Ω read

$$\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) v_1 = \frac{\partial p}{\partial x_1}, \tag{2.1}$$

$$\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)v_2=\frac{\partial p}{\partial x_2}, \tag{2.2}$$

with the incompressibility condition

$$\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}=0, \tag{2.3}$$

where (x_1,x_2) denotes the Cartesian coordinate, (v_1,v_2) is the velocity field of the fluid, and p is its pressure.

The above governing equations are supplemented with the following boundary and interface conditions. The no-slip condition for the fluid on Γ_s reads

$$v_1=1. \tag{2.4}$$

The kinematic and hydrodynamic constraints on v_1, v_2 and p are as follows. The impenetrability of the fluid on its boundary gives

$$v_2=0 \qquad \text{on } \Gamma_s, \tag{2.5}$$

$$v_2\cos\alpha-v_1\sin\alpha=0 \qquad \text{on } \Gamma_f. \tag{2.6}$$

The balance of pressure and viscous stress across Γ_f can be expressed in the matrix form

$$\mathbb{T}\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix}=0, \tag{2.7}$$

where the total stress tensor

$$\mathbb{T}=\begin{pmatrix}\dfrac{\partial v_1}{\partial x_1}-p & \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_2}+\dfrac{\partial v_2}{\partial x_1}\right)\\ \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x_2}+\dfrac{\partial v_2}{\partial x_1}\right) & \dfrac{\partial v_2}{\partial x_2}-p\end{pmatrix}.$$

With these settings and notations, let us state the target of this paper. We are to show

THEOREM 2.1. *Under the assumption that the fluid interface is planar, the Stokes Equations (2.1)–(2.3) admit no solution* (v_1,v_2,p) *with*

$$v_1,v_2\in C^2(\Omega)\cap C(\Omega\cup\Gamma_s)\cap C^1(\Omega\cup\Gamma_f), \tag{2.8}$$

$$p\in C^1(\Omega)\cap C(\Omega\cup\Gamma_s\cup\Gamma_f) \tag{2.9}$$

that satisfies the interface and boundary conditions in Equations (2.4)–(2.7) simultaneously.

Since the velocity field may assume multiple values at the contact line due to the no-slip boundary condition [4], in Equations (2.8) and (2.9) we place no constraint on v_1, v_2 and p at the contact line—they are allowed to diverge or to be multi-valued at this location.

3. Complex form of governing equations and boundary conditions

The proof of Theorem 2.1 makes use of complex analysis. We begin with a classical result on the representation of the solutions to Stokes equations in terms of holomorphic functions. Then we convert the boundary conditions to appropriate conditions on these holomorphic functions.

For $v_1, v_2 \in C^2(\Omega)$ and $p \in C^1(\Omega)$, Stokes Equations (2.1)–(2.3) can be rewritten as

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial v}{\partial z} - p\right) = 0, \tag{3.1}$$

$$\operatorname{Re}\left(\frac{\partial v}{\partial z}\right) = 0, \tag{3.2}$$

where z and v are complex-valued variables defined as

$$z = x_1 + i x_2,$$
$$v = v_1 + i v_2.$$

In the following classical result [7] (see also Ref. [8, pp. 160]), the solutions to the above equations are represented using holomorphic functions, where $\mathcal{O}(U)$ denotes the collection of all holomorphic functions defined on $U \subseteq \mathbb{C}$.

LEMMA 3.1 (Goursat). *There exist complex-valued functions $v \in C^2(\Omega)$ and $p \in C^1(\Omega)$ that satisfy Equations (3.1) and (3.2) if and only if there exist holomorphic functions $g, h \in \mathcal{O}(\Omega)$ such that*

$$v(z,\bar{z}) = g(z) - z\overline{g'(z)} + \overline{h(z)}, \tag{3.3}$$

$$p(z,\bar{z}) = -2\operatorname{Re}(g'(z)) \tag{3.4}$$

hold for all $z \in \Omega$.

