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SINGULARITY FORMATION FOR A FLUID MECHANICS MODEL WITH NONLOCAL VELOCITY*

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Abstract. We study a 1D fluid mechanics model with nonlocal velocity. The equation can be viewed as a fractional porous medium flow, a 1D model of quasi-geostrophic equation, and also a special case of the Euler alignment system. For strictly positive smooth initial data, global regularity has been proved in [Do, Kiselev, Ryzhik and Tan, Arch. Ration. Mech. Anal., 228(1):1–37, 2018]. We construct a family of non-negative smooth initial data so that solution is not C^1-uniformly bounded. Our result indicates that strict positivity is a critical condition to ensure global regularity of the system. We also extend our construction to the corresponding models in multi-dimensions.

Keywords. porous medium flow; quasi-geostrophic equations; the Euler alignment equation; singularity formation.

AMS subject classifications. 35Q35; 35Q92.

1. Introduction

We are interested in the following 1D continuity equation

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{1.1}$$

with a nonlocal velocity field

$$u = H\Lambda^{\alpha-1}\rho, \quad 0<\alpha<2, \tag{1.2}$$

where H is the Hilbert transform, and $\Lambda^s = (-\Delta)^{s/2}$ denotes the nonlocal fractional Laplacian operator. The initial density is set to be non-negative

$$\rho(x,t)|_{t=0} = \rho_0(x) \geq 0. \tag{1.3}$$

The dynamics of ρ in the system (1.1)-(1.3) can be alternatively written as

$$\partial_t \rho + u\partial_x \rho = -\rho\Lambda^\alpha \rho. \tag{1.4}$$

It consists of a nonlocal transport term $u\partial_x\rho$, and a dissipation term $-\rho\Lambda^\alpha\rho$ which is nonlinear and nonlocal.

Without the dissipation term, the equation is an active scalar

$$\partial_t \rho + u\partial_x \rho = 0, \tag{1.5}$$

with the velocity u defined in (1.2). It arises as 1D simplified models for 2D surface quasi-geostrophic equations. For $\alpha = 1$, Equation (1.5) was studied by Córdoba, Córdoba and Fontelos [9], where a finite time loss of C^1 regularity is shown for some initial data. Silvestre and Vicol [17] proved the similar behavior for $\alpha \in (0,2)$. Both results indicate that the transport term intends to drive the dynamics into singularity in finite time.

With the dissipation term, the Equation (1.4) appears in many models in fluid mechanics. Since the dissipation term has a possible regularizing effect, the understanding of the competition between the transport term and the dissipation term attracted a lot of attention in recent years.

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Fractional porous medium flow. The main system (1.1)-(1.3) can be viewed as a porous medium equation with fractional potential pressure, where ρ represents the density of the fluid. It was introduced by Caffarelli and Vázquez [3], where an existence theory for weak solutions was established, for $\rho_0 \in L^1$. The regularizing effect was discussed in a series of successive works: [2] for $\alpha \in (0,1)\cup(1,2)$, and [4] for $\alpha=1$. Their result states that weak solutions of the system with any L^1 initial data instantly becomes Hölder continuous, and stays in C^γ for all time, with some $\gamma \in (0,1)$. Such regularizing effect is proved in higher dimensions as well.

For $\alpha=1$, Carrillo, Ferreira and Precioso [6] studied the system in the space of probability measures with bounded second moment. They established a global well-posedness theory by taking advantage of the gradient flow structure of the system in 1D.

The system is also related to a model for the motion of the dislocations in a solid proposed by Biler, Karch and Monneau in [1].

1D model of quasi-geostrophic equation. Chae, Córdoba, Córdoba and Fontelos [8] considered (1.1)-(1.3) with $\alpha=1$. They interpreted the system as a 1D simplified model of 2D quasi-geostrophic equation in atmospheric science, where $\theta(x,t)$, defined as $\rho(x,t)-\kappa$, represents the temperature of the air. The dynamics of θ reads

$$\partial_t\theta+\partial_x(\theta H\theta)=-\kappa\Lambda\theta.$$

They studied the system in the periodic domain $\mathbb{T}=[-1/2,1/2]$, and focused on propagation of regularity with smooth initial data. The result consists of two parts. First, they showed that if $\rho_0>0$ (or $\theta_0>-\kappa$ in their context), then all H^3 initial data stays in H^3 for all time. Second, they proved that the system loses C^1 regularity in finite time, with the initial data chosen as

$$\rho_0(x)=1-\cos(2\pi x), \quad x\in\mathbb{T}. \tag{1.6}$$

The main difference between the two types of initial data is that $\rho_0(x)=0$ is attained in the latter case. It indicates that the preservation of C^1 regularity critically depends on the strict positivity of the initial data.

In [7], Castro and Córdoba discussed the blowup phenomenon for more general initial data without strict positivity.

It is worth noting that $u=H\rho$ when $\alpha=1$. Some properties and identities of Hilbert transform were crucially used in their proof. So, the extension of the result to general $\alpha\in(0,2)$ is far from trivial.

Euler alignment system. System (1.1)-(1.3) is also related to a biologically-motivated complex interacting system modeling collective behaviors. The Cucker-Smale model [10] is an agent-based model governed by Newton's second law

$$\dot{x}_i=v_i, \quad m\dot{v}_i=F_i:=\frac{1}{N}\sum_{j=1}^N\psi(|x_i-x_j|)(v_j-v_i), \tag{1.7}$$

where $(x_i,v_i)_{i=1}^N$ represent the position and velocity of agent i. The force F_i describes the alignment interaction on velocity, where the influence function ψ characterizes the strength of the velocity alignment between two agents. Natually, it is a decreasing function of the distance between the agents.

