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LINEARIZED ASYMPTOTIC STABILITY OF RAREFACTION WAVES FOR GAS DYNAMICS IN THERMAL NONEQUILIBRIUM AND LIFE SPAN OF SOLUTIONS*

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Abstract. For the one-dimensional gas flow in vibrational nonequilibrium, the linearized asymptotic stability of rarefaction waves is obtained in this paper with convergence rate, and the life-span of the solution in terms of the rarefaction wave strength is also given when the initial data are perturbations of a smooth rarefaction wave of the equilibrium of the compressible Euler equations. The main feature of the problems is that the L^2-norm of the perturbations may grow in time.

Keywords. Thermal nonequilibrium; Rarefaction wave; Linearized asymptotic stability; Life-span.

AMS subject classifications. 35B35; 35B40.

1. Introduction

Gas dynamics in vibrational nonequilibrium in 1D is governed by the following equations in Lagrangian coordinates (see [25,27,34]):

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (e + \frac{u^2}{2})_t + (up)_x = 0, \\ q_t = \frac{Q-q}{\tau} = \frac{\chi}{\tau}, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^1$, $t \geq 0$, v, u, p, e, q and Q denote the specific volume, the velocity, the pressure, the internal energy, vibrational energy and local equilibrium value of q, respectively, $\tau \geq 0$, is local relaxation time. In the non-equilibrium thermal dynamical process discussed in this paper, it is assumed that the translational and rotational energy adjust quickly enough to keep in translational and rotational equilibrium, but the vibrational energy may take a longer time to adjust. In order to describe the process of translational and rotational motion and vibrational motions, we use e_1, T_1 and s_1 to denote the total of the translational and rotational energy of the molecules, the common temperature and the total entropy of translational and rotational modes, respectively. Similarly, q, T_2 and s_2 are introduced to describe the vibrational energy, vibrational temperature and vibrational entropy, respectively. For vibrational nonequilibrium mode, these two modes obey different thermodynamic laws:

$$\begin{cases} s = s_1 + s_2, \\ de_1 = T_1 ds_1 - p dv, \\ dq = T_2 ds_2. \end{cases} \tag{1.2}$$

In this paper, for the simplicity of presentation, we consider the case that the thermal

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dynamical variables satisfy the following constitutive relations [27]:

$$\begin{cases} e=e_1+q, e_1=\frac{\alpha}{2}pv=\frac{\alpha R}{2}T_1, \\ Q=\frac{\alpha_f}{2}pv=\frac{\alpha_f R}{2}T_1=\omega(T_1), \\ s_1=R(lnv+\frac{\alpha}{2}lne_1),\ s_2=\frac{\alpha_f R}{2}lnq,\ s=s_1+s_2, \\ q=\frac{\alpha_f R}{2}T_2=\omega(T_2), \\ \chi=Q-q, \end{cases} \tag{1.3}$$

where α, α_f are positive constants, denoting the degrees of the freedom adjusting instantaneously and taking longer to relax, respectively. $R>0$ is a constant and we take $R=1$ for convenience.

It is expected that, if the relaxation time τ is taken to be so short that $q=\frac{\alpha_f}{2}pv$ is an adequate approximation to the last equation of (1.1), the solution to the vibrational nonequilibrium system (1.1) should well approximate the corresponding equilibrium system:

$$\begin{cases} v_t-u_x=0, \\ u_t+p_x=0, \\ (e_1+\frac{\alpha_f}{2}pv+\frac{u^2}{2})_t+(up)_x=0, \end{cases} \tag{1.4}$$

which can be obtained by letting $T_1=T_2$ or Q takes the value of q in (1.1). The equilibrium entropy s is then given by

$$\begin{aligned} s&=lnv+\frac{\alpha}{2}lne_1+\frac{\alpha_f}{2}lnQ \\ &=lnv+\frac{\alpha}{2}lne_1+\frac{\alpha_f}{2}ln(\frac{\alpha_f}{2}pv). \end{aligned} \tag{1.5}$$

Noting that $e_1=\frac{\alpha}{2}pv$, in this case, the equilibrium pressure p as a function of v and the equilibrium entropy s is given by

$$p=p(v,s)=Kv^{-\gamma}e^{(\gamma-1)s}, \tag{1.6}$$

where

$$\gamma=1+\frac{2}{\alpha+\alpha_f},$$

$$K=\left(\left(\frac{\alpha}{2}\right)^{-\frac{\alpha}{2}}\left(\frac{\alpha_f}{2}\right)^{-\frac{\alpha_f}{2}}\right)^{(\gamma-1)}.$$

