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SHARP INTERFACE LIMIT OF A DIFFUSE INTERFACE MODEL FOR TUMOR-GROWTH*

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Abstract. We consider the asymptotic limit of a diffuse interface model for tumor-growth when a parameter ε proportional to the thickness of the diffuse interface goes to zero. An approximate solution which shows explicitly the behavior of the true solution for small ε will be constructed by using the matched expansion method. Based on the energy method, and a spectral condition in particular, we establish a smallness estimate of the difference between the approximate solution and the true solution.

Keywords. sharp interface limit; spectral condition; matched asymptotic expansion.

AMS subject classifications. 35Q30; 76D03.

1. Introduction

In this paper we consider the singular limit, as $\varepsilon\to 0$, of the solutions of the following system for $(u^\varepsilon,\sigma^\varepsilon)$:

$$\begin{cases} u_t^\varepsilon-\Delta\mu^\varepsilon=2\sigma^\varepsilon+u^\varepsilon-\mu^\varepsilon, & \text{in } \Omega\times(0,T),\\ \sigma_t^\varepsilon-\Delta\sigma^\varepsilon=-(2\sigma^\varepsilon+u^\varepsilon-\mu^\varepsilon), & \text{in } \Omega\times(0,T),\\ \varepsilon\mu^\varepsilon=-\varepsilon^2\Delta u^\varepsilon+f'(u^\varepsilon), & \text{in } \Omega\times(0,T),\\ u^\varepsilon(x,0)=u_0^\varepsilon(x),\quad \sigma^\varepsilon(x,0)=\sigma_0^\varepsilon(x), & \text{on } \Omega\times\{0\},\\ \frac{\partial u^\varepsilon}{\partial \mathbf{n}}=\frac{\partial \mu^\varepsilon}{\partial \mathbf{n}}=\frac{\partial \sigma^\varepsilon}{\partial \mathbf{n}}=0, & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{1.1}$$

Here $\Omega\subset\mathbb{R}^N$ is a bounded smooth domain, $\mathbf{n}$ is the unit outer normal to $\partial\Omega$, ε^2 is the diffusivity corresponding to the surface energy, u^ε is the tumor cell concentration, μ^ε is the chemical potential, σ^ε is the nutrient concentration, and f is a double equal-well potential taking its global minimum value 0 at $u=\pm1$. Without loss of generality we take $f(u)=(u^2-1)^2$.

The morphological evolution of tumor progression has been an area of intense research interest recently (see, for instance [5, 9, 15, 16, 20, 22, 25–30] and the references therein). System (1.1) is introduced to study the evolution of a growing solid tumor which coexists with the host tissue. The dynamics can be divided into two stages. During the first stage, two species are segregated according to the initial data and an interface appears around the common boundary of two species. After a very fast time the dynamics enters the second stage in which the interface begins to evolve, which takes a much longer time than the first one.

In this paper we are interested in the latter stage and assume that the interface has been formed initially, while the study on the generation of interface will be left in the forthcoming paper. There are two well-known approaches to describe the motion of the interface so far. The classical modeling approach is the so-called sharp interface approach which treats the interface between two phases as a $N-1$ dimensional sufficiently smooth surface with zero width. The second modeling approach (the so-called diffuse interface approach) treats the tumor/host tissue interface as a transition layer

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with finite (small) width. Comparing with the sharp interface model, the diffuse interface model has many advantages in numerical simulations of the interfacial motion(see, for instance, [11] and the references therein). Equation (1.1) is a diffuse interface model related to the dynamics of tumor growth which consists of advection-reaction-diffusion equation coupled with the Cahn-Hilliard equation.

One of the important and natural problems is to investigate whether the diffuse interface model can be related to the corresponding sharp interface model in the asymptotic limit (i.e., the sharp interface limit) when the interfacial width tends to zero. Some formal asymptotic analyses regarding the sharp interface limits of some different tumor-growth models can be found in [17, 18, 21] for instance. However, to the best of our knowledge, only a few rigorous results are set up for such coupled systems. The authors in [32] re-wrote (1.1) as a gradient flow and used the techniques related to gradient flow to prove that (1.1) converges to the corresponding sharp interface model in the sense of Γ-convergence as $\varepsilon \to 0$. One can see [12] for some rigorous sharp interface limit for a model which is introduced in [8] and coupled with the velocity field in a simplified case. More recently, the authors in [31] consider the Cahn-Hilliard-Darcy system (first neglecting the nutrient σ^ε) that models the tumor growth and prove that weak solutions tend to varifold solutions of a corresponding sharp interface model when the interface thickness goes to zero. One can see [1, 10, 23, 33] for instance for more works on the convergence in the sense of Γ-convergence or varifold solutions, and [2] for the sharp interface limit of the Stokes-Allen-Cahn system.

This paper focuses on the rigorous analysis of the sharp interface limit of the local classical solutions to (1.1). In [3] the authors proved that the classical solutions of the Cahn-Hilliard equation tend to solutions of the Mullins-Sekerka problem (also called the Hele-Shaw problem) assuming the classical solutions of the latter exist. By employing the method used in [3], we will show more explicitly the asymptotic behaviors of the local classical solutions $(u^\varepsilon, \sigma^\varepsilon, \mu^\varepsilon)$ in the pointwise sense when ε goes to zero and establish the convergence which is stronger than the convergence in the framework of Γ-limit and varifold solutions in some sense. In particular we can characterize the evolution laws of $(u^\varepsilon, \sigma^\varepsilon, \mu^\varepsilon)$ in the transition region, a small neighbourhood of Γ, in which the behaviors are different from the ones in the two phase spaces.

The sharp interface model of (1.1) is the following two-phase flow (Theorem 5.8 and Theorem 5.9 in [32]):

$$\begin{cases} -\Delta\mu + \mu = 2\sigma \pm 1, & \text{in } \ \Omega_\pm, \\ \partial_t \sigma - \Delta\sigma + 2\sigma = \mu \mp 1, & \text{in } \ \Omega_\pm, \\ [\mu] = [\sigma] = 0, & \text{on } \ \Gamma, \\ [\frac{\partial \mu}{\partial \mathbf{n}}] = -2V, & \text{on } \ \Gamma, \\ [\frac{\partial \sigma}{\partial \mathbf{n}}] = 0, & \text{on } \ \Gamma, \\ \mu = \kappa \int_{-1}^{1} \sqrt{2f(u)}du, & \text{on } \ \Gamma, \\ \frac{\partial \mu}{\partial \mathbf{n}} = \frac{\partial \sigma}{\partial \mathbf{n}} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here Γ is a closed sharp interface, Ω_- and Ω_+ are the interior and exterior of Γ in Ω respectively, $\mathbf{n}$ is the unit outer normal to Γ from Ω_- to Ω_+ or to $\partial\Omega$, V is the normal velocity of the sharp interface Γ, κ is the mean curvature of Γ and $[f]$ denotes the jump condition of f from Ω_+ to Ω_- defined by $[f] = f|_{\Omega_+} - f|_{\Omega_-}$. We can note that if $\sigma = 0$ then (1.2) would be the Hele-Shaw problem in [3].

Now we explain the strategy of our proof. Firstly, we use the Hilbert expansion method to construct an explicit approximate solution to (1.1) around the local classical solution of (1.2) by assuming the latter exists and this process also can recover the sharp interface model (1.2). This method has been used in [3, 4, 6, 13, 19, 34] and the references therein. In this paper the Hilbert expansions will be performed in the two phase spaces $\Omega_\pm$, the transition layer and the region near the boundary $\partial\Omega$. A kind of inner-outer matching condition will be imposed to ensure the outer expansions in $\Omega_\pm$ and inner expansions in the transition layer match in the overlapped region. The same approach applies to the case wherein the outer expansions in $\Omega_\pm$ and the boundary layer expansion near the boundary $\partial\Omega$ should match. Such a method is also called the matched asymptotic expansion method.

To deal with the difficulty coming from the coupling term $2\sigma^\varepsilon+u^\varepsilon-\mu^\varepsilon$ in (1.1), we introduce the auxiliary function $\varphi^\varepsilon=u^\varepsilon+\sigma^\varepsilon$. Our first result is the existence of an approximate solution.

THEOREM 1.1. *Given a smooth solution (μ,σ,Γ) in $\Omega_\pm\times[0,T]$ to (1.2), then for any positive integer $k\geq 2$, there exists $(\varphi^A,\sigma^A,\mu^A,u^A)$ which satisfies*

$$\begin{cases} \partial_t\varphi^A-\Delta\mu^A-\Delta\sigma^A=0, & in\ \ \Omega\times(0,T),\\ \partial_t\sigma^A-\Delta\sigma^A=-(2\sigma^A+u^A-\mu^A), & in\ \ \Omega\times(0,T),\\ \mu^A=-\varepsilon\Delta u^A+\varepsilon^{-1}f'(u^A)+\mathfrak{R}_1, & in\ \ \Omega\times(0,T),\\ u^A=\varphi^A-\sigma^A+\mathfrak{R}_2, & in\ \ \Omega\times(0,T),\\ \frac{\partial\varphi^A}{\partial n}=\frac{\partial\mu^A}{\partial n}=\frac{\partial\sigma^A}{\partial n}=0, & on\ \ \partial\Omega\times(0,T), \end{cases} \tag{1.3}$$

where u^A is close to ±1 in $\Omega_\pm$ respectively, and

$$\mathfrak{R}_1=\varepsilon^{k-2}O(1),\ \mathfrak{R}_2=\varepsilon^{k-1}O(1),$$

where $O(1)$ are uniformly bounded functions in ε.

Secondly, based on the energy method, we derive the smallness of the error between the approximate solution and the true solution. Consequently we can prove rigorously that (1.1) converges to (1.2) as $\varepsilon\to0$. To estimate the error we mainly need to prove that the following inequality

$$\int_\Omega\left(|\nabla v|^2+\frac{1}{\varepsilon^2}f''(u^A)v^2\right)dx\geq -C\int_\Omega v^2dx$$

holds for any small ε and $v\in H^1(\Omega)$. This inequality has been proved in [7] and used to prove the convergence of the Cahn-Hilliard equation to the Hele-Shaw model in [3]. According to the structure of u^A there holds

$$u^A\sim\pm1,\ \text{in}\ \Omega_\pm,$$

then $\frac{1}{\varepsilon^2}f''(u^A)v^2$ is non-negative in $\Omega_\pm$ due to $f''(\pm1)>0$ and thus we only need to control $\frac{1}{\varepsilon^2}f''(u^A)v^2$ in the transition layer. To deal with the singularity we will draw the support from the diffusion term and use the estimates of the first eigenvalue, the corresponding eigenfunction and the second eigenvalue of the following Neumann eigenvalue problem

$$\mathcal{L}_f q:=-\frac{d^2q}{dz^2}+f''(\theta)q=\lambda q,\quad z\in I_\varepsilon=(-\frac{1}{\varepsilon},\frac{1}{\varepsilon});\quad q'(\pm\frac{1}{\varepsilon})=0, \tag{1.4}$$

where

$$\theta''(z)=f'(\theta(z)),\ \ \theta(\pm\infty)=\pm1,\quad \theta(0)=0. \tag{1.5}$$

The problem has been studied in [7] (also by a new method in [14]).

Then we apply the energy method to estimate the errors $\varphi^\varepsilon-\varphi^A$ and $\sigma^\varepsilon-\sigma^A$. Our main conclusion is:

THEOREM 1.2. *Given a classical solution* (μ,σ,Γ) *of (1.2) in* $\Omega\times(0,T)$ *which satisfies*

$$dist(\Gamma,\partial\Omega)>0,\ t\in[0,T]. \tag{1.6}$$

Then there exist $\varepsilon_0>0$ *and* $C>0$ *such that the following hold: for any* $0<\varepsilon<\varepsilon_0$*, there exist* $u_0^\varepsilon(x)$*,* $\sigma_0^\varepsilon(x)$ *such that if* $(u^\varepsilon,\sigma^\varepsilon)$ *is the solution of (1.1) in* $\Omega\times(0,T)$*, we have*

$$\left\|u^\varepsilon-u^A\right\|_{C^{4,1}(\overline{\Omega}\times[0.T])}+\left\|\sigma^\varepsilon-\sigma^A\right\|_{C^{2,1}(\overline{\Omega}\times[0.T])}+\left\|\mu^\varepsilon-\mu^A\right\|_{C^{2,1}(\overline{\Omega}\times[0.T])}\le C\varepsilon,$$

where (u^A,σ^A,μ^A) *is the approximate solution in Theorem 1.1 for* k *fairly large.*

Aa a corollary we have

COROLLARY 1.1. *Given a classical solution* (μ,σ,Γ) *of (1.2) as in Theorem 1.2, there exist* $u_0^\varepsilon(x)$*,* $\sigma_0^\varepsilon(x)$ *such that the solution* $(u^\varepsilon,\sigma^\varepsilon)$ *of (1.1) in* $\Omega\times(0,T)$ *satisfies, as* $\varepsilon\to0$*,*

$$\left\|u^\varepsilon-(\pm1)\right\|_{C(\Omega_\pm\backslash\Gamma(\delta))}\to0,\quad \left\|u^\varepsilon-\theta\left(\frac{d^{(0)}}{\varepsilon}+d^{(1)}\right)\right\|_{C(\Gamma(\delta))}\to0, \tag{1.7}$$

$$\left\|\sigma^\varepsilon-\sigma\right\|_{C(\overline{\Omega}\times[0.T])}+\left\|\mu^\varepsilon-\mu\right\|_{C(\overline{\Omega}\times[0.T])}\to0, \tag{1.8}$$

here and in what follows δ *is a small positive constant satisfying* $\delta<\frac{1}{2}dist(\Gamma,\partial\Omega)$ *for* $t\in[0,T]$.

REMARK 1.1. $u_0^\varepsilon(x)$ and $\sigma_0^\varepsilon(x)$ will be defined in (3.10). The initial data like this form is often called the sharp interface initial data since we assume the interface has been formed initially. One can refer to [3, 24] for this kind of data.

We organize this paper as follows. In Section 2, we do the $\varepsilon^k(k=0,1)$-order asymptotic expansion and get all the zeroth order terms and some first order terms. In Section 3 we turn to establish a spectral condition and give a smallness estimate on the error between the approximate solution and the true solution, and then we complete the proof of Theorem 1.2 and Corollary 1.1. The general $\varepsilon^k(k\ge2)$-order asymptotic expansion and the construction of approximate solutions will be left to Section 4.

Through this paper C denotes a generic positive constant independent of small ε.

2. Matched asymptotic expansion

In this section we will perform a matched asymptotic expansion for $(u^\varepsilon,\sigma^\varepsilon,\mu^\varepsilon)$ and get all the zeroth order terms and some first order terms of the approximate solution. In particular we can deduce the sharp interface model (1.2).

Let Γ^ε be a smooth surface centered in the transition layer. For any $t\in[0,T]$ for fixed $T>0$, let $d^\varepsilon(x,t)$ be the signed distance from x to Γ^ε. Then d^ε is smooth and $|\nabla d^\varepsilon|=1$ in a neighborhood of Γ^ε. We assume

$$d^\varepsilon(x,t)=d^{(0)}(x,t)+\varepsilon d^{(1)}(x,t)+\varepsilon^2d^{(2)}(x,t)+\cdots, \tag{2.1}$$

where $d^{(i)}(i\ge0)$ is to be determined later.

Since

$$1=|\nabla d^\varepsilon|^2=|\nabla d^{(0)}|^2+2\varepsilon\nabla d^{(0)}\cdot\nabla d^{(1)}+\sum_{k=2}^{+\infty}\varepsilon^k\left(\sum_{i=0}^{k}\nabla d^{(i)}\cdot\nabla d^{(k-i)}\right)$$

$$=1+2\varepsilon\nabla d^{(0)}\cdot\nabla d^{(1)}+\sum_{k=2}^{+\infty}\varepsilon^k\left(\sum_{i=0}^{k}\nabla d^{(i)}\cdot\nabla d^{(k-i)}\right),$$

then $|\nabla d^{(0)}|=1$ and

$$\nabla d^{(0)}\cdot\nabla d^{(k)}=\mathcal{D}_{k-1}(k\geq 1)\triangleq\begin{cases}0, & k=1,\\ -\frac{1}{2}\sum\limits_{i=1}^{k-1}\nabla d^{(i)}\cdot\nabla d^{(k-i)}, & k\geq 2.\end{cases}\tag{2.2}$$

Let $\Gamma=\{(x,t):d^{(0)}(x,t)=0\}$ and $\Omega_\pm=\{(x,t):d^{(0)}(x,t)\gtrless 0\}$. As $|\nabla d^{(0)}|=1$, then $d^{(0)}$ is the signed distance to Γ, $V=-d_t^{(0)}$ and $\kappa=-\Delta d^{(0)}$. Defining

$$\Gamma(\delta)=\{(x,t)\in\Omega\times(0,T):|d^{(0)}(x,t)|<\delta\}.\tag{2.3}$$

Now we do the outer expansion in $\Omega_\pm$, the inner expansion in $\Gamma(\delta)$ and the boundary layer expansion in $\partial\Omega(\delta)=\{(x,t):\mathrm{dist}(x,\partial\Omega)<\delta,x\in\Omega,t\in[0,T]\}$. For clarity we only match zero-order and ε-order, and solve all the zero-order terms and some 1-order terms in this section. Matching $\varepsilon^k(k\geq 2)$-order and all the remaining terms will be presented in Section 4.

2.1. Outer expansion in $\Omega_\pm$. In $\Omega_\pm$, we set

$$u^\varepsilon=u_\pm^{(0)}+\varepsilon u_\pm^{(1)}+\varepsilon^2u_\pm^{(2)}+\cdots,\tag{2.4}$$

$$\mu^\varepsilon=\mu_\pm^{(0)}+\varepsilon \mu_\pm^{(1)}+\varepsilon^2\mu_\pm^{(2)}+\cdots,\tag{2.5}$$

$$\sigma^\varepsilon=\sigma_\pm^{(0)}+\varepsilon \sigma_\pm^{(1)}+\varepsilon^2\sigma_\pm^{(2)}+\cdots.\tag{2.6}$$

Moreover, by using the Taylor expansion we write in $\Omega_\pm$

$$f'(u^\varepsilon)=f'(u_\pm^{(0)})+\varepsilon f''(u_\pm^{(0)})u_\pm^{(1)}+\cdots+\varepsilon^k\Big(f''(u_\pm^{(0)})u_\pm^{(k)}+g\big(u_\pm^{(0)},\cdots,u_\pm^{(k-1)}\big)\Big)+\cdots,$$

where $g\big(u_\pm^{(0)},\cdots,u_\pm^{(k-1)}\big)$ depends on $u_\pm^{(0)},\cdots,u_\pm^{(k-1)}$.