The macroscopic representation of Cucker-Smale model (1.7), derived through a kinetic system (see [12]), is called Euler alignment system. In 1D, it reads

$$\partial_t\rho+\partial_x(\rho u)=0, \tag{1.8}$$

$$\partial_t u+u\partial_x u=\int_{\mathbb{R}}\psi(|x-y|)(u(y,t)-u(x,t))\rho(y,t)dy. \tag{1.9}$$

For the case when ψ is Lipschitz, the system was studied in [5, 18]. A critical threshold phenomenon was discovered: preservation of C^1 regularity depends on the choice of initial data. Subcritical initial data lead to global regularity, while supercritical initial data lead to fintie time shock formation.

Another case is when ψ is singular, taking the form

$$\psi(|x|)=\frac{c_\alpha}{|x|^{1+\alpha}},\quad 0<\alpha<2, \tag{1.10}$$

with c_α be a positive constant such that

$$\Lambda^\alpha f=c_\alpha\ P.V.\int_{\mathbb{R}}\frac{f(x)-f(y)}{|x-y|^{1+\alpha}}dy.$$

One interesting feature of such choice of ψ is that, Equation (1.9) becomes closely related to the Burgers equation with fractional dissipation

$$\partial_t u+u\partial_x u=-\Lambda^\alpha u, \tag{1.11}$$

by enforcing $\rho\equiv 1$. Kiselev, Nazarov and Shterenberg [13] studied (1.11): when $0<\alpha<1$, there exists initial data leading to finite time blow up; when $\alpha\in[1,2)$, all smooth initial data lead to global regularity.

The Euler alignment system (1.8)-(1.9) with singular influence function (1.10) was studied in [11] in the periodic domain. It was shown that all smooth initial data $\rho_0>0$ leads to global regularity. In particular, in the range of $\alpha\in(0,1)$, the behavior of the solution is very different from the Burgers equation with fractional dissipation, despite their similarity. The global regularity result is extended to more general singular influence function in [14]. Moreover, it is shown that the C^1 norm of the density ρ is uniformly bounded for all time. For $\alpha\in[1,2)$, global regularity was independently shown by Shvydkoy and Tadmor in [15] through a different approach. Their result can also be extended for $\alpha\in(0,1)$ in [16].

As discussed in [11], a useful reformulation of the Euler alignment system for ρ and $G=\partial_x u-\Lambda^\alpha\rho$ has the form

$$\partial_t\rho+\partial_x(\rho u)=0,\quad \partial_t G+\partial_x(Gu)=0,\quad \partial_x u=\Lambda^\alpha\rho+G. \tag{1.12}$$

In particular, if we pick the initial data such that $G_0(x)=\partial_x u_0(x)-\Lambda^\alpha\rho_0(x)\equiv 0$, then $G\equiv 0$ for all $t>0$, and the dynamics of ρ becomes our main system (1.1)-(1.2).

Therefore, the result in [11] implies that for $\alpha\in(0,2)$, system (1.1)-(1.2) with smooth initial data $\rho_0>0$ stays smooth for all time. It serves as an extension to the first part of the result in [8] with general α.

The main result. In this paper, we focus on (1.1)-(1.2) with non-negative initial data ρ_0 which is not strictly positive. We construct initial data which lead to singularity formations.

THEOREM 1.1. *Consider the system* (1.1)-(1.3) *in the periodic domain* $\mathbb{T}$. *There exists a family of smooth initial data* ρ_0 *such that the solution* $\rho(\cdot,t)$ *is not bounded in* C^1 *uniformly in* t.

Theorem 1.1 says that the solution will lose C^1 regularity as time approaches infinity. Note that this type of singularity does not happen when $\rho_0 > 0$ (see [14]). Hence, the non-vacuum assumption is critical to ensure global regularity.

Theorem 1.1 extends the blow up result in [8] to the general case $\alpha \in (0,2)$. However, it only guarantees singularity formations as time approaches infinity. Whether the blow up happens in finite time is still an open problem, which requires future investigations.

As a direct consequence, we have the following result for Euler alignment system.

COROLLARY 1.1. *Consider the initial value problem of Euler alignment system* (1.8)-(1.9) *with singular influence function* ψ *defined in* (1.10). *There exists smooth initial data* $\rho_0 \geq 0$ *and* u_0 *such that the solution loses uniform* C^1 *regularity.*

The choice of initial data could be ρ_0 from Theorem 1.1, and $u_0 = H\Lambda^{\alpha-1}\rho_0$.

The rest of the paper is organized as follows. In Section 2, we show apriori bounds for the system with some proposed symmetry. In Section 3, we obtain an enhanced estimate on the velocity u, which plays an essential role in proving the singularity formation. Theorem 1.1 is then proved in Section 4. In Section 5, we extend the result to systems in multi-dimensional spaces. Finally, in Section 6, we make some remarks on related topics for further investigation.

2. Apriori estimates

In this section, we derive some useful estimates for our main system (1.1)-(1.3), which will help us to construct initial data and obtain finite time blow up.

We first propose the following even-symmetry condition to ρ_0

$$\rho_0(x) = \rho_0(-x). \tag{H1}$$

Since we consider periodic data, ρ_0 can be determined by its value in $x \in [0,1/2]$. We also note that periodicity and even-symmetry are preserved in time.

2.1. Maximum principle. Let us assume the initial data is bounded, satisfying

$$0 \leq \rho_0(x) \leq \bar{\rho}, \quad \forall\, x \in \mathbb{T}. \tag{H2}$$

Then, $\rho(\cdot,t)$ satisfies (H2) for all $t \geq 0$, due to maximum principle.

PROPOSITION 2.1 (Maximum principle). *Let* ρ *be a smooth solution of* (1.1) *with initial data* ρ_0 *satisfying* (H2). *Then,* $\rho(\cdot,t)$ *satisfies* (H2) *for all* $t \geq 0$.