System (1.4) is strictly hyperbolic with three characteristic speeds:

$$\lambda_1(v,s)=-\sqrt{K\gamma v^{-(\gamma+1)}}e^{\frac{(\gamma-1)s}{2}},\ \lambda_2=0,\ \lambda_3=\sqrt{K\gamma v^{-(\gamma+1)}}e^{\frac{(\gamma-1)s}{2}}.$$

System (1.1) is a hyperbolic system with relaxation, which may induce certain dissipative effects. Extensive study has been made in literature on the global regularity and long-time behavior for some inhomogeneous quasilinear hyperbolic systems, mainly due to the dissipative effects induced by the inhomogeneous terms through the strong coupling with the flux functions. A typical version for such a coupling is the Shizuta-Kawashima condition, [22]. However, as one of the most important physical systems of hyperbolic conservation laws with relaxation, (1.1) does not satisfy those conditions of coupling, particulary, the Shizuta-Kawashima condition [22], as pointed out in [34]. This implies that, the dissipation of the relaxation has no effect on some variables for system (1.1), the relaxation term does not have, on all the equilibrium characteristic directions, a positive projection on the linear level. Therefore, compared with many dissipative hyperbolic systems, the dissipation of (1.1) induced by the relaxation is extremely weak, too weak to dominate the hyperbolicity. This makes it a challenging task to investigate the global regularity and long-time and small-relaxation behavior of solutions, which is important to understand the physical process from thermal non-equilibrium to equilibrium. In this direction, the global existence and the pointwise behavior of smooth solutions are obtained in [34] for the initial value problem of (1.1), while the pointwise behavior of smooth solutions is obtained too for gas flows with several thermal nonequilibrium modes in [35]. For the thermal non-equilibrium gas flow in three space dimensions, the pointwise estimates for the Green's functions linearized around a constant state is given in [36], and the global existence of the smooth irrotational flow was proved in [7].

It should be noted that all results mentioned above concerning with system (1.1) concentrate on the global well-posedness for small perturbations around constant equilibrium states. As one of most important physical models of hyperbolic conservation laws with relexation, it is important to understand the nonlinear wave propagations such as shock waves and rarefaction waves, the most important nonlinear waves in gas dynamics [3]. In this direction, the nonlinear asymptotic stability of shock profile, a traveling wave solution for (1.1), has been proved in [18]. In this paper we turn our attention to the asymptotic behavior of the solution to the Cauchy problem for the system (1.1) being a perturbation around a rarefaction wave. This topic has been one of the fundamental and important issues in fluid mechanics since rarefaction wave is one elementary wave of gas dynamics. For instance, the results on the nonlinear asymptotic stability of rarefaction waves for compressible Navier-Stokes equations for large time or small viscosity can be found in [4, 6, 8–12, 14, 19–21, 30, 31, 31, 38] and general viscous conservation laws ([16, 29, 33]). The related results for Boltzmann equations can be found in [15] and [31]. For some model problems of hyperbolic system with relaxation, the nonlinear asymptotic stability of rarefaction waves as either time goes to infinity or the relaxation parameter goes to zero has been studied in [5, 13, 17, 26, 32, 37]. It should be noted that, in most of these works, the systems discussed are 2×2 so that the equilibrium equation is scalar, except some for the Broadwell model of the discrete Boltzmann equation which is semilinear. Unlike (1.1), all these systems satisfy the Shizuta-Kawashima condition.

One may find discussions on the general structure of hyperbolic systems with relaxation in [2] including the dissipative structure and entropy.