Substituting (2.4)-(2.6) into (1.1) and collecting all terms of the zeroth order we have

$$\begin{aligned}\partial_t u_\pm^{(0)}-\Delta\mu_\pm^{(0)}&=2\sigma_\pm^{(0)}+u_\pm^{(0)}-\mu_\pm^{(0)},\\ \partial_t \sigma_\pm^{(0)}-\Delta\sigma_\pm^{(0)}&=-\big(2\sigma_\pm^{(0)}+u_\pm^{(0)}-\mu_\pm^{(0)}\big),\\ f'(u_\pm^{(0)})&=0.\end{aligned}$$

We take

$$u_\pm^{(0)}=\pm 1.\tag{2.7}$$

Then

$$-\Delta\mu_\pm^{(0)}+\mu_\pm^{(0)}=2\sigma_\pm^{(0)}\pm 1,\tag{2.8}$$

$$\partial_t\sigma_\pm^{(0)}-\Delta\sigma_\pm^{(0)}+2\sigma_\pm^{(0)}=\mu_\pm^{(0)}\mp 1.\tag{2.9}$$

Substituting (2.4)-(2.6) into (1.1) and collecting terms of order ε we have

$$u_\pm^{(1)} = \frac{\mu_\pm^{(0)}}{f''(u_\pm^{(0)})},$$
$$-\Delta\mu_\pm^{(1)} + \mu_\pm^{(1)} = 2\sigma_\pm^{(1)} + u_\pm^{(1)} - \partial_t u_\pm^{(1)},$$
$$\partial_t\sigma_\pm^{(1)} - \Delta\sigma_\pm^{(1)} + 2\sigma_\pm^{(1)} = \mu_\pm^{(1)} - u_\pm^{(1)}.$$

2.2. Inner expansion in $\Gamma(\delta)$. Let $z = \frac{d^\varepsilon}{\varepsilon} \in (-\infty, +\infty)$. In $\Gamma(\delta)$, we set

$$u^\varepsilon(x,t) = \widetilde{u}^\varepsilon(x,t,z) = \widetilde{u}^{(0)}(x,t,z) + \varepsilon\widetilde{u}^{(1)}(x,t,z) + \varepsilon^2\widetilde{u}^{(2)}(x,t,z) + \cdots, \tag{2.10}$$
$$\mu^\varepsilon(x,t) = \widetilde{\mu}^\varepsilon(x,t,z) = \widetilde{\mu}^{(0)}(x,t,z) + \varepsilon\widetilde{\mu}^{(1)}(x,t,z) + \varepsilon^2\widetilde{\mu}^{(2)}(x,t,z) + \cdots, \tag{2.11}$$
$$\sigma^\varepsilon(x,t) = \widetilde{\sigma}^\varepsilon(x,t,z) = \widetilde{\sigma}^{(0)}(x,t,z) + \varepsilon\widetilde{\sigma}^{(1)}(x,t,z) + \varepsilon^2\widetilde{\sigma}^{(2)}(x,t,z) + \cdots, \tag{2.12}$$

and the following inner-outer matching conditions: there exists a fixed positive constant ν such that as $z \to \pm\infty$ there hold

$$D_x^\alpha D_t^\beta D_z^\gamma \left(\widetilde{u}^{(i)}(x,t,z) - u_\pm^{(i)}(x,t)\right) = O(e^{-\nu|z|}), \tag{2.13}$$
$$D_x^\alpha D_t^\beta D_z^\gamma \left(\widetilde{\mu}^{(i)}(x,t,z) - \mu_\pm^{(i)}(x,t)\right) = O(e^{-\nu|z|}), \tag{2.14}$$
$$D_x^\alpha D_t^\beta D_z^\gamma \left(\widetilde{\sigma}^{(i)}(x,t,z) - \sigma_\pm^{(i)}(x,t)\right) = O(e^{-\nu|z|}) \tag{2.15}$$

for $(x,t) \in \Gamma(\delta)$ and $0 \le \alpha,\beta,\gamma \le N$ with N depending on the order of expansions we need.

Using the Taylor expansion and (2.10) one has

$$f'(\widetilde{u}^\varepsilon) = f'(\widetilde{u}^{(0)}) + \varepsilon f''(\widetilde{u}^{(0)})\widetilde{u}^{(1)} + \cdots + \varepsilon^k\left(f''(\widetilde{u}^{(0)})\widetilde{u}^{(k)} + \widetilde{g}\big(\widetilde{u}^{(0)}, \cdots, \widetilde{u}^{(k-1)}\big)\right) + \cdots, \tag{2.16}$$

where $g\big(\widetilde{u}^{(0)}, \cdots, \widetilde{u}^{(k-1)}\big)$ depends on $\widetilde{u}^{(0)}, \cdots, \widetilde{u}^{(k-1)}$.

Noting that

$$\widetilde{u}_t^\varepsilon(x,t,z) = \partial_t\widetilde{u}^\varepsilon + \varepsilon^{-1}\partial_z\widetilde{u}^\varepsilon\partial_t d^\varepsilon,$$
$$\Delta_x\widetilde{\mu}^\varepsilon(x,t,z) = \varepsilon^{-2}\partial_{zz}\widetilde{\mu}^\varepsilon + 2\varepsilon^{-1}\nabla_x\partial_z\widetilde{\mu}^\varepsilon \cdot \nabla_x d^\varepsilon + \varepsilon^{-1}\partial_z\widetilde{\mu}^\varepsilon\Delta_x d^\varepsilon + \Delta_x\widetilde{\mu}^\varepsilon,$$

then in the new coordinates (x,t,z) the first equation in (1.1) becomes

$$\begin{aligned} -\partial_{zz}\widetilde{\mu}^\varepsilon &+ \varepsilon\left(\partial_z\widetilde{u}^\varepsilon\partial_t d^\varepsilon - 2\nabla_x\partial_z\widetilde{\mu}^\varepsilon \cdot \nabla_x d^\varepsilon - \partial_z\widetilde{\mu}^\varepsilon\Delta_x d^\varepsilon\right) \\ &+ \varepsilon^2\left(\partial_t\widetilde{u}^\varepsilon - \Delta_x\widetilde{\mu}^\varepsilon - \big(2\widetilde{\sigma}^\varepsilon + \widetilde{u}^\varepsilon - \widetilde{\mu}^\varepsilon\big)\right) = 0. \end{aligned} \tag{2.17}$$

Similarly, the second equation and the third equation in (1.1) become respectively

$$\begin{aligned} -\partial_{zz}\widetilde{\sigma}^\varepsilon &+ \varepsilon\left(\partial_z\widetilde{\sigma}^\varepsilon\partial_t d^\varepsilon - 2\nabla_x\partial_z\widetilde{\sigma}^\varepsilon \cdot \nabla_x d^\varepsilon - \partial_z\widetilde{\sigma}^\varepsilon\Delta_x d^\varepsilon\right) \\ &+ \varepsilon^2\left(\partial_t\widetilde{\sigma}^\varepsilon - \Delta_x\widetilde{\sigma}^\varepsilon + 2\widetilde{\sigma}^\varepsilon + \widetilde{u}^\varepsilon - \widetilde{\mu}^\varepsilon\right) = 0, \end{aligned} \tag{2.18}$$
$$-\partial_{zz}\widetilde{u}^\varepsilon + f'(\widetilde{u}^\varepsilon) - \varepsilon\left(2\nabla_x\partial_z\widetilde{u}^\varepsilon \cdot \nabla_x d^\varepsilon + \partial_z\widetilde{u}^\varepsilon\Delta_x d^\varepsilon + \widetilde{\mu}^\varepsilon\right) - \varepsilon^2\Delta_x\widetilde{u}^\varepsilon = 0. \tag{2.19}$$

In general, the inner-outer exponentially decaying matching conditions (2.13)-(2.15) may not necessarily hold. To ensure them, we will modify (2.17)-(2.19) as follows, motivated by [3]. We choose a smooth non-decreasing function η such that $\eta(z)=0$ for $z\leq -1$, $\eta(z)=1$ for $z\geq 1$ and define

$$\eta^{\pm}(z)=\eta(-M\pm z), \quad z\in\mathbb{R},$$

where the constant $M=\|d^{(1)}\|_{C^0(\Gamma(\delta))}+2$.

The system (2.17)-(2.19) is then modified as follows

$$\begin{aligned}
&-\partial_{zz}\widetilde{\mu}^{\varepsilon}+\varepsilon\Big(\partial_z\widetilde{u}^{\varepsilon}\partial_t d^{\varepsilon}-2\nabla_x\partial_z\widetilde{\mu}^{\varepsilon}\cdot\nabla_x d^{\varepsilon}-\partial_z\widetilde{\mu}^{\varepsilon}\Delta_x d^{\varepsilon}\Big)\\
&\qquad+\varepsilon^2\Big(\partial_t\widetilde{u}^{\varepsilon}-\Delta_x\widetilde{\mu}^{\varepsilon}-\big(2\widetilde{\sigma}^{\varepsilon}+\widetilde{u}^{\varepsilon}-\widetilde{\mu}^{\varepsilon}\big)\Big)\\
&\qquad+\eta''p^{\varepsilon}(d^{\varepsilon}-\varepsilon z)+\varepsilon\eta' g^{\varepsilon}(d^{\varepsilon}-\varepsilon z)-\varepsilon^2(s_+^{\varepsilon}\eta^+ + s_-^{\varepsilon}\eta^-)=0, \qquad (2.20)\\
&-\partial_{zz}\widetilde{\sigma}^{\varepsilon}+\varepsilon\Big(\partial_z\widetilde{\sigma}^{\varepsilon}\partial_t d^{\varepsilon}-2\nabla_x\partial_z\widetilde{\sigma}^{\varepsilon}\cdot\nabla_x d^{\varepsilon}-\partial_z\widetilde{\sigma}^{\varepsilon}\Delta_x d^{\varepsilon}\Big)\\
&\qquad+\varepsilon^2\Big(\partial_t\widetilde{\sigma}^{\varepsilon}-\Delta_x\widetilde{\sigma}^{\varepsilon}+2\widetilde{\sigma}^{\varepsilon}+\widetilde{u}^{\varepsilon}-\widetilde{\mu}^{\varepsilon}\Big)\\
&\qquad+\eta''q^{\varepsilon}(d^{\varepsilon}-\varepsilon z)+\varepsilon\eta' h^{\varepsilon}(d^{\varepsilon}-\varepsilon z)-\varepsilon^2(r_+^{\varepsilon}\eta^+ + r_-^{\varepsilon}\eta^-)=0, \qquad (2.21)\\
&-\partial_{zz}\widetilde{u}^{\varepsilon}+f'(\widetilde{u}^{\varepsilon})-\varepsilon\big(2\nabla_x\partial_z\widetilde{u}^{\varepsilon}\cdot\nabla_x d^{\varepsilon}+\partial_z\widetilde{u}^{\varepsilon}\Delta_x d^{\varepsilon}+\widetilde{\mu}^{\varepsilon}\big)-\varepsilon^2\Delta_x\widetilde{u}^{\varepsilon}\\
&\qquad+\varepsilon\eta' l^{\varepsilon}(d^{\varepsilon}-\varepsilon z)=0, \qquad (2.22)
\end{aligned}$$

where

$$\begin{aligned}
&g^{\varepsilon}(x,t)=\sum_{i=0}^{+\infty}\varepsilon^i g^{(i)}(x,t),\quad h^{\varepsilon}(x,t)=\sum_{i=0}^{+\infty}\varepsilon^i h^{(i)}(x,t),\quad l^{\varepsilon}(x,t)=\sum_{i=0}^{+\infty}\varepsilon^i l^{(i)}(x,t),\\
&p^{\varepsilon}(x,t)=\sum_{i=0}^{+\infty}\varepsilon^i p^{(i)}(x,t),\quad q^{\varepsilon}(x,t)=\sum_{i=0}^{+\infty}\varepsilon^i q^{(i)}(x,t),\\
&s_{\pm}^{\varepsilon}(x,t)=\sum_{i=0}^{+\infty}\varepsilon^i s_{\pm}^{(i)}(x,t)=\sum_{i=0}^{+\infty}\varepsilon^i\big(\partial_t u_{\pm}^{(i)}-\Delta\mu_{\pm}^{(i)}-\big(2\sigma_{\pm}^{(i)}+u_{\pm}^{(i)}-\mu_{\pm}^{(i)}\big)\big)(x,t),\\
&r_{\pm}^{\varepsilon}(x,t)=\sum_{i=0}^{+\infty}\varepsilon^i r_{\pm}^{(i)}(x,t)=\sum_{i=0}^{+\infty}\varepsilon^i\big(\partial_t\sigma_{\pm}^{(i)}-\Delta\sigma_{\pm}^{(i)}-\big(2\sigma_{\pm}^{(i)}+u_{\pm}^{(i)}-\mu_{\pm}^{(i)}\big)\big)(x,t).
\end{aligned}$$

REMARK 2.1. As $z=\frac{d^{\varepsilon}}{\varepsilon}$, then $d^{\varepsilon}-\varepsilon z=0$. Moreover, since $M=\|d^{(1)}\|_{C^0(\Gamma(\delta))}+2$, we can find $r_+^{\varepsilon}\eta^+ + r_-^{\varepsilon}\eta^- =0$ for $(x,t)\in\Gamma(\delta)$ and small ε. One can refer to Remark 4.2 in [3] for the details. Therefore (2.17)-(2.19) are the same as (2.20)-(2.22) respectively. However, through the modifications, we have changed the equation of every order such that the matching conditions (2.13)-(2.15) hold.

For clarity we divide into two subsections to proceed.

2.2.1. Matching zeroth order. Substituting (2.1) and (2.10)-(2.12) into (2.20)-(2.22) and collecting all terms of the zeroth order we have

$$\partial_{zz}\widetilde{\mu}^{(0)}=\eta''p^{(0)}d^{(0)}, \qquad (2.23)$$

$$\partial_{zz}\widetilde{\sigma}^{(0)}=\eta''q^{(0)}d^{(0)}, \qquad (2.24)$$

$$\partial_{zz}\widetilde{u}^{(0)} = f'(\widetilde{u}^{(0)}). \tag{2.25}$$

Now we argue in this subsection according to the following order:

$$(\widetilde{\mu}^{(0)}, \widetilde{\sigma}^{(0)}) \to (p^{(0)}, q^{(0)}, [\mu^{(0)}], [\sigma^{(0)}]) \to \widetilde{u}^{(0)}.$$

- $(\widetilde{\mu}^{(0)}, \widetilde{\sigma}^{(0)})$

From (2.23) we can write

$$\widetilde{\mu}^{(0)}(z,x,t) = \eta(z)p^{(0)}(x,t)d^{(0)}(x,t) + a(x,t)z + b(x,t)$$

for some $a(x,t)$ and $b(x,t)$. Since the inner-outer matching condition $\widetilde{\mu}^{(0)}(z,x,t) \to \mu_{\pm}^{(0)}(x,t)$ as $z \to \pm\infty$ needs to be satisfied, we obtain

$$a(x,t) = 0, \ b(x,t) = \mu_{-}^{(0)}(x,t)$$

and

$$p^{(0)}(x,t)d^{(0)}(x,t) = \mu_{+}^{(0)}(x,t) - \mu_{-}^{(0)}(x,t). \tag{2.26}$$

Thus we have

$$\widetilde{\mu}^{(0)}(x,t,z) = \eta(z)\mu_{+}^{(0)}(x,t) + \big(1-\eta(z)\big)\mu_{-}^{(0)}(x,t) \tag{2.27}$$

and

$$D_x^{\alpha} D_t^{\beta} D_z^{\gamma} \Big(\widetilde{\mu}^{(0)}(x,t,z) - \mu_{\pm}^{(0)}(x,t)\Big) = O(e^{-\nu|z|})$$

for any $\alpha, \beta, \gamma \in \mathbb{N}$ and $\nu > 0$.

Similarly, we get

$$q^{(0)}(x,t)d^{(0)}(x,t) = \sigma_{+}^{(0)}(x,t) - \sigma_{-}^{(0)}(x,t) \tag{2.28}$$

and

$$\widetilde{\sigma}^{(0)}(x,t,z) = \eta(z)\sigma_{+}^{(0)}(x,t) + \big(1-\eta(z)\big)\sigma_{-}^{(0)}(x,t) \tag{2.29}$$

which implies

$$D_x^{\alpha} D_t^{\beta} D_z^{\gamma} \Big(\widetilde{\sigma}^{(0)}(x,t,z) - \sigma_{\pm}^{(0)}(x,t)\Big) = O(e^{-\nu|z|})$$

for any $\alpha, \beta, \gamma \in \mathbb{N}$ and $\nu > 0$.

- $(p^{(0)}, q^{(0)}, [\mu^{(0)}], [\sigma^{(0)}])$

Moreover, according to (2.26) and (2.28) there hold on Γ

$$[\mu^{(0)}] \triangleq \mu_{+}^{(0)} - \mu_{-}^{(0)} = 0, \quad [\sigma^{(0)}] \triangleq \sigma_{+}^{(0)} - \sigma_{-}^{(0)} = 0. \tag{2.30}$$

And we can define smooth functions $p^{(0)}$ and $q^{(0)}$ in $\Gamma(\delta)$ as follows

$$p^{(0)} = \begin{cases} \frac{\mu_{+}^{(0)} - \mu_{-}^{(0)}}{d^{(0)}}, & \text{in } \Gamma(\delta)\backslash\Gamma, \\ \nabla_x d^{(0)} \cdot \nabla_x(\mu_{+}^{(0)} - \mu_{-}^{(0)}), & \text{on } \Gamma, \end{cases} \tag{2.31}$$

and

$$q^{(0)}=\begin{cases}\frac{\sigma_+^{(0)}-\sigma_-^{(0)}}{d^{(0)}}, & \text{in } \Gamma(\delta)\backslash\Gamma,\\ \nabla_x d^{(0)}\cdot\nabla_x(\sigma_+^{(0)}-\sigma_-^{(0)}), & \text{on } \Gamma.\end{cases} \tag{2.32}$$

- $\widetilde{u}^{(0)}$

By (2.25) and the inner-outer matching condition, $\widetilde{u}^{(0)}$ satisfies

$$\partial_{zz}\widetilde{u}^{(0)}=f'(\widetilde{u}^{(0)}),\quad \widetilde{u}^{(0)}(\pm\infty)=u_\pm^{(0)}=\pm1,\quad \widetilde{u}^{(0)}(0)=0,$$

where the condition $\widetilde{u}^{(0)}(0)=0$ is imposed to ensure $\widetilde{u}^{(0)}$ is unique. Therefore $\widetilde{u}^{(0)}$ is independent of (x,t) and then $\widetilde{u}^{(0)}(x,t,z)=\theta(z)$ which is defined in (1.5). In fact, we can get

$$\widetilde{u}^{(0)}(x,t,z)=\theta(z)=\tanh(\sqrt{2}z) \tag{2.33}$$

and for $k\in\mathbb{N}\cup\{0\}$

$$\frac{d^k}{dz^k}\big(\theta(z)+1\big)=O(e^{-\sqrt{2}|z|}),\ \text{as } z\to-\infty;\ \frac{d^k}{dz^k}\big(\theta(z)-1\big)=O(e^{-\sqrt{2}|z|}),\ \text{as } z\to+\infty,$$

which implies

$$D_x^\alpha D_t^\beta D_z^\gamma\Big(\widetilde{u}^{(0)}(x,t,z)-u_\pm^{(0)}(x,t)\Big)=O(e^{-\sqrt{2}|z|})$$

for any $\alpha,\beta,\gamma\in\mathbb{N}$.

