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INVISCID LIMIT FOR AXISYMMETRIC NAVIER-STOKES-BOUSSINESQ SYSTEM*

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Abstract. Consideration herein is the inviscid limit of the 3-D incompressible axisymmetric Navier-Stokes-Boussinesq system with partial viscosity. We obtain uniform estimates of the solutions of this system with respect to the viscosity. We then provide a strong convergence result in the H^{s-2} norm of the viscous solutions of this Navier-Stokes-Boussinesq system to the one of Euler-Boussinesq equations.

Keywords. Inviscid limit; Naiver-Stokes-Boussinesq system; Axisymmetric velocity.

AMS subject classifications. 76B03; 76D03; 76D09.

1. Introduction

In this paper we deal with the 3-D incompressible anisotropic Navier-Stokes-Boussinesq equations

$$\begin{cases} \partial_t \rho_\mu + u_\mu \cdot \nabla \rho_\mu - \Delta \rho_\mu = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \\ \partial_t u_\mu + u_\mu \cdot \nabla u_\mu - \mu \partial_z^2 u_\mu + \nabla p_\mu = \rho_\mu e_z, \\ \operatorname{div} u_\mu = 0, \\ u_\mu|_{t=0} = u^0, \qquad \rho_\mu|_{t=0} = \rho^0. \end{cases} \tag{1.1}$$

These equations include the temperature ρ_μ (or the density in the modeling of geophysical fluids), the solenoidal velocity field $u_\mu = (u_{\mu 1}, u_{\mu 2}, u_{\mu 3})^T$, and the fluid pressure p_μ. The term $\rho_\mu e_z$ with $e_z = (0,0,1)^T$ takes into account the influence of the gravity and the stratification on the motion of the fluid. And the partial viscosity coefficient μ is a positive constant. Note that when the initial density ρ^0 is identically a nonnegative constant, the system (1.1) reduces, in general, to the following 3-D incompressible anisotropic Navier-Stokes system

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta_h u - \mu \partial_z^2 u + \nabla p = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u^0, \end{cases} \tag{1.2}$$

where the usual Laplacian in the classical Navier-Stokes equations is substituted by the anisotropic Laplacian $\nu \Delta_h + \mu \partial_z^2$ with $\nu, \mu \geq 0$, which appears in geophysical fluids (see for instance [9]). The system (1.2) has been extensively studied by many mathematicians recently (see [4, 8, 10, 18, 26, 33, 37] etc.).

The system (1.1) is an anisotropic version of the classical n-D incompressible Navier-

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Stokes-Boussinesq equations $(n=2,3)$

$$\begin{cases} \partial_t\rho+u\cdot\nabla\rho-\kappa\Delta\rho=0, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^n \\ \partial_t u+u\cdot\nabla u-\mu\Delta u+\nabla p=\rho e_z, \\ \operatorname{div} u=0, \\ u|_{t=0}=u^0, \qquad \rho|_{t=0}=\rho^0, \end{cases} \tag{1.3}$$

which are widely used to model the dynamics of the atmosphere or the ocean [29], and arise from the density-dependent fluid equations by using the Boussinesq approximation which consists in neglecting the density dependence in all the terms but the one involving the gravity. This Boussinesq approximation can be rigorously justified from compressible fluid equations by a simultaneous low Mach number/Froude number limit (see [15]).

In two dimensions, the standard energy method enables us to establish the global existence of regular solutions of (1.3) for the case where μ and κ are positive constants. But, for the case $\mu=\kappa=0$, the global well-posedness of (1.3) for some non constant ρ_0 is still a challenging open problem. For the case $\mu>0$, $\kappa=0$, or $\kappa>0,\mu\geq 0$, the global well-posedness of (1.3) was independently obtained [1,7,19,20,25], see also [21,22] for the global well-posedness in the critical spaces.

In three dimensions, R. Danchin and M. Paicu [14] proved the global existence of Leray weak solution of the system (1.3) and its global well-posedness with small initial data, and also obtained an existence and uniqueness result for small initial data belonging to some critical Lorentz spaces in [13]. As the outstanding open problem in the 3-D incompressible Navier-Stokes equations, there are few results about the global well-posedness of (1.3) for large initial data in 3-D.

Many mathematicians are devoted to the study of some special large initial data which may globally generate the smooth solution of the 3-D incompressible Navier-Stokes or Euler systems. There is an interesting case of the global well-posedness result for both the three-dimensional Navier-Stokes and Euler systems (see [34]) corresponding to large initial data but with special geometry, called axisymmetric without swirl, which means that they have, in cylindrical coordinates (e_r,e_θ,e_z), the following structure: $v(t,x)=v^r(t,r,z)e_r+v^z(t,r,z)e_z$. Note that we assume that the velocity is invariant by rotation around the vertical axis (axisymmetric flow) and that the angular component v^θ of the vector field v is identically zero (without swirl).

Inspired by this, more recent works are devoted to the study of the three-dimensional axisymmetric Boussinesq system for different viscosities, here the velocity field v is axisymmetric without swirl, and the axisymmetric scalar temperature ρ means $\rho(t,x)=\rho(t,r,z)$ independent of the angle θ in cylindrical coordinates. In [23], H. Abidi, T. Hmidi and S. Keraani proved the global well-posedness for the Navier-Stokes-Boussinesq system (1.3) with $\mu>0$ and $\kappa=0$ with smooth axisymmetric initial data without swirl. In [30,31] global well-posedness of (1.3) with axisymmetric initial data without swirl was established in the case when the viscosity only occurs in the horizontal direction. For the case $\mu=0$ and $\kappa>0$, the system (1.3) reduces to the 3-D incompressible Euler-Boussinesq system

$$\begin{cases} \partial_t\rho+u\cdot\nabla\rho-\Delta\rho=0, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^3 \\ \partial_t u+u\cdot\nabla u+\nabla p=\rho e_z, \\ \operatorname{div} u=0, \\ u_{|t=0}=u^0, \qquad \rho_{|t=0}=\rho^0, \end{cases} \tag{1.4}$$

which couples the Euler equation with a transport-diffusion equation governing the temperature. Recently, under the assumptions that $s>\frac{5}{2}$, $u^0\in H^s$ is an axisymmetric divergence-free vector field without swirl, and ρ^0 is an axisymmetric function belonging to $H^{s-2}\cap L^m$ with $m>6$ and such that $r^2\rho^0\in L^2$, T. Hmidi and F. Rousset [24] proved the global well-posedness for the three-dimensional Euler-Boussinesq system (1.4). In [35], S. Sokrani investigated the global well-posedness of the 3-D incompressible Navier-Stokes-Boussinesq system with partial viscosity and axisymmetric data without swirl.

In this paper we study the persistence of the Sobolev regularity H^s with $s>\frac{5}{2}$ for the 3-D incompressible anisotropic Navier-Stokes-Boussinesq Equations (1.1) uniformly with respect to the viscosity μ, and then investigate its inviscid limit problem (towards (1.4) as the partial viscosity coefficient μ goes to zero).

Our main results are as follows.

Theorem 1.1 (Uniform boundedness of the velocity). *Let $\mu\in(0,1]$, $s>\frac{5}{2}$, $u^0\in H^s$ be an axisymmetric divergence-free vector field without swirl and let ρ^0 be an axisymmetric function belonging to $H^{s-2}\cap L^m$ with $m>6$ and such that $r^2\rho^0\in L^2$. Then there exists a unique global solution (u_μ,ρ_μ) to the system (1.1) satisfying*

$$(u_\mu,\rho_\mu)\in\mathcal{C}(\mathbb{R}_+;H^s)\times\left(\mathcal{C}(\mathbb{R}_+;H^{s-2}\cap L^m)\cap\widetilde{L}^1_{loc}(\mathbb{R}_+;H^s)\right),\quad r^2\rho_\mu\in\mathcal{C}(\mathbb{R}_+;L^2). \tag{1.5}$$

Moreover, there holds

$$\|\rho_\mu\|_{\widetilde{L}^\infty_t H^{s-2}}+\|\rho_\mu\|_{\widetilde{L}^1_t H^s}+\|u_\mu\|_{\widetilde{L}^\infty_t H^s}\le\phi_5(t), \tag{1.6}$$

with

$$\phi_k(t)=C_0\underbrace{\exp(\ldots\exp}_{k\,times}(C_0t^3)\ldots),$$

where C_0 depends only on the involved norms of the initial data and not on the viscosity μ.

The proof relies on the uniform estimate of the Lipschitz norm of the velocity. For this purpose we use the method developed in [24] for the inviscid case. However, the situation in the viscous case is more complicated because of the appearance of dissipative term. We especially have to check that it doesn't undermine some geometric properties of the vorticity.

Remark 1.1. Under the assumptions in Theorem 1.1, there exists a unique global solution (u,ρ) to the system (1.4) satisfying (1.5) and (1.6).

In effect, in view of the proof of (1.6), all the estimates in it are independent of the viscosity coefficient μ, so we may repeat the argument in the proof of (1.6) in Theorem 1.1 to get the same estimates for the solution of the system (1.4).

Our second main result deals with the inviscid limit.

Theorem 1.2 (Rate convergence). *Let (ρ_μ,u_μ) and (ρ,u) be respectively the solution of the Navier-Stokes-Boussinesq Equations (1.1) and Euler-Boussinesq systems (1.4) with the same initial data (ρ^0,u^0) satisfying the conditions of Theorem 1.1. Then we have the rate of convergence*

$$\|u_\mu-u\|_{\widetilde{L}^\infty_t H^{s-2}}+\|\rho_\mu-\rho\|_{\widetilde{L}^\infty_t H^{s-2}}+\|\rho_\mu-\rho\|_{\widetilde{L}^1_t H^s}\le(\mu t)\phi_6(t). \tag{1.7}$$

The proofs of Theorems 1.1 and 1.2 are completed in Sections 4-5. We now present a summary of the principal difficulties we encounter in our analysis as well as a sketch of the key ideas used in our proof.

Note that in view of the proof for the 3-D axisymmetric Euler equations, the crucial part of the proof of Theorems 1.1 is to get an a priori estimate of ω_μ in L^∞, where ω_μ is the angular component, the only component, of the vorticity ($\operatorname{curl} u_\mu$) of the velocity field u_μ, i.e., $\nabla\times u_\mu=\omega_\mu e_\theta$, which satisfies

$$\nabla\partial_t\omega_\mu+u_\mu\cdot\nabla\omega_\mu-\mu\partial_z^2\omega_\mu=-\partial_r\rho_\mu+\frac{u_\mu^r}{r}\omega_\mu. \tag{1.8}$$

In view of the method in [3], the quantity $\|\omega_\mu(t)\|_{L^\infty}$ may be bounded if we control the quantity $\|\frac{u_\mu^r}{r}\|_{L^\infty}$, which conversely can be bounded by the Lorentz norm $\|\frac{\omega_\mu}{r}\|_{L^{3,1}}$.

In effect, according to (1.8), the evolution of the quantity $\frac{\omega_\mu}{r}$ is governed by the equation

$$(\partial_t+u_\mu\cdot\nabla-\mu\partial_z^2)\frac{\omega_\mu}{r}=-\frac{1}{r}\partial_r\rho_\mu. \tag{1.9}$$

As mentioned in [24], the first main difficulty is to find some strong *a priori* estimates on ρ_μ to control the forcing term $-\frac{1}{r}\partial_r\rho_\mu$ in the right-hand side of (1.9), which can be thought as a Laplacian of ρ_μ because of the appearance of the the singularity $\frac{1}{r}$ on the axis $r=0$, and thus one may try to use smoothing effects of the diffusion system of ρ_μ to control it. Unfortunately, because of the lack of the complete Laplacian of the velocity u_μ, when we want to use this argument to deal with the advection term in the system, it is not sufficient to obtain an estimate for $\frac{1}{r}\partial_r\rho_\mu$ in $L^1_{loc}(L^p)$ by considering the convection term as a source term and by using the maximal smoothing effect of the heat equation. To handle it, we turn to use more carefully the structure of the coupling between the two equations of (1.1) in order to find suitable *a priori* estimates for (u_μ,ρ_μ). Indeed, in order to cancel the source term on the right-hand side in (1.9), we apply the operator $\frac{\partial_r}{r}\Delta^{-1}$ to the equation of the density to show

$$(\partial_t+u_\mu\cdot\nabla)(\frac{1}{r}\partial_r\Delta^{-1}\rho_\mu)=\frac{1}{r}\partial_r\rho_\mu-[\frac{1}{r}\partial_r\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu, \tag{1.10}$$

where the term $\frac{1}{r}\partial_r\rho_\mu$ appears in the right-hand side of (1.10) with the opposite sign of the one in the right-hand side of (1.9). Motivated by [24], we introduce a good unknown Γ_μ as

$$\Gamma_\mu:=\frac{\omega_\mu}{r}+\frac{\partial_r}{r}\Delta^{-1}\rho_\mu,$$

which, thanks to (1.9) and (1.10), satisfies

$$\partial_t\Gamma_\mu+u_\mu\cdot\nabla\Gamma_\mu-\mu\partial_z^2\Gamma_\mu=-[\frac{\partial_r}{r}\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu-\mu\partial_z^2(\frac{\partial_r}{r}\Delta^{-1}\rho_\mu), \tag{1.11}$$

with the commutator term $[\frac{\partial_r}{r}\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu=\frac{\partial_r}{r}\Delta^{-1}(u_\mu\cdot\nabla\rho_\mu)-u_\mu\cdot\nabla(\frac{\partial_r}{r}\Delta^{-1}\rho_\mu)$. Thus the basic energy estimate of the Equation (1.11) gives us that for every $p\in(1,+\infty)$

$$\|\Gamma_\mu(t)\|_{L^p}\lesssim\|\Gamma^0\|_{L^p}+\|[\frac{\partial_r}{r}\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu\|_{L^1_t(L^p)}+\mu\|\partial_z\rho_\mu\|_{L^2_t(L^p)}. \tag{1.12}$$

Compared with the estimate of Γ_μ in [24] for the axisymmetric Euler-Boussinesq equations (where the control of $\|\Gamma_\mu\|_{L^{3,1}}$ can be obtained directly by using the interpolation inequality), the additional term $\mu\|\partial_z\rho_\mu\|_{L^2_t(L^p)}$ in (1.12) can not, in general, give the estimate of $\mu\|\partial_z\rho_\mu\|_{L^2_t(L^{3,1})}$ according to Lemma 2.4, more precisely, $L^2_t((L^2,L^p]_{(\theta,1)})$ can not be embedded in $\left[L^2_t(L^2),L^2_t(L^p)\right]_{(\theta,1)}$ for any $p>3$. For this reason, we do not directly estimate $\|\Gamma_\mu\|_{L^{3,1}}$ and then $\|\frac{\omega_\mu}{r}\|_{L^{3,1}}$ but use the interpolation inequality to bound

$$\|\Gamma_\mu\|_{L^{3,1}}\lesssim\|\Gamma_\mu\|_{L^2}^{\frac{2(p-3)}{3(p-2)}}\|\Gamma_\mu\|_{L^p}^{\frac{p}{3(p-2)}}\quad\text{for some}\quad p>3. \tag{1.13}$$

Note that $H^{s-2}(\mathbb{R}^3)\hookrightarrow L^2\cap L^p(\mathbb{R}^3)$ for $s>\frac{5}{2}$ and some $3<p$ near 3, where $\|\Gamma_\mu\|_{L^2}$ and $\|\Gamma_\mu\|_{L^p}$ are controlled by (1.12), we may achieve (1.13) from the initial data viewpoint since $u^0\in H^s$ and $\rho^0\in H^{s-2}\cap L^m$ with $m>6$ and $r^2\rho^0\in L^2$. In this process, the complicated commutator estimate $\|[\frac{\partial_r\Delta^{-1}}{r},v_\mu\cdot\nabla]\rho_\mu\|_{L^1_t(L^p)}$ will be treated more carefully. For this, we give Proposition 3.2 in Section 3, and then deduce (1.6). In order to prove Theorem 1.2, we use the uniform bounds of the velocity in H^s combined with some smoothing effects on the viscosity and vorticity, which will be done in the last section.