In this paper, we are interested in the behavior of solutions to the Cauchy problem of (1.1) for the thermal non-equilibrium gas dynamics, which are perturbations of a rarefaction wave for the equilibrium system. It is well-known that there are two families of rarefaction waves for the Euler system (1.4), see [23]. For illustration, we only discuss the 1-rarefaction wave and assume $s_+ = s_- = \bar{s}$. Actually the case of 3-rarefaction wave can be dealt with similarly. A centered $1-$rarefaction wave of (1.4) connecting two constant states (v_-, u_-, s_-) and (v_+, u_+, s_+) with

$$v_+ > v_- > 0,\ s_+ = s_- = \bar{s},\ u_+ - u_- = -\int_{v_-}^{v_+} \lambda_1(z, \bar{s})dz,\ p_\pm = K v_\pm^{-\gamma} e^{(\gamma-1)\bar{s}}$$

is a self-similar solution in the form $(v^r,u^r,s^r)(x/t)$ to (1.4) given by [23],

$$\begin{cases} s^r(\frac{x}{t}) = s_+ = s_- = \bar{s}, \\ \lambda_1(v^r,\bar{s}) = \frac{x}{t}, \\ u^r(\frac{x}{t}) = u_\pm - \int_{v_\pm}^{v^r(\frac{x}{t})} \lambda_1(z,\bar{s})dz, \end{cases} \tag{1.7}$$

for $\lambda_1(v_-,\bar{s}) \le \frac{x}{t} \le \lambda_1(v_+,\bar{s})$ and

$$(u,v,s)(x,t) = \begin{cases} (u_-,v_-,s_-), \ \frac{x}{t} < \lambda_1(v_-), \\ (u_+,v_+,s_+), \ \frac{x}{t} > \lambda_1(v_+). \end{cases} \tag{1.8}$$

The corresponding pressure is given by

$$p^r(\frac{x}{t}) = K\left(v^r(\frac{x}{t})\right)^{-\gamma} e^{(\gamma-1)s}. \tag{1.9}$$

The centered rarefaction wave is only Lipschitz continuous but not smooth, a smooth rarefaction wave which is time-asymptotically equivalent is constructed as follows (cf. [20]): For two constants $w_+ > w_-$, let w^R be the globally defined smooth solution to the following Cauchy problem for Burgers' equation:

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = 0, & x \in \mathbb{R}^1, t > 0, \\ w(x,0) = w_0(x) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \cdot \kappa \int_0^{\epsilon x} (1+y^2)^{-2} dy, & x \in \mathbb{R}^1, \end{cases} \tag{1.10}$$

where $\epsilon > 0$ and κ is a constant such that $\kappa \int_0^\infty (1+y^2)^{-2}dy = 1$. As in [4,20], the smooth rarefaction wave $(u^R,p^R,s^R)(x,t)$ for (1.4) is obtained as follows:

$$\begin{cases} s^R(x,t) = s_+ = s_- = \bar{s}, \\ w^R(x,1+t) = \lambda_1(v^R,\bar{s}) = -\sqrt{K\gamma(v^R)^{-(\gamma+1)}} e^{\frac{(\gamma-1)\bar{s}}{2}}, \\ u^R(x,t) = u_\pm + \int_{v_\pm}^{v^R(x,t)} (-\lambda_1(z,\bar{s}))dz, \\ p^R(x,t) = K(v^R)^{-\gamma} e^{(\gamma-1)\bar{s}}, \\ w_\pm = \lambda_1(v_\pm,\bar{s}), \ \lambda_{1x}(v^R,\bar{s}) > 0. \end{cases} \tag{1.11}$$

As mentioned earlier, the dissipation of (1.1) is very weak. Also, unlike the traveling wave solution of a shock profile which solves the non-equilibrium system (1.1) exactly, the rarefaction wave is only a solution of the equilibrium system which does not solve (3.1), there is a non-integrable-in-time error. This prevents us from obtaining the bounds on the L^2 norm of $||(u-u^R,p-p^R,\chi,s-s^R)||$. Therefore, we turn to investigating the linearized stability of the rarefaction wave first. That is to consider the asymptotic behavior of solutions to the Cauchy problem for linearized system of (3.1):

$$\begin{cases} (p^R+\phi)_t + \beta\frac{p^R}{v^R}\psi_x + \beta\frac{p^R}{v^R}u_x^R + a_0\frac{u_x^R}{v^R}\phi - a_1\frac{p^R}{v^R}u_x^R\zeta - a_1\frac{u_x^R}{(v^R)^2}\chi = -\frac{2\chi}{\tau\alpha v^R}, \\ (u^R+\psi)_t + (p^R+\phi)_x = 0, \\ \chi_t + \frac{\alpha_f}{\alpha}(p^R u_x^R + p^R\psi_x + \phi u_x^R) = -\frac{\alpha+\alpha_f}{\alpha}\frac{\chi}{\tau}, \\ (s^R+\zeta)_t = 0, \end{cases} \tag{1.12}$$

where

$$(\phi,\psi,\chi,\zeta) = (p-p^R, u-u^R, \chi, s-s^R),$$

$$\beta = \frac{\alpha+2}{\alpha}, a_0 = \frac{2(\alpha+2)(\alpha+\alpha_f+1)}{\alpha(\alpha+\alpha_f+2)}, a_1 = \frac{2(\alpha+2)}{\alpha(\alpha+\alpha_f+2)}. \tag{1.13}$$