Thanks to these representations, the regularity requirements (2.8) and (2.9) are equivalent to

$$g \in \mathcal{O}(\Omega) \cap C^1(\Omega \cup \Gamma_s) \cap C^2(\Omega \cup \Gamma_f), \tag{3.5}$$

$$h \in \mathcal{O}(\Omega) \cap C(\Omega \cup \Gamma_s) \cap C^1(\Omega \cup \Gamma_f). \tag{3.6}$$

Next, we convert the boundary conditions (2.4)–(2.7) into their respective complex forms in terms of z, v and p. These conditions could be rewritten as

$$\operatorname{Re}(v) = 1 \qquad \text{on } \Gamma_s, \tag{3.7}$$

$$\operatorname{Im}(v) = 0 \qquad \text{on } \Gamma_s, \tag{3.8}$$

$$\operatorname{Im}\left(e^{-i\alpha} v\right) = 0 \qquad \text{on } \Gamma_f, \tag{3.9}$$

$$\operatorname{Re}\left(\frac{\partial v}{\partial z} - p\right) = e^{-2i\alpha}\frac{\partial v}{\partial \bar{z}} \qquad \text{on } \Gamma_f. \tag{3.10}$$

In particular, Equation (3.10) is equivalent to

$$\operatorname{Re}\left(\frac{\partial v}{\partial z} - e^{-2i\alpha}\frac{\partial v}{\partial \bar{z}} - p\right) = 0 \qquad \text{on } \Gamma_f, \tag{3.11}$$

$$\operatorname{Im}\left(e^{-2i\alpha}\frac{\partial v}{\partial \bar{z}}\right) = 0 \qquad \text{on } \Gamma_f. \tag{3.12}$$

Now, we rewrite the above equivalent conditions (3.7)–(3.9), (3.11) and (3.12) in terms of the holomorphic functions g and h by using the representations (3.3) and (3.4). For one thing, we have $z=\bar{z}$ on Γ_s. Thus at this boundary,

$$\mathrm{Re}(v)=\mathrm{Re}\left(g-z\overline{g'}+\bar{h}\right)=\mathrm{Re}(g-zg'+h),$$

where we have also used Equations (3.3) and (3.5). As a result, Equation (3.7) can be expressed as

$$\mathrm{Re}(g-zg'+h)=1 \qquad \text{on } \Gamma_s. \tag{3.13}$$

Similarly, Equation (3.8) can be converted into

$$\mathrm{Im}(g+zg'-h)=0 \qquad \text{on } \Gamma_s. \tag{3.14}$$

On the interface Γ_f, we have $z=e^{i\alpha}|z|$. It follows that

$$\bar{z}=e^{-i\alpha}|z|=e^{-2i\alpha}\left(e^{i\alpha}|z|\right)=e^{-2i\alpha}z.$$

Thanks to this relation, Equations (3.9), (3.11) and (3.12) can be rewritten in a similar way as

$$\mathrm{Im}\left(e^{-i\alpha}(g+zg')-e^{i\alpha}h\right)=0 \qquad \text{on } \Gamma_f, \tag{3.15}$$

$$\mathrm{Re}\left(2g'+zg''-e^{2i\alpha}h'\right)=0 \qquad \text{on } \Gamma_f, \tag{3.16}$$

$$\mathrm{Im}\left(zg''-e^{2i\alpha}h'\right)=0 \qquad \text{on } \Gamma_f. \tag{3.17}$$

Equations (3.13)–(3.17) are the conditions that g and h need to satisfy on Γ_s and Γ_f, respectively.

4. Proof of the main result

In this section, we show that holomorphic functions g and h satisfying the above conditions on Γ_s and Γ_f do not exist, as is stated in the following lemma.

Lemma 4.1. *There do not exist holomorphic functions g and h, with regularities (3.5) and (3.6), that satisfy Equations (3.13)–(3.17) simultaneously.*

The proof of this lemma proceeds in three steps. First, we derive a relation between g and h by using Equations (3.15) and (3.16) with the help of the Schwarz reflection principle [1, pp. 172] and the identity theorem [1, pp. 127]. Then we obtain the general solution to g by using Equations (3.13) and (3.14), together with, again, the Schwarz reflection principle and the identity theorem. Lastly, we show that, as per all its possible solutions, g violates the condition (3.17).