The rest of the paper is organized as follows. In Section 2 we recall some basic ingredients of Littlwood-Paley theory. Following basic definitions above, in Section 3, we derive some qualitative and analytic properties of the flow associated with an axisymmetric vector field and an axisymmetric scalar function. In Section 4 we first give some necessary global a priori estimates, and then prove Theorem 1.1. The proof of Theorem 1.2 is completed in Section 5.

Let us complete this section with the notations we are going to use in this context.

Notations: Let A,B be two operators, we denote by $[A,B]=AB-BA$, the commutator between A and B. For $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\le Cb$ and C_0 denotes a positive constant depending on the initial data only.

For X a Banach space and I an interval of $\mathbb{R}$, we denote by $\mathcal{C}(I;X)$ the set of continuous functions on I with values in X, and by $\mathcal{C}_b(I;X)$ the subset of bounded functions of $\mathcal{C}(I;X)$. For $q\in[1,+\infty]$, the notation $L^q(I;X)$ stands for the set of measurable functions on I with values in X, such that $t\longmapsto\|f(t)\|_X$ belongs to $L^q(I)$. We also denote by $v_h=(v_1,v_2)^T$ the horizontal components of the vector field v, and $x=(x_h,x_3)^T\in\mathbb{R}^3$ with $x_h=(x_1,x_2)^T$. The operator $\mathcal{R}_{ij}$ $(i,j=1,2,3)$ means the Riesz transform: $\mathcal{R}_{ij}=\partial_i\partial_j\Delta^{-1}$.

2. Littlewood-Paley analysis and Lorentz spaces

The proof of Theorem 1.1 requires Littlewood-Paley decomposition. Let us briefly explain how it may be built in the case $x\in\mathbb{R}^3$ (see e.g. [5]). Let φ be a smooth function supported in the annulus $\mathcal{C}\overset{\text{def}}{=}\{\xi\in\mathbb{R}^3,\frac{3}{4}\le|\xi|\le\frac{8}{3}\}$ and $\chi(\xi)$ be a smooth function supported in the ball $\mathcal{B}\overset{\text{def}}{=}\{\xi\in\mathbb{R}^3,\ |\xi|\le\frac{4}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1\quad\text{for}\quad\xi\neq0\quad\text{and}\quad\chi(\xi)+\sum_{q\ge0}\varphi(2^{-q}\xi)=1\quad\text{for all}\quad\xi\in\mathbb{R}^3.$$

Then for $u\in\mathcal{S}'(\mathbb{R}^3)$, we set

$$\forall\ q\in\mathbb{N},\quad\Delta_q u\overset{\text{def}}{=}\varphi(2^{-q}\mathrm{D})u,\quad\Delta_{-1}u\overset{\text{def}}{=}\chi(D)u\quad\text{and}\quad S_q u\overset{\text{def}}{=}\sum_{-1\le q'\le q-1}\Delta_{q'}u,$$

we have the formal Littlewood-Paley decomposition

$$u=\sum_{q\geq -1}\Delta_q u \quad \forall u\in\mathcal{S}'(\mathbb{R}^3).$$

Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\Delta_j\Delta_k u\equiv 0 \quad\text{if}\quad |j-k|\geq 2 \quad\text{and}\quad \Delta_j(S_{k-1}u\Delta_k u)\equiv 0 \quad\text{if}\quad |j-k|\geq 5.$$

We recall now the definition of inhomogeneous Besov spaces and Bernstein-type inequalities from [5].

DEFINITION 2.1 (Definition 2.15 of [5]). *Let $(p,r)\in[1,+\infty]^2$, $s\in\mathbb{R}$ and $u\in\mathcal{S}'(\mathbb{R}^3)$, we set*

$$\|u\|_{B^s_{p,r}}\overset{def}{=}\Big(2^{js}\|\Delta_j u\|_{L^p}\Big)_{\ell^r}.$$

We define $B^s_{p,r}(\mathbb{R}^3)\overset{def}{=}\big\{u\in\mathcal{S}'(\mathbb{R}^3)\,\big|\,\|u\|_{B^s_{p,r}}<\infty\big\}$.

For the convenience of the reader, in what follows, we recall some basic facts on Littlewood-Paley theory, one may check [5] for more details.

LEMMA 2.1 (Bernstein inequalities, [5]). *Let $\mathcal{B}$ be a ball and $\mathcal{C}$ an annulus of $\mathbb{R}^3$. A constant C exists so that for any positive real number δ, any non-negative integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers (a,b) with $b\geq a\geq 1$, there hold*

$$\begin{aligned}
&\mathrm{Supp}\ \hat{u}\subset\delta\mathcal{B}\Rightarrow\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^b}\leq C^{k+1}\delta^{k+N(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a},\\
&\mathrm{Supp}\ \hat{u}\subset\delta\mathcal{C}\Rightarrow C^{-1-k}\delta^k\|u\|_{L^a}\leq\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^a}\leq C^{1+k}\delta^k\|u\|_{L^a},\\
&\mathrm{Supp}\ \hat{u}\subset\delta\mathcal{C}\Rightarrow\|\sigma(D)u\|_{L^b}\leq C_{\sigma,m}\delta^{m+N(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a}.
\end{aligned}\tag{2.1}$$

We also recall Bony's decomposition from [6]:

$$uv=T_uv+T'_vu=T_uv+T_vu+R(u,v),$$

where

$$\begin{aligned}
&T_uv\overset{\text{def}}{=}\sum_{j\in\mathbb{Z}}S_{j-1}u\Delta_jv, \qquad T'_vu\overset{\text{def}}{=}\sum_{j\in\mathbb{Z}}S_{j+2}v\Delta_ju,\\
&R(u,v)\overset{\text{def}}{=}\sum_{j\in\mathbb{Z}}\Delta_ju\widetilde{\Delta}_jv \quad\text{with}\quad \widetilde{\Delta}_jv\overset{\text{def}}{=}\sum_{|j'-j|\leq 1}\Delta_{j'}v.
\end{aligned}$$

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we need to use Chemin-Lerner-type spaces $\widetilde{L}^\lambda_T(B^s_{p,r}(\mathbb{R}^3))$ from [5].

DEFINITION 2.2. *Let $(r,\lambda,p)\in[1,+\infty]^3$ and $T\in(0,+\infty]$. We define $\widetilde{L}^\lambda_T(B^s_{pr}(\mathbb{R}^d))$ the space of all functions u satisfying*

$$\|u\|_{\widetilde{L}^\lambda_T(B^s_{p,r})}\overset{def}{=}\Big(\sum_{j\geq -1}2^{jrs}\Big(\int_0^T\|\Delta_ju(t)\|^\lambda_{L^p}\,dt\Big)^{\frac{r}{\lambda}}\Big)^{\frac{1}{r}}<\infty.$$

with the usual change if $r=\infty$. *For short, we just denote this space by* $\widetilde{L}^\lambda_T(B^s_{p,r})$.

The relationships between these spaces are detailed in the following lemma, which is a direct consequence of the Minkowski inequalities.

LEMMA 2.2. *Let* $s\in\mathbb{R},\epsilon>0$, $r\geq 1$ *and* $(p_1,p_2)\in[1,\infty]^2$. *Then we have the following embeddings*

$$L^r_T B^s_{p_1,p_2}\hookrightarrow \widetilde{L}^r_T B^s_{p_1,p_2}\hookrightarrow L^r_T B^{s-\epsilon}_{p_1,p_2}, \text{if} \quad r\leq p_2,$$

$$L^r_T B^{s+\epsilon}_{p_1,p_2}\hookrightarrow \widetilde{L}^r_T B^s_{p_1,p_2}\hookrightarrow L^r_T B^{s}_{p_1,p_2}, \text{if} \quad r\geq p_2.$$

To prove Theorem 1.1, we also need to use Lorentz space $L^{p,q}(\mathbb{R}^3)$. For the convenience of the readers, we recall some basic facts on $L^{p,q}(\mathbb{R}^3)$ from [17,27,32]:

DEFINITION 2.3 (Definition 1.4.6 of [17]). *For a measurable function* f *on* $\mathbb{R}^3$, *we define its non-increasing rearrangement by*

$$f^*(t)\stackrel{def}{=}\inf\Big\{s>0, \mu\big(\{x, |f(x)|>s\}\big)\leq t\Big\},$$

where μ *denotes the usual Lebesgue measure. For* $(p,q)\in[1,+\infty]^2$, *the Lorentz space* $L^{p,q}(\mathbb{R}^3)$ *is the set of functions* f *such that* $\|f\|_{L^{p,q}}<\infty$, *with*

$$\|f\|_{L^{p,q}}\stackrel{def}{=}\begin{cases}\Big(\displaystyle\int_0^\infty\big(t^{\frac{1}{p}}f^*(t)\big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}, & for 1\leq q<\infty\\ \sup\limits_{t>0} t^{\frac{1}{p}}f^*(t), & for q=\infty.\end{cases}$$

We remark that Lorentz spaces can also be defined by real interpolation from Lebesgue spaces (see for instance Definition 2.3 of [27]):

$$(L^{p_0},L^{p_1})_{(\beta,q)}=L^{p,q},$$

where $1\leq p_0<p<p_1\leq\infty$, β satisfies $\frac{1}{p}=\frac{1-\beta}{p_0}+\frac{\beta}{p_1}$ and $1\leq q\leq\infty$.

To establish some functional inequalities involving Lorentz spaces the following classical calculus will be very useful.

LEMMA 2.3 (see pages 18-20 of [27]). *Let* $1<p<\infty$ *and* $1\leq q\leq\infty$, *we have the following assertions.*

- *For the Riesz transform* $\mathcal{R}_{ij}=\partial_i\partial_j\Delta^{-1}$, $i,j=1,2$, *there holds*

$$\|\mathcal{R}_{ij}f\|_{L^{p,q}}\lesssim\|f\|_{L^{p,q}}.$$

- *If* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ *and* $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, *then*

$$\|fg\|_{L^{p,q}}\lesssim\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}.$$

- *If* $1<p<\infty$, $\frac{1}{p}+1=\frac{1}{p_1}+\frac{1}{p_2}$ *and* $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, *then*

$$\|f*g\|_{L^{p,q}}\lesssim\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}},$$

  for $p=\infty$, *and* $\frac{1}{q_1}+\frac{1}{q_2}=1$, *then*

$$\|f*g\|_{L^\infty}\lesssim\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}.$$

- *For* $1\le p\le\infty$ *and* $1\le q_1\le q_2\le\infty$, *we have*

$$L^{p,q_1}\hookrightarrow L^{p,q_2} \qquad \text{and} \qquad L^{p,p}=L^p.$$

The following Lions-Peetre formula for the space-time interpolation has made a special contribution to the proof of Theorem 1.1.

LEMMA 2.4 ([11,28]). *Let* $(A_0,A_1,\mathcal{A})$ *be an interpolation triple. Then*

(i) for $p_0,p_1\in[1,+\infty]$, $\theta\in(0,1)$, *there holds*

$$\left[L^{p_0}(A_0),L^{p_1}(A_1)\right]_{(\theta,q)}=L^q((A_0,A_1]_{(\theta,q)})$$

provided $q=p(\theta)$ *with* $\frac{1}{p(\theta)}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$;

(ii) for $1\le q\le p\le+\infty$, $\theta\in(0,1)$, *there holds*

$$\left[L^{p}(A_0),L^{p}(A_1)\right]_{(\theta,q)}\hookrightarrow L^p((A_0,A_1]_{(\theta,q)}),$$

and the reverse inclusion holds for $1\le p\le q\le+\infty$.

In order to estimate the convection terms in (1.1) and (1.4), we need to state some useful commutator estimates.

LEMMA 2.5 (Lemmas 2.7 and 2.8 in [24]).

(1) Given $(p,t,q,m)\in[1,+\infty]^4$ *such that*

$$1+\frac{1}{p}=\frac{1}{t}+\frac{1}{q}+\frac{1}{m},\quad p\ge t \quad \text{and} \quad q>3(1-\frac{1}{t}).$$

Let f, g *and* h *be three functions such that* $\nabla f\in L^q$, $g\in L^m$ *and* $x\mathcal{F}^{-1}h\in L^t$. *Then*

$$\|[h(D),f]g\|_{L^p}\le C\|x\mathcal{F}^{-1}h\|_{L^t}\|\nabla f\|_{L^q}\|g\|_{L^m}. \tag{2.2}$$

where C *is a constant independent of* f, g, *and* h.

(2) Given $(p,t,m)\in[1,+\infty]^3$ *such that* $\frac{1}{p}=\frac{1}{t}+\frac{1}{m}$. *Then there exists* $C>0$ *such that for* $\nabla f\in L^t$, $g\in L^m$ *and for every* $q\in\mathbb{N}\cup\{0\}$

$$\|[\Delta_q,f]g\|_{\dot W^{1,p}}\le C\|\nabla f\|_{L^t}\|g\|_{L^m} \tag{2.3}$$

with the definition $\|\phi\|_{\dot W^{1,p}}=\|\nabla\phi\|_{L^p}$.

3. Some estimates on axisymmetric functions

This section is concerned with the study of actions of some operators over axisymmetric functions.

Let's recall first the identity about the action of the operator $\frac{\partial_r}{r}\Delta^{-1}$ over axisymmetric functions in [24].