The basic idea for the linearization is to take u, p, χ and s as independent variables, and view v as a function of p, χ and s. Then an implicit function F was introduced to linearize v by implicit function theorem (Section 3). The details of this derivation of the linearization can be found in Section 3. Let

$$\delta = |u_+ - u_-| + |v_+ - v_-|.$$

We then consider the Cauchy problem of (1.12) with the initial data

$$(\phi,\psi,\chi,\zeta)(x,0) = (\phi_0,\psi_0,\chi_0,\zeta_0)(x),\ x\in\mathbb{R}. \tag{1.14}$$

Then we have the following theorem on the linearized stability of the rarefaction wave.

THEOREM 1.1. *Let (u^R,p^R,s^R) be the smooth 1-rarefaction wave of the equilibrium system (1.4) given by (1.11) with $\epsilon = \delta^{1/2}$. For the Cauchy problem of the linearized system (1.12) and (1.14) with the initial data satisfying $(\phi_0,\psi_0,\chi_0,\zeta_0)\in H^1(\mathbb{R}^1)$, it holds that, for all $t>0$,*

$$||(\phi,\psi,\chi)(\cdot,t)||_{L^\infty(\mathbb{R}^1)} \le C(1+t)^{-1/16}\left(\delta^{1/4} ln(2+t)+1\right),\ \|\zeta(\cdot,t)\|_{L^\infty(\mathbb{R}^1)} \le C, \tag{1.15}$$

for some constant C independent of t, provided the wave strength $\delta = |u_+ - u_-| + |v_+ - v_-|$ is suitably small.

Unlike the estimates for perturbation of rarefaction waves for other models, for instance, cf. [19, 29], where the L^2 of the perturbations are bounded in time. The interesting feature for the thermal non-equilibrium system (1.1) is that the L^2 norm of the perturbation (ϕ,ψ,χ) actually grows in time at the rate of $(ln(1+t))^{\frac{1}{2}}$ (Section 3.2). Whereas the L^2 norm of the first derivatives of the perturbation, (ϕ_x,ψ_x,χ_x), are found to decay at the rate bounded by $(1+t)^{-\frac{1}{8}}(ln(1+t))^{\frac{1}{2}}$ (Section 3.3). Hence by Sobolev's inequality, we find the decay of the perturbation in L^∞ is bounded by $(1+t)^{-1/16}(ln(1+t))^{1/2}$ on the linear level.

We now turn to the original nonlinear problem (1.1). We consider the Cauchy problem of (1.1) with the initial data

$$(v,u,p,q)(x,0) = (v_0,\ u_0,\ p_0,q_0)(x),\ x\in\mathbb{R}^1, \tag{1.16}$$

satisfying

$$\underline{v} \le v_0(x) \le \bar{v},\ \underline{p} \le p_0(x) \le \bar{p},\ \underline{q} \le q_0(x) \le \bar{q},\ x\in\mathbb{R}^1,$$

for some positive constants $\underline{v}$, $\bar{v}$, $\underline{p}$, $\bar{p}$, $\underline{q}$, $\bar{q}$.

The initial data of χ and s are given through (1.3), i. e.,

$$\chi(x,0) = Q(x,0) - q_0(x) = (\frac{\alpha_f}{2}p_0v_0 - q_0)(x) =: \chi_0(x),\ x\in\mathbb{R}^1,$$

$$s(x,0) = s_1(x,0) + s_2(x,0) =: s_0(x), x\in\mathbb{R}^1,$$

with

$$s_1(x,0) = \left(lnv_0 + \frac{\alpha}{2}ln(\frac{\alpha}{2}p_0v_0)\right)(x),\ s_2(x,0) = \frac{\alpha_f R}{2}lnq_0(x).$$

The second result in this paper is concerned with the life-span of smooth solutions in terms of the rarefaction wave strength δ for the nonlinear problem with the initial data being a small perturbation of a rarefaction wave:

THEOREM 1.2. *Suppose the constant states $(v_\pm,u_\pm,p_\pm,s_\pm)$ are connected by a centered 1-rarefaction wave (v^r,u^r,p^r,s^r) defined by (1.7) and (1.9). Let (v^R,u^R,p^R,s^R) be the smooth 1-rarefaction wave (v^R,u^R,p^R,s^R) given by (1.11) with $\epsilon=\delta^{1/2}$. Then there exist positive constants δ_0 and δ_1 such that, if*

$$||(v_0-v_0^R,p_0-p_0^R,u_0-u_0^R,\chi_0,s_0-\bar{s})||^2_{H^2(\mathbb{R}^1)}\le\delta_1^2,$$

and

$$\delta=:|v_+-v_-|+|u_+-u_-|<\delta_1,$$

then the initial value problem (1.1) and (1.16) admits a unique solution in the time interval $[0,\ \delta_0\delta^{-3/2}]$ satisfying

$$\begin{aligned}&(v-v^R,u-u^R,p-p^R,\chi,s-s^R)\in C^1((0,T_1);H^1(\mathbb{R}^1))\cap C((0,T_1);H^2(\mathbb{R}^1));\\&((u-u^R)_x,(p-p^R)_x)\in L^2((0,T_1);H^1(\mathbb{R}^1)),\chi\in L^2((0,T_1);H^2(\mathbb{R}^1))\end{aligned}$$

and the following estimate

$$\begin{aligned}&||(v-v^R,p-p^R,u-u^R,\chi,s-\bar{s})(\cdot,t)||^2_{H^2(\mathbb{R}^1)}\\&\le C||(v-v^R,p-p^R,u-u^R,\chi,s-\bar{s})(\cdot,0)||^2_{H^2(\mathbb{R}^1)}+C\delta\ln(1+\delta^{-3/2}),\ 0\le t\le T_1\end{aligned}\tag{1.17}$$

holds if $T_1\le\delta_0\delta^{-3/2}$ for some constant $C>0$ independent of t.

This theorem gives the life span of the solution for the original nonlinear problem in terms of the strength of the rarefaction wave, δ, in the of $O(\delta^{-3/2})$, for the initial data being small perturbations of a smooth rarefaction wave.

REMARK 1.1. When the strength of the rarefaction wave $\delta=0$, Theorem 1.2 shows that the life span of the solution becomes infinite, reducing to the case when the initial data are small perturbations of a constant equilibrium state as studied in [34]. However, for a non-trivial rarefaction wave strength $\delta>0$, it is not clear whether a global-in-time solution would be possible for small perturbations. It seems that this problem may not be handled by the L^2-method and the relative-entropy argument used in this paper. This is due to the following reason: the rarefaction wave of the equilibrium system (1.4) does not solve the non-equilibrium system (1.1), an error term of the form $p^Ru_x^R/v^R$ appears. On the linear level, this error may induce a growth of the L^2-norm of the perturbation in time at the rate of $(ln(1+t))^{\frac{1}{2}}$ and the slow decay of the L^2-norm for the first derivatives at the order of $(1+t)^{-\frac{1}{8}}(ln(1+t))^{\frac{1}{2}}$ due to the slow decay of the L^2-norm of u_x^R of the order of $(1+t)^{-1/2}$ whose square is not integrable in $[0,\infty)$ for time. This difficulty is overcome for other models, for instance, cf. [19,29], satisfying the classical Shizuta-Kawashima dissipation condition by using strong dissipations. As mentioned earlier, the non-equilibrium system (1.1) fails to satisfy the Shizuta-Kawashima dissipation condition so that the dissipation induced by the relaxation is too weak to control the time-growth of the L^2-norm of the perturbation. In order to investigate the global existence problem, it seems the more accurate pointwise estimates are needed by investigating the Green's function of the linearized system around a rarefaction wave.

However, this would be extremely challenging as few results are available in the analysis of Green functions for systems linearized around a rarefaction wave, for dissipative hyperbolic systems or viscous conservation laws.