Step 1. Relation between h and g. To begin with, let $q_1=e^{-i\alpha}(g+zg')-e^{i\alpha}h$. Note that (i) $q_1\in\mathcal{O}(\Omega)\cap C(\Omega\cup\Gamma_f)$ by Equations (3.5) and (3.6), (ii) $\mathrm{Im}(q_1)=0$ on Γ_f by Equation (3.15), and (iii) Γ_f is part of a straight line by the assumption upon the geometry. Thanks to these properties, the Schwarz reflection principle applies. It follows that q_1 can be analytically extended across Γ_f, such that

$$q_1=e^{-i\alpha}(g+zg')-e^{i\alpha}h\in\mathcal{O}(U_f), \tag{4.1}$$

where $U_f=\{z\in\mathbb{C}:|z|>0\wedge\arg z\in(0,2\alpha)\}$.

Next, let $z(s) = e^{i\alpha}s$ be the arc-length parametrization of Γ_f. Taking the derivative of Equation (3.15) with respect to s yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{Im}\left(e^{-i\alpha}(g+zg') - e^{i\alpha}h\right)_{z=e^{i\alpha}s}\right) = 0,$$

from which, as well as property (4.1), we have

$$\mathrm{Im}\left(\frac{\partial}{\partial z}\left(e^{-i\alpha}(g+zg') - e^{i\alpha}h\right)_{z=e^{i\alpha}s}\frac{\mathrm{d}z}{\mathrm{d}s}\bigg|_{z=e^{i\alpha}s}\right) = 0,$$

i.e.,

$$\mathrm{Im}\left(2g' + zg'' - e^{2i\alpha}h'\right)_{z=e^{i\alpha}s} = 0.$$

In other words,

$$\mathrm{Im}\left(2g' + zg'' - e^{2i\alpha}h'\right) = 0 \qquad \text{on } \Gamma_f. \tag{4.2}$$

Combined with Equation (3.16), Equation (4.2) implies that

$$2g' + zg'' - e^{2i\alpha}h' = 0 \qquad \text{on } \Gamma_f. \tag{4.3}$$

Thereafter, note that

$$2g' + g'' - e^{2i\alpha}h' = \left(g + zg' - e^{2i\alpha}h\right)' = \left(e^{i\alpha}q_1\right)'. \tag{4.4}$$

It follows from property (4.1) that

$$2g' + zg'' - e^{2i\alpha}h' \in \mathcal{O}(U_f).$$

Since we also have $\Gamma_f \subseteq U_f$, the identity theorem applies: Equation (4.3) holds not only on Γ_f, but in the whole of U_f, i.e.,

$$2g' + zg'' - e^{2i\alpha}h' = 0 \qquad \text{in } U_f. \tag{4.5}$$

This result, together with Equation (4.4), implies that

$$g + zg' - e^{2i\alpha}h = C_1 \qquad \text{in } U_f$$

for some $C_1 \in \mathbb{C}$. With the help of Equation (3.15), it is straightforward to see that $C_1 = K_1 e^{i\alpha}$ for some $K_1 \in \mathbb{R}$. As a result, the last equation from above becomes

$$g + zg' - e^{2i\alpha}h = K_1 e^{i\alpha} \qquad \text{in } U_f,$$

or, taking the continuity of g, g' and h in $\Omega \cup \Gamma_s$ into consideration,

$$g + zg' - e^{2i\alpha}h = K_1 e^{i\alpha} \qquad \text{in } U_f \cup \Gamma_s.$$

Thus,

$$h = e^{-2i\alpha}(g + zg') - K_1 e^{-i\alpha} \qquad \text{in } U_f \cup \Gamma_s. \tag{4.6}$$

Equation (4.6) reveals the relation between h and g. This completes Step 1.