PROPOSITION 3.1 (Proposition 2.9 in [24]). *For every axisymmetric smooth scalar function* f, *we have*

$$\frac{\partial_r}{r}\Delta^{-1}f=\frac{x_2^2}{r^2}\mathcal{R}_{11}f(x)+\frac{x_1^2}{r^2}\mathcal{R}_{22}f(x)-2\frac{x_1x_2}{r^2}\mathcal{R}_{12}f(x) \tag{3.1}$$

with $\mathcal{R}_{ij}=\partial_{ij}(-\Delta)^{-1}$ *for* $i,j=1,2$.

With the aid of the identity (3.1) and the commutator estimate (2.2), similar to but more complicated than the proof of Theorem 3.1 in [24], we have

PROPOSITION 3.2. *Let* $2 \le p < 6$, v *be an axisymmetric smooth and divergence-free without swirl vector field,* $\mathrm{curl}v = \omega e_\theta$ *and* ρ *an axisymmetric smooth scalar function. Then we have, with the notation* $x_h = (x_1, x_2)$, *that*

$$\|[\frac{\partial_r}{r}\Delta^{-1}, v\cdot\nabla]\rho\|_{L^p} \lesssim \|\omega/r\|_{L^p}\big(\|\rho\|_{L^{3,1}} + \|\rho\|_{L^6} + \|\rho x_h\|_{L^2\cap B^0_{\infty,1}}\big) + \|\omega/r\|_{L^2}\big(\|\rho\|_{B^0_{\frac{6p}{6-p},2}} + \|\rho\|_{L^2} + \|\rho x_h\|_{L^{\frac{6p}{6-p}}}\big). \tag{3.2}$$

Proof. Since the functions ρ and $v\cdot\nabla\rho$ are axisymmetric, an application of the identity (3.1) of Proposition 3.1 shows that

$$\frac{\partial_r}{r}\Delta^{-1}\rho(x) = \sum_{i,j=1}^{2} a_{ij}(x)\mathcal{R}_{ij}\rho(x)$$

and

$$\frac{\partial_r}{r}\Delta^{-1}(v\cdot\nabla\rho)(x) = \sum_{i,j=1}^{2} a_{ij}(x)\mathcal{R}_{ij}(v\cdot\nabla\rho)(x)$$

with

$$a_{11}(x) = \frac{x_2^2}{r^2}, a_{12}(x) = a_{21}(x) = -\frac{x_1x_2}{r^2}, a_{22}(x) = \frac{x_1^2}{r^2}. \tag{3.3}$$

Hence, since the velocity v is divergence-free,

$$[\frac{\partial_r}{r}\Delta^{-1}, v\cdot\nabla]\rho(x) = \sum_{i,j=1}^{2} a_{ij}(x)\,\mathrm{div}\Big([\mathcal{R}_{ij}, v\cdot\nabla]\rho\Big),$$

which immediately, according to the fact $|a_{ij}(x)| \le 1$ ($\forall\, x\in\mathbb{R}^3$, $i,j = 1,2$), gives us that

$$\|[\frac{\partial_r}{r}\Delta^{-1}, v\cdot\nabla]\rho\|_{L^p} \le \sum_{i,j=1}^{2} \|\mathrm{div}([\mathcal{R}_{ij}, v]\rho)\|_{L^p}. \tag{3.4}$$

Let's now bound the L^p norm of terms in $\mathrm{div}([\mathcal{R}_{ij}, v]\rho) = \sum_{k=1}^{3}\partial_k([\mathcal{R}_{ij}, v^k]\rho)$ step by step.

Since an application of the Biot-Savart law shows

$$v^1 = \Delta^{-1}\big(\cos(\theta)\partial_3\omega\big) = \Delta^{-1}\partial_3\big(x_1\frac{\omega}{r}\big), \quad v^2 = \Delta^{-1}\big(\sin(\theta)\partial_3\omega\big) = \Delta^{-1}\partial_3\big(x_2\frac{\omega}{r}\big),$$

the terms $\partial_1([\mathcal{R}_{ij}, v^1]\rho)$ and $\partial_2([\mathcal{R}_{ij}, v^2]\rho)$ can be treated in same way and hence, we shall prove the estimate of the first one only.

• Estimates of $\partial_1([\mathcal{R}_{ij}, v^1]\rho)$. Before proceeding, let us split the term $\partial_1([\mathcal{R}_{ij}, v^1]\rho)$ into thirteen terms by using Bony's decomposition $\partial_1([\mathcal{R}_{ij}, v^1]\rho) = \sum_{\ell=1}^{13} I_\ell$ with

$$I_1 = \mathcal{R}_{13}(\omega/r)\mathcal{L}^1_{ij}\rho,\ I_2 = \partial_3\Delta^{-1}(\omega/r)\partial_1\mathcal{L}^1_{ij}\rho,\ I_3 = \partial_1\sum_{q\ge 0}[\mathcal{R}_{ij}, S_{q-1}(\mathcal{L}(\omega/r))]\Delta_q\rho,$$

$$I_4=\sum_{q\geq 0}\partial_1\mathcal{R}_{ij}(\Delta_q(\mathcal{L}(\omega/r))S_{q-1}\rho),\ I_5=-\sum_{q\geq 0}\partial_1\{\Delta_q(\mathcal{L}(\omega/r))\mathcal{R}_{ij}S_{q-1}\rho\},$$
$$I_6=\partial_1\sum_{q\geq 1}[\mathcal{R}_{ij},\Delta_q(\mathcal{L}(\omega/r))]\tilde{\Delta}_q\rho,\ I_7=\sum_{-1\leq q\leq 0}[\partial_1\mathcal{R}_{ij},\Delta_q(\mathcal{L}(\omega/r))]\tilde{\Delta}_q\rho,$$
$$I_8=-\sum_{-1\leq q\leq 0}\partial_1\mathcal{L}\Delta_q(\omega/r)\mathcal{R}_{ij}\tilde{\Delta}_q\rho,\ I_9=\partial_1\sum_{q\geq 0}[\mathcal{R}_{ij},S_{q-1}(\partial_3\Delta^{-1}(\omega/r))]\Delta_q(x_1\rho),$$
$$I_{10}=\partial_1\sum_{q\geq 1}[\mathcal{R}_{ij},\Delta_q(\partial_3\Delta^{-1}(\omega/r))]\tilde{\Delta}_q(x_1\rho),$$
$$I_{11}=\partial_1\sum_{-1\leq q\leq 0}[\mathcal{R}_{ij},\Delta_q(\partial_3\Delta^{-1}(\omega_/r))]\tilde{\Delta}_q(x_1\rho),$$
$$I_{12}=\sum_{q\geq 0}[\mathcal{R}_{ij},\Delta_q(\partial_{13}\Delta^{-1}(\omega/r))]S_{q-1}(x_1\rho),$$
$$I_{13}=\sum_{q\geq 0}[\mathcal{R}_{ij},\Delta_q(\partial_3\Delta^{-1}(\omega/r))]\partial_1S_{q-1}(x_1\rho),$$

where $\mathcal{L}^1_{ij}=-2\partial_1\Delta^{-1}\mathcal{R}_{ij}+\delta_{i1}\partial_j\Delta^{-1}+\delta_{j1}\partial_i\Delta^{-1}$ with δ_{ij} denotes the Kronecker symbol, $\mathcal{L}_{ij}=-2\mathcal{R}_{ij}\Delta^{-1}$ and $\mathcal{L}=-2\mathcal{R}_{13}\Delta^{-1}$. We estimate them term by term.

For I_1, since $\mathcal{R}_{13}$ is a Riesz operator and the operator $\mathcal{L}^1_{ij}$ has a convolution kernel whose behavior looks like $\frac{1}{|x|^2}$ $(\in L^{3/2,\infty})$, we deduce from Lemma 2.3 that

$$\|I_1\|_{L^p}=\|\mathcal{R}_{13}(\omega/r)\mathcal{L}^1_{ij}\rho\|_{L^p}\leq\|\mathcal{R}_{13}(\omega/r)\|_{L^p}\|\mathcal{L}^1_{ij}\rho\|_{L^\infty}\lesssim\|\omega/r\|_{L^p}\|\rho\|_{L^{3,1}}.$$

Similarly, for $I_2=\partial_3\Delta^{-1}(\omega/r)\partial_1\mathcal{L}^1_{ij}\rho$, $\partial_1\mathcal{L}^1_{ij}$ is a Riesz operator and the behavior of $\partial_3\Delta^{-1}$ is similar to the one of $\mathcal{L}^1_{ij}$, then we use Hölder's inequality and Sobolev embeddings to show

$$\begin{aligned}\|I_2\|_{L^p}\leq\|\partial_3\Delta^{-1}(\omega/r)\|_{L^6}\|\partial_1\mathcal{L}^1_{ij}\rho\|_{L^{\frac{6p}{6-p}}}&\lesssim\|\nabla\partial_3\Delta^{-1}(\omega/r)\|_{L^2}\|\rho\|_{L^{\frac{6p}{6-p}}}\\&\lesssim\|\omega/r\|_{L^2}\|\rho\|_{L^{\frac{6p}{6-p}}}.\end{aligned}$$

In order to estimate I_3, we first get that there exists a function $\psi\in\mathcal{S}(\mathbb{R}^3)$ such that

$$I_3=\sum_{q\geq 0}\partial_1\{[\psi_q(D),S_{q-1}(\mathcal{L}(\omega/r))]\Delta_q\rho\}\tag{3.5}$$

with $\psi_q=2^{3q}\psi(2^q\cdot)$. As for $2\leq p<\infty$, we have $B^0_{p,2}\hookrightarrow L^p$ (see [36]), then by using Bernstein inequalities (2.1), it follows

$$\begin{aligned}\|I_3\|^2_{L^p}&\lesssim\sum_{q\geq 0}\|\partial_1\{[\psi_q(D),S_{q-1}(\mathcal{L}(\omega/r))]\Delta_q\rho\}\|^2_{L^p}\\&\lesssim\sum_{q\geq 0}2^{2q}\|[\psi_q(D),S_{q-1}(\mathcal{L}(\omega/r))]\Delta_q\rho\|^2_{L^p}.\end{aligned}$$

Thanks to Lemmas 2.3 and 2.5, we find

$$\begin{aligned}&\|\partial_1\{[\psi_q(D),S_{q-1}(\mathcal{L}(\omega/r))]\Delta_q\rho\}\|_{L^p}\lesssim 2^q||x\psi_q||_{L^1}||\nabla\mathcal{L}(\omega/r)||_{L^6}||\Delta_q\rho||_{L^{\frac{6p}{6-p}}}\\&\lesssim||\nabla^2\mathcal{L}(\omega/r)||_{L^2}||\Delta_q\rho||_{L^{\frac{6p}{6-p}}}\lesssim||\omega/r||_{L^2}||\Delta_q\rho||_{L^{\frac{6p}{6-p}}},\end{aligned}$$

which leads to

$$\|I_3\|_{L^p} \lesssim \|\omega/r\|_{L^2} \|\rho\|_{B^0_{\frac{6p}{6-p},2}}.$$

By using the Hölder inequality and Bernstein's inequalities (2.1) again, we show

$$\|I_4\|_{L^p}^2 + \|I_5\|_{L^p}^2 \lesssim \sum_{q\geq 0} 2^{2q} \|\Delta_q(\mathcal{L}(\omega/r)) S_{q-1}\rho\|_{L^p}^2 + \sum_{q\geq 0} 2^{2q} \|\Delta_q(\mathcal{L}(\omega/r)) \mathcal{R}_{ij} S_{q-1}\rho\|_{L^p}^2$$
$$\lesssim \sum_{q\geq 0} 2^{2q} \|\Delta_q(\mathcal{L}(\omega/r))\|_{L^p}^2 (\|S_{q-1}\rho\|_{L^\infty}^2 + \|\mathcal{R}_{ij} S_{q-1}\rho\|_{L^\infty}^2)$$
$$\lesssim \|\omega/r\|_{L^p}^2 \sum_{q\geq 0} 2^{-2q} (\|S_{q-1}\rho\|_{L^\infty}^2 + \|\mathcal{R}_{ij} S_{q-1}\rho\|_{L^\infty}^2) \lesssim \|\omega/r\|_{L^p}^2 \|\rho\|_{L^m}^2,$$

for $3 < m$, and in particular,

$$\|I_4\|_{L^p} + \|I_5\|_{L^p} \lesssim \|\omega/r\|_{L^p} \|\rho\|_{L^6}.$$

To estimate I_6, using Bernstein's inequalities (2.1) yields, $\forall\, k \in \mathbb{N} \cup \{-1\}$,

$$\|\Delta_k I_6\|_{L^p} \lesssim 2^k \sum_{q\geq k-4} \|[\mathcal{R}_{ij}, \Delta_q(\mathcal{L}(\omega/r))] \tilde{\Delta}_q \rho\|_{L^p}.$$

While

$$\|[\mathcal{R}_{ij}, \Delta_q(\mathcal{L}(\omega/r))] \tilde{\Delta}_q \rho\|_{L^p}$$
$$\lesssim \|\Delta_q(\mathcal{L}(\omega/r))\|_{L^6} \|\tilde{\Delta}_q \rho\|_{L^{\frac{6p}{6-p}}} + \|\Delta_q(\mathcal{L}(\omega/r))\|_{L^6} \|\mathcal{R}_{ij} \tilde{\Delta}_q \rho\|_{L^{\frac{6p}{6-p}}}$$
$$\lesssim 2^{-q} \|\omega/r\|_{L^2} \|\tilde{\Delta}_q \rho\|_{L^{\frac{6p}{6-p}}},$$

it follows from the embedding $B^0_{p,2} \hookrightarrow L^p$ that

$$\|I_6\|_{L^p} \lesssim \|\omega/r\|_{L^2} \|\rho\|_{B^0_{\frac{6p}{6-p},2}}.$$

Let's now turn to handle I_7. In view of Lemma 2.5, we deduce that

$$\|[\partial_1 \mathcal{R}_{ij}, \Delta_q(\mathcal{L}(\omega/r))] \tilde{\Delta}_q \rho\|_{L^p} \lesssim \|xh\|_{L^{\frac{3p}{p+3}}} \|\nabla \mathcal{L}(\omega/r)\|_{L^6} \|\tilde{\Delta}_q \rho\|_{L^2},$$

where $\widehat{h}(\xi) = \xi_1 \frac{\xi_i \xi_j}{|\xi|^2} \Phi(\xi)$ and $\Phi \in \mathcal{D}(\mathbb{R}^3)$. An application of the well-known Mikhlin-Hormander theorem shows that

$$|h(x)| \lesssim (1+|x|)^{-4}, \quad \forall x \in \mathbb{R}^3,$$

which leads to $xh \in L^{\frac{3p}{p+3}}$ and then

$$\|[\partial_1 \mathcal{R}_{ij}, \Delta_q(\mathcal{L}(\omega/r))] \tilde{\Delta}_q \rho\|_{L^p} \lesssim \|\nabla^2 \mathcal{L}(\omega/r)\|_{L^2} \|\tilde{\Delta}_q \rho\|_{L^2} \lesssim \|\omega/r\|_{L^2} \|\tilde{\Delta}_q \rho\|_{L^2}.$$