2.2.2. Matching the first order. Substituting (2.1) and (2.10)-(2.12) into (2.20)-(2.22) and collecting all the terms of ε-order we have

$$\begin{aligned}&-\partial_{zz}\widetilde{\mu}^{(1)}+\Big(\partial_z\widetilde{u}^{(0)}\partial_t d^{(0)}-2\nabla_x\partial_z\widetilde{\mu}^{(0)}\cdot\nabla_x d^{(0)}-\partial_z\widetilde{\mu}^{(0)}\Delta_x d^{(0)}\Big)\\&\qquad+\eta''\big(p^{(1)}d^{(0)}+p^{(0)}d^{(1)}\big)-\eta''zp^{(0)}+\eta' g^{(0)}d^{(0)}=0,\end{aligned} \tag{2.34}$$

$$\begin{aligned}&-\partial_{zz}\widetilde{\sigma}^{(1)}+\Big(\partial_z\widetilde{\sigma}^{(0)}\partial_t d^{(0)}-2\nabla_x\partial_z\widetilde{\sigma}^{(0)}\cdot\nabla_x d^{(0)}-\partial_z\widetilde{\sigma}^{(0)}\Delta_x d^{(0)}\Big)\\&\qquad+\eta''\big(q^{(1)}d^{(0)}+q^{(0)}d^{(1)}\big)-\eta''zq^{(0)}+\eta' h^{(0)}d^{(0)}=0,\end{aligned} \tag{2.35}$$

$$\begin{aligned}&-\partial_{zz}\widetilde{u}^{(1)}+f''(\widetilde{u}^{(0)})\widetilde{u}^{(1)}-\Big(2\nabla_x\partial_z\widetilde{u}^{(0)}\cdot\nabla_x d^{(0)}+\partial_z\widetilde{u}^{(0)}\Delta_x d^{(0)}+\widetilde{\mu}^{(0)}\Big)\\&\qquad+\eta' l^{(0)}d^{(0)}=0.\end{aligned} \tag{2.36}$$

Next we argue according to the following order:

$$\Big(\widetilde{\mu}^{(1)},g^{(0)},\Big[\frac{\partial\mu^{(0)}}{\partial\mathbf{n}}\Big]\Big)\to\Big(\widetilde{\sigma}^{(1)},h^{(0)},\Big[\frac{\partial\sigma^{(0)}}{\partial\mathbf{n}}\Big]\Big)\to\Big(\widetilde{u}^{(1)},l^{(0)},\mu_\pm^{(0)}|_\Gamma\Big).$$

- $\Big(\widetilde{\mu}^{(1)},g^{(0)},\Big[\frac{\partial\mu^{(0)}}{\partial\mathbf{n}}\Big]\Big)$

For $(x,t)\in\Gamma(\delta)$, we write (2.34) as

$$-\Big(\widetilde{\mu}^{(1)}-\eta\big(p^{(1)}d^{(0)}+p^{(0)}d^{(1)}\big)\Big)_{zz}=\eta''zp^{(0)}-\eta' g^{(0)}d^{(0)}-\partial_z\widetilde{u}^{(0)}\partial_t d^{(0)}$$

$$+2\nabla_x\partial_z\widetilde{\mu}^{(0)}\cdot\nabla_x d^{(0)}+\partial_z\widetilde{\mu}^{(0)}\Delta_x d^{(0)}$$
$$\triangleq\Theta_{0,1}. \tag{2.37}$$

It follows from Lemma 4.3 in [3] and direct computations that if

$$\int_{-\infty}^{+\infty}\Theta_{0,1}(x,t,z)dz=0, \tag{2.38}$$

then (2.37) has a bounded solution

$$\widetilde{\mu}^{(1)}(x,t,z)=\eta(z)\big(p^{(1)}d^{(0)}+p^{(0)}d^{(1)}\big)(x,t)$$
$$+\int_{-\infty}^{z}\int_{z'}^{+\infty}\Theta_{0,1}(z'',x,t)dz''dz'+\mu_-^{(1)}(x,t)$$

which and $\widetilde{\mu}^{(1)}(+\infty,x,t)=\mu_+^{(1)}(x,t)$ imply

$$\big(p^{(1)}d^{(0)}+p^{(0)}d^{(1)}\big)(x,t)=\mu_+^{(1)}(x,t)-\int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{0,1}(z'',x,t)dz''dz'-\mu_-^{(1)}(x,t).$$

Hence we obtain

$$\widetilde{\mu}^{(1)}(x,t,z)=\eta(z)\mu_+^{(1)}(x,t)+\big(1-\eta(z)\big)\mu_-^{(1)}(x,t)$$
$$-\eta(z)\int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{0,1}(z'',x,t)dz''dz'+\int_{-\infty}^{z}\int_{z'}^{+\infty}\Theta_{0,1}(z'',x,t)dz''dz'$$

and

$$\big[\mu^{(1)}\big]\triangleq\mu_+^{(1)}-\mu_-^{(1)}=p^{(0)}d^{(1)}+\int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{0,1}(z'',x,t)dz''dz' \quad \text{on}\quad \Gamma.$$

According to the results obtained in Subsection 2.2.1 one has for any $\alpha,\beta,\gamma\in\mathbb{N}$,

$$D_x^\alpha D_t^\beta D_z^\gamma\Theta_{0,1}=O(e^{-\nu|z|}),\quad \text{for some}\quad \nu>0,$$

and thus

$$D_x^\alpha D_t^\beta D_z^\gamma\Big(\widetilde{\mu}^{(1)}(x,t,z)-\mu_\pm^{(1)}(x,t)\Big)=O(e^{-\nu|z|}),\quad \text{for some}\quad \nu>0.$$

Moreover, by (2.38), (2.30), (2.31) and direct computations we can get

$$\left[\frac{\partial\mu^{(0)}}{\partial\mathbf{n}}\right]\triangleq\nabla_x\big(\mu_+^{(0)}-\mu_-^{(0)}\big)\cdot\nabla_x d^{(0)}=2d_t^{(0)}\triangleq-2V,\quad \text{on }\Gamma, \tag{2.39}$$

and define a smooth function $g^{(0)}$ in $\Gamma(\delta)$

$$g^{(0)}=\begin{cases}\frac{(\mu_+^{(0)}-\mu_-^{(0)})\triangle_x d^{(0)}+2\nabla_x(\mu_+^{(0)}-\mu_-^{(0)})\cdot\nabla_x d^{(0)}-p^0-2\partial_t d^{(0)}}{d^{(0)}}, & \text{in}\quad \Gamma(\delta)\backslash\Gamma,\\ \nabla_x d^{(0)}\cdot\nabla_x\big((\mu_+^{(0)}-\mu_-^{(0)})\Delta_x d^{(0)}+2\nabla_x(\mu_+^{(0)}-\mu_-^{(0)})\cdot\nabla_x d^{(0)} \\ \qquad -p^0-2\partial_t d^{(0)}\big), & \text{on}\quad \Gamma.\end{cases} \tag{2.40}$$

- $\Big(\widetilde{\sigma}^{(1)},h^{(0)},\big[\frac{\partial\sigma^{(0)}}{\partial\mathbf{n}}\big]\Big)$

Applying a similar argument as above to (2.35), we obtain that if

$$\int_{-\infty}^{+\infty} \Theta_{0,2}(x,t,z)dz = 0, \tag{2.41}$$

then (2.35) has a bounded solution

$$\begin{aligned}\widetilde{\sigma}^{(1)}(x,t,z) =& \eta(z)\sigma_+^{(1)}(x,t) + \big(1-\eta(z)\big)\sigma_-^{(1)}(x,t)\\ &- \eta(z)\int_{-\infty}^{+\infty}\int_{z'}^{+\infty} \Theta_{0,2}(x,t,z'')dz''dz' + \int_{-\infty}^{z}\int_{z'}^{+\infty} \Theta_{0,2}(x,t,z'')dz''dz',\end{aligned}$$

and for any $\alpha,\beta,\gamma\in\mathbb{N}$,

$$D_x^\alpha D_t^\beta D_z^\gamma \Big(\widetilde{\sigma}^{(1)}(x,t,z) - \sigma_\pm^{(1)}(x,t)\Big) = O(e^{-\nu|z|}), \quad \text{for some} \quad \nu>0,$$

and

$$\big[\sigma^{(1)}\big] \triangleq \sigma_+^{(1)} - \sigma_-^{(1)} = q^{(0)}d^{(1)} + \int_{-\infty}^{+\infty}\int_{z'}^{+\infty} \Theta_{0,2}(x,t,z'')dz''dz', \quad \text{on} \ \ \Gamma,$$

where

$$\Theta_{0,2} \triangleq \eta'' z q^{(0)} - \eta' h^{(0)} d^{(0)} - \partial_z\widetilde{\sigma}^{(0)}\partial_t d^{(0)} + 2\nabla_x\partial_z\widetilde{\sigma}^{(0)}\cdot\nabla_x d^{(0)} + \partial_z\widetilde{\sigma}^{(0)}\triangle_x d^{(0)}.$$

Moreover, it follows from (2.41), (2.30), (2.32) and direct computations that

$$\left[\frac{\partial\sigma^{(0)}}{\partial\mathbf{n}}\right] \triangleq \nabla\big(\sigma_+^{(0)} - \sigma_-^{(0)}\big)\cdot\nabla d^{(0)} = 0, \quad \text{on } \Gamma, \tag{2.42}$$

and

$$h^{(0)} = \begin{cases} \frac{(\sigma_+^{(0)}-\sigma_-^{(0)})(\triangle_x d^{(0)}-\partial_t d^{(0)})+2\nabla_x(\sigma_+^{(0)}-\sigma_-^{(0)})\cdot\nabla_x d^{(0)}-q^0}{d^{(0)}}, & \text{in } \ \Gamma(\delta)\backslash\Gamma,\\ \nabla_x d^{(0)}\cdot\nabla_x\big((\sigma_+^{(0)}-\sigma_-^{(0)})(\Delta_x d^{(0)}-\partial_t d^{(0)}) \\ \qquad +2\nabla_x(\sigma_+^{(0)}-\sigma_-^{(0)})\cdot\nabla_x d^{(0)}-q^0\big), & \text{on } \ \Gamma.\end{cases} \tag{2.43}$$

- $(\widetilde{u}^{(1)}, l^{(0)}, \mu_\pm^{(0)}|_\Gamma)$

Based on the method of variation of constants for ODE and direct computations (or Lemma 4.3 in [3]), we find that if

$$\int_{-\infty}^{+\infty} \Theta_{0,3}(x,t,z)\theta'(z)dz = 0, \tag{2.44}$$

then the solution to (2.36) with $\widetilde{u}^{(1)}(0,x,t)=1$ is

$$\widetilde{u}^{(1)}(x,t,z) = \frac{\theta'(z)}{\theta'(0)} + \theta'(z)\int_0^z (\theta'(\varsigma))^{-2}\int_\varsigma^{+\infty}\Theta_{0,3}(x,t,\tau)\theta'(\tau)d\tau d\varsigma$$

where

$$\Theta_{0,3} := 2\nabla_x\partial_z\widetilde{u}^{(0)}\cdot\nabla_x d^{(0)} + \partial_z\widetilde{u}^{(0)}\Delta_x d^{(0)} + \widetilde{\mu}^{(0)} - \eta' l^0 d^{(0)} = \theta'\Delta_x d^{(0)} + \widetilde{\mu}^{(0)} - \eta' l^0 d^{(0)},$$

and for any $\alpha,\beta,\gamma\in\mathbb{N}$,

$$D_x^\alpha D_t^\beta D_z^\gamma\Big(\widetilde{u}^{(1)}(x,t,z)-u_\pm^{(1)}(x,t)\Big)=O(e^{-\nu|z|}),\quad \text{for some}\quad \nu>0.$$

Due to (2.44) there holds

$$\begin{aligned}\mu_\pm^{(0)}(x,t)&=-\Delta_x d^{(0)}\int_{-\infty}^{+\infty}(\theta'(z))^2dz=\kappa\int_{-\infty}^{+\infty}(\theta'(z))^2dz\\&=2\kappa\int_{-\infty}^{+\infty}f(\theta(z))dz=\kappa\int_{-1}^{1}\sqrt{2f(u)}du,\quad \text{on}\quad \Gamma,\end{aligned}\tag{2.45}$$

where we have used $\theta'(z)=\sqrt{2f(\theta(z))}$.

Furthermore, by (2.44) a smooth function $l^{(0)}$ is defined as follows:

$$l^{(0)}=\begin{cases}\frac{1}{d^{(0)}\int_{-\infty}^{+\infty}\eta'\theta'dz}\Big(\Delta_x d^{(0)}\int_{-\infty}^{+\infty}(\theta')^2dz+\int_{-\infty}^{+\infty}\widetilde{\mu}^{(0)}\theta'dz\Big), & \text{in}\quad \Gamma(\delta)\backslash\Gamma,\\[2ex] \frac{1}{\int_{-\infty}^{+\infty}\eta'\theta'dz}\nabla_x d^{(0)}\cdot\nabla_x\Big(\Delta_x d^{(0)}\int_{-\infty}^{+\infty}(\theta')^2dz+\int_{-\infty}^{+\infty}\widetilde{\mu}^{(0)}\theta'dz\Big), & \text{on}\quad \Gamma.\end{cases}\tag{2.46}$$

2.3. Boundary layer expansion in $\partial\Omega(\delta)$. Let $d_B(x)<0$ be the signed distance from x to $\partial\Omega$ and $z=\frac{d_B(x)}{\varepsilon}\in(-\infty,0]$. In $\partial\Omega(\delta)=\{x\in\overline{\Omega}:-\delta<d_B(x)\le 0\}$, we set

$$u^\varepsilon(x,t)=u_B^\varepsilon(x,t,z)=u_B^{(0)}(x,t,z)+\varepsilon u_B^{(1)}(x,t,z)+\varepsilon^2u_B^{(2)}(x,t,z)+\cdots,\tag{2.47}$$

$$\mu^\varepsilon(x,t)=\mu_B^\varepsilon(x,t,z)=\mu_B^{(0)}(x,t,z)+\varepsilon \mu_B^{(1)}(x,t,z)+\varepsilon^2\mu_B^{(2)}(x,t,z)+\cdots,\tag{2.48}$$

$$\sigma^\varepsilon(x,t)=\sigma_B^\varepsilon(x,t,z)=\sigma_B^{(0)}(x,t,z)+\varepsilon \sigma_B^{(1)}(x,t,z)+\varepsilon^2\sigma_B^{(2)}(x,t,z)+\cdots,\tag{2.49}$$

and the following boundary-outer matching conditions: there exists a fixed positive constant ν such that as $z\to-\infty$ there hold

$$D_x^\alpha D_t^\beta D_z^\gamma\Big(u_B^{(i)}(x,t,z)-u_+^{(i)}(x,t)\Big)=O(e^{-\nu|z|}),\tag{2.50}$$

$$D_x^\alpha D_t^\beta D_z^\gamma\Big(\mu_B^{(i)}(x,t,z)-\mu_+^{(i)}(x,t)\Big)=O(e^{-\nu|z|}),\tag{2.51}$$

$$D_x^\alpha D_t^\beta D_z^\gamma\Big(\sigma_B^{(i)}(x,t,z)-\sigma_+^{(i)}(x,t)\Big)=O(e^{-\nu|z|}),\tag{2.52}$$

for $(x,t)\in\partial\Omega(\delta)\times[0,T]$ and $0\le\alpha,\beta,\gamma\le N$ with N depending on actual expansion order.

Using the Taylor expansion and (2.47) one has

$$f'(u_B^\varepsilon)=f'(u_B^{(0)})+\varepsilon f''(u_B^{(0)})u_B^{(1)}+\cdots+\varepsilon^k\Big(f''(u_B^{(0)})u_B^{(k)}+g_{_B}(u_B^{(0)},\cdots,u_B^{(k-1)})\Big)+\cdots,$$

here the function $g_{_B}(u_B^{(0)},\cdots,u_B^{(k-1)})$ depends on $u_B^{(0)},\cdots,u_B^{(k-1)}$.

We write (1.1) in $\partial\Omega(\delta)$ in the new coordinates (x,t,z) as follows

$$\begin{aligned}&-\partial_{zz}\mu_B^\varepsilon-\varepsilon\Big(2\nabla_x\partial_z\mu_B^\varepsilon\cdot\nabla_x d_B+\partial_z\mu_B^\varepsilon\Delta_x d_B\Big)+\varepsilon^2\Big(\partial_t u_B^\varepsilon\\&\qquad\qquad\qquad\qquad\qquad -\Delta_x\mu_B^\varepsilon-\big(2\sigma_B^\varepsilon+u_B^\varepsilon-\mu_B^\varepsilon\big)\Big)=0,\end{aligned}\tag{2.53}$$

$$-\partial_{zz}\sigma_B^\varepsilon-\varepsilon\Big(2\nabla_x\partial_z\sigma_B^\varepsilon\cdot\nabla_x d_B+\partial_z\sigma_B^\varepsilon\Delta_x d_B\Big)+\varepsilon^2\Big(\partial_t\sigma_B^\varepsilon$$

$$-\Delta_x\sigma_B^\varepsilon+2\sigma_B^\varepsilon+u_B^\varepsilon-\mu_B^\varepsilon\Big)=0, \tag{2.54}$$

$$-\partial_{zz}u_B^\varepsilon+f'(u_B^\varepsilon)-\varepsilon\big(2\nabla_x\partial_z u_B^\varepsilon\cdot\nabla_x d_B+\partial_z u_B^\varepsilon\Delta_x d_B+\mu_B^\varepsilon\big)-\varepsilon^2\Delta_x u_B^\varepsilon=0. \tag{2.55}$$

Moreover, homogeneous Neumann boundary conditions in (1.1) imply that, on $\partial\Omega\times[0,T]$,

$$\partial_z\mu_B^\varepsilon(x,t,0)+\varepsilon\nabla_x\mu_B^\varepsilon(x,t,0)\cdot\nabla_x d_B(x,t)=0, \tag{2.56}$$

$$\partial_z\sigma_B^\varepsilon(x,t,0)+\varepsilon\nabla_x\sigma_B^\varepsilon(x,t,0)\cdot\nabla_x d_B(x,t)=0, \tag{2.57}$$

$$\partial_z u_B^\varepsilon(x,t,0)+\varepsilon\nabla_x u_B^\varepsilon(x,t,0)\cdot\nabla_x d_B(x,t)=0. \tag{2.58}$$

Firstly substituting (2.47)-(2.49) into (2.53)-(2.55) and (2.56)-(2.58), and collecting all the terms of the zeroth order we have

$$-\partial_{zz}\mu_B^{(0)}=0,$$
$$-\partial_{zz}\sigma_B^{(0)}=0,$$
$$-\partial_{zz}u_B^{(0)}+f'(u_B^{(0)})=0,$$

and on $\partial\Omega\times[0,T]$

$$\partial_z\mu_B^{(0)}(x,t,0)=0,$$
$$\partial_z\sigma_B^{(0)}(x,t,0)=0,$$
$$\partial_z u_B^{(0)}(x,t,0)=0.$$

Therefore we can take

$$\mu_B^{(0)}=\mu_+^{(0)},\quad \sigma_B^{(0)}=\sigma_+^{(0)},\quad u_B^{(0)}=u_+^{(0)}=1, \tag{2.59}$$

which satisfy (2.50)-(2.52) in the case of $i=0$.