Proof. Suppose $\rho(x,t) \leq \bar{\rho}$ does not hold for all (x,t). Then, there exist x_0, t_0 and $\varepsilon_0 > 0$ such that

$$\rho(x_0,t_0) = \bar{\rho}, \quad \rho(x,t_0) \leq \bar{\rho}, \quad \forall\, x \in \mathbb{T}, \quad \text{and} \quad \rho(x_0,t_0+\varepsilon) > \bar{\rho}, \quad \forall\, \varepsilon \in (0,\varepsilon_0). \tag{2.1}$$

So the violation first occurs at x_0 at time t_0+.

Since $\rho(\cdot,t_0)$ attains its maximum at x_0, we know $\partial_x\rho(x_0,t_0) = 0$, and

$$\Lambda^\alpha \rho(x_0,t_0) > 0,$$

unless $\rho(x,t_0) = \bar{\rho}$ is a constant, in which case $\rho(x,t) = \bar{\rho}$ for all time, and (H2) holds.

From (1.4) we obtain

$$\partial_t\rho(x_0,t_0)=-u(x_0,t_0)\partial_x\rho(x_0,t_0)-\rho(x_0,t_0)\Lambda^\alpha\rho(x_0,t_0)=\bar{\rho}\Lambda^\alpha\rho(x_0,t_0)<0.$$

This contradicts with (2.1). Therefore, $\rho(x,t)\leq\bar{\rho}$ holds for all $x\in\mathbb{T}$ and $t\geq 0$.

Positivity preserving property $\rho(x,t)\geq 0$ can be proved similarly. □

2.2. Conservation of mass. We denote m as the initial mass

$$m=\int_{\mathbb{T}}\rho_0(x)dx. \tag{2.2}$$

Integrating the continuity Equation (1.1) in x, we get

$$\frac{d}{dt}\int_{\mathbb{T}}\rho(x,t)dx=-\int_{\mathbb{T}}\partial_x(\rho(x,t)u(x,t))=0.$$

This implies the conservation of total mass.

Moreover, the mass in any interval is conserved along the characteristic flow.

Proposition 2.2 (Conservation of mass). *Let ρ be a strong solution of the continuity Equation (1.1). Let $X_1(t),X_2(t)$ be two characteristic paths starting at x_1 and x_2, respectively.*

$$\frac{d}{dt}X_i(t)=u(X_i(t),t),\quad X_i(0)=x_i,\quad i=1,2.$$

Then, the mass in the interval $[X_1(t),X_2(t)]$ is conserved in time, namely

$$\int_{X(t;x_1)}^{X(t;x_2)}\rho(x,t)dx=\int_{x_1}^{x_2}\rho_0(x)dx,\quad\forall\ t\geq 0. \tag{2.3}$$

The proof can be found, for instance, in [19, Lemma 5.1].

2.3. Preservation of monotonicity. We make another assumption on ρ_0.

$$\rho_0(0)=0,\quad \partial_x\rho_0(x)\geq 0,\ \forall\ x\in[0,1/2], \tag{H3}$$

namely ρ_0 is increasing in $[0,1/2]$. See Figure 2.1 for an illustration of the initial data ρ_0 satisfying (H1)-(H3).

The following proposition shows that such monotonicity is preserved in time.

Proposition 2.3 (Monotonicity). *Assume that ρ_0 is smooth and satisfies (H1)-(H3). Let ρ be a smooth solution of (1.1)-(1.3). Then, $\rho(\cdot,t)$ satisfies (H3) for any $t\geq 0$.*

Proof. Let us denote $\zeta:=\partial_x\rho$, and write down its dynamics by differentiating (1.4) in x

$$\partial_t\zeta=-u\partial_x\zeta-2\zeta\partial_x u-\rho\partial_x^2u=-u\partial_x\zeta-2\zeta\Lambda^\alpha\rho-\rho\Lambda^\alpha\zeta.$$

Along each characteristic path, we have

$$(\partial_t+u\partial_x)\zeta=-2\zeta\Lambda^\alpha\rho-\rho\Lambda^\alpha\zeta. \tag{2.4}$$

By periodicity and (H1), we know $\zeta(\cdot,t)$ is odd, and so

$$\zeta(0,t)=\zeta(1/2,t)=0.$$

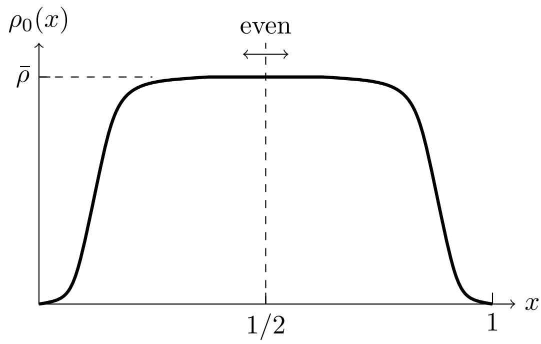

FIG. 2.1. *The choice of initial data ρ_0, satisfying* (H1)-(H3)

Our goal is to prove $\zeta(x,t) \geq 0$, for all $x \in (0,1/2)$ and $t \geq 0$. Assume the argument is false, then there exists at time t_0, a characteristic path $X(t)$, and an $\varepsilon_0 > 0$, such that the solution satisfies

$$\zeta(X(t_0),t_0) = 0, \quad \zeta(x,t_0) \geq 0, \quad \forall\, x \in \mathbb{T}, \quad \text{and} \quad \zeta(X(t_0+\varepsilon),t_0+\varepsilon) < 0, \quad \forall\, \epsilon \in (0,\varepsilon_0). \tag{2.5}$$

so that the break down happens at $(X(t_0),t_0+)$. The dynamics (2.4) at $(X(t_0),t_0)$ becomes

$$\frac{d}{dt}\zeta(X(t),t)\bigg|_{t=t_0} = -\rho(X(t_0),t_0)\Lambda^\alpha \zeta(X(t),t_0). \tag{2.6}$$

From Proposition 2.1, we know $\rho(X(t_0),t_0) \geq 0$.