Therefore we get that

$$\|I_7\|_{L^p} \leq \sum_{-1\leq q\leq 0} \|[\partial_1 \mathcal{R}_{ij}, \Delta_q(\mathcal{L}(\omega/r))] \tilde{\Delta}_q \rho\|_{L^p} \lesssim \|\omega/r\|_{L^2} \|\rho\|_{L^2}.$$

We directly estimate by the Hölder inequality and Sobolev embeddings

$$\begin{aligned}||I_8||_{L^p} &\lesssim \sum_{-1\le q\le 0} ||\partial_1 \mathcal{L}\Delta_q(\omega/r)\mathcal{R}_{ij}\tilde{\Delta}_q\rho||_{L^p} \lesssim ||\partial_1\mathcal{L}(\omega/r)||_{L^6}||\rho||_{L^{\frac{6p}{6-p}}}\\ &\lesssim ||\nabla\partial_1\mathcal{L}(\omega/r)||_{L^2}||\rho||_{L^{\frac{6p}{6-p}}} \lesssim ||\omega/r||_{L^2}||\rho||_{L^{\frac{6p}{6-p}}}.\end{aligned}$$

For I_9, as in (3.5), $I_9=\sum_{q\ge 0}\partial_1\{[\psi_q(D),S_{q-1}(\partial_3\Delta^{-1}(\omega/r))]\Delta_q(x_1\rho)\}$, thanks to Berstein's inequalities (2.1), we write that

$$\begin{aligned}&\|\partial_1\{[\psi_q(D),S_{q-1}(\partial_3\Delta^{-1}(\omega/r))]\Delta_q(x_1\rho)\}\|_{L^p}\\ \lesssim& 2^q\|\{[\psi_q(D),S_{q-1}(\partial_3\Delta^{-1}(\omega/r))]\Delta_q(x_1\rho)\}\|_{L^p},\end{aligned}$$

which follows from Lemmas 2.3 and 2.5 that

$$\begin{aligned}&\|\partial_1\{[\psi_q(D),S_{q-1}(\partial_3\Delta^{-1}(\omega/r))]\Delta_q(x_1\rho)\}\|_{L^p}\\ \lesssim& 2^q\|x\psi_q\|_{L^1}2^{-q}\|\nabla\partial_3\Delta^{-1}(\omega/r)\|_{L^p}\|\Delta_q(x_1\rho)\|_{L^\infty}\lesssim\|\omega/r\|_{L^p}\|\Delta_q(x_1\rho)\|_{L^\infty}\end{aligned}$$

Thus, we deduce

$$\|I_9\|_{L^p}\le\sum_{q\ge 0}\|\partial_1\{[\psi_q(D),S_{q-1}(\partial_3\Delta^{-1}(\omega/r))]\Delta_q(x_1\rho)\}\|_{L^p}\lesssim\|\omega/r\|_{L^p}\|x_1\rho\|_{B^0_{\infty,1}}.$$

Note that $I_{10}=\partial_1\sum_{q\ge 1}[\mathcal{R}_{ij},\Delta_q(\partial_3\Delta^{-1}(\omega/r))]\tilde{\Delta}_q(x_1\rho)$, without using the structure of the commutator, one has $\forall\, k\in\mathbb{N}\cup\{-1\}$

$$\begin{aligned}\|\Delta_k I_{10}\|_{L^p}&\lesssim 2^k\sum_{q\ge k-4}\|[\mathcal{R}_{ij},\Delta_q(\partial_3\Delta^{-1}(\omega/r))]\tilde{\Delta}_q(x_1\rho)\|_{L^p}\\ \lesssim& 2^k\sum_{q\ge k-4}\|\Delta_q(\partial_3\Delta^{-1}(\omega/r))\|_{L^p}\Big(\|\tilde{\Delta}_q(x_1\rho)\|_{L^\infty}+\|\mathcal{R}_{ij}\tilde{\Delta}_q(x_1\rho)\|_{L^\infty}\Big)\\ \lesssim& 2^k\sum_{q\ge k-4}2^{-q}\|\omega/r\|_{L^p}\|\tilde{\Delta}_q(x_1\rho)\|_{L^\infty}.\end{aligned}$$

Hence, we infer

$$\begin{aligned}\|I_{10}\|_{L^p}&\lesssim 2^k\sum_{q\ge k-4}\|[\mathcal{R}_{ij},\Delta_q(\partial_3\Delta^{-1}(\omega/r))]\tilde{\Delta}_q(x_1\rho)\|_{L^p}\\ \lesssim& 2^k\sum_{q\ge k-4}\|\Delta_q(\partial_3\Delta^{-1}(\omega/r))\|_{L^p}\Big(\|\tilde{\Delta}_q(x_1\rho)\|_{L^\infty}+\|\mathcal{R}_{ij}\tilde{\Delta}_q(x_1\rho)\|_{L^\infty}\Big)\\ \lesssim&\|\omega/r\|_{L^p}\sum_{k\ge -1}\sum_{q\ge k-4}2^{k-q}\|\tilde{\Delta}_q(x_1\rho)\|_{L^\infty}\lesssim\|\omega/r\|_{L^p}\|x_1\rho\|_{B^0_{\infty,1}}.\end{aligned}$$

While the continuity of the Riesz transform on L^λ for $\forall\, 1<\lambda<\infty$ shows us that

$$\begin{aligned}||[\mathcal{R}_{ij},\Delta_q(\partial_3\Delta^{-1}(\omega/r))]\tilde{\Delta}_q(x_1\rho)||_{L^p}&\lesssim||\Delta_q(\partial_3\Delta^{-1}(\omega/r))||_{L^6}||\tilde{\Delta}_q(x_1\rho)||_{L^{\frac{6p}{6-p}}}\\ &\lesssim||\omega/r||_{L^2}||x_1\rho||_{L^{\frac{6p}{6-p}}},\end{aligned}$$

and then

$$\|I_{11}\|_{L^p}\lesssim||\omega/r||_{L^2}||x_1\rho||_{L^{\frac{6p}{6-p}}}.$$

Similarly, we may readily deduce

$$\|I_{12}\|_{L^p}+\|I_{13}\|_{L^p}\lesssim ||\omega/r||_{L^p}||x_1\rho||_{L^2\cap B^0_{\infty,1}}.$$

Therefore, we obtain

$$\begin{aligned}\|\partial_1([\mathcal{R}_{ij},v^1]\rho)\|_{L^p}\lesssim&\|\omega/r\|_{L^p}\big(\|\rho\|_{L^{3,1}}+\|\rho\|_{L^6}+||x_1\rho||_{L^2\cap B^0_{\infty,1}}\big)\\&+||\omega/r||_{L^2}\big(||\rho||_{B^0_{\frac{6p}{6-p},2}}+||\rho||_{L^2}+||x_1\rho||_{L^{\frac{6p}{6-p}}}\big).\end{aligned}\tag{3.6}$$

We also can show that the same estimate of $\|\partial_2([\mathcal{R}_{ij},v^2]\rho)\|_{L^p}$ is true, and let us now turn to estimate the term $\partial_3([\mathcal{R}_{ij},v^3]\rho)$ which has a different structure from $\partial_1([\mathcal{R}_{ij},v^1]\rho)$.

- Estimate of $\partial_3([\mathcal{R}_{ij},v^3]\rho)$. We first have the decomposition

$$\begin{aligned}-\partial_3([\mathcal{R}_{ij},v^3]\rho)=&\sum_{k=1}^{2}\partial_3(\partial_k\Delta^{-1}(\omega/r)[\mathcal{R}_{ij},x_k]\rho)+2\partial_3([\mathcal{R}_{ij},x_k]\rho)\\&+2\partial_3([\mathcal{R}_{ij},\Delta^{-1}\mathcal{R}_{33}(\omega/r)]\rho)+\sum_{k=1}^{2}\partial_3([\mathcal{R}_{ij},\partial_k\Delta^{-1}(\omega/r)](x_k\rho)\\=&I+II+III.\end{aligned}$$

To estimate the first term I, we use the form

$$\partial_3(\partial_k\Delta^{-1}(\frac{\omega}{r})[\mathcal{R}_{ij},x_k]\rho)=\mathcal{R}_{3k}(\frac{\omega}{r})\mathcal{L}^k_{ij}\rho+\partial_k\Delta^{-1}(\frac{\omega}{r})\partial_3\mathcal{L}^k_{ij}\rho$$

to give

$$\begin{aligned}\|I\|_{L^p}\le&\sum_{i=1}^{2}\big(||\mathcal{L}^k_{ij}\rho||_{L^\infty}||\mathcal{R}_{13}(\omega/r)||_{L^p}+||\partial_k\Delta^{-1}(\omega/r)||_{L^6}||\partial_3\mathcal{L}^k_{ij}\rho||_{L^{\frac{6p}{6-p}}}\big)\\\lesssim&||\omega/r||_{L^p}\|\rho\|_{L^{3,1}}+||\omega/r||_{L^2}||\rho||_{L^{\frac{6p}{6-p}}}.\end{aligned}$$

As the operator $\Delta^{-1}\mathcal{R}_{33}$ has the same properties as $\mathcal{L}=-2\partial_{13}\Delta^{-2}$, then the estimates of the terms II and III are similar to the ones of I_ℓ for $3\le\ell\le 13$. Hence, one finds

$$\begin{aligned}\|\partial_3([\mathcal{R}_{ij},v^3]\rho)\|_{L^p}\lesssim&\|\omega/r\|_{L^p}\big(\|\rho\|_{L^{3,1}}+\|\rho\|_{L^6}+||x_1\rho||_{L^2\cap B^0_{\infty,1}}\big)\\&+||\omega/r||_{L^2}\big(||\rho||_{B^0_{\frac{6p}{6-p},2}}+||\rho||_{L^2}+||x_1\rho||_{L^{\frac{6p}{6-p}}}\big).\end{aligned}\tag{3.7}$$

Combining (3.6) with (3.7) and (3.4) leads to (3.2), which concludes the proof of the proposition. ∎

In order to bound $||x_h\rho||_{L^1_t(B^0_{\infty,1})}$ in Proposition 4.3 in Section 4, we need also to estimate the commutator about the Littlewood-Paley operator Δ_q.

Proposition 3.3. *Under the assumptions in Proposition 3.2, there holds for every* $q\in\mathbb{N}\cup\{-1\}$

$$\|[\Delta_q,v\cdot\nabla]\rho\|_{L^p}\lesssim\|\omega/r\|_{L^2}\|\rho\|_{L^{\frac{6p}{6-p}}}+||\omega/r||_{L^p}||x_h\rho||_{L^\infty}+||\omega/r||_{L^p}||\rho||_{L^3}\tag{3.8}$$

and

$$\|[\Delta_q,v\cdot\nabla]\rho\|_{L^2}\lesssim\|\omega/r\|_{L^{3,1}}\big(\|\rho\|_{L^2}+\|\rho x_h\|_{L^6}\big).\tag{3.9}$$

Proof. Since the velocity v is divergence-free, we rewrite $[\Delta_q, v\cdot\nabla]\rho$ as the form

$$[\Delta_q, v\cdot\nabla]\rho = \sum_{j=1}^{3}\partial_j[\Delta_q, v^j]\rho = I + II + III.$$

For I, we decompose it into the following four terms:

$$I = \partial_1\big([\Delta_q, \mathcal{L}(\omega/r)]\rho\big) + \partial_1\big([\Delta_q, \Delta^{-1}\partial_3(\omega/r)](x_1\rho)\big) - \big(\mathcal{R}_{13}(\omega/r)\big)2^{2q}\varphi_1(2^q.)\star\rho$$
$$-\{\Delta^{-1}\partial_3(\omega/r)\}2^{3q}(\partial_1\varphi_1)(2^q.)\star\rho = \sum_{\ell=1}^{4} I_\ell,$$

with $\mathcal{L} = -2\mathcal{R}_{13}\Delta^{-1}$ and $\varphi_1(x) = x_1\varphi(x) \in \mathcal{S}(\mathbb{R}^3)$.
Thanks to (2.3), Sobolev embeddings, and Lemma 2.3, we deduce

$$\|I_1\|_{L^p} \lesssim \|\nabla\mathcal{L}(\omega/r)\|_{L^6}\|\rho\|_{L^{\frac{6p}{6-p}}} \lesssim \|\nabla^2\mathcal{L}(\omega/r)\|_{L^2}\|\rho\|_{L^{\frac{6p}{6-p}}} \lesssim \|\omega/r\|_{L^2}\|\rho\|_{L^{\frac{6p}{6-p}}},$$

$$\|I_2\|_{L^p} \lesssim \|\nabla\Delta^{-1}\partial_3(\omega/r)\|_{L^p}\|x_1\rho\|_{L^\infty} \lesssim \|\omega/r\|_{L^p}\|x_1\rho\|_{L^\infty}$$

and

$$\begin{aligned}
&\|I_3\|_{L^p} + \|I_4\|_{L^p}\\
&\lesssim \|\mathcal{R}_{13}(\omega/r)\|_{L^p}2^{2q}\|\varphi_1(2^q.)\star\rho\|_{L^\infty} + \|\Delta^{-1}\partial_3(\omega/r)\|_{L^6}2^{3q}\|(\partial_1\varphi_1)(2^q.)\star\rho\|_{L^{\frac{6p}{6-p}}}\\
&\lesssim \|\omega/r\|_{L^p}\|\varphi_1\|_{L^{\frac{3}{2}}}\|\rho\|_{L^3} + \|\omega/r\|_{L^2}\|\partial_1\varphi_1\|_{L^1}\|\rho\|_{L^{\frac{6p}{6-p}}}\\
&\lesssim \|\omega/r\|_{L^p}\|\rho\|_{L^3} + \|\omega/r\|_{L^2}\|\rho\|_{L^{\frac{6p}{6-p}}}.
\end{aligned}$$

Thus it follows

$$\|I\|_{L^p} \lesssim \|\omega/r\|_{L^2}\|\rho\|_{L^{\frac{6p}{6-p}}} + \|\omega/r\|_{L^p}\|x_1\rho\|_{L^\infty} + \|\omega/r\|_{L^p}\|\rho\|_{L^3}.$$

In the same way, we may get that

$$\|II\|_{L^p} \lesssim \|\omega/r\|_{L^2}\|\rho\|_{L^{\frac{6p}{6-p}}} + \|\omega/r\|_{L^p}\|x_2\rho\|_{L^\infty} + \|\omega/r\|_{L^p}\|\rho\|_{L^3}.$$

Let's turn to estimate III. In fact, we first show

$$\begin{aligned}
-III =& \partial_3\{[\Delta_q, \nabla_h\Delta^{-1}(\omega/r)](x_h\rho)\} + 2\partial_3\{[\Delta_q, \Delta^{-1}\mathcal{R}_{33}(\omega/r)]\}\\
&+ 2^{-q}(\partial_3\nabla_h\Delta^{-1}(\omega/r))(2^{3q}\varphi_h(2^q.)\star\rho) + \nabla_h\Delta^{-1}(\omega/r)(2^{3q}(\partial_3\varphi_h)(2^q.)\star\rho)\\
=& \sum_{\ell=1}^{4} III_\ell,
\end{aligned}$$

with $\varphi_h(x) = x_h\varphi(x)$.