Next, substituting (2.47)-(2.49) into (2.53)-(2.55) and (2.56)-(2.58), and collecting all the terms of ε-order we have

$$-\partial_{zz}\mu_B^{(1)}=0,$$
$$-\partial_{zz}\sigma_B^{(1)}=0,$$
$$-\partial_{zz}u_B^{(1)}+f''(1)u_B^{(1)}=\mu_B^{(0)},$$

and on $\partial\Omega\times[0,T]$

$$\partial_z\mu_B^{(1)}(x,t,0)=-\frac{\partial\mu_+^{(0)}}{\partial\mathbf{n}},$$
$$\partial_z\sigma_B^{(1)}(x,t,0)=-\frac{\partial\sigma_+^{(0)}}{\partial\mathbf{n}},$$
$$\partial_z u_B^{(1)}(x,t,0)=0,$$

where $\mathbf{n}$ is the unit outer normal vector on $\partial\Omega$. Thus, we can take

$$\mu_B^{(1)}=\mu_+^{(1)},\quad \sigma_B^{(1)}=\sigma_+^{(1)},\quad u_B^{(1)}=u_+^{(1)},$$

which satisfy (2.50)-(2.52) in the case of $i=1$ and imply that, on $\partial\Omega\times[0,T]$,

$$\frac{\partial\mu_+^{(0)}}{\partial\mathbf{n}}=\frac{\partial\sigma_+^{(0)}}{\partial\mathbf{n}}=0. \tag{2.60}$$

2.4. Solving the leading order terms. $u_{\pm}^{(0)}$ and $\widetilde{u}^{(0)}$ are determined by (2.7) and (2.33) respectively. Collecting (2.8), (2.9), (2.30), (2.39), (2.42), (2.45) and (2.60) one has

$$\begin{cases} -\Delta\mu_{\pm}^{(0)}+\mu_{\pm}^{(0)}=2\sigma_{\pm}^{(0)}\pm 1, & \text{in } \Omega_{\pm},\\ \partial_t\sigma_{\pm}^{(0)}-\Delta\sigma_{\pm}^{(0)}+2\sigma_{\pm}^{(0)}=\mu_{\pm}^{(0)}\mp 1, & \text{in } \Omega_{\pm},\\ [\mu^{(0)}]=[\sigma^{(0)}]=0, & \text{on } \Gamma,\\ \left[\frac{\partial\mu^{(0)}}{\partial\mathbf{n}}\right]=-2V, & \text{on } \Gamma,\\ \left[\frac{\partial\sigma^{(0)}}{\partial\mathbf{n}}\right]=0, & \text{on } \Gamma,\\ \mu_{\pm}^{(0)}=\kappa\int_{-1}^{1}\sqrt{2f(u)}du, & \text{on } \Gamma,\\ \frac{\partial\mu_{+}^{(0)}}{\partial\mathbf{n}}=\frac{\partial\sigma_{+}^{(0)}}{\partial\mathbf{n}}=0, & \text{on } \partial\Omega, \end{cases}$$

which is just the sharp interface model (1.2) of (1.1). Therefore we recover the sharp interface model by the above matched asymptotic expansion method.

Let (1.2) has a local smooth solution (μ,σ,Γ) which satisfies (1.6). Let $\mu_{\pm}^{(0)}=\mu|_{\Omega_{\pm}},\sigma_{\pm}^{(0)}=\sigma|_{\Omega_{\pm}}$ and $d^{(0)}$ be the signed distance to Γ. Then $\Gamma=\{(x,t)\in\Omega\times(0,T): d^{(0)}(x,t)=0\}$ and $\Omega_{\pm}=\{(x,t)\in\Omega\times(0,T): d^{(0)}(x,t)\gtrless 0\}$. $p^{(0)},q^{(0)},g^{(0)},h^{(0)}$ and $l^{(0)}$ are determined by (2.31), (2.32), (2.40), (2.43) and (2.46) respectively. $\widetilde{\mu}^{(0)}$ and $\widetilde{\sigma}^{(0)}$ are determined by (2.27) and (2.29) respectively. And $\mu_B^{(0)},\sigma_B^{(0)},u_B^{(0)}$ are determined by (2.59). Moreover, the inner-outer matching conditions (2.13)-(2.15) and the boundary-outer matching conditions (2.50)-(2.52) hold for $i=0$.

REMARK 2.2. We can extend $(u_{\pm}^{(0)},\mu_{\pm}^{(0)},\sigma_{\pm}^{(0)})$ smoothly from $\Omega_{\pm}$ to Ω as Remark 4.1 in [3].

3. Spectral condition and error estimates

For clarity we leave the higher order expansions and the complete construction of the approximate solution $(u^A,\mu^A,\sigma^A,\varphi^A)$ to Section 4, see (4.38) for the explicit form. In this section we firstly establish a spectral condition and estimate the error between the approximate solution and the true solution. Then Theorem 1.2 and Corollary 1.1 can be proved.

3.1. Spectral condition.

THEOREM 3.1 (Spectral condition). *There exist two positive constants ε_0 and C such that for any $0<\varepsilon<\varepsilon_0$, $v\in H^1(\Omega)$ and $w\in H^2(\Omega)$ with $\Delta w=v$ there holds*

$$\int_{\Omega}\left(\varepsilon|\nabla v|^2+\frac{1}{\varepsilon}f''(u^A)v^2\right)dx\geq -C\int_{\Omega}|\nabla w|^2dx, \tag{3.1}$$

where u^A is the approximate solution which will be constructed in Section 4.5.

Thanks to Theorem 3.1 in [7] we only need to prove the following lemma.

LEMMA 3.1. *There exist two positive constants ε_0 and C such that for any $0<\varepsilon<\varepsilon_0$ and $v\in H^1(\Omega)$ there holds*

$$\int_{\Omega}\left(|\nabla v|^2+\frac{1}{\varepsilon^2}f''(u^A)v^2\right)dx\geq -C\int_{\Omega}v^2dx. \tag{3.2}$$

In fact, if

$$\int_\Omega \Big(|\nabla v|^2 + \frac{1}{\varepsilon^2} f''(u^A) v^2\Big) dx \geq 0,$$

then (3.1) holds obviously. If

$$\int_\Omega \Big(|\nabla v|^2 + \frac{1}{\varepsilon^2} f''(u^A) v^2\Big) dx \leq 0,$$

and $w \in H^2(\Omega)$ with $\Delta w = v$, then the assumptions of Theorem 3.1 in [7] hold, hence we have

$$-\varepsilon \int_\Omega v^2 dx \geq -C \int_\Omega |\nabla w|^2 dx$$

which together with (3.2) immediately implies (3.1).

Theorem 3.1 and Lemma 3.1 have been used to prove the convergence of the Cahn-Hilliard equation to the Hele-Shaw model in [3]. However the proof of Lemma 3.1 was established in another paper [7]. Here we write the details in a concise way which shows clearly how to deal with the singular terms.

In order to prove Lemma 3.1 we show the following proposition on the spectral analysis of Neumann eigenvalue problem (1.4). The proof has been given in [7] and proved by a new method in [14].

Proposition 3.1 ([7]).

(1) *(Estimate of the first eigenvalue of $\mathcal{L}_f$)*

$$\lambda_1^f \triangleq \inf_{\|q\|=1} \int_{I_\varepsilon} \Big((q')^2 + f''(\theta) q^2\Big) dz = O(e^{-\frac{C}{\varepsilon}}), \tag{3.3}$$

here C is a positive constant independent of small ε and $\|q\| = \big(\int_{I_\varepsilon} q^2(z)dz\big)^{\frac{1}{2}}$.

(2) *(Estimate of the second eigenvalue of $\mathcal{L}_f$)*

$$\lambda_2^f \triangleq \inf_{\|q\|=1, q\perp q_1^f} \int_{I_\varepsilon} \Big((q')^2 + f''(\theta) q^2\Big) dz \geq c_f > 0, \tag{3.4}$$

here $q \perp q_1^f \Leftrightarrow \int_{I_\varepsilon} q q_1^f dz = 0$, q_1^f is the normalized eigenfunction corresponding to λ_1^f and c_f is a positive constant independent of small ε.

(3) *(Characterization of the first normalized eigenfunction of $\mathcal{L}_f$)*

$$\|q_1^f - \alpha\theta'\|^2 = O(e^{-\frac{C}{\varepsilon}}), \tag{3.5}$$

here $\alpha = \frac{1}{\|\theta'\|}$.

Let

$$d^{[k]}(x,t) = d^{(0)}(x,t) + \varepsilon d^{(1)}(x,t) + \varepsilon^2 d^{(2)}(x,t) + \cdots + \varepsilon^k d^{(k)}(x,t),$$

then $d^{[k]}(x,t) (k \geq 1)$ is a k-th approximate of the signed distance from x to the interface $\Gamma_k^\varepsilon \triangleq \{(x,t) : d^{[k]}(x,t) = 0\}$ in the following sense.

LEMMA 3.2. *For every fixed $t\in[0,T]$, let $r_t(x)$ be the signed distance from x to Γ_k^ε. Then for small ε*

$$\left\|r_t(x)-d^{[k]}(x,t)\right\|_{C^1(\Gamma_t(\delta))}=O(\varepsilon^{k+1}). \tag{3.6}$$

Proof. Noting that $|\nabla d^{[k]}|^2=1+O(\varepsilon^{k+1})$, then for small ε one gets

$$|\nabla d^{[k]}|-1=\frac{|\nabla d^{[k]}|^2-1}{|\nabla d^{[k]}|+1}=O(\varepsilon^{k+1}).$$

Since $r_t(x)$ is the signed distance, then $|\nabla r_t|=1$ and ∇r_t is parallel to $\nabla d^{[k]}$. Consequently, we obtain

$$\begin{aligned}|\nabla r_t(x)-\nabla d^{[k]}(x,t)|^2&=|\nabla r_t(x)|^2-2\nabla d^{[k]}(x,t)\cdot\nabla r_t(x)+|\nabla d^{[k]}(x,t)|^2\\&=1-2|\nabla d^{[k]}(x,t)|+|\nabla d^{[k]}(x,t)|^2\\&=\left(1-|\nabla d^{[k]}(x,t)|\right)^2\\&=O(\varepsilon^{2(k+1)}).\end{aligned}$$

Choosing $x_0\in\Gamma_k^\varepsilon$, i.e., $r_t(x_0)=d^{[k]}(x_0,t)=0$, then

$$\begin{aligned}|r_t(x)-d^{[k]}(x,t)|&=|r_t(x)-d^{[k]}(x,t)-r_t(x_0)+d^{[k]}(x_0,t)|\\&=\left|\int_0^1\Big(\nabla r_t(t'x+(1-t')x_0)-\nabla d^{[k]}\big((t'x+(1-t')x_0),t\big)\Big)\cdot(x-x_0)dt'\right|\\&\le C\varepsilon^{k+1},\end{aligned}$$

thus the proof of this lemma is complete. ∎

Let $s_t(x)$ be the projection of x on Γ_k^ε along the normal of Γ_k^ε. Then the transformation $x\longmapsto(r_t(x),s_t(x))$ is a diffeomorphism in $\Gamma_k^\varepsilon(\delta)$ for small δ. Let $J(r_t,s_t)=\det\frac{\partial x^{-1}(r_t,s_t)}{\partial(r_t,s_t)}$ be the Jacobian of the transformation, then $J|_{\Gamma_k^\varepsilon}=1$ and $\frac{\partial J}{\partial r_t}|_{\Gamma_k^\varepsilon}=0$. Thus

$$0<C_1\le J(r_t,s_t)\le C_2,\quad \left|J_{r_t}(r_t,s_t)\triangleq\frac{\partial J}{\partial r_t}(r_t,s_t)\right|\le C\left|r_t\right|. \tag{3.7}$$

In view of (2.36) and the similar arguments as those on page 199 in [3] we obtain

LEMMA 3.3. *In $\Gamma(\delta)$, $\widetilde{u}^{(1)}$ can be expressed as*

$$\widetilde{u}^{(1)}(x,t,z)\Big|_{z=\frac{r_t(x)}{\varepsilon}}=\bar{p}(s_t(x))\theta_1\left(\frac{r_t(x)}{\varepsilon}\right)+\bar{q}(x)=\bar{p}(s_t(x))\theta_1(z)+\bar{q}(x),$$

where $\theta_1\in L^\infty(\mathbb{R})$, $\bar{p}\in L^\infty(\Gamma(\delta))$ and

$$\int_{-\infty}^{+\infty}f'''\big(\theta(z)\big)\theta_1(z)\big(\theta'(z)\big)^2dz=0,\quad \left|\bar{q}(x)\right|\le C(\varepsilon+\left|r_t(x)\right|)\le C\varepsilon(1+|z|).$$

Now we focus on the proof of Lemma 3.1.

Proof. **(Proof of Lemma 3.1.)** For clarity we divide the proof into three steps.

Step 1. Noting that $f''(\pm 1)>0$, then for small ε there holds

$$\begin{aligned}
&\int_\Omega \Big(|\nabla v|^2+\frac{1}{\varepsilon^2}f''(u^A)v^2\Big)dx\\
\geq &\int_{\Gamma_k^\varepsilon(\delta)} \Big(|\nabla v|^2+\frac{1}{\varepsilon^2}f''(u^A)v^2\Big)dx\\
=&\varepsilon^{-2}\int_{\Gamma_k^\varepsilon}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \Big(|\partial_z v|^2+f''(u^A)v^2\Big)J(r_t(x),s_t(x))dzdS\\
\geq&\varepsilon^{-2}\int_{\Gamma_k^\varepsilon}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \Big(|\partial_z v|^2+f''\big(\theta(z)\big)v^2\Big)J(r_t(x),s_t(x))dzdS\\
&+\varepsilon^{-1}\int_{\Gamma_k^\varepsilon}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\widetilde{u}^{(1)}(x,t,z)v^2J(r_t(x),s_t(x))dzdS-C\int_\Omega v^2dx,
\end{aligned}$$

where the following fact is used: in $\Gamma_k^\varepsilon(\delta)$, there holds

$$\begin{aligned}
f''(u^A)&=f''\Big(\theta(z)+\varepsilon\widetilde{u}^{(1)}(x,t,z)+O(\varepsilon^2)\Big)\Big|_{z=\frac{d[k]}{\varepsilon}}\\
&=f''\big(\theta(z)\big)+\varepsilon f'''\big(\theta(z)\big)\widetilde{u}^{(1)}(x,t,z)+O(\varepsilon^2)\Big|_{z=\frac{d[k]}{\varepsilon}}\\
&=f''\big(\theta(z)\big)+\varepsilon f'''\big(\theta(z)\big)\widetilde{u}^{(1)}(x,t,z)+O(\varepsilon^2)\Big|_{z=\frac{r_t(x)}{\varepsilon}}.
\end{aligned}$$

Set $\hat{v}=vJ^{\frac{1}{2}}$, from Lemma 5.8 in [14] (the proof is presented in the Appendix for completeness of the paper), we deduce that

$$\begin{aligned}
&\int_{\Gamma_k^\varepsilon}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \Big(|\partial_z v|^2+f''\big(\theta(z)\big)v^2\Big)J(r_t(x),s_t(x))dzdS\\
\geq&\frac{3}{4}\int_{\Gamma_k^\varepsilon}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \Big(|\partial_z \hat{v}|^2+f''\big(\theta(z)\big)\hat{v}^2\Big)dzdS-C\varepsilon^2\int_\Omega v^2dx.
\end{aligned}\tag{3.8}$$

Consequently

$$\int_\Omega \Big(|\nabla v|^2+\frac{1}{\varepsilon^2}f''(u^A)v^2\Big)dx\geq \varepsilon^{-2}\int_{\Gamma_k^\varepsilon} IdS+\varepsilon^{-1}\int_{\Gamma_k^\varepsilon} IIdS-C\int_\Omega v^2dx, \tag{3.9}$$

where

$$I=\frac{3}{4}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\Big(|\partial_z\hat{v}|^2+f''\big(\theta(z)\big)\hat{v}^2\Big)dz,\ II=\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\widetilde{u}^{(1)}(x,t,z)\hat{v}^2dz.$$

Step 2. To deal with the term I, we decompose $\hat{v}=\gamma q_1^f+p_1$, here $p_1\perp q_1^f$ and then $\|\hat{v}\|^2=\gamma^2+\|p_1\|^2$. Thus

$$\begin{aligned}
I=&\frac{3}{4}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\Big(|\partial_z\hat{v}|^2+f''\big(\theta(z)\big)\hat{v}^2\Big)dz\\
=&\frac{3}{4}\gamma^2\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\Big(\big((q_1^f)'\big)^2+f''\big(\theta(z)\big)(q_1^f)^2\Big)dz+\frac{3}{4}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\Big((\partial_z p_1)^2+f''\big(\theta(z)\big){p_1}^2\Big)dz
\end{aligned}$$

$$\begin{aligned}&\geq \frac{3}{4}\gamma^2\lambda_1^f+\frac{3}{4}\lambda_2^f\|p_1\|^2\geq -C\varepsilon^2\gamma^2+\frac{3}{4}\lambda_2^f\|p_1\|^2\\&\geq -C\varepsilon^2\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}v^2Jdz+\frac{3}{4}\lambda_2^f\|p_1\|^2,\end{aligned}$$

where we use (3.3) and (3.4).

Step 3. To estimate II, we use Lemma 3.3 to deduce that

$$\begin{aligned}&\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\widetilde{u}^{(1)}(x,t,z)(\theta'(z))^2dz\\
=&\bar{p}(s_t(x))\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\theta_1(z)(\theta'(z))^2dz+\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\bar{q}(x)(\theta'(z))^2dz\\
=&-\bar{p}(s_t(x))\int_{-\infty}^{-\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\theta_1(z)(\theta'(z))^2dz-\bar{p}(s_t(x))\int_{\frac{\delta}{\varepsilon}}^{+\infty} f'''\big(\theta(z)\big)\theta_1(z)(\theta'(z))^2dz\\
&+\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\bar{q}(x)(\theta'(z))^2dz\\
=&O(\varepsilon).\end{aligned}$$

Hence there holds

$$\begin{aligned}II=&\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\widetilde{u}^{(1)}(x,t,z)\hat{v}^2dzdS\\
=&\gamma^2\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\widetilde{u}^{(1)}(q_1^f)^2dz+2\gamma\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\widetilde{u}^{(1)}q_1^fp_1dz\\
&+\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\widetilde{u}^{(1)}{p_1}^2dz\\
=&\gamma^2\alpha^2\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} f'''\big(\theta(z)\big)\widetilde{u}^{(1)}(\theta')^2dz+O(e^{-\frac{C}{\varepsilon}})\gamma^2+O(1)\|p_1\|\gamma+O(1)\|p_1\|^2\\
=&O(\varepsilon)\gamma^2+O(e^{-\frac{C}{\varepsilon}})\gamma^2+O(1)\gamma\|p_1\|+O(1)\|p_1\|^2\\
\geq&\big(O(e^{-\frac{C}{\varepsilon}})+O(\varepsilon)\big)\gamma^2-\frac{1}{8\varepsilon}\lambda_2^f\|p_1\|^2\\
\geq&-C\varepsilon\gamma^2-\frac{1}{8\varepsilon}\lambda_2^f\|p_1\|^2\\
\geq&-C\varepsilon\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}v^2Jdz-\frac{1}{8\varepsilon}\lambda_2^f\|p_1\|^2,\end{aligned}$$

where we have used $\hat{v}=\gamma q_1^f+p_1$ and (3.5).

Finally, we obtain

$$\varepsilon^{-2}\int_{\Gamma_k^\varepsilon}IdS+\varepsilon^{-1}\int_{\Gamma_k^\varepsilon}IIdS\geq -C\int_\Omega v^2dx$$

which together with (3.9) implies (3.2). Thus the lemma is concluded. ∎

3.2. Error estimates. Let $u^{err}=u^\varepsilon-u^A, \mu^{err}=\mu^\varepsilon-\mu^A, \sigma^{err}=\sigma^\varepsilon-\sigma^A, \varphi^{err}=\varphi^\varepsilon-\varphi^A$ with $\varphi^\varepsilon=u^\varepsilon+\sigma^\varepsilon$ and impose

$$u_0^\varepsilon(x)=\varphi^A(x,0)-\sigma_0^\varepsilon(x),\ \sigma_0^\varepsilon(x)=\sigma^A(x,0). \tag{3.10}$$

Here $(u^A,\mu^A,\sigma^A,\varphi^A)$ is defined in (4.38).