If $\rho(X(t_0),t_0) = 0$, then $\rho(X(t),t) = 0$ for any $t \geq t_0$ as

$$\rho(X(t),t) = \rho(X(t_0),t_0)\exp\left(-\int_{t_0}^t \Lambda^\alpha \rho(X(s),s)ds\right) = 0.$$

This implies $\zeta(X(t),t) = 0$ for all $t \geq t_0$. It contradicts with (2.5).

If $\rho(X(t_0),t_0) > 0$, we estimate $\Lambda^\alpha \zeta(X(t_0),t_0)$ as follows. Denote $x_0 = X(t_0)$.

$$\begin{aligned}
\Lambda^\alpha \zeta(x_0,t_0) &= c_\alpha \int_{\mathbb{R}} \frac{\zeta(x_0,t_0) - \zeta(y,t_0)}{|x_0-y|^{1+\alpha}}dy = -c_\alpha \sum_{l\in\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\zeta(y,t_0)}{|x_0-y-l|^{1+\alpha}}dy \\
&= -c_\alpha \left[\sum_{l\in\mathbb{Z}} \int_0^{1/2} \frac{\zeta(-y,t_0)}{|x_0+y-l|^{1+\alpha}}dy + \sum_{l\in\mathbb{Z}} \int_0^{1/2} \frac{\zeta(y,t_0)}{|x_0-y-l|^{1+\alpha}}dy\right] \\
&= -c_\alpha \int_0^{1/2} \zeta(y,t_0) \sum_{l\in\mathbb{Z}} \left(\frac{1}{|x_0-y-l|^{1+\alpha}} - \frac{1}{|x_0+y-l|^{1+\alpha}}\right) dy.
\end{aligned}$$

From (2.5) and the following Lemma 2.1, we conclude that $\Lambda^\alpha \zeta(x_0,t_0) < 0$ and hence

$$\frac{d}{dt}\zeta(X(t),t)\bigg|_{t=t_0} > 0.$$

This contradicts with the last inequality in (2.5). ∎

LEMMA 2.1. *Suppose $x,y\in(0,1/2)$ and $\alpha>0$. Then*

$$\sum_{l\in\mathbb{Z}}\left(\frac{1}{|x-y-l|^{1+\alpha}}-\frac{1}{|x+y-l|^{1+\alpha}}\right)>0.$$

Proof. We first consider the case when $y\le x$. The sum can be rewritten as

$$\sum_{l\ge1}\left[\left(\frac{1}{(l-1+x-y)^{1+\alpha}}-\frac{1}{(l-x-y)^{1+\alpha}}\right)-\left(\frac{1}{(l-1+x+y)^{1+\alpha}}-\frac{1}{(l-x+y)^{1+\alpha}}\right)\right].$$

Define

$$H_l(z)=\frac{1}{(l-1+x-z)^{1+\alpha}}-\frac{1}{(l-x-z)^{1+\alpha}}.$$

Then, the sum can be represented as

$$\sum_{l\ge1}(H_l(y)-H_l(-y)).$$

Since we have

$$H_l'(z)=(1+\alpha)\left[\frac{1}{(l-1+x-z)^{2+\alpha}}-\frac{1}{(l-x-z)^{2+\alpha}}\right]>0,\quad \forall\, z\in(-1/2,1/2),$$

we get $H_l(y)-H_l(-y)\ge0$ for any $y\in[0,x]$. It implies that the sum is non-negative.

The case when $y>x$ can be treated in the same way. ∎

2.4. An estimate on velocity. The velocity u defined in (1.2) can be expressed in the integral form as follows:

$$u(x,t)=c_\alpha\int_{\mathbb{R}}\frac{\rho(y,t)-\rho(x,t)}{sgn(x-y)|x-y|^\alpha}dy. \tag{2.7}$$

Fix $x\in[0,1/2]$ and $t\ge0$. We decompose the integrand and use (H1) to get

$$\begin{aligned}\frac{1}{c_\alpha}u(x,t)&=\int_0^\infty\frac{\rho(y,t)-\rho(x,t)}{|x+y|^\alpha}dy+\int_0^x\frac{\rho(y,t)-\rho(x,t)}{|x-y|^\alpha}dy-\int_x^\infty\frac{\rho(y,t)-\rho(x,t)}{|x-y|^\alpha}dy\\&=\int_0^x(\rho(y,t)-\rho(x,t))\left(\frac{1}{(x+y)^\alpha}+\frac{1}{(x-y)^\alpha}\right)dy\\&\quad+\int_x^\infty(\rho(y,t)-\rho(x,t))\left(\frac{1}{(x+y)^\alpha}-\frac{1}{(y-x)^\alpha}\right)dy=:I+II.\end{aligned}$$

Due to monotonicity condition of $\rho(\cdot,t)$ (H3), we know that the first term $I\le0$. For the second term II, observe that

$$\frac{1}{(x+y)^\alpha}-\frac{1}{(y-x)^\alpha}<0,\quad\forall\, y>x>0.$$

So, the integral in II can be decomposed into two parts:

$$\int_x^\infty=\sum_{l=0}^\infty\int_{l+x}^{l+1-x}+\sum_{l=1}^\infty\int_{l-x}^{l+x}.$$

Again, condition (H3) implies that for the first part $\rho(y,t)-\rho(x,t)\geq 0$, and for the second part $\rho(y,t)-\rho(x,t)\leq 0$. Let us denote $II=II_1+II_2$ where II_1 and II_2 represents the corresponding integrals. Then, $II_1\leq 0$ and $II_2\geq 0$.

The next lemma shows $I+II_2\leq 0$, at least when x is sufficiently small.