Thanks to (2.3) and Sobolev embeddings, we find that

$$\begin{aligned}
\|III_1\|_{L^p} + \|III_2\|_{L^p} \lesssim& \|\nabla^2\Delta^{-1}(\omega/r)\|_{L^p}\|x_h\rho\|_{L^\infty} + \|\nabla\Delta^{-1}\mathcal{R}_{33}(\omega/r)\|_{L^6}\|\rho\|_{L^{\frac{6p}{6-p}}}\\
\lesssim& \|\omega/r\|_{L^p}\|x_h\rho\|_{L^\infty} + \|\omega/r\|_{L^2}\|\rho\|_{L^{\frac{6p}{6-p}}}.
\end{aligned}$$

The estimates of III_3 and III_4 follow from Bernstein's inequality (2.1) to show

$$\begin{aligned}
&\|III_3\|_{L^p}+\|III_4\|_{L^p}\\
\lesssim&2^{-q}||\omega/r||_{L^p}2^{3q}||\varphi_h(2^q.)\star\rho||_{L^\infty}+||\nabla_h\Delta^{-1}(\omega/r)||_{L^6}2^{3q}||(\partial_3\varphi_h)(2^q.)\star\rho||_{L^{\frac{6p}{6-p}}}\\
\lesssim&||\omega/r||_{L^p}||\varphi_h||_{L^{\frac{3}{2}}}||\rho||_{L^3}+||\omega/r||_{L^2}||\partial_3\phi_h||_{L^1}||\rho||_{L^{\frac{6p}{6-p}}}\\
\lesssim&||\omega/r||_{L^p}||\rho||_{L^3}+||\omega/r||_{L^2}||\rho||_{L^{\frac{6p}{6-p}}}.
\end{aligned}$$

This completes the proof of (3.8).

The second inequality (3.9) is in Proposition 3.2, [24], we omit its proof.

This finishes the proof of the proposition. □

4. Proof of Theorem 1.1

4.1. A priori estimate. The existence and uniqueness of the solution to the system (1.1) was obtained in [35], we just need to give some necessary *a priori* estimates for the proof of Theorem 1.1.

In the rest of this paper, we always denote

$$\phi_k(t)=C_0\underbrace{\exp(\dots\exp}_{k\,\text{times}}(C_0t^3)\dots),$$

where C_0 depends on the involved norms of the initial data and its value may vary from line to line up to some absolute constants and independent of μ.

Let's first recall the following proposition obtained (with a slight modification) in [24].

Proposition 4.1 (Propositions 4.1 and 4.2 of [24]). *Let (u_μ,ρ_μ) be a smooth solution of (1.1), then*

(1) for $p\in(1,+\infty)$, $q\in[1,+\infty]$, and $t\in\mathbb{R}_+$, we have

$$\|\rho_\mu(t)\|_{L^\infty_tL^2}+\|\nabla\rho_\mu\|_{L^2_tL^2}\le 2\|\rho^0\|_{L^2}\quad\text{and}\quad\|\rho_\mu\|_{L^\infty_tL^{p,q}}\le\|\rho^0\|_{L^{p,q}};$$

(2) for $\rho^0\in L^2$, $u^0\in L^2$, and $t\in\mathbb{R}_+$, we have

$$\|u_\mu(t)\|_{L^2}\le C_0(1+t);$$

(3) for $\rho^0\in L^2$ and $t\in\mathbb{R}_+$, we have

$$||\rho_\mu(t)||_{L^\infty}\le C_0(1+t^{-\frac{3}{4}});$$

(4) for $\rho^0\in L^2$, $x_h\rho^0\in L^2$, and $t\in\mathbb{R}_+$, we have

$$||x_h\rho_\mu||_{L^\infty_tL^2}+||\nabla(x_h\rho_\mu)||_{L^2_tL^2}\le C_0(1+t^{\frac{5}{4}});$$

(5) for $\rho^0\in L^m\cap L^2$ with $m>6$, $x_h\rho^0\in L^2$, and $t\in\mathbb{R}_+$, we have

$$||x_h\rho_\mu(t)||_{L^\infty}\le C_0(t^{\frac{1}{4}}+t^{-\frac{3}{4}});$$

(6) for $|x_h|^2\rho^0\in L^2$, $\rho^0\in L^2$, and $t\in\mathbb{R}_+$, we have

$$|||x_h|^2\rho_\mu||_{L^\infty_tL^2}+||\nabla(|x_h|^2\rho_\mu)||_{L^2_tL^2}\le C_0(1+t^{\frac{5}{2}});$$

(7) for $\rho^0\in L^6$, $|x_h|^2\rho^0\in L^2$, $\rho^0\in L^2$, *and* $t\in\mathbb{R}_+$, *we have*

$$|||x_h|^2\rho_\mu(t)||_{L^6}\le C_0(t^{\frac{13}{6}}+t^{-\frac{1}{2}}),$$

where constants C_0 *depend only on the norm of the initial data involved in the estimates, and are independent of* μ.

In order to get the further estimate about $\||x_h|^2\rho_\mu(t)\|_{L^\infty}$, we recall the following Nash-De Giorgi estimate for the convection-diffusion equation.

Lemma 4.1 (Lemma A.1 in [24]). *Consider the equation*

$$\begin{cases} \partial_t f+(v\cdot\nabla)f-\Delta f=\partial_i F+G, \quad \forall t>0, x\in\mathbb{R}^3,\\ f(0,x)=f_0(x). \end{cases} \tag{4.1}$$

Let $(p,q,p_1,q_1)\in[1,+\infty]^4$ *and* $r\in[2,+\infty]$, *such that*

$$\frac{2}{p}+\frac{3}{q}<1,\ \frac{2}{p_1}+\frac{3}{q_1}<2.$$

There exists $C>0$ *such that for every smooth divergence-free vector field* v, *for every* $F\in L^p_TL^q$ *and for every* $f\in L^r$, *the solution of (4.1) satisfies the estimate: for every* $t\in]0,T]$,

$$\begin{aligned}||f(t)||_{L^\infty}\le C(1+t^{-\frac{3}{2r}})||f_0||_{L^r}+C(1+\sqrt{T}^{1-(\frac{2}{p}+\frac{3}{q})})||F||_{L^p_TL^q}\\ +C(1+\sqrt{T}^{2-(\frac{2}{p_1}+\frac{3}{q_1})})||G||_{L^{p_1}_TL^{q_1}}.\end{aligned} \tag{4.2}$$

With this lemma in hand, we may deduce that

Proposition 4.2. *Under the assumptions of Proposition 4.1,* $\rho^0\in L^m\cap L^2$ *with* $m>6$ *and* $|x_h|^2\rho^0\in L^2$, *there exists a constant* $C_0>0$ *such that* $\forall t\in\mathbb{R}_+$,,

(1) for $p\in[2,\infty]$,

$$\|x_h\rho_\mu\|_{L^p}\le C_0(t^{\frac{5}{4}}+t^{-\frac{3}{4}}), \tag{4.3}$$

(2)

$$|||x_h|^2\rho_\mu(t)||_{L^\infty}\le C_0(t^{\frac{-3}{4}}+t^{\frac{3}{4}}). \tag{4.4}$$

Proof. The first inequality (4.3) can be immediately obtained by using the interpolation inequality, Young's inequality and Proposition 4.1.

For the second estimate (4.4), denoting $g_\mu:=|x_h|^2\rho_\mu$ and $f_\mu:=x_h\rho_\mu$, from the ρ_μ equation in (1.1), we know that g_μ satisfies the convection-diffusion equation

$$\partial_t g_\mu+u_\mu\cdot\nabla g_\mu-\Delta g_\mu=2u_{\mu,h}\cdot f_\mu-2(\partial_1\rho_\mu+\partial_2\rho_\mu)-4\partial_1(x_1\rho_\mu)-4\partial_2(x_2\rho_\mu)+8\rho_\mu.$$

Then by Lemma 4.1 and Proposition 4.1, we obtain

$$\begin{aligned}\|g_\mu(t)\|_{L^\infty}\lesssim&(1+t^{\frac{-3}{4}})\|g^0\|_{L^2}+(1+t^{\frac{1}{4}})\|\rho_\mu u_{\mu,h}\|_{L^\infty_t(L^2)}+(1+t^{\frac{1}{2}})\|\rho_\mu\|_{L^\infty_t(L^\infty)}\\ &+(1+t^{\frac{1}{2}})\|x_h\rho_\mu\|_{L^\infty_t(L^\infty)}+(1+t^{\frac{1}{4}})\|\rho_\mu\|_{L^\infty_t(L^2)}\\ \le& C_0(t^{\frac{-3}{4}}+t^{\frac{1}{2}}+t^{-\frac{1}{4}}+t^{\frac{3}{4}}+t^{\frac{1}{4}})\le C_0(t^{\frac{-3}{4}}+t^{\frac{3}{4}}),\end{aligned}$$

which completes the proof of the proposition. ∎

Let's now turn to get the estimate of $||x_h\rho_\mu||_{L^1_t B^0_{\infty,1}}$.

PROPOSITION 4.3. *Under the assumption of Proposition 4.2, there holds* $\forall\, t\in\mathbb{R}_+$

$$||x_h\rho_\mu||_{L^1_t B^0_{\infty,1}} \leq C_0(1+t^{\frac{9}{4}})+C_0\int_0^t(\tau^2+\tau^{-\frac{3}{4}})\log(2+||\frac{\omega_\mu}{r}||_{L^\infty_\tau L^2})d\tau. \tag{4.5}$$

Proof. Thanks to Proposition 4.1 and Bernstein inequalities (2.1), we find that

$$\begin{aligned}||x_h\rho_\mu||_{L^1_t B^0_{\infty,1}} &= \int_0^t \sum_{-1\leq q\leq N(\tau)} ||\Delta_q(x_h\rho_\mu)(\tau)||_{L^\infty}\,d\tau + \int_0^t \sum_{q>N(\tau)} ||\Delta_q(x_h\rho_\mu)(\tau)||_{L^\infty}\,d\tau\\ &\lesssim \int_0^t (\tau^{\frac{1}{4}}+\tau^{\frac{-3}{4}})(2+N(\tau))\,d\tau + \int_0^t \sum_{q>N(\tau)} 2^{\frac{3}{2}}||\Delta_q(x_h\rho_\mu)(\tau)||_{L^2}\,d\tau\end{aligned}$$

for any positive fixed integer $N(\tau)$ which will be determined later on. In order to control the last sum in the above inequality, we localize in frequency the equation of $f_\mu = x_h\rho_\mu$

$$\partial_t f_\mu + u_\mu\cdot\nabla f_\mu - \Delta f_\mu = u_{\mu,h}\rho_\mu - 2\nabla_h\rho_\mu \overset{\text{def}}{=} F_\mu$$

to give

$$\partial_t f_{\mu,q} + u_\mu\cdot\nabla f_{\mu,q} - \Delta f_{\mu,q} = -[\Delta_q, u_\mu\cdot\nabla]f_\mu + F_{\mu,q}, \tag{4.6}$$

where $f_{\mu,q} \overset{\text{def}}{=} \Delta_q f_\mu$, $q\in\mathbb{N}\cup\{-1\}$.

A standard energy estimate of the system (4.6) yields that $\forall\, q\geq 0$

$$\|f_{\mu,q}(t)\|_{L^2} \lesssim e^{-ct2^{2q}}||f_{\mu,q}(0)||_{L^2} + \int_0^t e^{-c(t-\tau)2^{2q}}\big(||[\Delta_q,u_\mu\cdot\nabla]f_\mu||_{L^2}+||F_{\mu,q}||_{L^2}\big)d\tau. \tag{4.7}$$

To estimate the commutator in the right-hand side, we can use Propositions 3.3 and 4.2 to give

$$\begin{aligned}||[\Delta_q,u_\mu\cdot\nabla]f_\mu(\tau)||_{L^2} &\lesssim ||(\frac{\omega_\mu}{r})(\tau)||_{L^2}(|||x_h|^2\rho_\mu(\tau)||_{L^\infty}+||x_h\rho_\mu||_{L^3})\\ &\lesssim ||(\frac{\omega_\mu}{r})(\tau)||_{L^2}(\tau^{\frac{5}{4}}+\tau^{-\frac{3}{4}}),\end{aligned}$$

which shows us, when setting $\kappa(\tau) := \tau^{\frac{5}{4}}+\tau^{-\frac{3}{4}}$, that

$$\begin{aligned}&\int_0^t \sum_{q>N(\tau)} 2^{\frac{3}{2}q}||\Delta_q(x_h\rho_\mu)(\tau)||_{L^2}d\tau\\ &\lesssim ||f^0||_{L^2}+||F_\mu||_{L^1_tL^2}+\int_0^t||\frac{\omega_\mu}{r}||_{L^\infty_\tau L^2}\Big(\sum_{q>N(\tau)}2^{q\frac{3}{2}}\int_0^\tau e^{-c(\tau-\tau')2^{2q}}\kappa(\tau')d\tau'\Big)d\tau.\end{aligned} \tag{4.8}$$

Moreover, thanks to Proposition 4.1, we immediately get

$$||F_\mu||_{L^1_tL^2} \leq \sqrt{t}\|\nabla\rho_\mu\|_{L^2_tL^2}+\|u_\mu\|_{L^\infty_tL^2}\|\rho_\mu\|_{L^1_tL^\infty} \lesssim 1+t^2.$$