By (1.1) and (1.3) there hold

$$\begin{cases} \partial_t\varphi^{err}-\Delta\mu^{err}-\Delta\sigma^{err}=0, & \text{in } \Omega\times(0,T),\\ \partial_t\sigma^{err}-\Delta\sigma^{err}=-(2\sigma^{err}+u^{err}-\mu^{err}), & \text{in } \Omega\times(0,T),\\ \mu^{err}=-\varepsilon\Delta u^{err}+\frac{1}{\varepsilon}f''(u^A)u^{err}+\frac{1}{\varepsilon}\mathbb{F}-\mathfrak{R}_1, & \text{in } \Omega\times(0,T),\\ u^{err}=\varphi^{err}-\sigma^{err}-\mathfrak{R}_2, & \text{in } \Omega\times(0,T),\\ \varphi^{err}(x,0)=\sigma^{err}(x,0)=0, & \text{on } \Omega\times\{0\},\\ \frac{\partial\varphi^{err}}{\partial\mathbf{n}}=\frac{\partial\mu^{err}}{\partial\mathbf{n}}=\frac{\partial\sigma^{err}}{\partial\mathbf{n}}=0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{3.11}$$

where

$$\mathbb{F}=f'(u^{err}+u^A)-f'(u^A)-f''(u^A)u^{err}=4(u^{err})^3+8u^A(u^{err})^2.$$

THEOREM 3.2. *For small ε and large k, there exist $\gamma=\gamma(k)\in(1,k)$ which is an increasing function of k and a positive constant C such that*

$$\|u^{err}\|_{L^p(\Omega\times(0.T))}+\|\varphi^{err}\|_{L^p(\Omega\times(0.T))}+\|\sigma^{err}\|_{L^p(\Omega\times(0.T))}\le C\varepsilon^\gamma.$$

Proof. For the sake of simplicity, we omit the superscript "err" in the proof of this theorem.

Noting that

$$\int_\Omega\varphi(t,x)dx=\int_0^t\int_\Omega\partial_t\varphi(t,x)dx=\int_0^t\int_\Omega(\Delta\mu+\Delta\sigma)(t,x)dx=0,$$

then there exists a function $\psi(t,\cdot)(t\in(0,T))$ which satisfies

$$\begin{cases} -\Delta\psi=\varphi, & \text{in } \Omega,\\ \frac{\partial\psi}{\partial\mathbf{n}}=0, & \text{on } \partial\Omega,\\ \int_\Omega\psi(t,x)dx=0,\ t\in(0,T). \end{cases} \tag{3.12}$$

Multiplying the first equation in (3.11) by ψ and integrating by parts we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\psi|^2dx+\int_\Omega\left(\varepsilon|\nabla\varphi|^2+\frac{1}{\varepsilon}f''(u^A)\varphi^2\right)dx\\ &\quad-\int_\Omega\left(\varepsilon\nabla\varphi\nabla\sigma+\frac{1}{\varepsilon}f''(u^A)\varphi\sigma\right)dx+\frac{1}{\varepsilon}\int_\Omega\mathbb{F}\varphi dx+\int_\Omega\nabla\psi\nabla\sigma dx\\ &=\int_\Omega\omega_6^A\varphi dx, \end{aligned} \tag{3.13}$$

where

$$\omega_6^A=\omega_4^A+\varepsilon\Delta\omega_5^A-\varepsilon^{-1}f''(u^A)\omega_5^A=O(\varepsilon^{k-2}).$$

Multiplying the second equation in (3.11) by σ and integrating by parts we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \sigma^2 dx+\int_\Omega |\nabla\sigma|^2 dx+\int_\Omega \sigma^2 dx+\int_\Omega\Big(\varepsilon|\nabla\sigma|^2+\frac{1}{\varepsilon}f''(u^A)\sigma^2\Big)dx\\&\qquad-\int_\Omega\Big(\varepsilon\nabla\varphi\nabla\sigma+\frac{1}{\varepsilon}f''(u^A)\varphi\sigma\Big)dx-\frac{1}{\varepsilon}\int_\Omega \mathbb{F}\sigma dx+\int_\Omega\nabla\psi\nabla\sigma dx\\&=\int_\Omega \omega_7^A\sigma dx,\end{aligned}\tag{3.14}$$

where

$$\omega_7^A=-\omega_4^A-\omega_5^A-\varepsilon\Delta\omega_5^A+\varepsilon^{-1}f''(u^A)\omega_5^A=O(\varepsilon^{k-2}).$$

Combining (3.13) and (3.14) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\psi|^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega\sigma^2dx+\int_\Omega|\nabla\sigma|^2dx+\int_\Omega\sigma^2dx\\&\quad+\int_\Omega\Big(\varepsilon|\nabla(\varphi-\sigma)|^2+\frac{1}{\varepsilon}f''(u^A)(\varphi-\sigma)^2\Big)dx+2\int_\Omega\nabla\psi\nabla\sigma dx+\frac{1}{\varepsilon}\int_\Omega(\varphi-\sigma)\mathbb{F}dx\\&=\int_\Omega\omega_6^A\varphi dx+\int_\Omega\omega_7^A\sigma dx.\end{aligned}\tag{3.15}$$

Moreover, we can easily find that

$$\begin{aligned}(\varphi-\sigma)\mathbb{F}=&4(\varphi-\sigma)^4+(8u^A+12\omega_5^A)(\varphi-\sigma)^3+(16u^A\omega_5^A+12(\omega_5^A)^2)(\varphi-\sigma)^2\\&+(8u^A(\omega_5^A)^2+4(\omega_5^A)^3)(\varphi-\sigma)\\\geq&-\widetilde{C}_p|\varphi-\sigma|^p+\omega_8^A|\varphi-\sigma|,\\\geq&-C_p|\varphi|^p-C_p|\sigma|^p+\omega_8^A|\varphi-\sigma|,\quad\forall p\in(1,3],\end{aligned}\tag{3.16}$$

where the positive constants $\widetilde{C}_p,C_p$ depend on p and $\omega_8^A=O(\varepsilon^{k-1})$. Plugging (3.16) into (3.15), and using Young's inequality and the Sobolev inequality, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\psi|^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega\sigma^2dx+\frac{1}{2}\int_\Omega|\nabla\sigma|^2dx+\int_\Omega\sigma^2dx\\&\quad+\int_\Omega\Big(\varepsilon|\nabla(\varphi-\sigma)|^2+\frac{1}{\varepsilon}f''(u^A)(\varphi-\sigma)^2\Big)dx\\&\leq C\int_\Omega|\nabla\psi|^2dx+C\int_\Omega\sigma^2dx+\frac{C_p}{\varepsilon}\int_\Omega|\varphi|^pdx+\frac{C_p}{\varepsilon}\int_\Omega|\sigma|^pdx\\&\quad+\Big(\int_\Omega|\varphi|^pdx\Big)^{\frac{1}{p}}\Big(\int_\Omega|\omega_9^A|^qdx\Big)^{\frac{1}{q}}+\Big(\int_\Omega|\sigma|^pdx\Big)^{\frac{1}{p}}\Big(\int_\Omega|\omega_{10}^A|^qdx\Big)^{\frac{1}{q}},\end{aligned}\tag{3.17}$$

where $\omega_9^A=O(\varepsilon^{k-2}),\omega_{10}^A=O(\varepsilon^{k-2})$ and $\frac{1}{p}+\frac{1}{q}=1$.

According to the spectral condition in Theorem 3.1 one has for small ε

$$\int_\Omega\Big(\varepsilon|\nabla(\varphi-\sigma)|^2+\frac{1}{\varepsilon}f''(u^A)(\varphi-\sigma)^2\Big)dx\geq-C\int_\Omega|\nabla(\psi-\varrho)|^2dx,\tag{3.18}$$

where ϱ is the solution of the following equation

$$\begin{cases}-\Delta\varrho=\sigma, & \text{in }\ \Omega,\\ \varrho=0, & \text{on }\partial\Omega,\end{cases}$$

and C is a positive constant independent of t. By the Poincaré inequality, we deduce

$$\int_\Omega |\nabla \varrho|^2 dx \leq C \int_\Omega \sigma^2 dx. \tag{3.19}$$

Plugging (3.18)-(3.19) into (3.17) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\psi|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \sigma^2 dx + \frac{1}{2}\int_\Omega |\nabla\sigma|^2 dx + \int_\Omega \sigma^2 dx\\ \leq & C\int_\Omega |\nabla\psi|^2 dx + C\int_\Omega \sigma^2 dx + \frac{C_p}{\varepsilon}\int_\Omega |\varphi|^p dx + \frac{C_p}{\varepsilon}\int_\Omega |\sigma|^p dx\\ &+ \left(\int_\Omega |\varphi|^p dx\right)^{\frac{1}{p}}\left(\int_\Omega |\omega_9^A|^q dx\right)^{\frac{1}{q}} + \left(\int_\Omega |\sigma|^p dx\right)^{\frac{1}{p}}\left(\int_\Omega |\omega_{10}^A|^q dx\right)^{\frac{1}{q}}.\end{aligned}$$

It follows from the Grönwall inequality that for any $t\in(0,T]$

$$\begin{aligned}&\sup_{0\leq t'\leq t}\left(\|\nabla\psi\|^2_{L^2(\Omega)} + \|\sigma\|^2_{L^2(\Omega)}\right) + \|\sigma\|^2_{L^2(\Omega\times(0,t))} + \|\nabla\sigma\|^2_{L^2(\Omega\times(0,t))}\\ \leq & C\Big(\varepsilon^{-1}\|\varphi\|^p_{L^p(\Omega\times(0,t))} + \varepsilon^{-1}\|\sigma\|^p_{L^p(\Omega\times(0,t))} + \|\varphi\|_{L^p(\Omega\times(0,t))}\|\omega_9^A\|_{L^q(\Omega\times(0,t))}\\ &\quad + \|\sigma\|_{L^p(\Omega\times(0,t))}\|\omega_{10}^A\|_{L^q(\Omega\times(0,t))}\Big).\end{aligned} \tag{3.20}$$

Furthermore, by (3.17) and (3.20), we obtain

$$\begin{aligned}&\|\nabla(\varphi-\sigma)\|^2_{L^2(\Omega\times(0.t))}\\ \leq & \varepsilon^{-2}\left(-\int_0^t\int_\Omega f''(u^A)(\varphi-\sigma)^2 dx dt'\right) + C\varepsilon^{-2}\left(\|\varphi\|^p_{L^p(\Omega\times(0,t))} + \|\sigma\|^p_{L^p(\Omega\times(0,t))}\right)\\ &+ C\varepsilon^{-1}\left(\|\varphi\|_{L^p(\Omega\times(0,t))}\|\omega_9^A\|_{L^q(\Omega\times(0,t))} + \|\sigma\|_{L^p(\Omega\times(0,t))}\|\omega_{10}^A\|_{L^q(\Omega\times(0,t))}\right)\\ \leq & \varepsilon^{-2}\|\varphi-\sigma\|^2_{L^p(\Omega\times(0,t))}\|f''(u^A)\|_{L^\infty(\Omega\times(0,t))}\text{measure}\{f''(u^A)<0\}^{1-\frac{2}{p}}\\ &+ C\varepsilon^{-2}\left(\|\varphi\|^p_{L^p(\Omega\times(0,t))} + \|\sigma\|^p_{L^p(\Omega\times(0,t))}\right) + C\varepsilon^{-1}\|\varphi\|_{L^p(\Omega\times(0,t))}\|\omega_7^A\|_{L^q(\Omega\times(0,t))}\\ \leq & C\varepsilon^{-1-\frac{2}{p}}\left(\|\varphi\|^2_{L^p(\Omega\times(0,t))} + \|\sigma\|^2_{L^p(\Omega\times(0,t))}\right) + C\varepsilon^{-2}\left(\|\varphi\|^p_{L^p(\Omega\times(0,t))} + \|\sigma\|^p_{L^p(\Omega\times(0,t))}\right)\\ &+ C\varepsilon^{-1}\left(\|\varphi\|_{L^p(\Omega\times(0,t))}\|\omega_9^A\|_{L^q(\Omega\times(0,t))} + \|\sigma\|_{L^p(\Omega\times(0,t))}\|\omega_{10}^A\|_{L^q(\Omega\times(0,t))}\right).\end{aligned}$$

Thus, there holds

$$\begin{aligned}&\|\nabla\varphi\|^2_{L^2(\Omega\times(0,t))}\\ \leq & \|\nabla(\varphi-\sigma)\|^2_{L^2(\Omega\times(0,t))} + \|\nabla\sigma\|^2_{L^2(\Omega\times(0,t))}\\ \leq & C\varepsilon^{-1-\frac{2}{p}}\left(\|\varphi\|^2_{L^p(\Omega\times(0,t))} + \|\sigma\|^2_{L^p(\Omega\times(0,t))}\right) + C\varepsilon^{-2}\left(\|\varphi\|^p_{L^p(\Omega\times(0,t))} + \|\sigma\|^p_{L^p(\Omega\times(0,t))}\right)\\ &+ C\varepsilon^{-1}\left(\|\varphi\|_{L^p(\Omega\times(0,t))}\|\omega_9^A\|_{L^q(\Omega\times(0,t))} + \|\sigma\|_{L^p(\Omega\times(0,t))}\|\omega_{10}^A\|_{L^q(\Omega\times(0,t))}\right).\end{aligned}$$

Applying the Sobolev imbedding and the Hölder inequality we get

$$\|\varphi\|^p_{L^p(\Omega\times(0,t))} + \|\sigma\|^p_{L^p(\Omega\times(0,t))}$$

$$
\begin{aligned}
&= \int_0^t \Big(\|\varphi\|_{L^p(\Omega)}^p + \|\sigma\|_{L^p(\Omega)}^p \Big)(t')dt' \\
&\le C\int_0^t \|\varphi\|_{L^2(\Omega)}^{\theta p} \|\nabla\varphi\|_{L^2(\Omega)}^{(1-\theta)p}(t')dt' + C\int_0^t \|\sigma\|_{L^2(\Omega)}^{\theta p} \|\nabla\sigma\|_{L^2(\Omega)}^{(1-\theta)p}(t')dt' \\
&\quad + C\int_0^t \|\sigma\|_{L^2(\Omega)}^{p}(t')dt' \\
&\le C\int_0^t \|\nabla\psi\|_{L^2(\Omega)}^{\frac{\theta p}{2}} \|\nabla\varphi\|_{L^2(\Omega)}^{(1-\frac{\theta}{2})p}(t')dt' + C\int_0^t \|\sigma\|_{L^2(\Omega)}^{\theta p} \|\nabla\sigma\|_{L^2(\Omega)}^{(1-\theta)p}(t')dt' \\
&\quad + C\int_0^t \|\sigma\|_{L^2(\Omega)}^{p}(t')dt' \\
&\le C\Big(\int_0^t \|\nabla\psi\|_{L^2(\Omega)}^{\frac{\alpha\theta p}{2}}(t')dt'\Big)^{\frac{1}{\alpha}} \|\nabla\varphi\|_{L^2(\Omega\times(0,t))}^{(1-\frac{\theta}{2})p} \\
&\quad + C\Big(\int_0^t \|\sigma\|_{L^2(\Omega)}^{\beta\theta p}(t')dt'\Big)^{\frac{1}{\beta}} \|\nabla\sigma\|_{L^2(\Omega\times(0,t))}^{(1-\theta)p} + C\int_0^t \|\sigma\|_{L^2(\Omega)}^{p}(t')dt' \\
&\le C\Big(\sup_{0\le t'\le t} \|\nabla\psi\|_{L^2(\Omega)}\Big)^{\frac{\theta p}{2}} \|\nabla\varphi\|_{L^2(\Omega\times(0,t))}^{(1-\frac{\theta}{2})p} \\
&\quad + C\Big(\sup_{0\le t'\le t} \|\sigma\|_{L^2(\Omega)}\Big)^{\theta p} \|\nabla\sigma\|_{L^2(\Omega\times(0,t))}^{(1-\theta)p} + CT\Big(\sup_{0\le t'\le t} \|\sigma\|_{L^2(\Omega)}\Big)^{p},
\end{aligned}
\tag{3.21}
$$

where $\frac{1}{p}=\frac{\theta}{2}+\frac{(N-2)(1-\theta)}{2N}$, $\frac{1}{\alpha}+\frac{(1-\frac{\theta}{2})p}{2}=1$ and $\frac{1}{\beta}+\frac{(1-\theta)p}{2}=1$.

Setting $\Theta(t)=\|\varphi\|_{L^p(\Omega\times(0,t))}+\|\sigma\|_{L^p(\Omega\times(0,t))}(t\in(0,T])$, then with the help of (3.20)-(3.21), we derive a recursive inequality for $\Theta(t)$:

$$
\begin{aligned}
\Theta^p(t) \le & \Big(\varepsilon^{-1}\Theta^p(t)+\Theta(t)\|\omega_{11}^A\|_{L^q(\Omega\times(0,t))}\Big)^{\frac{\theta p}{4}} \\
&\cdot\Big(\varepsilon^{-1-\frac{2}{p}}\Theta^2(t)+\varepsilon^{-2}\Theta^p(t)+\varepsilon^{-1}\Theta(t)\|\omega_{11}^A\|_{L^q(\Omega\times(0,t))}\Big)^{\left(\frac{1}{2}-\frac{\theta}{4}\right)p} \\
&+C\Big(\varepsilon^{-1}\Theta^p(t)+\Theta(t)\|\omega_{11}^A\|_{L^q(\Omega\times(0,t))}\Big)^{\frac{p}{2}}
\end{aligned}
\tag{3.22}
$$

where $\omega_{11}^A=O(\varepsilon^{k-2})$. Noting that if we fix $p\in(2,3]$, then

$$
\frac{\theta p}{4}+2\Big(\frac{1}{2}-\frac{\theta}{4}\Big)=1+\frac{\theta(p-2)}{4}>1,\quad \frac{p}{2}>1.
$$

By the continuity argument, for small ε and large k there exists $\gamma=\gamma(k)\in(1,k)$ which tends to $+\infty$ as $k\to+\infty$ such that

$$
\Theta(T)\le\varepsilon^{\gamma}.
$$

Therefore it holds that

$$
\|\varphi\|_{L^p(\Omega\times(0,T))}+\|\sigma\|_{L^p(\Omega\times(0,T))}\le\varepsilon^{\gamma},
$$

and

$$
\|u\|_{L^p(\Omega\times(0,T))}=\|\varphi-\sigma+\omega_5^A\|_{L^p(\Omega\times(0,T))}
$$

$$\le \|\varphi\|_{L^p(\Omega\times(0,T))} + \|\sigma\|_{L^p(\Omega\times(0,T))} + \|\omega_5^A\|_{L^p(\Omega\times(0,T))}$$
$$\le C\varepsilon^\gamma.$$

The proof is completed. ∎

In order to establish higher order regularity estimates of u^{err}, φ^{err} and σ^{err}, we consider the system for (u^{err}, σ^{err})

$$\begin{cases} \partial_t u^{err} - \Delta\mu^{err} + \mu^{err} = 2\sigma^{err} + u^{err} + \partial_t \omega_5^A, & \text{in } \Omega\times(0,T), \\ \partial_t \sigma^{err} - \Delta\sigma^{err} + 2\sigma^{err} = \mu^{err} - u^{err}, & \text{in } \Omega\times(0,T), \end{cases}$$

which is a Cahn-Hilliard equation coupled linearly with a heat equation. Using similar arguments as those in Theorem 2.3 in [3] and the boot-strap method we can give the desired conclusions. Here we omit the details. In particular we have

THEOREM 3.3. *For small ε and large k,*

$$\|u^{err}\|_{C^{4,1}(\overline{\Omega}\times[0.T])} + \|\varphi^{err}\|_{C^{2,1}(\overline{\Omega}\times[0.T])} + \|\sigma^{err}\|_{C^{2,1}(\overline{\Omega}\times[0.T])} + \|\mu^{err}\|_{C^{2,1}(\overline{\Omega}\times[0.T])} \le C\varepsilon.$$

Theorem 1.2 can be obtained with the aid of Theorem 3.3 if we take initial data $(u_0^\varepsilon(x), \sigma_0^\varepsilon(x))$ as (3.10). Next we give a proof of Corollary 1.1.