LEMMA 2.2. *There exists a $\delta=\delta(\alpha)>0$, such that for all $x\in[0,\delta]$, $I+II_2\leq 0$.*

Proof. Let us first write

$$II_2=\int_{-x}^{x}(\rho(x,t)-\rho(y,t))\sum_{l=1}^{\infty}\left(\frac{1}{(y+l-x)^\alpha}-\frac{1}{(y+l+x)^\alpha}\right)dy.$$

Using mean value theorem, we have for $y\in(-x,x)$,

$$\frac{1}{(y+l-x)^\alpha}-\frac{1}{(y+l+x)^\alpha}\leq \alpha(l-2x)^{-1-\alpha}\cdot(2x).$$

Therefore,

$$\sum_{l=1}^{\infty}\frac{1}{(y+l-x)^\alpha}-\frac{1}{(y+l+x)^\alpha}\leq 2\alpha x\left[(1-2x)^{-1-\alpha}+\int_1^\infty(z-2x)^{-1-\alpha}dz\right]\leq Cx.$$

For $x\leq 1/4$, the last inequality holds with the choice of $C=2^{\alpha+1}(1+2\alpha)$.

Now, let us put together I and II_2.

$$\begin{aligned}I+II_2=&\int_0^x(\rho(x,t)-\rho(y,t))\left[-\frac{1}{(x+y)^\alpha}-\frac{1}{(x-y)^\alpha}\right.\\&\left.+\sum_{l=1}^{\infty}\left(\frac{1}{(y+l-x)^\alpha}-\frac{1}{(y+l+x)^\alpha}+\frac{1}{(-y+l-x)^\alpha}-\frac{1}{(-y+l+x)^\alpha}\right)\right]dy\\ \leq&\int_0^x(\rho(x,t)-\rho(y,t))\left[-\frac{1}{(x-y)^\alpha}+0+2Cx\right]dy\\ \leq&(-x^{-\alpha}+2Cx)\int_0^x(\rho(x,t)-\rho(y,t))dy.\end{aligned}$$

We pick a small enough δ as follows

$$\delta=\min\left\{\frac{1}{4},\left(\frac{1}{3C}\right)^{\frac{1}{1+\alpha}}\right\},\tag{2.8}$$

Then, for any $x\in(0,\delta]$, we have $-x^{-\alpha}+2Cx\leq -Cx<0$.

Also, the monotonicity condition (H3) implies that

$$\int_0^x(\rho(x,t)-\rho(y,t))dy\geq 0.$$

Therefore, conclude that $I+II_2\leq 0$ for all $x\in[0,\delta]$. ◻

Lemma 2.2 directly implies the following estimate on u.

THEOREM 2.1. *Let ρ be a smooth solution of (1.1)-(1.3), with periodic initial data ρ_0 satisfying (H1)-(H3). Let δ be defined as (2.8). Then, the velocity*

$$u(x,t)\leq 0,\quad \forall\ x\in[0,\delta],\quad t\geq 0.$$

One may remove the smallness assumption on x in Theorem 2.1 by a more careful estimate on II_2. For our purpose, it is enough to consider small x.

3. An enhanced estimate on velocity

In order to show singularity formations, we need a stronger estimate on the velocity. Recall

$$u(x,t)=(I+II_2)+II_1.$$

Lemma 2.2 ensures $I+II_2\le 0$. The estimate $II_1\le 0$ simply follows for (H3).

We aim to improve our estimate on

$$II_1=-\sum_{l=0}^{\infty}\int_{l+x}^{l+1-x}(\rho(y,t)-\rho(x,t))\left(\frac{1}{(y-x)^\alpha}-\frac{1}{(y+x)^\alpha}\right)dy.$$

An easy observation is that, if $\rho(x,t)=\bar{\rho}$, then $II_1=0$. In this case, it is not possible to get any improvement. Therefore, we obtain an enhanced estimate when $\rho(x,t)$ is small.

THEOREM 3.1. *Let ρ be a smooth solution of (1.1)-(1.3), with periodic initial data ρ_0 satisfying (H1)-(H3), Let δ be defined as (2.8). Then, there exists a positive constant $A=A(\alpha,m,\bar{\rho})>0$, for any (x,t) satisfying $x\in[0,\delta]$ and*

$$\rho(x,t)\le\frac{m}{2},\tag{3.1}$$

the velocity

$$u(x,t)\le -Ax.\tag{3.2}$$

Let us explain the main idea of the proof. We focus on a better bound on $[x,1/2]$, and use the rough bound by zero for the rest of the integrand.

$$II_1\le-\int_x^{1/2}(\rho(y,t)-\rho(x,t))\left(\frac{1}{(y-x)^\alpha}-\frac{1}{(y+x)^\alpha}\right)dy.$$

Denote the term that we are concerned with, as III.

$$III=\int_x^{1/2}(\rho(y,t)-\rho(x,t))h(x,y)dy,\quad h(x,y)=\frac{1}{(y-x)^\alpha}-\frac{1}{(y+x)^\alpha}.$$

To obtain a lower bound on III, we need several observations. First, for a fixed $x\in[0,\delta]$, $h(x,y)\ge 0$ for any $y\in(x,1/2]$. Moreover,

$$\partial_y h(x,y)=-\alpha\left[\frac{1}{(y-x)^{\alpha+1}}-\frac{1}{(y+x)^{\alpha+1}}\right]\le 0.\tag{3.3}$$

Next, we apply (H2) (H3), and get

$$0\overset{(H3)}{\le}\rho(y,t)-\rho(x,t)\overset{(H2)}{\le}\bar{\rho}-\rho(x,t),\quad\forall\, y\in(x,1/2].\tag{3.4}$$

Moreover, the assumption (3.1) implies

$$\int_x^{1/2}(\rho(y,t)-\rho(x,t))\,dy\overset{(H3)}{\ge}\int_0^{1/2}(\rho(y,t)-\rho(x,t))\,dy=\frac{m}{2}-\frac{\rho(x,t)}{2}\overset{(3.1)}{\ge}\frac{m}{4}.\tag{3.5}$$

The following lemma is helpful to get a positive lower bound of III.