Thus, from (4.8), one finds

$$\begin{aligned}||x_h\rho_\mu||_{L^1_tB^0_{\infty,1}} \lesssim &(1+t^2)+\int_0^t(\tau^{\frac14}+\tau^{-\frac34})N(\tau)d\tau\\ &+\int_0^t||\frac{\omega_\mu}{r}||_{L^\infty_\tau L^2}\Big(\sum_{q>N(\tau)}2^{q\frac32}\int_0^\tau e^{-c(\tau-\tau')2^{2q}}(\tau'^{\frac54}+\tau'^{-\frac34})d\tau'\Big)d\tau\\ \lesssim &(1+t^2)+\int_0^t(\tau^{\frac14}+\tau^{-\frac34})N(\tau)d\tau\\ &+\int_0^t||\frac{\omega_\mu}{r}||_{L^\infty_\tau L^2}\Big(\tau^{\frac54}2^{-\frac12 N(\tau)}+\sum_{q>N(\tau)}2^{q\frac32}\int_0^\tau e^{-c(\tau-\tau')2^{2q}}\tau'^{-\frac34}d\tau'\Big)d\tau.\end{aligned} \tag{4.9}$$

For the term $\sum_{q>N(\tau)}2^{q\frac32}\int_0^\tau e^{-c(\tau-\tau')2^{2q}}\tau'^{-\frac34}d\tau'$, a change of variables shows us that

$$\begin{aligned}&\sum_{q>N(\tau)}2^{q\frac32}\int_0^\tau e^{-c(\tau-\tau')2^{2q}}\tau'^{-\frac34}d\tau'=\sum_{q>N(\tau)}2^qe^{-c\tau2^{2q}}\int_0^{2^{2q}\tau}e^{c\tau'}\tau'^{-\frac34}d\tau'\\ &=\sum_{q\in B_1(\tau)}2^qe^{-c\tau2^{2q}}\int_0^{2^{2q}\tau}e^{c\tau'}\tau'^{-\frac34}d\tau'+\sum_{q\in B_2(\tau)}2^qe^{-c\tau2^{2q}}\int_0^{2^{2q}\tau}e^{c\tau'}\tau'^{-\frac34}d\tau'\\ &=:I(\tau)+II(\tau)\end{aligned} \tag{4.10}$$

with

$$B_2(\tau)=\{q|q>N(\tau)\text{ and }\tau2^{2q}\geq1\}\quad\text{and}\quad B_2(\tau)=\{q|q>N(\tau)\text{ and }\tau2^{2q}\leq1\}.$$

Using integration by parts, one can see

$$I(\tau)\lesssim\tau^{-\frac34}\sum_{q>N(\tau)}2^{-\frac12}\lesssim\tau^{-\frac34}2^{-\frac12N(\tau)}. \tag{4.11}$$

While for the second term $II(\tau)$, one finds

$$II(\tau)\lesssim\sum_{q\in B_2(\tau)}2^q\lesssim2^{-\frac12N(\tau)}\sum_{2^{-2q}\leq\tau^{-1}}2^{\frac32q}\lesssim2^{-\frac12N(\tau)}(1+\tau^{-\frac34}). \tag{4.12}$$

Hence, plugging (4.11) and (4.12) into (4.10) yields

$$\sum_{q>N(\tau)}2^{q\frac32}\int_0^\tau e^{-c(\tau-\tau')2^{2q}}\tau'^{-\frac34}d\tau'\lesssim2^{-\frac12N(\tau)}(1+\tau^{-\frac34}). \tag{4.13}$$

Therefore, combining (4.13) with (4.9), we obtain

$$||x_h\rho_\mu||_{L^1_tB^0_{\infty,1}}\lesssim(1+t^2)+\int_0^t\big(\tau^{\frac54}+\tau^{-\frac34}\big)\big(N(\tau)+2+||\frac{\omega_\mu}{r}||_{L^\infty_\tau L^2}2^{-\frac12N(\tau)}\big)d\tau. \tag{4.14}$$

Choose $N(\tau)$ in (4.14) such that

$$N(\tau)=2[\log_2(2+||\frac{\omega_\mu}{r}||_{L^\infty_\tau L^2})],$$

we have

$$\|x_h\rho_\mu\|_{L^1_tB^0_{\infty,1}}\leq C_0(1+t^{\frac{9}{4}})+C_0\int_0^t(\tau^2+\tau^{-\frac{3}{4}})\log(2+\|\frac{\omega_\mu}{r}\|_{L^\infty_\tau L^2})\,d\tau.$$

This completes the proof of the proposition. □

Based on Propositions 4.3 and 3.2, we may get the control of $\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^2}$.

Proposition 4.4. *Under the assumptions in Proposition 4.1, let $u^0\in L^2$ be an axisymmetric vector field such that $\frac{\omega^0}{r}\in L^2$ and $\rho^0\in L^2\cap L^m$ for $m>6$, axisymmetric and such that $|x_h|^2\rho^0\in L^2$. Then, we have for every $t\in\mathbb{R}_+$*

$$\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^2}\leq \phi_2(t). \tag{4.15}$$

Proof. Recall that the equation of the scalar component of the vorticity $\nabla\times u_\mu=\omega_\mu e_\theta$ is given by

$$\partial_t\omega_\mu+u_\mu\cdot\nabla\omega_\mu-\mu\partial_z^2\omega_\mu=\frac{u^r_\mu}{r}\omega_\mu-\partial_r\rho_\mu. \tag{4.16}$$

It follows that the evolution of the quantity $\frac{\omega_\mu}{r}$ is governed by the equation

$$(\partial_t+u_\mu\cdot\nabla-\mu\partial_z^2)\frac{\omega_\mu}{r}=-\frac{\partial_r\rho_\mu}{r}. \tag{4.17}$$

On the other hand, applying the operator $\frac{\partial_r}{r}\Delta^{-1}$ to the equation of the density yields

$$(\partial_t+u_\mu\cdot\nabla)(\frac{1}{r}\partial_r\Delta^{-1}\rho_\mu)-\frac{\partial_r\rho}{r}=-[\frac{1}{r}\partial_r\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu. \tag{4.18}$$

Setting $\Gamma_\mu:=\frac{\omega_\mu}{r}+\frac{1}{r}\partial_r\Delta^{-1}\rho_\mu$, we infer from (4.17) and (4.18) that the new unknown Γ_μ satisfies

$$(\partial_t+u_\mu\cdot\nabla-\mu\partial_z^2)\Gamma_\mu=-[\frac{1}{r}\partial_r\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu-\mu\partial_z^2\frac{1}{r}\partial_r\Delta^{-1}\rho_\mu. \tag{4.19}$$

The basic L^2 energy estimate of (4.19) yields

$$\begin{aligned}&\|\Gamma_\mu\|_{L^\infty_tL^2}+\mu^{\frac12}\|\partial_z\Gamma_\mu\|_{L^2_tL^2}\\ &\lesssim\|\Gamma^0\|_{L^2}+\|[\frac{1}{r}\partial_r\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu\|_{L^1_tL^2}+\mu^{\frac12}\|\partial_z\frac{1}{r}\partial_r\Delta^{-1}\rho_\mu\|_{L^2_tL^2}\\ &\lesssim\|\Gamma^0\|_{L^2}+\|[\frac{1}{r}\partial_r\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu\|_{L^1_tL^2}+\mu^{\frac12}\|\partial_z\rho_\mu\|_{L^2_tL^2}.\end{aligned} \tag{4.20}$$

From Proposition 3.2, we estimate the commutator in (4.20) to get

$$\|[(\frac{\partial_r}{r}\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu\|_{L^2}\lesssim\|\omega_\mu/r\|_{L^2}\big(\|\rho_\mu\|_{L^2}+\|\nabla\rho_\mu\|_{L^2}+\|\rho_\mu x_h\|_{L^2\cap B^0_{\infty,1}}\big),$$

which along with (4.20) implies

$$\begin{aligned}&\|\Gamma_\mu\|_{L^\infty_tL^2}+\mu^{\frac12}\|\partial_z\Gamma_\mu\|_{L^2_tL^2}\\ &\lesssim\|\Gamma^0\|_{L^2}+\|\omega_\mu/r\|_{L^2}\big(\|\rho_\mu\|_{L^2}+\|\nabla\rho_\mu\|_{L^2}+\|\rho_\mu x_h\|_{L^2\cap B^0_{\infty,1}}\big)+\mu^{\frac12}\|\partial_z\rho_\mu\|_{L^2_tL^2}.\end{aligned} \tag{4.21}$$

On the other hand, applying Proposition 4.1 shows us that

$$\|\rho_\mu\|_{L^\infty_t L^2}+\|\nabla\rho_\mu\|_{L^2_t L^2}\lesssim\|\rho^0\|_{L^2},\quad \|\rho_\mu x_h\|_{L^1_t L^2}\lesssim t+t^{\frac{9}{4}}, \tag{4.22}$$

which follows

$$\begin{aligned}&\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^2}+\mu\|\frac{\partial_z\omega_\mu}{r}\|_{L^2_t L^2}\\ \le&\|\Gamma_\mu\|_{L^\infty_t L^2}+\mu\|\partial_z\Gamma_\mu\|_{L^2_t L^2}+\|\frac{1}{r}\partial_r\Delta^{-1}\rho_\mu\|_{L^\infty_t L^2}+\mu\|\frac{1}{r}\partial_r\Delta^{-1}\partial_z\rho_\mu\|_{L^2_t L^2}\\ \lesssim&\|\Gamma_\mu\|_{L^\infty_t L^2}+\mu\|\partial_z\Gamma_\mu\|_{L^2_t L^2}+\|\rho_\mu\|_{L^\infty_t L^2}+\mu\|\partial_z\rho_\mu\|_{L^2_t L^2},\end{aligned}$$

and then

$$\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^2}+\mu\|\frac{\partial_z\omega_\mu}{r}\|_{L^2_t L^2}\lesssim\|\Gamma_\mu\|_{L^\infty_t L^2}+\mu\|\partial_z\Gamma_\mu\|_{L^2_t L^2}+\|\rho^0\|_{L^2}. \tag{4.23}$$

Combining (4.21) with (4.22) and (4.23) implies

$$\begin{aligned}&\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^2}+\mu\|\frac{\partial_z\omega_\mu}{r}\|_{L^2_t L^2}\\ \lesssim&\|\Gamma^0\|_{L^2}+\|\rho^0\|_{L^2}+\|\omega_\mu/r\|_{L^2}\big(\|\rho_\mu\|_{L^2}+\|\nabla\rho_\mu\|_{L^2}+\|\rho_\mu x_h\|_{L^2\cap B^0_{\infty,1}}\big),\end{aligned}$$

which, from the Grönwall inequality, gives that

$$\begin{aligned}\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^2}+\mu\|\frac{\partial_z\omega_\mu}{r}\|_{L^2_t L^2}\le& C\big(\|\Gamma^0\|_{L^2}+\|\rho^0\|_{L^2}\big)\\ &\times e^{C(\|\rho_\mu\|_{L^1_t L^2}+\|\nabla\rho_\mu\|_{L^1_t L^2}+\|\rho_\mu x_h\|_{L^1_t(L^2\cap B^0_{\infty,1})})}.\end{aligned}$$

Therefore, an application of Proposition 4.1 yields

$$\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^2}+\mu\|\frac{\partial_z\omega_\mu}{r}\|_{L^2_t L^2}\le C\big(\|\Gamma^0\|_{L^2}+\|\rho^0\|_{L^2}\big)e^{C_0(\sqrt{t}+t^{\frac{9}{4}}+\|\rho_\mu x_h\|_{L^1_t B^0_{\infty,1}})},$$

which along with Proposition 4.3 gives rise to

$$\begin{aligned}\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^2}+\mu\|\frac{\partial_z\omega_\mu}{r}\|_{L^2_t L^2}\le& C\big(\|\Gamma^0\|_{L^2}+\|\rho^0\|_{L^2}\big)e^{C_0(1+t^{\frac{9}{4}})}\\ &\times\exp\Big\{C_0\int_0^t(\tau^2+\tau^{-\frac{3}{4}})\log(2+\|\frac{\omega_\mu}{r}\|_{L^\infty_\tau L^2})d\tau\Big\},\end{aligned}$$

and then

$$\log(2+\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^2})\le\log C_0+C_0(1+t^{\frac{9}{4}})+C_0\int_0^t(\tau^2+\tau^{-\frac{3}{4}})\log(2+\|\frac{\omega_\mu}{r}\|_{L^\infty_\tau L^2})d\tau. \tag{4.24}$$

Therefore, applying the Grönwall inequality to (4.24) gives the desired estimate. □

Proposition 4.5. *Under the assumptions in Proposition 4.1, let $\frac{5}{2}<s_0<3$, $3<p<+\infty$ satisfy $0<\frac{3}{2}-\frac{3}{p}\le s_0$, $u^0\in L^2$ be an axisymmetric vector field without swirl such that $\frac{\omega^0}{r}\in L^2\cap L^p$ and $\rho^0\in B^{s_0}_{2,1}\cap L^m$, for $m>6$, axisymmetric and such that $|x_h|^2\rho^0\in L^2$. Then, we have for every $t\in\mathbb{R}_+$*

$$\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^p}+\|\frac{u^r_\mu}{r}\|_{L^\infty_t L^\infty}+\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^{3,1}}+\|\rho_\mu\|_{\widetilde{L}^1_t B^2_{p,\infty}}\le\phi_2(t) \tag{4.25}$$

and

$$\|\omega_\mu(t)\|_{L^\infty}\le\phi_3(t). \tag{4.26}$$

Proof. Let's first give some estimates of Γ_μ in the L^p framework. Multiplying the Γ_μ Equation (4.19) by $|\Gamma_\mu|^{p-2}\Gamma_\mu$ and then integrating it over $\mathbb{R}^3$ yields that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}||\Gamma_\mu||_{L^p}^p+\mu\|\partial_z|\Gamma_\mu|^{\frac{p}{2}}\|_{L^2}^2&\lesssim||[\frac{1}{r}\partial_r\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu||_{L^p}||\Gamma_\mu||_{L^p}^{p-1}\\&\quad-\mu\int_{\mathbb{R}^3}\partial_z^2(\frac{1}{r}\partial_r\Delta^{-1}\rho_\mu)|\Gamma_\mu|^{p-2}\Gamma_\mu dx\\&\lesssim||[\frac{1}{r}\partial_r\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu||_{L^p}||\Gamma_\mu||_{L^p}^{p-1}+\mu\int_{\mathbb{R}^3}|\partial_z\frac{1}{r}\partial_r\Delta^{-1}\rho_\mu||\partial_z|\Gamma_\mu|^{\frac{p}{2}}||\Gamma_\mu|^{\frac{p}{2}-1}dx,\end{aligned}$$