The rest of this paper is organized as follows. In Section 2, we give a lemma on the properties of smooth rarefaction waves which can be found in [19,20]. In Section 3, we first give the details of the linearization of the original nonlinear problem around the smooth rarefaction wave and prove Theorem 1.1, the linearized stability. Section 4 is dedicated to certify Theorem 1.2 of the life span of the original nonlinear problem. We first use the entropy relative to the smooth rarefaction wave to obtain the basic energy. Second, in order to get the higher-order energy, we take the equivalent form (3.1) instead of (1.1). We then establish L^2-estimates for the first and second derivatives.

Throughout this paper, we use the following notations:

$$\mathbb{R}^1 := (-\infty,+\infty),\ \mathbb{R}^- := (-\infty,0),\ \mathbb{R}^+ := (0,+\infty),\ ||\cdot|| \equiv ||\cdot||_{L^2(\mathbb{R}^1)}\ ||\cdot||_{L^\infty} \equiv ||\cdot||_{L^\infty(\mathbb{R}^1)}.$$

We also use $\int$ to denote $\int_{-\infty}^{\infty}$ unless otherwise stated. Moreover, C will be used as a generic constant independent of time t.

2. A lemma on the smooth rarefaction wave

Lemma 2.1. *The smooth solution (v^R,u^R,p^R,s^R) constructed in (1.11) for $\epsilon=\delta^{1/2}$ has the following properties:*

(1) $v_t^R = u_x^R > 0$ for each $(x,t)\in\mathbb{R}^1\times\mathbb{R}_+$;

(2) For any p with $1\le p\le +\infty$, there exists a constant $C_p>0$ depending only on p such that

$$||(v_x^R,u_x^R,p_x^R)||_{L^p(\mathbb{R}^1)} \le C_p min\{\delta^{3/2-1/2p},\delta^{1/p}(1+t)^{-1+1/p}\},$$
$$||(v_{xx}^R,u_{xx}^R,p_{xx}^R)||_{L^p(\mathbb{R}^1)} \le C_p min\{\delta^{2-1/2p},\delta^{1/8-1/8p}(1+t)^{-5/4+1/4p}\};$$

(3) There exists a positive constant C such that

$$||(v_{xx}^R,u_{xx}^R,p_{xx}^R)||_{L^\infty(\mathbb{R}^1)} \le C\delta^{1/2}||(v_x^R,u_x^R,p_x^R)||_{L^\infty(\mathbb{R}^1)},$$
$$||(v_{xxx}^R,u_{xxx}^R,p_{xxx}^R)||_{L^\infty(\mathbb{R}^1)} \le C\delta||(v_x^R,u_x^R,p_x^R)||_{L^\infty(\mathbb{R}^1)},$$

and

$$|(v_t^R,u_t^R,p_t^R)| \le C|(v_x^R,u_x^R,p_x^R)|.$$

(4) There exists a positive constant C such that

$$||(v^R-v^r,u^R-u^r,p^R-p^r)||_{L^\infty(\mathbb{R}^1)} \le C min\{\delta,\delta^{1/4}(1+t)^{-1/3}(ln(2+t)+|ln\delta|)\}.$$

The proof of this lemma can be found in [20] (see also [4]).

3. Proof of Theorem 1.1

3.1. Linearization. In this subsection, we derive the linearized system. By (1.2) and (1.3), we can see that the original system (1.1) is equivalent to

$$\begin{cases} p_t+\beta\frac{p}{v}u_x = -\frac{2\chi}{\tau\alpha v}, \\ u_t+p_x=0, \\ \chi_t+\frac{\alpha_f}{\alpha}pu_x = -\frac{\alpha+\alpha_f}{\alpha}\frac{\chi}{\tau}, \\ s_t=\frac{\chi^2}{\tau p v q}=\frac{2\chi^2}{\alpha_f\tau T_1T_2}. \end{cases} \tag{3.1}$$

We may take p,u,χ,s as basic unknowns. Also, we observe that

$$\begin{cases} p_t^R+(1+\frac{2}{\alpha+\alpha_f})\frac{p^R}{v^R}u_x^R=0,\\ u_t^R+p_x^R=0,\\ \chi^R=0,\\ s^R=\bar{s}. \end{cases} \tag{3.2}$$

Set

$$(\phi,\psi,\chi,\zeta)=(p-p^R,u-u^R,\chi,s-s^R). \tag{3.3}$$

Now, we linearize the system (3.1) around $(p^R,u^R,0,s^R=\bar{s})$. We first rewrite the first equation in (3.1) as

$$(p^R+\phi)_t+\frac{(\alpha+2)}{\alpha}\frac{(p^R+\phi)}{v}(u^R+\psi)_x=-\frac{2\chi}{\tau\alpha v}.$$