LEMMA 3.1. *Let f be a positive decreasing function on $[a,b]$. λ and M are positive constants such that $\lambda < M(b-a)$. Then,*

$$\min_{\omega}\left\{\int_a^b \omega(x)f(x)dx \;\middle|\; 0\le\omega(x)\le M, \int_a^b \omega(x)dx \ge \lambda\right\} = M\int_{b-\frac{\lambda}{M}}^b f(x)dx.$$

The minimum is attained at

$$\omega_{\min}(x) = \begin{cases} 0 & a\le x < b-\frac{\lambda}{M} \\ M & b-\frac{\lambda}{M}\le x\le b \end{cases}.$$

Proof. First, it is easy to check that $\omega_{\min}$ satisfies

$$0\le\omega(x)\le M, \quad \int_a^b \omega(x)dx \ge \lambda. \tag{3.6}$$

We will prove that for any ω which satisfies (3.6), $\int_a^b(\omega(x)-\omega_{\min}(x))f(x)dx \ge 0$. Compute

$$\int_a^b(\omega(x)-\omega_{\min}(x))f(x)dx = \int_a^{b-\frac{\lambda}{M}} \omega(x)f(x)dx + \int_{b-\frac{\lambda}{M}}^b (\omega(x)-M)f(x)dx.$$

From the first condition in (3.6), we know $\omega(x)\ge 0$ and $\omega(x)-M\le 0$. Together with the assumption that f is positive and decreasing, we obtain

$$\begin{aligned}\int_a^b(\omega(x)-\omega_{\min}(x))f(x)dx &\ge f(b-\frac{\lambda}{M})\int_a^{b-\frac{\lambda}{M}}\omega(x)dx + f(b-\frac{\lambda}{M})\int_{b-\frac{\lambda}{M}}^b(\omega(x)-M)dx \\ &= f(b-\frac{\lambda}{M})\left[\int_a^b \omega(x)dx - M\cdot\frac{\lambda}{M}\right] \ge f(b-\frac{\lambda}{M})(\lambda-\lambda)=0.\end{aligned}$$

Hence, we conclude

$$\min_{\omega \text{ satisfies } (3.6)} \int_a^b \omega(x)f(x)dx = \int_a^b \omega_{\min}(x)f(x)dx = M\int_{b-\frac{\lambda}{M}}^b f(x)dx.$$

∎

Putting together (3.3), (3.4) and (3.5), we can apply Lemma 3.1 with

$$f(y)=h(x,y),\quad \omega(y)=\rho(y,t)-\rho(x,t),\quad \lambda=\frac{m}{4},\quad M=1-\rho(x,t),\quad a=x,\quad b=\frac{1}{2}.$$

Then,

$$III \ge (1-\rho(x,t))\int_{\frac12-\frac{m}{4(\bar\rho-\rho(x,t))}}^{\frac12} h(x,y)dy \ge (1-\rho(x,t))\int_{\frac12-\frac{m}{4\bar\rho}}^{\frac12} h(x,y)dy.$$

Using the mean value theorem, we have

$$h(x,y)\ge\frac{\alpha}{(y+x)^{1+\alpha}}\cdot(2x)\ge 2\alpha x.$$

Finally, we obtain

$$III \geq \frac{1}{2} \cdot \frac{m}{4\bar{\rho}} \cdot (2\alpha x) = \frac{\alpha m}{4\bar{\rho}} x,$$

and therefore

$$II_1 \leq -\frac{\alpha m}{4\bar{\rho}} x.$$

We end up with the improved estimate (3.2) with $A = \frac{\alpha m}{4\bar{\rho}}$.

4. Singularity formation

In this section, we prove Theorem 1.1: for any smooth initial data satisfying (H1)-(H3), the solution loses uniform C^1 regularity.

We will argue by contradiction. Suppose the solution is uniformly C^1 for all time, then there exists $\varepsilon > 0$ such that

$$\rho(\varepsilon, t) \leq \frac{m}{2}, \quad \forall\ t \geq 0. \tag{4.1}$$

Without loss of generality, we assume $\varepsilon \leq \delta$. In fact, if $\varepsilon > \delta$, $u(\delta, t) \leq u(\varepsilon, t) \leq \frac{m}{2}$ by (H3). We can then take $\varepsilon = \delta$.

Let us denote $X(t;x)$ be the characteristic path initiated at x, satisfying

$$\frac{d}{dt} X(t;x) = u(X(t;x), t), \quad X(0;x) = x.$$

By symmetry, we know $u(0,t) = 0$ and hence $X(t;0) = 0$ for all $t \geq 0$.

Define $m(x,t)$ to be the mass in the interval $[0,x]$ at time t:

$$m(x,t) := \int_0^x \rho(x,t)\, dx.$$

We apply Proposition 2.2 and get

$$m(X(t;x), t) = m(x, 0). \tag{4.2}$$

Let $x_0 = \inf\{x \geq 0 \ :\ \rho_0(x) > 0\}$. By (H1) and (H3), we have

$$\mathrm{supp}(\rho_0) = (x_0, 1 - x_0).$$

We shall proceed with two cases.

Case 1: $x_0 < \varepsilon$**.** By the definition of x_0, we know $\rho_0(\varepsilon) > 0$. Moreover, $m(\varepsilon, 0) > 0$.

By Theorem 2.1, we know $X(t;\varepsilon) \leq \varepsilon$ for any $t \geq 0$. Then, the assumption (4.1) ensures that $\rho(X(t;\varepsilon)) \leq \frac{m}{2}$ for all time. This allows us to use the enhanced estimate, Theorem 3.1, and get

$$u(X(t;\varepsilon), t) \leq -AX(t;\varepsilon),$$

where $A > 0$ does not depend on ε or t.