Thanks to the Hölder and Young inequalities, we infer that

$$\frac{1}{p}\frac{d}{dt}||\Gamma_\mu||_{L^p}^p+\mu\|\nabla|\Gamma_\mu|^{\frac{p}{2}}\|_{L^2}^2\lesssim||[\frac{1}{r}\partial_r\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu||_{L^p}||\Gamma_\mu||_{L^p}^{p-1}+\mu\|\nabla\rho_\mu\|_{L^p}^2||\Gamma_\mu||_{L^p}^{p-2},$$

and then deduce that

$$\|\Gamma_\mu\|_{L_t^\infty L^p}\lesssim\|\Gamma^0\|_{L^p}+\|[\frac{1}{r}\partial_r\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu\|_{L_t^1L^p}+\mu^{\frac{1}{2}}\|\nabla\rho_\mu\|_{L_t^2L^p}. \tag{4.27}$$

In view of the definition of Γ_μ,

$$\|\frac{\omega_\mu}{r}||_{L_t^\infty L^p}\le\|\Gamma_\mu||_{L_t^\infty L^p}+\|\frac{1}{r}\partial_r\Delta^{-1}\rho_\mu\|_{L_t^\infty L^p}\lesssim\|\Gamma_\mu||_{L_t^\infty L^p}+\|\rho_\mu\|_{L_t^\infty L^p}.$$

which along with (4.27) implies

$$\|\frac{\omega_\mu}{r}\|_{L_t^\infty L^p}\le C_0\big(1+||[\frac{1}{r}\partial_r\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu||_{L_t^1L^p}+\mu^{\frac{1}{2}}\|\nabla\rho_\mu\|_{L_t^2L^p}\big). \tag{4.28}$$

Let's now handle the estimate of $||[\frac{1}{r}\partial_r\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu||_{L_t^1L^p}$. From Proposition 3.2, we have

$$\begin{aligned}\|[(\frac{\partial_r}{r}\Delta^{-1},u_\mu\cdot\nabla]\rho_\mu\|_{L^p}\lesssim&\|\omega_\mu/r\|_{L^p}\big(\|\rho_\mu\|_{L^{3,1}}+\|\rho_\mu\|_{L^6}+||\rho_\mu x_h||_{L^2\cap B^0_{\infty,1}}\big)\\&+||\omega_\mu/r||_{L^2}\big(||\rho_\mu||_{B^0_{\frac{6p}{6-p},2}}+||\rho_\mu||_{L^2}+||\rho_\mu x_h||_{L^{\frac{6p}{6-p}}}\big).\end{aligned} \tag{4.29}$$

In the second term on the right-hand side in (4.29), applying Propositions 4.1, 4.2, and 4.4 gives rise to

$$||\omega_\mu/r||_{L^2}\big(||\rho_\mu||_{B^0_{\frac{6p}{6-p},2}}+||\rho_\mu||_{L^2}+||\rho_\mu x_h||_{L^{\frac{6p}{6-p}}}\big)\lesssim\phi_2(t)\big(||\rho_\mu||_{B^0_{\frac{6p}{6-p},2}}+1+t^{\frac{5}{4}}+t^{-\frac{3}{4}}\big),$$

which follows

$$\int_0^t||\frac{\omega_\mu}{r}||_{L^2}\big(||\rho_\mu||_{B^0_{\frac{6p}{6-p},2}}+||\rho_\mu||_{L^2}+||\rho_\mu x_h||_{L^{\frac{6p}{6-p}}}\big)d\tau\lesssim\int_0^t\phi_2(\tau)\|\rho_\mu\|_{B^0_{\frac{6p}{6-p},2}}d\tau+\phi_2(t). \tag{4.30}$$

The control of the Besov norm $||\rho_\mu||_{B^0_{\frac{6p}{6-p},2}}$ remains. In fact, similar to the proof of (4.7), a standard energy estimate in local frequency of the ρ_μ equation shows us that $\forall\, q\geq 0$

$$||\Delta_q\rho_\mu(t)||_{L^2}\lesssim e^{-ct2^{2q}}||\Delta_q\rho^0||_{L^2}+\int_0^t e^{-c(t-\tau)2^{2q}}||[\Delta_q,u_\mu\cdot\nabla]\rho_\mu||_{L^2}d\tau, \tag{4.31}$$

which follows, from Propositions 3.3, 4.1 and 4.4, that

$$\begin{aligned}\|\rho_\mu\|_{\widetilde{L}^1_tB^2_{2,\infty}}&\lesssim\|\rho^0\|_{L^2}(1+t)+\sup_q||[\Delta_q,u_\mu\cdot\nabla]\rho_\mu||_{L^1_tL^2}\\&\lesssim\|\rho^0\|_{L^2}(1+t)+\int_0^t\|\omega_\mu/r\|_{L^2}(\|\rho_\mu||_{L^3}+||x_h\rho_\mu\|_{L^\infty})d\tau\\&\lesssim\|\rho^0\|_{L^2}(1+t)+\int_0^t\phi_2(\tau)(1+\tau^{\frac{5}{4}}+\tau^{-\frac{3}{4}})d\tau\lesssim\phi_2(t).\end{aligned} \tag{4.32}$$

As a consequence, we get from (4.30) that

$$\begin{aligned}&\int_0^t||\omega_\mu/r||_{L^2}\big(||\rho_\mu||_{B^0_{\frac{6p}{6-p},2}}+||\rho_\mu||_{L^2}+||\rho_\mu x_h||_{L^{\frac{6p}{6-p}}}\big)d\tau\\&\lesssim\phi_2(t)(1+\|\rho_\mu\|_{L^1_tB^0_{\frac{6p}{6-p},2}})\lesssim\phi_2(t)(1+\|\rho_\mu\|_{\widetilde{L}^1_tB^2_{2,\infty}})\lesssim\phi_2(t).\end{aligned} \tag{4.33}$$

An application of (4.31) shows us that

$$\begin{aligned}\|\rho_\mu\|_{\widetilde{L}^\infty_tB^{s_0-2}_{2,1}}&\lesssim\|\rho^0\|_{B^{s_0-2}_{2,1}}+\sum_q 2^{(s_0-3)q}||[\Delta_q,u_\mu\cdot\nabla]\rho_\mu||_{L^2_tL^2}\\&\lesssim\|\rho^0\|_{B^{s_0-2}_{2,1}}+\Big(\int_0^t(\sup_q||[\Delta_q,u_\mu\cdot\nabla]\rho_\mu||_{L^2})^2d\tau\Big)^{\frac{1}{2}}.\end{aligned}$$

Hence, thanks to (3.9), (4.15), Proposition 4.1, and the interpolation inequality, we find

$$\begin{aligned}\|\rho_\mu\|_{\widetilde{L}^\infty_tB^{s_0-2}_{2,1}}&\lesssim\|\rho^0\|_{B^{s_0-2}_{2,1}}+\|\omega/r\|_{L^\infty_tL^{3,1}}\big(\|\rho\|_{L^2_tL^2}+\|\rho x_h\|_{L^2_tL^6}\big)\\&\lesssim 1+(1+t^{\frac{5}{4}})\|\frac{\omega_\mu}{r}\|^{\frac{2(p-3)}{3(p-2)}}_{L^\infty_tL^2}\|\frac{\omega_\mu}{r}\|^{\frac{p}{3(p-2)}}_{L^\infty_tL^p}\lesssim 1+\phi_2(t)\|\frac{\omega_\mu}{r}\|^{\frac{p}{3(p-2)}}_{L^\infty_tL^p}.\end{aligned} \tag{4.34}$$

On the other hand, localizing the ρ_μ equation in frequency in the L^p framework yields for $q\in\mathbb{N}\cup\{-1\}$

$$\frac{1}{p}\frac{d}{dt}\|\Delta_q\rho_\mu\|^p_{L^p}-\int_{\mathbb{R}^3}\Delta_q\Delta\rho_\mu|\Delta_q\rho_\mu|^{p-2}\Delta_q\rho_\mu\,dx=-\int_{\mathbb{R}^3}[\Delta_q,u_\mu\cdot\nabla]\rho_\mu|\Delta_q\rho_\mu|^{p-2}\Delta_q\rho_\mu\,dx. \tag{4.35}$$

By using the inequality in [12]

$$-\int_{\mathbb{R}^3}\Delta_q\Delta f|\Delta_q f|^{p-2}\Delta_q f\,dx\geq\begin{cases}c2^{2q}\|\Delta_q f\|^p_{L^p}, & \text{if}\quad q\geq 0,\\ 0, \quad \text{if}\quad q=-1,\end{cases}$$

we get from (4.35) that

$$\|\Delta_q\rho_\mu\|_{L^\infty_tL^p}+c2^{2q}\|\Delta_q\rho_\mu\|_{L^1_tL^p}\leq\|\Delta_q\rho^0\|_{L^p}+C\|[\Delta_q,u_\mu\cdot\nabla]\rho_\mu\|_{L^1_tL^p}\quad(\forall\, q\geq 0),$$

$$\|\Delta_{-1}\rho_\mu\|_{L^\infty_t L^p} \leq \|\Delta_{-1}\rho^0\|_{L^p} + C\|[\Delta_{-1}, u_\mu\cdot\nabla]\rho_\mu\|_{L^1_t L^p}. \tag{4.36}$$

Therefore, thanks to (4.33) and Propositions 3.3, 4.1 and 4.4, one can obtain from (4.36) that

$$\|\rho_\mu\|_{\widetilde{L}^1_t B^2_{p,\infty}} \lesssim \phi_2(t) + \int_0^t ||\omega/r||_{L^p}\big(||x_h\rho||_{L^\infty} + ||\rho||_{L^3}\big). \tag{4.37}$$

By using the interpolation inequality, (4.32) and (4.34), we deduce that

$$\begin{aligned}\|\nabla\rho_\mu\|_{L^2_t L^p} &\lesssim \|\rho_\mu\|^{\frac{1}{2}}_{\widetilde{L}^\infty_t B^0_{p,1}}\|\rho_\mu\|^{\frac{1}{2}}_{\widetilde{L}^1_t B^2_{p,\infty}} \lesssim \|\rho_\mu\|^{\frac{1}{2}}_{\widetilde{L}^\infty_t B^{s_0-2}_{2,1}}\|\rho_\mu\|^{\frac{1}{2}}_{\widetilde{L}^1_t B^2_{p,\infty}}\\ &\lesssim \phi_2(t) + \phi_2(t)\|\frac{\omega_\mu}{r}\|^{\frac{p}{6(p-2)}}_{L^\infty_t L^p}.\end{aligned} \tag{4.38}$$

Inserting (4.29), (4.33) and (4.38) into (4.28) yields

$$\begin{aligned}\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^p} &\lesssim \phi_2(t) + \phi_2(t)\|\frac{\omega_\mu}{r}\|^{\frac{p}{6(p-2)}}_{L^\infty_t L^p}\\ &+ \int_0^t \|\frac{\omega_\mu}{r}\|_{L^p}\big(\|\rho_\mu\|_{L^{3,1}} + \|\rho_\mu\|_{L^6} + ||\rho_\mu x_h||_{L^2\cap B^0_{\infty,1}} + ||x_h\rho||_{L^\infty} + ||\rho||_{L^3}\big)d\tau,\end{aligned}$$

and then it follows from Young's inequality that

$$\begin{aligned}&\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^p}\\ \lesssim&\phi_2(t) + \int_0^t \|\frac{\omega_\mu}{r}\|_{L^p}\big(\|\rho_\mu\|_{L^{3,1}} + \|\rho_\mu\|_{L^6} + ||\rho_\mu x_h||_{L^2\cap B^0_{\infty,1}} + ||x_h\rho||_{L^\infty} + ||\rho||_{L^3}\big)d\tau.\end{aligned} \tag{4.39}$$

Thanks to Propositions 4.1, 4.3 and 4.4, it shows

$$\|\rho_\mu\|_{L^{3,1}} + \|\rho_\mu\|_{L^6} + ||\rho_\mu x_h||_{L^2} + ||x_h\rho||_{L^\infty} + ||\rho||_{L^3} \leq C_0(t^{\frac{5}{4}} + t^{-\frac{3}{4}}),$$

and then follows from (4.39) that

$$\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^p} \lesssim \phi_2(t) + \int_0^t \|\frac{\omega_\mu}{r}(\tau)\|_{L^p}\big(C_0(\tau^{\frac{5}{4}} + \tau^{-\frac{3}{4}}) + ||x_h\rho_\mu(\tau)||_{B^0_{\infty,1}}\big)d\tau. \tag{4.40}$$

Applying Grönwall's inequality to (4.40) leads to

$$\|\frac{\omega_\mu}{r}\|_{L^\infty_t L^p} \lesssim \phi_2(t)\exp\{C_0(1 + \tau^{\frac{9}{4}} + \|x_h\rho_\mu\|_{L^1_t B^0_{\infty,1}})\}.$$

which follows from (4.15) and (4.5) that

$$||\frac{\omega_\mu}{r}||_{L^\infty_t L^p} \leq \phi_2(t) \tag{4.41}$$

and then

$$||\nabla\rho_\mu||_{L^\infty_t L^p} \leq \phi_2(t), \quad \|\rho_\mu\|_{\widetilde{L}^1_t B^2_{p,\infty}} \lesssim \phi_2(t), \tag{4.42}$$

where we have used the inequality (4.37).

Thanks to the fact $\|\frac{u_\mu^r}{r}\|_{L^\infty}\lesssim\|\frac{\omega_\mu}{r}\|_{L^{3,1}}$ in [2], we have

$$\|\frac{u_\mu^r}{r}\|_{L^\infty}\lesssim\|\frac{\omega_\mu}{r}\|_{L^{3,1}}\lesssim\|\frac{\omega_\mu}{r}\|_{L^2}^{\frac{2(p-3)}{3(p-2)}}\|\frac{\omega_\mu}{r}\|_{L^p}^{\frac{p}{3(p-2)}},$$

then a consequence of (4.15) and (4.41) gives

$$\|\frac{u_\mu^r}{r}\|_{L^\infty}\le\phi_2(t). \tag{4.43}$$

From the maximum principle for the Equation (4.16), we obtain that

$$\|\omega_\mu(t)\|_{L^\infty}\le\|\omega^0\|_{L^\infty}+\int_0^t\|\frac{u_\mu^r}{r}(\tau)\|_{L^\infty}\|\omega_\mu(\tau)\|_{L^\infty}d\tau+\|\nabla\rho_\mu\|_{L_t^1L^\infty},$$

which follows from the Grönwall inequality that

$$\|\omega_\mu(t)\|_{L^\infty}\le\big(\|\omega^0\|_{L^\infty}+\|\nabla\rho_\mu\|_{L_t^1L^\infty}\big)e^{\|\frac{u_\mu^r}{r}\|_{L_t^1L^\infty}}\lesssim\big(\|\omega^0\|_{L^\infty}+\|\rho_\mu\|_{L_t^1B_{p,1}^{1+\frac{3}{p}}}\big)e^{\|\frac{u_\mu^r}{r}\|_{L_t^1L^\infty}}.$$

By combining Lemma 2.2 and inequalities (4.42) and (4.43), we deduce

$$\|\omega_\mu(t)\|_{L^\infty}\le C_0\big(1+\|\rho_\mu\|_{\widetilde{L}_t^1B_{p,\infty}^2}\big)\exp\{\|\frac{u_\mu^r}{r}\|_{L_t^1L^\infty}\}\lesssim\phi_3(t).$$

This ends the proof of the proposition. ∎

4.2. Proof of Theorem 1.1. Let's now achieve the proof of Theorem 1.1 by giving the persistence of the initial regularity uniformly on the viscosity.