Step 2. The general solution to g. To begin with, substitute Equation (4.6) into the boundary conditions (3.13) and (3.14), and we have, on Γ_s,

$$\begin{aligned}1 &= \operatorname{Re}\left(\left(1+e^{-2i\alpha}\right)g-\left(1-e^{-2i\alpha}\right)zg'-K_1e^{-i\alpha}\right)\\ &=(1+\cos 2\alpha)\operatorname{Re}(g)+\sin 2\alpha\operatorname{Im}(g)-(1-\cos 2\alpha)\operatorname{Re}(zg')+\sin 2\alpha\operatorname{Im}(zg')-K_1\cos\alpha\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}0 &= \operatorname{Im}\left(\left(1-e^{-2i\alpha}\right)(g+zg')+K_1e^{-i\alpha}\right)\\ &=(1-\cos 2\alpha)\operatorname{Im}(g+zg')+\sin 2\alpha\operatorname{Re}(g+zg')-K_1\sin\alpha,\end{aligned} \tag{4.8}$$

respectively.

Next, let $q_2=\left(1-e^{-2i\alpha}\right)(g+zg')+K_1e^{-i\alpha}$. Note that (i) $q_2\in\mathcal{O}(\Omega)\cap C(\Omega\cup\Gamma_s)$ by Equations (3.5) and (3.6), (ii) $\operatorname{Im}(q_2)=0$ on Γ_s by Equation (4.8), and (iii) Γ_s is part of a straight line by the assumption upon the geometry. Thanks to these properties, the Schwarz reflection principle applies. It follows that q_2 can be analytically extended across Γ_s, such that

$$q_2=\left(1-e^{-2i\alpha}\right)(g+zg')+Ke^{-i\alpha}\in\mathcal{O}(U_w),$$

where $U_w=\{z\in\mathbb{C}:|z|>0\wedge\arg z\in(-\alpha,\alpha)\}$. This result obviously implies that

$$(zg)'=g+zg'\in\mathcal{O}(U_w),$$

which, furthermore, implies that g itself can be analytically extended to U_w, i.e.,

$$g\in\mathcal{O}(U_w). \tag{4.9}$$

Thereafter, let $z=s$ be the arc-length parametrization of Γ_s. It follows from Equation (4.9) that

$$\operatorname{Re}(zg')_{z=s}=s\operatorname{Re}(g')_{z=s}=s\operatorname{Re}\left(\frac{\mathrm{d}}{\mathrm{d}s}g|_{z=s}\right)=s\frac{\mathrm{d}}{\mathrm{d}s}\operatorname{Re}(g)_{z=s},$$

and similarly,

$$\operatorname{Im}(zg')_{z=s}=s\frac{\mathrm{d}}{\mathrm{d}s}\operatorname{Im}(g)_{z=s}.$$

With these results, Equations (4.7) and (4.8) become

$$s\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\operatorname{Re}(g)\\ \operatorname{Im}(g)\end{pmatrix}_{z=s}+\begin{pmatrix}0 & 0\\ \cot\alpha & 1\end{pmatrix}\begin{pmatrix}\operatorname{Re}(g)\\ \operatorname{Im}(g)\end{pmatrix}_{z=s}=\begin{pmatrix}-\dfrac{1}{2}\\ \dfrac{K_1+\cos\alpha}{2\sin\alpha}\end{pmatrix}, \tag{4.10}$$

whose general solution reads

$$\begin{aligned}\operatorname{Re}(g)_{z=s}&=-\frac{1}{2}\log s+K_2,\\ \operatorname{Im}(g)_{z=s}&=\frac{K_3}{s}+\frac{\cot\alpha}{2}\log s+\frac{K_1-2K_2\cos\alpha}{2\sin\alpha},\end{aligned}$$

for some $K_2,K_3\in\mathbb{R}$. Therefore, all the possible solutions to g must follow

$$g=i\left(\frac{K_3}{z}+\frac{e^{i\alpha}}{2\sin\alpha}\log z+\frac{K_1-2K_2e^{i\alpha}}{2\sin\alpha}\right)\qquad\text{on }\Gamma_s. \tag{4.11}$$