Then, we can integrate along the characteristic path, and get

$$X(t;\varepsilon) \leq \varepsilon e^{-At}.$$

A simple estimate yields

$$m(X(t;\varepsilon),t)=\int_0^{X(t;\varepsilon)}\rho(x,t)dx \overset{\text{(H3)}}{\leq} X(t;\varepsilon)\rho(X(t;\varepsilon),t)\leq \frac{\varepsilon m}{2}e^{-At}.$$

This contradicts with the mass conservation (4.2) if we pick t large enough, more precisely,

$$t>\frac{1}{A}\log\frac{\varepsilon m}{2m(\varepsilon,0)}. \tag{4.3}$$

REMARK 4.1. If $\rho_0(x)=0$ only at a single point $x=0$, then $x_0=0$. No matter what ε is, we are always under this case. Therefore, we have already shown the singularity formation. Note that the initial data (1.6) lie in this category.

Case 2: $x_0\geq\varepsilon$. If $x_0>0$, namely $\rho_0(x)=0$ in an interval $[-x_0,x_0]$, it is possible that $\varepsilon\leq x_0$. Then, $m(\varepsilon,0)=0$. Consequently, the right-hand side of (4.3) is not bounded any more.

To obtain a contradiction, we first examine the characteristic path starting at x_0. Since $\rho_0(x_0)=0$, it is easy to see that $\rho_0(X(t;x_0),t)=0$ at any time. We can apply the enhanced estimate (3.2) at $(X(t;x_0),t)$, and obtain

$$X(t;x_0)\leq x_0e^{-At}.$$

Then, there exists a finite time T_* such that $X(t;x_0)\leq\varepsilon$. For instance, one can take

$$T_*=\frac{1}{A}\log\frac{x_0}{\varepsilon}.$$

Now, we consider the characteristic path that goes through the point (ε,T_*+1). If the flow is smooth, we can track back and find a unique point x_* such that $\varepsilon=X(T_*+1;x_*)$. See Figure 4.1 for an illustration.

Moreover, as $X(T_*+1;x_0)<\varepsilon$, we have $x_0<x_*$. By the definition of x_0, we know $\rho_0(x_*)>0$ and hence $m(x_*,0)>0$.

Now, we can repeat the argument in case 1 along $X(t;x_*)$. First, apply the enhanced estimate (3.2) at $(X(t;x_*),t)$ for $t\geq T_*+1$ and get

$$X(t;x_*)\leq\varepsilon e^{-A(t-(T_*+1))},\quad \forall\, t\geq T_*+1.$$

Next, we estimate the mass

$$m(X(t;x_*),t)\leq X(t;x_*)\rho(X(t;x_*),t)\leq\frac{\varepsilon m}{2}e^{-A(t-(T_*+1))},\quad \forall\, t\geq T_*+1.$$

Finally, take t large enough

$$t>\frac{1}{A}\log\frac{\varepsilon m}{2m(x_*,0)}+(T_*+1).$$

Then, $m(X(t;x_*),t)<m(x_*,0)$, which contradicts with the mass conservation (4.2).

5. Extension to systems in multi-dimensions

In this section, we extend our main result to systems in higher dimensions. The main idea is to consider $\rho_0(\mathbf{x})=\rho_0(x_1)$ and reduce the system to 1D so that our construction can be used.

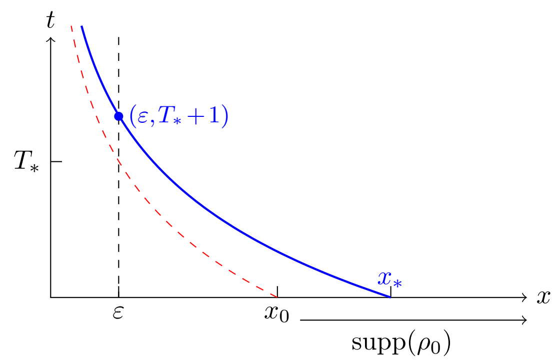

FIG. 4.1. *The characteristic path that leads to a contradiction.*

5.1. Fractional porous medium flow. Let us recall the fractional porous medium flow in multi-dimensions

$$\partial_t\rho+\nabla\cdot(\rho\mathbf{u})=0,\quad \mathbf{u}=\nabla\Lambda^{\alpha-2}\rho, \tag{5.1}$$

with $\mathbf{x}=(x_1,\cdots,x_n)\in\mathbb{T}^n$ and $0<\alpha<2$.

Fix any time t and drop the time dependence for simplicity. Assume $\rho(\mathbf{x})=\rho(x_1)$, namely ρ is a constant in $(x_2,\cdots,x_n)$ variables. We calculate the velocity field u, starting with

$$\Lambda^{\alpha-2}\rho=c_{n,\alpha}\int_{\mathbb{R}^n}\rho(\mathbf{x}-\mathbf{y})\frac{1}{|\mathbf{y}|^{n+\alpha-2}}d\mathbf{y}.$$

Then, we obtain u by taking the gradient of the potential

$$u_i(\mathbf{x})=\partial_{x_i}\Lambda^{\alpha-2}\rho=c_{n,\alpha}\int_{\mathbb{R}^n}(\rho(x_1-y_1)-\rho(x_1))\frac{y_i}{|\mathbf{y}|^{n+\alpha}}d\mathbf{y}.$$

For $i=2,\cdots,n$, we have

$$u_i(\mathbf{x})=c_{n,\alpha}\int_{\mathbb{R}}(\rho(x_1-y_1)-\rho(x_1))\left[\int_{\mathbb{R}^{n-1}}\frac{y_i}{|\mathbf{y}|^{n+\alpha}}dy_2\cdots dy_n\right]dy_1=0. \tag{5.2}$$

The last eqaulity is due to oddness of the inside integral with respect to y_i.