Proof. **(Proof of Theorem 1.1.)** By a standard energy estimate for the system (1.1), we have

$$\|u_\mu\|_{\widetilde{L}_t^\infty H^s}\lesssim\|u^0\|_{H^s}+\|\rho\|_{\widetilde{L}_t^1H^s}+\int_0^t\big(\sum_q2^{2qs}\|[\Delta_q,u_\mu\cdot\nabla]u_\mu\|_{L^2}^2d\tau\big)^{\frac{1}{2}} \tag{4.44}$$

and

$$\begin{aligned}&\|\rho_\mu\|_{\widetilde{L}_t^\infty H^{s-2}}+\|\rho_\mu\|_{\widetilde{L}_t^1H^s}\\ &\lesssim\|\rho^0\|_{H^{s-2}}+\|\Delta_{-1}\rho_\mu\|_{L_t^1L^2}+\int_0^t\big(\sum_q2^{2q(s-2)}\|[\Delta_q,u_\mu\cdot\nabla]\rho_\mu\|_{L^2}^2d\tau\big)^{\frac{1}{2}}.\end{aligned} \tag{4.45}$$

We recall the proof of Lemma 2.100 in [5] to estimate commutators in (4.45)

$$\big(\sum_q2^{2qs}\|[\Delta_q,u_\mu\cdot\nabla]u_\mu\|_{L^2}^2d\tau\big)^{\frac{1}{2}}\lesssim\|\nabla u_\mu\|_{L^\infty}\|u_\mu\|_{H^s}$$

and

$$\sum_q2^{2q(s-2)}\|[\Delta_q,u_\mu\cdot\nabla]\rho_\mu\|_{L^2}^2\lesssim\|\nabla u_\mu\|_{L^\infty}\|\rho_\mu\|_{H^{s-2}}+\|\nabla\rho_\mu\|_{L^\infty}\|u_\mu\|_{H^{s-2}}.$$

Hence, thanks to $\|\Delta_{-1}\rho_\mu\|_{L_t^1L^2}\lesssim t\|\rho^0\|_{L^2}$, one can show from (4.44) and (4.45) that

$$\begin{aligned}&\|\rho_\mu\|_{\widetilde{L}_t^\infty H^{s-2}}+\|\rho_\mu\|_{\widetilde{L}_t^1H^s}\\ &\lesssim\|\rho^0\|_{H^{s-2}}+t\|\rho^0\|_{L^2}+\int_0^t\big(\|\nabla u_\mu\|_{L^\infty}\|\rho_\mu\|_{H^{s-2}}+\|\nabla\rho_\mu\|_{L^\infty}(\|u_\mu\|_{L^2}+\|u_\mu\|_{H^s})\big)d\tau,\end{aligned}$$

and then

$$
\begin{aligned}
&\|\rho_\mu\|_{\widetilde{L}^\infty_t H^{s-2}}+\|\rho_\mu\|_{\widetilde{L}^1_t H^s}+\|u_\mu\|_{\widetilde{L}^\infty_t H^s}\\
\lesssim&\|\rho^0\|_{H^{s-2}}+t\|\rho^0\|_{L^2}+\|u^0\|_{H^s}+\int_0^t\|\nabla\rho_\mu\|_{L^\infty}\|u_\mu\|_{L^2}d\tau\\
&+\int_0^t\|\nabla u_\mu\|_{L^\infty}\|\rho_\mu\|_{H^{s-2}}+\int_0^t\big(\|\nabla\rho_\mu\|_{L^\infty}+\|\nabla u_\mu\|_{L^\infty}\big)\|u_\mu\|_{H^s}d\tau.
\end{aligned}
$$

Then, an application of the Grönwall inequality gives rise to

$$
\begin{aligned}
&\|\rho_\mu\|_{\widetilde{L}^\infty_t H^{s-2}}+\|\rho_\mu\|_{\widetilde{L}^1_t H^s}+\|u_\mu\|_{\widetilde{L}^\infty_t H^s}\\
\lesssim&\big(\|\rho^0\|_{H^{s-2}}+t\|\rho^0\|_{L^2}+\|u^0\|_{H^s}+\int_0^t\|\nabla\rho_\mu\|_{L^\infty}\|u_\mu\|_{L^2}d\tau\big)e^{C(\|\nabla u_\mu\|_{L^1_tL^\infty}+\|\nabla\rho_\mu\|_{L^1_tL^\infty})}.
\end{aligned}
\tag{4.46}
$$

Note that, from Proposition 4.1 and inequality (4.42), we have

$$\|\nabla\rho_\mu\|_{L^1_tL^\infty}\lesssim\|\rho_\mu\|_{\widetilde{L}^1_t B^2_{p,\infty}}\lesssim\phi_2(t),$$

and then

$$\int_0^t\|\nabla\rho_\mu\|_{L^\infty}\|u_\mu\|_{L^2}d\tau\le\phi_2(t),$$

which along with (4.46) implies

$$\|\rho_\mu\|_{\widetilde{L}^\infty_t H^{s-2}}+\|\rho_\mu\|_{\widetilde{L}^1_t H^s}+\|u_\mu\|_{\widetilde{L}^\infty_t H^s}\le\phi_3(t)e^{C\|\nabla u_\mu\|_{L^1_tL^\infty}}.\tag{4.47}$$

In view of the classical logarithmic Sobolev embedding inequality

$$\|\nabla u_\mu\|_{L^\infty}\lesssim\|u_\mu\|_{L^2}+\|\omega_\mu\|_{L^\infty}\log(e+\|u_\mu\|_{H^s})\quad(\forall\, s>\frac{5}{2}),$$

we deduce from the inequality (4.47) and Proposition 4.5 that $\forall t\in\mathbb{R}_+$

$$\|\nabla u_\mu(t)\|_{L^\infty}\le\phi_3(t)\big(1+\int_0^t\|\nabla u_\mu\|_{L^\infty}d\tau\big).$$

It follows from Grönwall inequality that $\forall t\in\mathbb{R}_+$

$$\|\nabla u_\mu(t)\|_{L^\infty}\le\phi_4(t).$$

Plugging this estimate into (4.47) gives

$$\|\rho_\mu\|_{\widetilde{L}^\infty_t H^{s-2}}+\|\rho_\mu\|_{\widetilde{L}^1_t H^s}+\|u_\mu\|_{\widetilde{L}^\infty_t H^s}\le\phi_5(t).$$

This ends the proof of the theorem. □

5. The rate convergence

With Theorem 1.1 in hand, we are now in a position to get the convergence rate of the solution of the Navier-Stokes-Boussinesq Equations (1.1) to the one of the Euler-Boussinesq Equations (1.4).

Proof. **(Proof of Theorem 1.2.)** Let (ρ_μ, u_μ, p_μ) and (ρ, u, p) be solutions of (1.1) and (1.4) respectively, and denote

$$\varrho_\mu := \rho_\mu - \rho, \quad z_\mu := u_\mu - u \quad \text{and} \quad P_\mu := p_\mu - p,$$

we can easily check that $(\varrho_\mu, z_\mu, P_\mu)$ satisfies the system

$$\begin{cases} \partial_t \varrho_\mu + u_\mu \cdot \nabla \varrho_\mu - \Delta \varrho_\mu = -z_\mu \cdot \nabla \rho, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \\ \partial_t z_\mu + u_\mu \cdot \nabla z_\mu + \nabla P_\mu = \mu \partial_z^2 u_\mu + \varrho_\mu e_z - z_\mu \cdot \nabla u, \\ \operatorname{div} z_\mu = 0, \\ z_{\mu|t=0} = 0, \qquad \varrho_{\mu|t=0} = 0. \end{cases}$$

Similar to inequalities (4.44) and (4.45), the energy estimate in localized frequency shows us that

$$\begin{aligned} \|z_\mu\|_{\widetilde{L}_t^\infty H^{s-2}} \lesssim & \|\varrho_\mu\|_{\widetilde{L}_t^1 H^{s-2}} + t\mu \|u_\mu\|_{L_t^\infty H^s} + \|z_\mu \cdot \nabla u\|_{L_t^1 H^{s-2}} \\ & + \int_0^t \Big(\sum_q 2^{2q(s-2)} \|[\Delta_q, u_\mu \cdot \nabla] z_\mu\|_{L^2}^2 d\tau\Big)^{\frac12} \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \|\varrho_\mu\|_{\widetilde{L}_t^\infty H^{s-2}} + \|\varrho_\mu\|_{\widetilde{L}_t^1 H^s} \lesssim & \|\Delta_{-1} \varrho_\mu\|_{L_t^1 L^2} + \|z_\mu \cdot \nabla \rho\|_{L_t^1 H^{s-2}} \\ & + \int_0^t \Big(\sum_q 2^{2q(s-2)} \|[\Delta_q, u_\mu \cdot \nabla] \varrho_\mu\|_{L^2}^2 d\tau\Big)^{\frac12}. \end{aligned} \tag{5.2}$$

As $s > \frac52$, according to Bony's decomposition and Bernstein's inequality, we get

$$\begin{aligned} \|z_\mu \cdot \nabla u\|_{H^{s-2}} & \lesssim \|z_\mu\|_{H^{s-2}} \|\nabla u\|_{L^\infty} + \|z_\mu\|_{L^3} \|\nabla u\|_{B_{6,2}^{s-2}} \\ & \lesssim \|z_\mu\|_{H^{s-2}} \|u\|_{H^s} \end{aligned} \tag{5.3}$$

and for some $\lambda > 3$ (with $H^{s-2} \hookrightarrow L^\lambda$)

$$\begin{aligned} \|z_\mu \cdot \nabla \rho\|_{L_t^1 H^{s-2}} & \lesssim \|z_\mu\|_{H^{s-2}} \|\nabla \rho\|_{L^\infty} + \|z_\mu\|_{L^\lambda} \|\nabla \rho\|_{B^{s-2}_{\frac{2\lambda}{\lambda-2},2}} \\ & \lesssim \|z_\mu\|_{H^{s-2}} \big(\|\nabla \rho\|_{L^\infty} + \|\rho\|_{H^{s+\frac{3}{\lambda}-1}}\big). \end{aligned} \tag{5.4}$$

Recalling the proof of Lemma 2.100, [5], we have

$$\begin{aligned} & \Big(\sum_q 2^{2q(s-2)} \|[\Delta_q, u_\mu \cdot \nabla] z_\mu\|_{L^2}^2 d\tau\Big)^{\frac12} \\ \lesssim & \begin{cases} \|\nabla u_\mu\|_{B^{\frac32}_{2,\infty} \cap L^\infty} \|z_\mu\|_{H^{s-2}} & \text{if } s < \frac92 \\ \|\nabla u_\mu\|_{L^\infty} \|z_\mu\|_{H^{s-2}} + \|\nabla z_\mu\|_{L^2} \|u_\mu\|_{B^{s-2}_{\infty,2}} & \text{if } s \geq \frac92, \end{cases} \end{aligned}$$

then $\forall s > \frac52$, we have

$$\Big(\sum_q 2^{2q(s-2)} \|[\Delta_q, u_\mu \cdot \nabla] z_\mu\|_{L^2}^2 d\tau\Big)^{\frac12} \lesssim \|u_\mu\|_{H^s} \|z_\mu\|_{H^{s-2}}. \tag{5.5}$$

Similarly, one can show

$$\sum_q 2^{2q(s-2)}\|[\Delta_q, u_\mu\cdot\nabla]\varrho_\mu\|_{L^2}^2 \lesssim \|u_\mu\|_{H^s}\|\varrho_\mu\|_{H^{s-2}}. \tag{5.6}$$

Plugging (5.3)-(5.6) into (5.1) and (5.2), we obtain from Minkowski's inequality and the fact $\|\Delta_{-1}\varrho_\mu\|_{L^1_tL^2}\lesssim\|\varrho_\mu\|_{L^1_tH^{s-2}}$, that

$$\|z_\mu\|_{\widetilde{L}^\infty_tH^{s-2}}+\|\varrho_\mu\|_{\widetilde{L}^\infty_tH^{s-2}}+\|\varrho_\mu\|_{\widetilde{L}^1_tH^s}\lesssim t\mu\|u_\mu\|_{L^\infty_tH^s}+\int_0^t\|\varrho_\mu\|_{H^{s-2}}\big(1+\|u_\mu\|_{H^s}\big)d\tau$$
$$+\int_0^t\|z_\mu\|_{H^{s-2}}\big(\|u\|_{H^s}+\|\nabla\rho\|_{L^\infty}+\|\rho\|_{H^{s+\frac{3}{\lambda}-1}}\big)d\tau,$$

which follows from the Grönwall inequality that

$$\begin{aligned}&\|z_\mu\|_{\widetilde{L}^\infty_tH^{s-2}}+\|\varrho_\mu\|_{\widetilde{L}^\infty_tH^{s-2}}+\|\varrho_\mu\|_{\widetilde{L}^1_tH^s}\\ \le& Ct\mu\|u_\mu\|_{L^\infty_tH^s}\exp\{C(t+t\|u_\mu\|_{L^\infty_tH^s}+t\|u\|_{L^\infty_tH^s}+\|\nabla\rho\|_{L^1_tL^\infty}+\|\rho\|_{L^1_tH^{s+\frac{3}{\lambda}-1}})\}.\end{aligned} \tag{5.7}$$

Thanks to the fact that $s>\frac{5}{2}$ and $\lambda>3$, we get from Remark 1.1 that

$$t\|u\|_{L^\infty_tH^s}+\|\nabla\rho\|_{L^1_tL^\infty}+\|\rho\|_{L^1_tH^{s+\frac{3}{\lambda}-1}}\lesssim t\|u\|_{L^\infty_tH^s}+\|\rho\|_{\widetilde{L}^1_tH^s}\le\phi_5(t),$$

which along with (1.6) implies

$$\|z_\mu\|_{\widetilde{L}^\infty_tH^{s-2}}+\|\varrho_\mu\|_{\widetilde{L}^\infty_tH^{s-2}}+\|\varrho_\mu\|_{\widetilde{L}^1_tH^s}\le(\mu t)\phi_6(t),$$

that is, (1.7) holds. This achieves the proof of the theorem. □

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