For the term

$$f(p,v,u_x):=\frac{p^R+\phi}{v}(u^R+\psi)_x,$$

using the Taylor's formula to obtain

$$f(p,v,u_x)=\frac{p^R}{v^R}u_x^R+\frac{1}{v^R}u_x^R\phi+\frac{p^R}{v^R}\psi_x-\frac{p^R}{(v^R)^2}u_x^R(v-v^R)+o(\phi^2,\psi_x^2,(v-v^R)^2),$$

where $o(\phi^2,\psi_x^2,(v-v^R)^2)$ is the higher order of $(\phi,\psi_x,v-v^R)$.

If one regards v as an implicit function of other three variables (p,χ,s), the implicit function theorem can be considered to estimate the term $v-v^R$. For this purpose, given F by

$$\begin{aligned} F:=&F(v,p,\chi,s)\\ =&s-(\frac{\alpha}{2}ln(\frac{\alpha}{2}pv)+lnv+\frac{\alpha_f}{2}ln(\frac{\alpha_f}{2}pv-\chi))\\ =&0, \end{aligned} \tag{3.4}$$

then we have

$$\begin{cases} F_s=1,\\ F_v=-\frac{\alpha+2}{2}\frac{1}{v}-\frac{(\frac{\alpha_f}{2})^2p}{\frac{\alpha_f}{2}pv-\chi}\neq 0, \quad \text{for small } \chi,\\ F_p=-\frac{\alpha}{2p}-\frac{(\frac{\alpha_f}{2})^2v}{\frac{\alpha_f}{2}pv-\chi},\\ F_\chi=\frac{\alpha_f}{2}\frac{1}{\frac{\alpha_f}{2}pv-\chi},\\ v_s=-\frac{F_s}{F_v},\\ v_p=-\frac{F_p}{F_v},\\ v_\chi=-\frac{F_\chi}{F_v}. \end{cases} \tag{3.5}$$

Hence

$$v-v^R=-\frac{(\alpha+\alpha_f)}{(\alpha+\alpha_f+2)}\frac{v^R}{p^R}\phi+\frac{2}{\alpha+\alpha_f+2}v^R\zeta+\frac{2}{(\alpha+\alpha_f+2)}\frac{\chi}{p^R}+o(\phi^2,\chi^2,\zeta^2)$$

by the Taylor's formula with higher order $o(\phi^2,\chi^2,\zeta^2)$, and finally

$$f(p,v,u_x)=\frac{p^R}{v^R}u_x^R+\frac{1}{v^R}u_x^R\phi+\frac{p^R}{v^R}\psi_x -\frac{p^R}{(v^R)^2}u_x^R\left(-\frac{\alpha+\alpha_f}{(\alpha+\alpha_f+2)}\frac{v^R}{p^R}\phi+\frac{2}{\alpha+\alpha_f+2}v^R\zeta+\frac{2}{(\alpha+\alpha_f+2)}\frac{\chi}{p^R}\right) +o(\phi^2,\psi_x^2,\chi^2,\zeta^2).$$

We can also obtain

$$\frac{2\chi}{\tau\alpha v}=\frac{2\chi}{\tau\alpha v^R}+o(\phi^2,\chi^2,\zeta^2).$$

Combining above expansions, the linearization of the first equation of (3.1) has the form

$$(p^R+\phi)_t+\frac{\alpha+2}{\alpha}\frac{p^R}{v^R}\psi_x+\frac{\alpha+2}{\alpha}\frac{p^R}{v^R}u_x^R+\frac{2(\alpha+2)(\alpha+\alpha_f+1)}{\alpha(\alpha+\alpha_f+2)}\frac{u_x^R}{v^R}\phi -\frac{2(\alpha+2)}{\alpha(\alpha+\alpha_f+2)}\frac{p^R}{v^R}u_x^R\zeta-\frac{2(\alpha+2)}{\alpha(\alpha+\alpha_f+2)}\frac{u_x^R}{(v^R)^2}\chi=-\frac{2\chi}{\tau\alpha v^R}$$

if we drop the higher-order term. Similarly, for the second equation of (3.1), it is easy to see

$$(u^R+\psi)_t+(p^R+\phi)_x=0.$$

and for the third equation of (3.1), we have

$$\chi_t+\frac{\alpha_f}{\alpha}(p^Ru_x^R+p^R\psi_x+\phi u_x^R)=-\frac{\alpha+\alpha_f}{\alpha}\frac{\chi}{\tau}.$$