Proof. (**Proof of Corollary 1.1.**) Recalling the definition of u^A in Section 4, we easily obtain (1.7). Now we prove (1.8). More concretely we only prove

$$\|\sigma^\varepsilon - \sigma\|_{C(\overline{\Omega}\times[0.T])} \to 0, \quad \text{as } \varepsilon\to 0.$$

The other one in (1.8) is similar.

The definition of σ^A in Section 4 yields

$$\text{the leading order of } \sigma^A = \begin{cases} \sigma_+^{(0)}, & \text{in } \Omega_+\backslash\Gamma(\delta), \\ \widetilde{\sigma}^{(0)}\zeta(\frac{d^{(0)}}{\delta}) + \sigma_+^{(0)}(1-\zeta(\frac{d^{(0)}}{\delta})), & \text{in } \big(\Gamma(\delta)\backslash\Gamma(\frac{\delta}{2})\big)\cap\Omega_+, \\ \widetilde{\sigma}^{(0)}, & \text{in } \Gamma(\frac{\delta}{2}), \\ \widetilde{\sigma}^{(0)}\zeta(\frac{d^{(0)}}{\delta}) + \sigma_-^{(0)}(1-\zeta(\frac{d^{(0)}}{\delta})), & \text{in } \big(\Gamma(\delta)\backslash\Gamma(\frac{\delta}{2})\big)\cap\Omega_-, \\ \sigma_-^{(0)}, & \text{in } \Omega_-\backslash\Gamma(\delta). \end{cases}$$

Based on the inner-outer matching condition we find

$$\left\|\left(\widetilde{\sigma}^{(0)}\zeta(\frac{d^{(0)}}{\delta}) + \sigma_\pm^{(0)}(1-\zeta(\frac{d^{(0)}}{\delta}))\right) - \sigma_\pm^{(0)}\right\|_{C\left(\left(\Gamma(\delta)\backslash\Gamma(\frac{\delta}{2})\right)\cap\Omega_\pm\right)} \le Ce^{-\frac{\delta}{4\varepsilon}} \to 0, \quad \text{as } \varepsilon\to 0.$$

Consequently we only need to prove

$$\|\widetilde{\sigma}^{(0)} - \sigma\|_{C(\Gamma(\frac{\delta}{2}))} \to 0, \quad \text{as } \varepsilon\to 0.$$

Note that

$$\widetilde{\sigma}^{(0)}(x,t,z) = \begin{cases} \sigma_+^{(0)}(x,t), & \text{in } \{(x,t): d^{[k]}(x,t)\ge\varepsilon\}, \\ \eta(z)\sigma_+^{(0)}(x,t) + \big(1-\eta(z)\big)\sigma_-^{(0)}(x,t), & \text{in } \{(x,t): -\varepsilon < d^{[k]}(x,t) < \varepsilon\}, \\ \sigma_-^{(0)}(x,t), & \text{in } \{(x,t): d^{[k]}(x,t)\le -\varepsilon\}, \end{cases}$$

and

$$\sigma(x,t)=\begin{cases}\sigma_+^{(0)}(x,t), & \text{in } \Omega_+,\\ \kappa\int_{-1}^{1}\sqrt{2f(u)}du, & \text{on } \Gamma,\\ \sigma_-^{(0)}(x,t), & \text{in } \Omega_-.\end{cases}$$

By mean value theorem we derive that in $\Gamma(\frac{\delta}{2})$, there hold

$$\left|\sigma_+^{(0)}(x,t)-\kappa\int_{-1}^{1}\sqrt{2f(u)}du\right|,\ \left|\sigma_-^{(0)}(x,t)-\kappa\int_{-1}^{1}\sqrt{2f(u)}du\right|\le C\left|d^{(0)}(x,t)\right|.$$

Moreover,

$$\begin{aligned}\{(x,t):d^{[k]}(x,t)\ge\varepsilon\}\cap\Omega_-\subseteq\{(x,t):|d^{(0)}(x,t)|\le C\varepsilon\},\\ \{(x,t):-\varepsilon<d^{[k]}(x,t)<\varepsilon\}\subseteq\{(x,t):|d^{(0)}(x,t)|\le C\varepsilon\},\\ \{(x,t):d^{[k]}(x,t)\le-\varepsilon\}\cap\Omega_+\subseteq\{(x,t):|d^{(0)}(x,t)|\le C\varepsilon\}.\end{aligned}$$

Thus there holds

$$\left\|\widetilde{\sigma}^{(0)}-\sigma\right\|_{C(\Gamma(\frac{\delta}{2}))}\le C\varepsilon,$$

which implies the desired result, and the proof of Corollary 1.1 is completed. □

4. Matching $\varepsilon^k(k\ge 2)$-order expansions and construction of an approximate solution

4.1. Matching k-th order($k\ge 2$) outer expansion. Substituting (2.4)-(2.6) into (1.1) and collecting all the terms of ε^k-order($k\ge 2$) we have

$$u_\pm^{(k)}=\frac{\mu_\pm^{(k-1)}+\Delta u_\pm^{(k-2)}-g\big(u_\pm^{(0)},\cdots,u_\pm^{(k-1)}\big)}{f''(u_\pm^{(0)})}, \tag{4.1}$$

$$-\Delta\mu_\pm^{(k)}+\mu_\pm^{(k)}=2\sigma_\pm^{(k)}+u_\pm^{(k)}-\partial_t u_\pm^{(k)}, \tag{4.2}$$

$$\partial_t\sigma_\pm^{(k)}-\Delta\sigma_\pm^{(k)}+2\sigma_\pm^{(k)}=\mu_\pm^{(k)}-u_\pm^{(k)}. \tag{4.3}$$

4.2. Matching k-th order($k\ge 2$) inner expansion. Substituting (2.1) and (2.10)-(2.12) into (2.20)-(2.22) and collecting all the terms of ε^k-order we have

$$\begin{aligned}&-\partial_{zz}\Big(\widetilde{\mu}^{(k)}-\eta\big(p^{(k)}d^{(0)}+p^{(0)}d^{(k)}\big)\Big)\\ &=-\sum_{i=0}^{k-1}\Big(\partial_z\widetilde{u}^{(i)}\partial_t d^{(k-1-i)}-2\nabla_x\partial_z\widetilde{\mu}^{(i)}\cdot\nabla_x d^{(k-1-i)}-\partial_z\widetilde{\mu}^{(i)}\triangle_x d^{(k-1-i)}\Big)\\ &\quad-\Big(\partial_t\widetilde{u}^{(k-2)}-\Delta_x\widetilde{\mu}^{(k-2)}-\big(2\widetilde{\sigma}^{(k-2)}+\widetilde{u}^{(k-2)}-\widetilde{\mu}^{(k-2)}\big)\Big)\\ &\quad-\eta''\sum_{i=1}^{k-1}p^{(i)}d^{(k-i)}+\eta'' zp^{(k-1)}-\eta'\sum_{i=0}^{k-1}g^{(i)}d^{(k-1-i)}+z\eta' g^{(k-2)}+\big(s_+^{(k-2)}\eta^++s_-^{(k-2)}\eta^-\big)\\ &\triangleq\Theta_{k-1,1},\end{aligned} \tag{4.4}$$

$$-\partial_{zz}\Big(\widetilde{\sigma}^{(k)}-\eta\big(q^{(k)}d^{(0)}+q^{(0)}d^{(k)}\big)\Big)$$

$$
\begin{aligned}
&= -\sum_{i=0}^{k-1}\Big(\partial_z\widetilde{\sigma}^{(i)}\partial_t d^{(k-1-i)} - 2\nabla_x\partial_z\widetilde{\sigma}^{(i)}\cdot\nabla_x d^{(k-1-i)} - \partial_z\widetilde{\sigma}^{(i)}\triangle_x d^{(k-1-i)}\Big)\\
&\quad -\Big(\partial_t\widetilde{\sigma}^{(k-2)} - \Delta_x\widetilde{\sigma}^{(k-2)} + 2\widetilde{\sigma}^{(k-2)} + \widetilde{u}^{(k-2)} - \widetilde{\mu}^{(k-2)}\Big)\\
&\quad -\eta''\sum_{i=1}^{k-1} q^{(i)}d^{(k-i)} + \eta'' z q^{(k-1)} - \eta'\sum_{i=0}^{k-1} h^{(i)}d^{(k-1-i)} + z\eta' h^{(k-2)}\\
&\quad +\big(r_+^{(k-2)}\eta^+ + r_-^{(k-2)}\eta^-\big)\\
&\triangleq \Theta_{k-1,2},
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
&\quad -\partial_{zz}\widetilde{u}^{(k)} + f''(\widetilde{u}^{(0)})\widetilde{u}^{(k)}\\
&= -\widetilde{g}\big(\widetilde{u}^{(0)},\cdots,\widetilde{u}^{(k-1)}\big) + 2\sum_{i=0}^{k-1}\nabla_x\partial_z\widetilde{u}^{(i)}\cdot\nabla_x d^{(k-1-i)}\\
&\quad +\sum_{i=0}^{k-1}\partial_z\widetilde{u}^{(i)}\triangle_x d^{(k-1-i)} + \widetilde{\mu}^{(k-1)} + \triangle_x\widetilde{u}^{(k-2)} - \eta'\sum_{i=0}^{k-1} l^{(i)}d^{(k-1-i)} + z\eta' l^{(k-2)}\\
&\triangleq \Theta_{k-1,3}.
\end{aligned} \tag{4.6}
$$

Step 1. By induction we assume that the inner-outer matching conditions (2.13)-(2.15) hold for the order up to $k-1$. It follows from Lemma 4.3 in [3] and direct computations that if

$$
\int_{-\infty}^{+\infty}\Theta_{k-1,1}(x,t,z)dz = 0, \tag{4.7}
$$

then (4.4) has a bounded solution

$$
\begin{aligned}
\widetilde{\mu}^{(k)}(x,t,z) &= \eta(z)\mu_+^{(k)}(x,t) + \big(1-\eta(z)\big)\mu_-^{(k)}(x,t)\\
&\quad -\eta(z)\int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{k-1,1}(x,t,z'')dz''dz' + \int_{-\infty}^{z}\int_{z'}^{+\infty}\Theta_{k-1,1}(x,t,z'')dz''dz'
\end{aligned} \tag{4.8}
$$

which satisfies

$$
\big(p^{(k)}d^{(0)} + p^{(0)}d^{(k)}\big)(x,t) = \mu_+^{(k)}(x,t) - \int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{k-1,1}(x,t,z'')dz''dz' - \mu_-^{(k)}(x,t), \tag{4.9}
$$

and

$$
\big[\mu^{(k)}\big] \triangleq \mu_+^{(k)} - \mu_-^{(k)} = p^{(0)}d^{(k)} + \int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{k-1,1}(x,t,z'')dz''dz', \quad \text{on } \Gamma, \tag{4.10}
$$

and for any $\alpha,\beta,\gamma\in\mathbb{N}$,

$$
D_x^\alpha D_t^\beta D_z^\gamma\Big(\widetilde{\mu}^{(k)}(x,t,z) - \mu_\pm^{(k)}(x,t)\Big) = O(e^{-\nu|z|}), \quad \text{for } |z|\gg 1 \text{ and some } \nu>0.
$$

From (4.9) we can take

$$p^{(k)} = \begin{cases} \frac{1}{d^{(0)}}\Big(\mu_+^{(k)}(x,t) - \mu_-^{(k)}(x,t) - \int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{k-1,1}(x,t,z'')dz''dz' \\ \qquad\qquad -p^{(0)}d^{(k)}\Big), \qquad \text{in } \Gamma(\delta)\backslash\Gamma, \\ \nabla_x d^{(0)}\cdot\nabla_x\Big(\mu_+^{(k)}(x,t) - \mu_-^{(k)}(x,t) - \int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{k-1,1}(x,t,z'')dz''dz' \\ \qquad\qquad -p^{(0)}d^{(k)}\Big), \qquad \text{on } \Gamma. \end{cases} \tag{4.11}$$

Moreover, by (4.7) we arrive at

$$\begin{aligned} &2\partial_t d^{(k-1)} \\ &= -g^{(0)}d^{(k-1)} - g^{(k-1)}d^{(0)} + 2\nabla_x\big(\mu_+^{(0)} - \mu_-^{(0)}\big)\cdot\nabla_x d^{(k-1)} - \big(u_+^{(k-1)} - u_-^{(k-1)}\big)\partial_t d^{(0)} \\ &\quad + 2\nabla_x\big(\mu_+^{(k-1)} - \mu_-^{(k-1)}\big)\cdot\nabla_x d^{(0)} + \big(\mu_+^{(k-1)} - \mu_-^{(k-1)}\big)\Delta_x d^{(0)} \\ &\quad - \sum_{i=1}^{k-2}\Big(\partial_z\widetilde{u}^{(i)}\partial_t d^{(k-1-i)} - 2\nabla_x\partial_z\widetilde{\mu}^{(i)}\cdot\nabla_x d^{(k-1-i)} - \partial_z\widetilde{\mu}^{(i)}\Delta_x d^{(k-1-i)}\Big) \\ &\quad - p^{(k-1)} - \sum_{i=1}^{k-2} g^{(i)}d^{(k-1-i)} + g^{(k-2)}\int_{-\infty}^{+\infty} z\eta' dz + \int_{-\infty}^{+\infty}\big(s_+^{(k-2)}\eta^+ + s_-^{(k-2)}\eta^-\big)dz \\ &\quad - \int_{-\infty}^{+\infty}\big(\partial_t\widetilde{u}^{(k-2)} - \Delta_x\widetilde{\mu}^{(k-2)} - (2\widetilde{\sigma}^{(k-2)} + \widetilde{u}^{(k-2)} - \widetilde{\mu}^{(k-2)})\big)dz. \end{aligned} \tag{4.12}$$

In particular, for $(x,t)\in\Gamma$ there holds

$$\begin{aligned} 2\partial_t d^{(k-1)} &= -g^{(0)}d^{(k-1)} + 2\big[\nabla_x\mu^{(0)}\big]\cdot\nabla_x d^{(k-1)} - \big[u^{(k-1)}\big]\partial_t d^{(0)} \\ &\quad + 2\left[\frac{\partial\mu^{(k-1)}}{\partial\mathbf{n}}\right] + \big[\mu^{(k-1)}\big]\Delta_x d^{(0)} \\ &\quad - \sum_{i=1}^{k-2}\Big(\partial_z\widetilde{u}^{(i)}\partial_t d^{(k-1-i)} - 2\nabla_x\partial_z\widetilde{\mu}^{(i)}\cdot\nabla_x d^{(k-1-i)} - \partial_z\widetilde{\mu}^{(i)}\Delta_x d^{(k-1-i)}\Big) \\ &\quad - p^{(k-1)} - \sum_{i=1}^{k-2} g^{(i)}d^{(k-1-i)} + g^{(k-2)}\int_{-\infty}^{+\infty} z\eta' dz \\ &\quad + \int_{-\infty}^{+\infty}\big(s_+^{(k-2)}\eta^+ + s_-^{(k-2)}\eta^-\big)dz \\ &\quad - \int_{-\infty}^{+\infty}\big(\partial_t\widetilde{u}^{(k-2)} - \Delta_x\widetilde{\mu}^{(k-2)} - (2\widetilde{\sigma}^{(k-2)} + \widetilde{u}^{(k-2)} - \widetilde{\mu}^{(k-2)})\big)dz. \end{aligned} \tag{4.13}$$

From (2.30) and (2.31), we deduce that $\big[\nabla_x\mu^{(0)}\big]$ is parallel to $\nabla_x d^{(0)}$ on Γ and $\big[\nabla_x\mu^{(0)}\big] = p^{(0)}\nabla_x d^{(0)}$ on Γ. Thus for $(x,t)\in\Gamma$ it holds

$$\big[\nabla_x\mu^{(0)}\big]\cdot\nabla_x d^{(k-1)} = \begin{cases} 0, & k=2, \\ -\frac{p^{(0)}}{2}\sum_{i=1}^{k-2}\nabla_x d^{(i)}\cdot\nabla_x d^{(k-1-i)}, & k\geq 3. \end{cases} \tag{4.14}$$

Combining (4.14), (4.1)($k\to k-1$), (4.10)($k\to k-1$), (4.11)($k\to k-1$) with (4.13) we obtain

$$\partial_t d^{(k-1)} = \frac{1}{2}\Big(p^{(0)}\Delta_x d^{(0)} - \nabla_x d^{(0)}\cdot\nabla_x p^{(0)} - g^{(0)}\Big)d^{(k-1)} + \frac{1}{2}\left[\frac{\partial\mu^{(k-1)}}{\partial\mathbf{n}}\right] + \Lambda_{k-2,1}, \quad \text{on } \Gamma, \tag{4.15}$$

where the function $\Lambda_{k-2,1}$ depends only on terms up to order $k-2$.

And by (4.12) one has

$$g^{(k-1)} = \begin{cases} \frac{1}{d^{(0)}}\Big(-2\partial_t d^{(k-1)} - g^{(0)}d^{(k-1)} + 2\nabla_x\big(\mu_+^{(0)}-\mu_-^{(0)}\big)\cdot\nabla_x d^{(k-1)} \\ \quad -\big(u_+^{(k-1)}-u_-^{(k-1)}\big)\partial_t d^{(0)} + 2\nabla_x\big(\mu_+^{(k-1)}-\mu_-^{(k-1)}\big)\cdot\nabla_x d^{(0)} \\ \quad +\big(\mu_+^{(k-1)}-\mu_-^{(k-1)}\big)\Delta_x d^{(0)} - p^{(k-1)} + \Lambda_{k-2,2}\Big), \qquad \text{in } \Gamma(\delta)\backslash\Gamma, \\ \nabla d^{(0)}\cdot\nabla\Big(-2\partial_t d^{(k-1)} - g^{(0)}d^{(k-1)} + 2\nabla_x\big(\mu_+^{(0)}-\mu_-^{(0)}\big)\cdot\nabla_x d^{(k-1)} \\ \quad -\big(u_+^{(k-1)}-u_-^{(k-1)}\big)\partial_t d^{(0)} + 2\nabla_x\big(\mu_+^{(k-1)}-\mu_-^{(k-1)}\big)\cdot\nabla_x d^{(0)} \\ \quad +\big(\mu_+^{(k-1)}-\mu_-^{(k-1)}\big)\Delta_x d^{(0)} - p^{(k-1)} + \Lambda_{k-2,2}\Big), \qquad \text{on } \Gamma, \end{cases} \tag{4.16}$$

where the function $\Lambda_{k-2,2}$ depends only on terms up to order $k-2$.