Lastly, thanks to Equation (4.9), as well as the fact that $\Gamma_s \subseteq U_w$, the identity theorem applies: Equation (4.11) holds not only on Γ_s, but in the whole of U_w, i.e.,

$$g = i\left(\frac{K_3}{z} + \frac{e^{i\alpha}}{2\sin\alpha}\log z + \frac{K_1 - 2K_2 e^{i\alpha}}{2\sin\alpha}\right) \qquad \text{in } U_w,$$

or, taking the continuity of g in $\Omega \cup \Gamma_f$ into consideration,

$$g = i\left(\frac{K_3}{z} + \frac{e^{i\alpha}}{2\sin\alpha}\log z + \frac{K_1 - 2K_2 e^{i\alpha}}{2\sin\alpha}\right) \qquad \text{in } U_w \cup \Gamma_f. \tag{4.12}$$

Equation (4.12) depicts the general solution to g. This completes Step 2.

Step 3. Violation of condition (3.17). The rest of this proof is straightforward: as Equation (4.12) gives all the possible solutions to g in $U_w \cup \Gamma_f$, it suffices to show that none of them comply with Equation (3.17), the only boundary condition that has not yet been considered.

Combining Equations (3.17) and (4.2) yields the following necessary condition

$$\operatorname{Im}(g') = 0 \qquad \text{on } \Gamma_f.$$

Let $z = e^{i\alpha}s$ be the arc-length parametrization of Γ_f. It follows from Equation (4.12), together with property (3.5), that

$$\begin{aligned}
\operatorname{Im}(g')_{z=e^{i\alpha}s} &= \operatorname{Im}\left(i\left(\frac{K_3}{z} + \frac{e^{i\alpha}}{2\sin\alpha}\log z + \frac{K_1 - 2K_2 e^{i\alpha}}{2\sin\alpha}\right)'\right)_{z=e^{i\alpha}s} \\
&= \operatorname{Im}\left(i\left(-\frac{K_3}{z^2} + \frac{e^{i\alpha}}{2\sin\alpha}\frac{1}{z}\right)\right)_{z=e^{i\alpha}s} \\
&= \frac{1}{s}\left(\frac{1}{2\sin\alpha} - \frac{K_3\cos 2\alpha}{s}\right).
\end{aligned}$$

It is now clear that, due to the arbitrariness of s, $\operatorname{Im}(g')_{z=e^{i\alpha}s} = 0$ shall never be possible no matter what value K_3 takes. In other words, there does not exist holomorphic functions g and h, with properties (3.5) and (3.6), that satisfy Equations (3.13)–(3.17) simultaneously. This completes Step 3, and also the proof of Lemma 4.1.

Theorem 2.1 follows from Lemmas 3.1 and 4.1 immediately.

5. Conclusions

Using techniques in complex analysis, in particular the Schwarz reflection principle and the identity theorem, we proved the non-existence of classical solutions to the two-dimensional steady Stokes flow, with a flat fluid interface, that satisfy the no-slip and impenetrability conditions at the solid wall, and the stress conditions at the fluid interface simultaneously.

This result generalizes the work in Ref. [11] by considering a more general space of classical solutions (2.8) and (2.9), with multi-values of the solution being allowed at the moving contact line. This space is more general than the one depicted by the Michell solution [15].

While we have focused on a vacuum-fluid-solid system in this paper, our method and result also apply to general fluid-fluid-solid systems considered in Ref. [11]. For those systems, the proof remains almost the same if the two fluids have the same viscosity, but becomes rather involved when their viscosities differ. Whatever the setup, (multi-valued) classical solutions do not exist if the interface is assumed to be flat. Therefore, if there does exist a (multi-valued) classical solution consistent with the no-slip condition, the interface must be curved. Yet this existence still remains an open problem.

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