For $i=1$,

$$u_1(\mathbf{x})=c_{n,\alpha}\int_{\mathbb{R}}(\rho(x_1-y_1)-\rho(x_1))y_1\left[\int_{\mathbb{R}^{n-1}}\frac{1}{|\mathbf{y}|^{n+\alpha}}dy_2\cdots dy_n\right]dy_1.$$

Compute the integral inside,

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}\frac{1}{|\mathbf{y}|^{n+\alpha}}dy_2\cdots dy_n&=\int_{\mathbb{R}^{n-1}}\left(y_1^2+y_2^2+\cdots+y_n^2\right)^{-\frac{n+\alpha}{2}}dy_2\cdots dy_n\\&=|y_1|^{-(n+\alpha)}\int_{\mathbb{R}^{n-1}}\left(1+y_2^2+\cdots+y_n^2\right)^{-\frac{n+\alpha}{2}}|y_1|^{n-1}dy_2\cdots dy_n\\&=|y_1|^{-1-\alpha}\omega_{n-1}\int_0^\infty(1+r^2)^{-\frac{n+\alpha}{2}}r^{n-2}dr=c'_{n,\alpha}|y_1|^{-1-\alpha}.\end{aligned}$$

Here, ω_n denotes the area of the unit sphere in n dimensions. The constant $c'_{n,\alpha}$ is clearly positive, finite, and only depends on n and α.

Then, we obtain

$$u_1(\mathbf{x}) = c_{n,\alpha} c'_{n,\alpha} \int_{\mathbb{R}} \frac{\rho(x_1 - y_1) - \rho(x_1)}{sgn(y_1)|y_1|^{\alpha}} dy_1. \tag{5.3}$$

So, $u_1(\mathbf{x}) = u_1(x_1)$ is also a constant in $(x_2, \cdots, x_n)$. Moreover, as a function of x_1, the expression of u_1 is the same as (2.7), except the constant c_α might be different.

From (5.2) and (5.3), we have

$$\nabla \cdot (\rho(\mathbf{x})\mathbf{u}(\mathbf{x})) = \partial_{x_1}(\rho(x_1)u_1(x_1)).$$

This implies that if $\rho_0(\mathbf{x}) = \rho_0(x_1)$, then $\rho(\mathbf{x}, t) = \rho(x_1, t)$. Moreover, (ρ, u_1) as functions of x_1, will be the solution of the 1D system (1.1)-(1.3). Hence, Theorem 1.1 can be extended to multi-dimensions, with the choice of initial data $\rho_0(\mathbf{x}) = \rho_0(x_1)$, where ρ_0 as a function of x_1 is chosen the same way as in the 1D case. The different constant in (5.3) mentioned above will only affect the choice of δ throughout the proof.

We summarize the discussion in the following theorem.

THEOREM 5.1. *Consider the initial value problem of system (5.1) in the periodic domain $\mathbb{T}^n$. There exists a family of smooth initial data ρ_0 such that the solution loses uniform C^1 regularity.*

5.2. Fractional Euler alignment system. The multi-dimensional Euler alignment system with singular influence function takes the form

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{\mathbf{u}(y,t) - \mathbf{u}(x,t)}{|y - x|^{n+\alpha}} \rho(y,t) dy. \tag{5.4}$$

Let $G = \nabla \cdot \mathbf{u} - \Lambda^\alpha \rho$. Then, the dynamics of G reads

$$\partial_t G + \nabla \cdot (G\mathbf{u}) = \mathrm{tr}(\nabla \mathbf{u}^{\otimes 2}) - (\nabla \cdot \mathbf{u})^2.$$

Note that in the 1D case, the right-hand side becomes $(\partial_x u)^2 - (\partial_x u)^2 = 0$. Then, the dynamics becomes (1.12), and as a special case of $G \equiv 0$, we reach our system (1.1)-(1.2).

However, the right-hand side is not necessarily zero in higher dimensions. This quantity is known as *spectral gap*. In particular, it destroys the maximum principle on G, and hence $G_0 \equiv 0$ does not imply $G(\cdot, t) \equiv 0$.

Therefore, fractional porous medium flow (5.1) is not a special case of the Euler alignment system, except in 1D. The global regularity on (5.4) for $\rho_0 > 0$ is an open problem. The main difficulty is the lack of apriori control of the spectral gap.

To construct $\rho_0 \geq 0$ which leads to singularity formations, we can avoid the difficulty by selecting a special family of initial data such that the spectral gap is zero for all time.

The choice of $(\rho_0, \mathbf{u}_0)$ is the same as Section 5.1:

$$\rho_0(\mathbf{x}) = \rho_0(x_1), \quad (u_0)_1(\mathbf{x}) = (u_0)_1(x_1), \quad (u_0)_i(\mathbf{x}) = 0, \ \ \forall\, i = 2, \cdots, n.$$

By the same argument, we know that such structure preserves in time. So,

$$\mathrm{tr}(\nabla \mathbf{u}^{\otimes 2}) - (\nabla \cdot \mathbf{u})^2 = (\partial_{x_1} u_1)^2 - (\partial_{x_1} u_1)^2 = 0.$$

Therefore, we pick ρ_0 the same as in Theorem 5.1, and $\mathbf{u}_0 = \nabla\Lambda^{\alpha-2}\rho_0$. The solution will form singularities the same way as (5.1).

COROLLARY 5.1. *Consider the initial value problem of system (5.4) in the periodic domain $\mathbb{T}^n$. There exists a family of smooth initial data $(\rho_0, \mathbf{u}_0)$ such that the solution loses uniform C^1 regularity.*

6. Further discussions

Theorem 1.1 shows singularity formations for equations (1.1)-(1.3). However, it does not specify whether the blow up happens in finite time or when time approaches infinity.

For the special case with $\alpha = 1$ and initial data (1.6), a finite time blowup was shown in [8]. Therefore, a reasonable conjecture would be, the singularity formations happen at a finite time.

The proof of the conjecture will require a stronger estimate on the velocity field

$$u(x,t) \leq -Cx^{\gamma},$$

with $\gamma < 1$. This will ensure the characteristic paths intersect in finite time, causing a blowup. To obtain the strong inequality, a delicate estimate to the singular integral near the singularity is required. We will leave it for future investigations.

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