Similarly, if

$$\int_{-\infty}^{+\infty}\Theta_{k-1,2}(x,t,z)dz = 0, \tag{4.17}$$

then (4.5) has a bounded solution

$$\begin{aligned}\widetilde{\sigma}^{(k)}(x,t,z) =& \eta(z)\sigma_+^{(k)}(x,t) + \big(1-\eta(z)\big)\sigma_-^{(k)}(x,t) \\ & -\eta(z)\int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{k-1,2}(z'',x,t)dz''dz' + \int_{-\infty}^{z}\int_{z'}^{+\infty}\Theta_{k-1,2}(x,t,z'')dz''dz'\end{aligned}$$

which satisfies

$$\big[\sigma^{(k)}\big] \triangleq \sigma_+^{(k)} - \sigma_-^{(k)} = q^{(0)}d^{(k)} + \int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{k-1,2}(x,t,z'')dz''dz', \quad \text{on } \Gamma,$$

and for any $\alpha,\beta,\gamma\in\mathbb{N}$,

$$D_x^\alpha D_t^\beta D_z^\gamma\Big(\widetilde{\sigma}^{(k)}(x,t,z) - \sigma_\pm^{(k)}(x,t)\Big) = O(e^{-\nu|z|}), \quad \text{for } |z|\gg 1 \text{ and some } \nu>0.$$

Furthermore we can obtain

$$q^{(k)} = \begin{cases} \frac{1}{d^{(0)}}\Big(\sigma_+^{(k)}(x,t) - \sigma_-^{(k)}(x,t) - \int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{k-1,2}(x,t,z'')dz''dz' \\ \qquad\qquad -q^{(0)}d^{(k)}\Big), \qquad \text{in } \Gamma(\delta)\backslash\Gamma, \\ \nabla_x d^{(0)}\cdot\nabla_x\Big(\sigma_+^{(k)}(x,t) - \sigma_-^{(k)}(x,t) - \int_{-\infty}^{+\infty}\int_{z'}^{+\infty}\Theta_{k-1,2}(x,t,z'')dz''dz' \\ \qquad\qquad -q^{(0)}d^{(k)}\Big), \qquad \text{on } \Gamma, \end{cases} \tag{4.18}$$

and

$$\left[\frac{\partial\sigma^{(k-1)}}{\partial\mathbf{n}}\right]=\Big(\nabla_x d^{(0)}\cdot\nabla_x q^{(0)}+h^{(0)}-q^{(0)}\Delta_x d^{(0)}\Big)d^{(k-1)}+\Lambda_{k-2,3},\quad \text{on}\quad \Gamma, \tag{4.19}$$

and

$$h^{(k-1)}=\begin{cases}\frac{1}{d^{(0)}}\Big(-h^{(0)}d^{(k-1)}-\big(\sigma_+^{(k-1)}-\sigma_-^{(k-1)}\big)\partial_t d^{(0)}+2\nabla_x\big(\sigma_+^{(k-1)}-\sigma_-^{(k-1)}\big)\cdot\nabla_x d^{(0)}\\ \qquad+\big(\sigma_+^{(k-1)}-\sigma_-^{(k-1)}\big)\Delta_x d^{(0)}-q^{(k-1)}+\Lambda_{k-2,4}\Big),\quad \text{in}\quad \Gamma(\delta)\setminus\Gamma\\ \\ \nabla_x d^{(0)}\cdot\nabla_x\Big(-h^{(0)}d^{(k-1)}-\big(\sigma_+^{(k-1)}-\sigma_-^{(k-1)}\big)\partial_t d^{(0)}\\ \qquad+2\nabla_x\big(\sigma_+^{(k-1)}-\sigma_-^{(k-1)}\big)\cdot\nabla_x d^{(0)}\\ \qquad+\big(\sigma_+^{(k-1)}-\sigma_-^{(k-1)}\big)\Delta_x d^{(0)}-q^{(k-1)}+\Lambda_{k-2,4}\Big),\qquad \text{on}\quad \Gamma,\end{cases} \tag{4.20}$$

where the functions $\Lambda_{k-2,3}$ and $\Lambda_{k-2,4}$ depend only on terms up to order $k-2$.

Step 2. Based on the method of variation of constants and direct computations (or Lemma 4.3 in [3]) we find that if

$$\int_{-\infty}^{+\infty}\Theta_{k-1,3}(x,t,z)\theta'(z)dz=0, \tag{4.21}$$

then (4.6) has a bounded solution $\widetilde{u}^{(k)}(x,t,z)$ satisfying $\widetilde{u}^{(k)}(0,x,t)=0$ and for any $\alpha,\beta,\gamma\in\mathbb{N}$,

$$D_x^\alpha D_t^\beta D_z^\gamma\Big(\widetilde{u}^{(k)}(x,t,z)-u_\pm^{(k)}(x,t)\Big)=O(e^{-\nu|z|}),\quad \text{for } |z|\gg 1 \text{ and }\ \text{some } \nu>0.$$

According to (4.8) and (4.21) we get

$$\begin{aligned}&\mu_+^{(k-1)}\int_{-\infty}^{+\infty}\eta(z)\theta'(z)dz+\mu_-^{(k-1)}\int_{-\infty}^{+\infty}\big(1-\eta(z)\big)\theta'(z)dz\\ &=-\Delta_x d^{(k-1)}\int_{-\infty}^{+\infty}\big(\theta'(z)\big)^2dz+l^{(0)}d^{(k-1)}\int_{-\infty}^{+\infty}\eta'(z)\theta'(z)dz\\ &\quad+l^{(k-1)}d^{(0)}\int_{-\infty}^{+\infty}\eta'(z)\theta'(z)dz+\Lambda_{k-2,5},\end{aligned} \tag{4.22}$$

where the function $\Lambda_{k-2,5}$ depends only on terms up to order $k-2$. Here we have used the fact that $\widetilde{u}^{(k-1)}$ actually depends only on terms up to order $k-2$. In particular, for $(x,t)\in\Gamma$ there holds

$$\begin{aligned}&\mu_+^{(k-1)}\int_{-\infty}^{+\infty}\eta(z)\theta'(z)dz+\mu_-^{(k-1)}\int_{-\infty}^{+\infty}\big(1-\eta(z)\big)\theta'(z)dz\\ &=-\Delta_x d^{(k-1)}\int_{-\infty}^{+\infty}\big(\theta'(z)\big)^2dz+l^{(0)}d^{(k-1)}\int_{-\infty}^{+\infty}\eta'(z)\theta'(z)dz+\Lambda_{k-2,5},\quad \text{on}\quad \Gamma.\end{aligned} \tag{4.23}$$

It follows from (4.10)$(k\to k-1)$ and (4.23) that

$$\mu_\pm^{(k-1)}=-\frac{1}{2}\Delta_x d^{(k-1)}\int_{-\infty}^{+\infty}\big(\theta'(z)\big)^2dz+\frac{1}{2}l^{(0)}d^{(k-1)}\int_{-\infty}^{+\infty}\eta'(z)\theta'(z)dz$$

$$+p^{(0)}d^{(k-1)}\int_{-\infty}^{+\infty}\Big(\frac{1}{2}-\eta(z)\pm\frac{1}{2}\Big)\theta'(z)dz+\Lambda^{\pm}_{k-2,6},\quad \text{on}\quad \Gamma, \tag{4.24}$$

where the functions $\Lambda^{\pm}_{k-2,6}$ depend only on terms up to order $k-2$.

And by (4.22) one has

$$l^{(k-1)}=\begin{cases}\frac{1}{d^{(0)}\int_{-\infty}^{+\infty}\eta'(z)\theta'(z)dz}\Big(\mu_+^{(k-1)}\int_{-\infty}^{+\infty}\eta(z)\theta'(z)dz+\mu_-^{(k-1)}\int_{-\infty}^{+\infty}(1-\eta(z))\theta'(z)dz\\ \quad +\Delta_x d^{(k-1)}\int_{-\infty}^{+\infty}(\theta'(z))^2dz-l^{(0)}d^{(k-1)}\int_{-\infty}^{+\infty}\eta'(z)\theta'(z)dz+\Lambda_{k-2,7}\Big),\quad \text{in}\ \Gamma(\delta)\backslash\Gamma,\\ \frac{1}{\int_{-\infty}^{+\infty}\eta'(z)\theta'(z)dz}\nabla_x d^{(0)}\cdot\nabla_x\Big(\mu_+^{(k-1)}\int_{-\infty}^{+\infty}\eta(z)\theta'(z)dz+\mu_-^{(k-1)}\int_{-\infty}^{+\infty}(1-\eta(z))\theta'(z)dz\\ \quad +\Delta_x d^{(k-1)}\int_{-\infty}^{+\infty}(\theta'(z))^2dz-l^{(0)}d^{(k-1)}\int_{-\infty}^{+\infty}\eta'(z)\theta'(z)dz+\Lambda_{k-2,7}\Big),\quad \text{on}\ \Gamma,\end{cases} \tag{4.25}$$

where the function $\Lambda_{k-2,7}$ depends only on terms up to order $k-2$.

4.3. Matching k-th order($k\geq 2$) boundary layer expansion. Substituting (2.47)-(2.49) into (2.53)-(2.55) and (2.56)-(2.58) and collecting all the terms of ε^k-order($k\geq 2$) we have

$$-\partial_{zz}\mu_B^{(k)}=\Xi_{k-1,1}, \tag{4.26}$$

$$-\partial_{zz}\sigma_B^{(k)}=\Xi_{k-1,2}, \tag{4.27}$$

$$-\partial_{zz}u_B^{(k)}+f''(1)u_B^{(k)}=\Xi_{k-1,3}, \tag{4.28}$$

and on $\partial\Omega\times[0,T]$

$$\partial_z\mu_B^{(0)}(x,t,0)=0,\quad \partial_z\mu_B^{(k)}(x,t,0)=-\nabla_x\mu_B^{(k-1)}(x,t,0)\cdot\nabla_x d_B(x,t), \tag{4.29}$$

$$\partial_z\sigma_B^{(0)}(x,t,0)=0,\quad \partial_z\sigma_B^{(k)}(x,t,0)=-\nabla_x\sigma_B^{(k-1)}(x,t,0)\cdot\nabla_x d_B(x,t), \tag{4.30}$$

$$\partial_z u_B^{(0)}(x,t,0)=0,\quad \partial_z u_B^{(k)}(x,t,0)=-\nabla_x u_B^{(k-1)}(x,t,0)\cdot\nabla_x d_B(x,t), \tag{4.31}$$

where the functions $\Xi_{k-1,1},\Xi_{k-1,2}$ and $\Xi_{k-1,3}$ depend only on the terms up to order $k-1$. More concretely, we can write

$$\begin{aligned}\Xi_{k-1,1}&=2\nabla_x\partial_z\mu_B^{(k-1)}\cdot\nabla_x d_B+\partial_z\mu_B^{(k-1)}\Delta_x d_B-\partial_t u_B^{(k-2)}+\Delta_x\mu_B^{(k-2)}\\ &\quad +2\sigma_B^{(k-2)}+u_B^{(k-2)}-\mu_B^{(k-2)},\\ \Xi_{k-1,2}&=2\nabla_x\partial_z\sigma_B^{(k-1)}\cdot\nabla_x d_B+\partial_z\sigma_B^{(k-1)}\Delta_x d_B-\partial_t\sigma_B^{(k-2)}+\Delta_x\sigma_B^{(k-2)}\\ &\quad -\big(2\sigma_B^{(k-2)}+u_B^{(k-2)}-\mu_B^{(k-2)}\big),\\ \Xi_{k-1,3}&=-g_B\big(u_B^{(0)},\cdots,u_B^{(k-1)}\big)+2\nabla_x\partial_z u_B^{(k-1)}\cdot\nabla_x d_B+\partial_z u_B^{(k-1)}\Delta_x d_B\\ &\quad +\mu_B^{(k-1)}+\Delta_x u_B^{(k-2)}.\end{aligned}$$

By induction we assume that the boundary-outer matching conditions (2.50)-(2.52) hold for the order up to $k-1$. Then by (4.26)-(4.27), we get that

$$\mu_B^{(k)}(x,t,z)=-\int_{-\infty}^{z}\int_{-\infty}^{z'}\Xi_{k-1,1}(x,t,z'')dz''dz'+\mu_+^{(k)}(x,t), \tag{4.32}$$

$$\sigma_B^{(k)}(x,t,z)=-\int_{-\infty}^{z}\int_{-\infty}^{z'}\Xi_{k-1,2}(x,t,z'')dz''dz'+\sigma_+^{(k)}(x,t), \tag{4.33}$$

and (2.51)-(2.52) for order k are satisfied by using induction arguments.

In order that $\mu_B^{(k)}$ defined by (4.32) satisfies (4.29)($k\geq 2$) and $\sigma_B^{(k)}$ defined by (4.33) satisfies (4.30)($k\geq 2$), we only need to assume on $\partial\Omega\times[0,T]$ that

$$\begin{aligned}
&\nabla_x d_B(x,t)\cdot\nabla_x\mu_+^{(k-1)}(x,t)\\
=&-\Big(2\nabla_x d_B(x,t)\cdot\nabla_x+\Delta_x d_B(x,t)\Big)\int_{-\infty}^0\int_{-\infty}^{z'}\Xi_{k-2,1}(x,t,z'')dz''dz'\\
&-\int_{-\infty}^0\Big(\partial_t u_B^{(k-2)}-\Delta_x\mu_B^{(k-2)}-2\sigma_B^{(k-2)}-u_B^{(k-2)}+\mu_B^{(k-2)}\Big)(x,t,z)dz\\
\triangleq&\Pi_{k-2,1}
\end{aligned}\tag{4.34}$$

and

$$\begin{aligned}
&\nabla_x d_B(x,t)\cdot\nabla_x\sigma_+^{(k-1)}(x,t)\\
=&-\Big(2\nabla_x d_B(x,t)\cdot\nabla_x+\Delta_x d_B(x,t)\Big)\int_{-\infty}^0\int_{-\infty}^{z'}\Xi_{k-2,2}(x,t,z'')dz''dz'\\
&-\int_{-\infty}^0\Big(\partial_t \sigma_B^{(k-2)}-\Delta_x\sigma_B^{(k-2)}-2\sigma_B^{(k-2)}-u_B^{(k-2)}+\mu_B^{(k-2)}\Big)(x,t,z)dz\\
\triangleq&\Pi_{k-2,2}.
\end{aligned}\tag{4.35}$$

In fact, for $(x,t)\in\partial\Omega\times[0,T]$ one has

$$\begin{aligned}
-\partial_z\mu_B^{(k)}(x,t,0)=&\int_{-\infty}^0\Xi_{k-1,1}(x,t,z)dz\\
=&\Big(2\nabla_x d_B(x,t)\cdot\nabla_x+\Delta_x d_B(x,t)\Big)\mu_B^{(k-1)}(x,t,0)\\
&-\Big(2\nabla_x d_B(x,t)\cdot\nabla+\Delta_x d_B(x,t)\Big)\mu_+^{(k-1)}(x,t)\\
&-\int_{-\infty}^0\Big(\partial_t u_B^{(k-2)}-\Delta_x\mu_B^{(k-2)}-2\sigma_B^{(k-2)}-u_B^{(k-2)}+\mu_B^{(k-2)}\Big)(x,t,z)dz\\
=&-\Big(2\nabla_x d_B(x,t)\cdot\nabla_x+\Delta_x d_B(x,t)\Big)\int_{-\infty}^0\int_{-\infty}^{z'}\Xi_{k-2,1}(x,t,z'')dz''dz'\\
&-\int_{-\infty}^0\Big(\partial_t u_B^{(k-2)}-\Delta_x\mu_B^{(k-2)}-2\sigma_B^{(k-2)}-u_B^{(k-2)}+\mu_B^{(k-2)}\Big)(x,t,z)dz
\end{aligned}$$

and

$$\begin{aligned}
\nabla_x\mu_B^{(k-1)}(x,t,0)\cdot\nabla_x d_B(x,t)=&-\nabla_x d_B(x,t)\cdot\nabla_x\Big(\int_{-\infty}^0\int_{-\infty}^{z'}\Xi_{k-2,1}(x,t,z'')dz''dz'\Big)\\
&+\nabla_x d_B(x,t)\cdot\nabla_x\mu_+^{(k-1)}(x,t).
\end{aligned}$$

Then we easily get (4.29)($k\geq 2$) with the help of (4.34) and the above equalities. The other cases can be done similarly.

Finally, we equip (4.28)($k\geq 2$) with the following boundary condition at $z=0$

$$\partial_z u_B^{(k)}(x,t,0)=-\nabla_x u_B^{(k-1)}(x,t,0)\cdot\nabla_x d_B(x,t)\tag{4.36}$$

for $(x,t)\in\partial\Omega(\delta)\times[0,T]$. Obviously (4.36) implies (4.31).

Since (4.28)($k\geq 1$) is a linear second-order ordinary differential equation with constant coefficients, we can solve it explicitly and conclude that there exists a unique solution $u_B^{(k)}$ which satisfies (4.36) and (2.50).

4.4. Solving expansions of k−th order($k\geq 1$).

Assuming $u_\pm^{(k-1)}$, $\mu_\pm^{(k-1)}$, $\sigma_\pm^{(k-1)}$, $d^{(k-1)}$, $p^{(k-1)}$, $q^{(k-1)}$, $g^{(k-1)}$, $h^{(k-1)}$, $l^{(k-1)}$, $\widetilde{u}^{(k-1)}$, $\widetilde{\mu}^{(k-1)}$, $\widetilde{\sigma}^{(k-1)}$, $\mu_B^{(k-1)}$, $\sigma_B^{(k-1)}$, $u_B^{(k-1)}$ are known and the inner-outer matching conditions (2.13)-(2.15), the boundary-outer matching conditions (2.50)-(2.52) hold for order up to $k-1$. Then $u_\pm^{(k)}$ are defined by (4.1). Combining (2.2), (4.2), (4.3), (4.15)($k-1\to k$), (4.19)($k-1\to k$), (4.24)($k-1\to k$),, (4.34)($k-1\to k$), (4.35)($k-1\to k$), we have

$$\begin{cases} -\Delta\mu_\pm^{(k)}+\mu_\pm^{(k)}=2\sigma_\pm^{(k)}+u_\pm^{(k)}-\partial_t u_\pm^{(k)}, & \text{in } \Omega_\pm,\\ \partial_t\sigma_\pm^{(k)}-\Delta\sigma_\pm^{(k)}+2\sigma_\pm^{(k)}=\mu_\pm^{(k)}-u_\pm^{(k)}, & \text{in } \Omega_\pm,\\ \nabla d^{(0)}\cdot\nabla d^{(k)}=\mathcal{D}_{k-1}, & \text{in } \Gamma(\delta),\\ \mu_\pm^{(k)}=-a_0\Delta d^{(k)}+a_1 d^{(k)}+\Lambda_{k-1,6}^\pm, & \text{on } \Gamma,\\ \left[\frac{\partial\sigma^{(k)}}{\partial\mathbf{n}}\right]=a_2 d^{(k)}+\Lambda_{k-1,3}, & \text{on } \Gamma,\\ \partial_t d^{(k)}=a_3 d^{(k)}+\frac{1}{2}\left[\frac{\partial\mu^{(k)}}{\partial\mathbf{n}}\right]+\Lambda_{k-1,1}, & \text{on } \Gamma,\\ \frac{\partial\mu_+^{(k)}}{\partial\mathbf{n}}=\Pi_{k-1,1}, & \text{on } \partial\Omega,\\ \frac{\partial\sigma_+^{(k)}}{\partial\mathbf{n}}=\Pi_{k-1,2}, & \text{on } \partial\Omega,\\ d^{(k)}(x,0)=0, & \text{on } \Gamma_0, \end{cases} \tag{4.37}$$

where a_0 is a positive constant, the functions a_1,a_2 and a_3 depend on $p^{(0)},q^{(0)},g^{(0)},h^{(0)},l^{(0)}$ and $d^{(0)}$, and $\Gamma_0=\Gamma|_{t=0}$. Giving an initial data $\sigma_\pm^{(k)}(x,0)$ and solving (4.37) leads to $\mu_\pm^{(k)},\sigma_\pm^{(k)}$ and $d^{(k)}$. Equation (4.37) is a "linearized" Hele-Shaw problem (P193 in [3]) coupled linearly with a heat equation satisfied by $\sigma_\pm^{(k)}$. The first and key strategy is to get the value of $d^{(k)}$ on Γ. Here we don't aim to show the lengthy details and one can refer to similar arguments in Section 6 of [3].

Then $p^{(k)},q^{(k)},g^{(k)},h^{(k)},l^{(k)}$ are determined by (4.11), (4.18), (4.16), (4.20), (4.25) respectively. Moreover $\widetilde{u}^{(k)},\widetilde{\mu}^{(k)},\widetilde{\sigma}^{(k)}$ are determined in Section 4.2, $\mu_B^{(k)},\sigma_B^{(k)},u_B^{(k)}$ are determined in Section 4.3, and the inner-outer matching conditions (2.13)-(2.15) and the boundary-outer matching conditions (2.50)-(2.52) hold for order k.

REMARK 4.1. We can extend $(u_\pm^{(k)},\mu_\pm^{(k)},\sigma_\pm^{(k)})$ smoothly from $\Omega_\pm$ to Ω as Remark 4.1 in [3].

4.5. Construction of the approximate solution. In this section we divide into two steps to construct an approximate solution and determine the system which is satisfied by the approximate solution.

Step 1. In $\Omega_+\cup\Omega_-$ we define

$$u_O^A(x,t)=\left(\sum_{i=0}^k\varepsilon^i u_+^{(i)}(x,t)\right)\chi_{\Omega_+}(x,t)+\left(\sum_{i=0}^k\varepsilon^i u_-^{(i)}(x,t)\right)\chi_{\Omega_-}(x,t),$$

$$\mu_O^A(x,t)=\left(\sum_{i=0}^k\varepsilon^i \mu_+^{(i)}(x,t)\right)\chi_{\Omega_+}(x,t)+\left(\sum_{i=0}^k\varepsilon^i \mu_-^{(i)}(x,t)\right)\chi_{\Omega_-}(x,t),$$

$$\sigma_O^A(x,t) = \left(\sum_{i=0}^{k} \varepsilon^i \sigma_+^{(i)}(x,t)\right)\chi_{\Omega_+}(x,t) + \left(\sum_{i=0}^{k} \varepsilon^i \sigma_-^{(i)}(x,t)\right)\chi_{\Omega_-}(x,t),$$

where $\chi_{\Omega_\pm}$ is the characteristic function of $\Omega_\pm$.

Thanks to the outer matching expansion procedure, we obtain in $\Omega_+ \cup \Omega_-$

$$\begin{aligned}
\partial_t u_O^A - \Delta \mu_O^A &= 2\sigma_O^A + u_O^A - \mu_O^A, \\
\partial_t \sigma_O^A - \Delta \sigma_O^A &= -(2\sigma_O^A + u_O^A - \mu_O^A), \\
\mu_O^A &= -\varepsilon \Delta u_O^A + \varepsilon^{-1} f'(u_O^A) + O(\varepsilon^k).
\end{aligned}$$

In $\Gamma(\delta)$ we define

$$\begin{aligned}
u_I^A(x,t) &= \sum_{i=0}^{k} \varepsilon^i \tilde{u}^{(i)}(x,t,z)\Big|_{z=\frac{d^{[k]}(x,t)}{\varepsilon}}, \\
\mu_I^A(x,t) &= \sum_{i=0}^{k} \varepsilon^i \tilde{\mu}^{(i)}(x,t,z)\Big|_{z=\frac{d^{[k]}(x,t)}{\varepsilon}}, \\
\sigma_I^A(x,t) &= \sum_{i=0}^{k} \varepsilon^i \tilde{\sigma}^{(i)}(x,t,z)\Big|_{z=\frac{d^{[k]}(x,t)}{\varepsilon}}.
\end{aligned}$$

It is direct to check that in $\Gamma(\delta)$

$$\begin{aligned}
\partial_t u_I^A - \Delta \mu_I^A &= 2\sigma_I^A + u_I^A - \mu_I^A + O(\varepsilon^{k-1}), \\
\partial_t \sigma_I^A - \Delta \sigma_I^A &= -(2\sigma_I^A + u_I^A - \mu_I^A) + O(\varepsilon^{k-1}), \\
\mu_I^A &= -\varepsilon \Delta u_I^A + \varepsilon^{-1} f'(u_I^A) + O(\varepsilon^k).
\end{aligned}$$

In $\partial\Omega(\delta)$ we define

$$\begin{aligned}
u_B^A(x,t) &= \sum_{i=0}^{k} \varepsilon^i u_B^{(i)}(x,t,z)\Big|_{z=\frac{d_B(x)}{\varepsilon}} - \varepsilon^k u_B^{(k)}(0,x,t), \\
\mu_B^A(x,t) &= \sum_{i=0}^{k} \varepsilon^i \mu_B^{(i)}(x,t,z)\Big|_{z=\frac{d_B(x)}{\varepsilon}} - \varepsilon^k \mu_B^{(k)}(0,x,t), \\
\sigma_B^A(x,t) &= \sum_{i=0}^{k} \varepsilon^i \sigma_B^{(i)}(x,t,z)\Big|_{z=\frac{d_B(x)}{\varepsilon}} - \varepsilon^k \sigma_B^{(k)}(0,x,t),
\end{aligned}$$

and we can find in $\partial\Omega(\delta)$ that

$$\begin{aligned}
\partial_t u_B^A - \Delta \mu_B^A &= 2\sigma_B^A + u_B^A - \mu_B^A + O(\varepsilon^{k-1}), \\
\partial_t \sigma_B^A - \Delta \sigma_B^A &= -(2\sigma_B^A + u_B^A - \mu_B^A) + O(\varepsilon^{k-1}), \\
\mu_B^A &= -\varepsilon \Delta u_B^A + \varepsilon^{-1} f'(u_B^A) + O(\varepsilon^{k-1})
\end{aligned}$$

and

$$\frac{\partial u_B^A}{\partial \mathbf{n}} = \frac{\partial \mu_B^A}{\partial \mathbf{n}} = \frac{\partial \sigma_B^A}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega \times (0,T).$$

Step 2. We define $(\overline{u^A},\overline{\mu^A},\overline{\sigma^A})$ as follows:

$$\overline{u^A}=\begin{cases} u_B^A, & \text{in } \partial\Omega(\frac{\delta}{2}),\\ u_B^A\zeta(\frac{d_B}{\delta})+u_O^A\big(1-\zeta(\frac{d_B}{\delta})\big), & \text{in } \partial\Omega(\delta)\backslash\partial\Omega(\frac{\delta}{2}),\\ u_O^A, & \text{in } \Omega\backslash\big(\partial\Omega(\delta)\cup\Gamma(\delta)\big),\\ u_I^A\zeta(\frac{d^{(0)}}{\delta})+u_O^A\big(1-\zeta(\frac{d^{(0)}}{\delta})\big), & \text{in } \Gamma(\delta)\backslash\Gamma(\frac{\delta}{2}),\\ u_I^A, & \text{in } \Gamma(\frac{\delta}{2}),\end{cases}$$

and $\overline{\mu^A},\overline{\sigma^A}$ are defined similarly, where

$$\zeta\in C_c^\infty(R),\quad \zeta=1\quad\text{for}\quad |\zeta|\le\frac{1}{2},\quad \zeta=0\quad\text{for}\quad |\zeta|\ge 1.$$

Based on the boundary-outer matching conditions (2.50)-(2.52) and inner-outer matching conditions (2.13)-(2.15), one has for small ε

$$\begin{aligned}\|\overline{u^A}-u_O^A\|_{C^2(\partial\Omega(\delta)\backslash\partial\Omega(\frac{\delta}{2}))}&=\|(u_B^A-u_O^A)\zeta(\frac{d_B}{\delta})\|_{C^2(\partial\Omega(\delta)\backslash\partial\Omega(\frac{\delta}{2}))}\\&=O(\varepsilon^2e^{-\frac{\nu\delta}{2\varepsilon}})+O(\varepsilon^k)\end{aligned}$$

and

$$\begin{aligned}\|\overline{u^A}-u_O^A\|_{C^2(\Gamma(\delta)\backslash\Gamma(\frac{\delta}{2}))}&=\|(u_I^A-u_O^A)\zeta(\frac{d^{(0)}}{\delta})\|_{C^2(\Gamma(\delta)\backslash\Gamma(\frac{\delta}{2}))}\\&=O(\varepsilon^2e^{-\frac{\nu\delta}{4\varepsilon}}).\end{aligned}$$

Similar estimates hold for $\overline{\mu^A}-\mu_O^A$ and $\overline{\sigma^A}-\sigma_O^A$.

Consequently $(\overline{u^A},\overline{\mu^A},\overline{\sigma^A})$ satisfies in $\Omega\times(0,T)$

$$\begin{aligned}\partial_t\overline{u^A}-\Delta\overline{\mu^A}&=2\overline{\sigma^A}+\overline{u^A}-\overline{\mu^A}+\omega_1^A,\\ \partial_t\overline{\sigma^A}-\Delta\overline{\sigma^A}&=-(2\overline{\sigma^A}+\overline{u^A}-\overline{\mu^A})+\omega_2^A,\\ \overline{\mu^A}&=-\varepsilon\Delta\overline{u^A}+\varepsilon^{-1}f'(\overline{u^A})+\omega_3^A\end{aligned}$$

and

$$\frac{\partial\overline{u^A}}{\partial\mathbf{n}}=\frac{\partial\overline{\mu^A}}{\partial\mathbf{n}}=\frac{\partial\overline{\sigma^A}}{\partial\mathbf{n}}=0,\quad\text{on}\quad\partial\Omega\times(0,T),$$

where $\omega_i^A=O(\varepsilon^{k-1})(i=1,2,3)$ which depends on $(\overline{u^A},\overline{\mu^A},\overline{\sigma^A})$.

Letting $\overline{\varphi^A}=\overline{u^A}+\overline{\sigma^A}$, we have

$$\begin{cases}\partial_t\overline{\varphi^A}-\Delta\overline{\mu^A}-\Delta\overline{\sigma^A}=\omega_1^A+\omega_2^A, & \text{in}\ \ \Omega\times(0,T),\\ \frac{\partial\overline{\varphi^A}}{\partial\mathbf{n}}=0, & \text{on } \partial\Omega\times(0,T).\end{cases}$$

Define the approximate solution $(\varphi^A,\mu^A,\sigma^A)$ as follows:

$$\begin{cases}\varphi^A(x,t)=\overline{\varphi^A}(x,t)-\frac{1}{|\Omega|}\int_0^t\int_\Omega\big(\omega_1^A+\omega_2^A\big)(x,t')dxdt',\\ \mu^A(x,t)=\overline{\mu^A}(x,t)-\tilde{\mu}^A(x,t),\\ \sigma^A(x,t)=\overline{\sigma^A}(x,t),\\ u^A(x,t)=\overline{u^A}(x,t)-\omega_2^A(x,t)-\tilde{\mu}^A(x,t),\end{cases}\tag{4.38}$$

where $\tilde{\mu}^A(x,t)$ satisfies

$$\begin{cases} \Delta\tilde{\mu}^A = \omega_1^A + \omega_2^A - \frac{1}{|\Omega|}\int_\Omega \left(\omega_1^A + \omega_2^A\right)(x,t)dx, & \text{in } \Omega\times(0,T), \\ \frac{\partial\tilde{\mu}^A}{\partial\mathbf{n}} = 0, & \text{on } \partial\Omega\times(0,T), \\ \int_\Omega \tilde{\mu}^A(x,t)dx = 0, & t\in(0,T). \end{cases}$$

Then it holds that

$$\begin{cases} \partial_t\varphi^A - \Delta\mu^A - \Delta\sigma^A = 0, & \text{in } \Omega\times(0,T), \\ \partial_t\sigma^A - \Delta\sigma^A = -(2\sigma^A + u^A - \mu^A), & \text{in } \Omega\times(0,T), \\ \mu^A = -\varepsilon\Delta u^A + \varepsilon^{-1}f'(u^A) + \omega_4^A, & \text{in } \Omega\times(0,T), \\ u^A = \varphi^A - \sigma^A + \omega_5^A, & \text{in } \Omega\times(0,T), \\ \frac{\partial\varphi^A}{\partial\mathbf{n}} = \frac{\partial\mu^A}{\partial\mathbf{n}} = \frac{\partial\sigma^A}{\partial\mathbf{n}} = 0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{4.39}$$

where $\omega_4^A = O(\varepsilon^{k-2})$ and

$$\omega_5^A = \frac{1}{|\Omega|}\int_0^t\int_\Omega \left(\omega_1^A + \omega_2^A\right)(x,t')dxdt' - \omega_2^A - \tilde{\mu}^A = O(\varepsilon^{k-1}).$$

Thus Theorem 1.1 is arrived by letting $\mathfrak{R}_1 = \omega_4^A$ and $\mathfrak{R}_2 = \omega_5^A$.

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Appendix. For completeness, here we give a proof of (3.8) which has been shown in [14].

Proof. Firstly we observe that

$$\begin{aligned} &\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \left((\partial_z v)^2 + f''\big(\theta(z)\big)v^2\right)Jdz \\ &\geq \int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \left((\partial_z\hat{v})^2 + f''\big(\theta(z)\big)\hat{v}^2\right)dz - \varepsilon\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \hat{v}\partial_z\hat{v}J_{r_t}J^{-1}dz - C\varepsilon^2\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} v^2dz. \end{aligned} \tag{5.1}$$

Let $\hat{v} = \gamma q_1^f + p_1$, then

$$\begin{aligned} &-\varepsilon\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \hat{v}\partial_z\hat{v}J_{r_t}J^{-1}dz \\ &= -\varepsilon\gamma\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \hat{v}(q_1^f)'J_{r_t}J^{-1}dz - \varepsilon\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \hat{v}\partial_z p_1 J_{r_t}J^{-1}dz \\ &= -\varepsilon\gamma\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \hat{v}(q_1^f - \alpha\theta')'J_{r_t}J^{-1}dz + \varepsilon\alpha\gamma\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \hat{v}\theta''J_{r_t}J^{-1}dz - \varepsilon\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} \hat{v}\partial_z p_1 J_{r_t}J^{-1}dz. \end{aligned} \tag{5.2}$$

We easily find

$$-\left(q_1^f - \alpha\theta'\right)'' + f''(\theta)(q_1^f - \alpha\theta') = \lambda_1^f q_1^f, \quad (q_1^f - \alpha\theta')'\Big|_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}} = -\alpha\theta''\Big|_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}.$$

Multiplying the above equation by $q_1^f-\alpha\theta'$, integrating by parts and using (3.5), we have

$$\begin{aligned}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\left|(q_1^f-\alpha\theta')'\right|^2dz=&-\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}f''(\theta)(q_1^f-\alpha\theta')^2dz+\lambda_1^f\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}q_1^f(q_1^f-\alpha\theta')dz\\&-\alpha^2\theta''(\frac{\delta}{\varepsilon})\theta'(\frac{\delta}{\varepsilon})+\alpha^2\theta''(-\frac{\delta}{\varepsilon})\theta'(-\frac{\delta}{\varepsilon})\\=&O(e^{-\frac{C}{\varepsilon}}).\end{aligned}\tag{5.3}$$

It follows from (5.3) that

$$\begin{aligned}\varepsilon\gamma\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\hat{v}(q_1^f-\alpha\theta')'J_{r_t}J^{-1}dz&\le C\varepsilon\gamma\Big(\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}|\hat{v}|^2dz\Big)^{\frac12}\Big(\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}|(q_1^f-\alpha\theta')'|^2dz\Big)^{\frac12}\\&\le O(e^{-\frac{C}{\varepsilon}})\Big(\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}|\hat{v}|^2dz\Big)\le O(e^{-\frac{C}{\varepsilon}})\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}v^2dz.\end{aligned}\tag{5.4}$$

And from (3.7) one has

$$\begin{aligned}\varepsilon\alpha\gamma\Big|\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\hat{v}\theta''J_{r_t}J^{-1}dz\Big|&\le C\varepsilon^2\alpha\gamma\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}|\hat{v}\theta''z|dz\\&\le C\varepsilon^2\alpha\gamma\Big(\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}|\hat{v}|^2dz\Big)^{\frac12}\Big(\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}|\theta''z|^2dz\Big)^{\frac12}\\&\le C\varepsilon^2\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}v^2dz.\end{aligned}\tag{5.5}$$

Using (3.4) we can arrive at

$$\begin{aligned}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\left|\partial_zp_1\right|^2dz&=\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\Big(\left|(\partial_zp_1\right|^2+f''(\theta)(p_1)^2\Big)dz-\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}f''(\theta)(p_1)^2dz\\&\le C\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\Big(\left|\partial_zp_1\right|^2+f''(\theta)(p_1)^2\Big)dz\\&=C\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\Big((\partial_z\hat{v})^2+f''(\theta)\hat{v}^2\Big)dz-C\lambda_1^f\gamma^2\\&\le C\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\Big((\partial_z\hat{v})^2+f''(\theta)\hat{v}^2\Big)dz+Ce^{-\frac{C_2}{\varepsilon}}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\hat{v}^2dz.\end{aligned}\tag{5.6}$$

By (5.6) and the Young's inequality, one has

$$\begin{aligned}\Big|-\varepsilon\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\hat{v}\partial_zp_1J_{r_t}J^{-1}dz\Big|&\le C\varepsilon\Big(\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}v^2dz\Big)^{\frac12}\Big(\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}(\partial_zp_1)^2dz\Big)^{\frac12}\\&\le C\varepsilon^2\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}v^2dz+\frac14\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\big((\partial_z\hat{v})^2+f''(\theta)\hat{v}^2\big)dz.\end{aligned}$$

Thus

$$-\varepsilon\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\hat{v}\partial_zp_1J_{r_t}J^{-1}dz\ge-C\varepsilon^2\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}v^2dz-\frac14\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\big((\partial_z\hat{v})^2+f''(\theta)\hat{v}^2\big)dz.\tag{5.7}$$

Substituting (5.4), (5.5) and (5.7) into (5.2) we have

$$-\varepsilon\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\hat{v}\partial_z\hat{v}J_{r_t}J^{-1}dz\geq -C\varepsilon^2\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}v^2dz-\frac{1}{4}\int_{-\frac{\delta}{\varepsilon}}^{\frac{\delta}{\varepsilon}}\left((\partial_z\hat{v})^2+f''(\theta)\hat{v}^2\right)dz$$

which together with (5.1) leads to (3.8).

Hence the proof of (3.8) is finished. □

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