Since $(s^R+\zeta)_t=\frac{\chi^2}{\tau p v q}$, the last equation of (3.1) has the following linearized equation

$$(s^R+\zeta)_t=0.$$

Due to the above linearization process, it is easy to get the linearized system (1.12). By means of (3.2) and (1.12), the linearized system takes the form

$$\begin{cases}\phi_t+\beta\frac{p^R}{v^R}\psi_x+\frac{2\alpha_f}{\alpha(\alpha+\alpha_f)}\frac{p^R}{v^R}u_x^R+a_0\frac{u_x^R}{v^R}\phi-a_1\frac{p^R}{v^R}u_x^R\zeta-a_1\frac{u_x^R}{(v^R)^2}\chi=-\frac{2\chi}{\tau\alpha v^R},\\ \psi_t+\phi_x=0,\\ \chi_t+\frac{\alpha_f}{\alpha}(p^Ru_x^R+p^R\psi_x+\phi u_x^R)=-\frac{\alpha+\alpha_f}{\alpha}\frac{\chi}{\tau},\\ \zeta_t=0,\end{cases}\tag{3.6}$$

where β,a_0,a_1 are given by (1.13).

Here we first give some estimates for ζ and ζ_x in L^2 norm by a simple observation which will be used in the following sections.

Lemma 3.1. *Suppose that the assumption in Theorem 1.1 holds. Then there exists a constant $C>0$ independent of time such that*

$$||(\zeta,\zeta_x)||^2(t)+\int_0^t\int u_x^R(\zeta^2,\zeta_x^2)(x,t')dxdt'\le C,\tag{3.7}$$

$$||\zeta||_{L^\infty(\mathbb{R}^1)}\le C.$$

Proof. Since $v_t^R=u_x^R$, the desired results can be obtained by a simple calculation. □

3.2. The estimate for $\|(\phi,\psi,\chi)\|_{L^2(\mathbb{R}^1)}$.

LEMMA 3.2. *Under the assumptions in Theorem 1.1, there exists a constant $C>0$ independent of time such that, for $t>0$,*

$$||(\phi,\psi,\chi)||^2(t)+\int_0^t\int(u_x^R\phi^2+\chi^2)(x,t')dxdt'\leq C\delta ln(1+t)+C.$$

Proof. For convenience, we rewrite (3.6) by considering a combination of the first and the third equations in (3.6) to obtain

$$\begin{cases}\frac{v^R}{p^R}\phi_t=-\beta\psi_x-\frac{2\alpha_f}{\alpha(\alpha+\alpha_f)}u_x^R-a_0\frac{u_x^R}{p^R}\phi+a_1u_x^R\zeta+a_1\frac{u_x^R}{p^Rv^R}\chi-\frac{2\chi}{\tau\alpha p^R},\\ \beta\psi_t=-\beta\phi_x,\\ (\frac{v^R}{p^R}\phi-\frac{\alpha+2}{\alpha_fp^R}\chi)_t=a_2\frac{u_x^R}{p^R}\phi+\gamma u_x^R+a_1u_x^R\zeta+a_4\frac{u_x^R}{p^Rv^R}\chi+b_0\frac{\chi}{\tau p^R},\end{cases}\tag{3.8}$$

where we define

$$a_2=\frac{2(\alpha+\alpha_f+1)}{\alpha+\alpha_f}-\frac{2(\alpha+2)(\alpha+\alpha_f+1)}{\alpha(\alpha+\alpha_f+2)}+\frac{\alpha+2}{\alpha},\quad a_3=\frac{\alpha+\alpha_f+1}{\alpha+\alpha_f},$$
$$a_4=\frac{2(\alpha+2)}{\alpha(\alpha+\alpha_f+2)}-\frac{(\alpha+2)(\alpha+\alpha_f+2)}{\alpha_f(\alpha+\alpha_f)},\quad b_0=\frac{\alpha+\alpha_f+2}{\alpha_f}.$$

Multiplying the equations in (3.8) by ϕ, ψ and $a_5(\phi-\frac{\alpha+2}{\alpha_fv^R}\chi)$, respectively, where

$$a_5=\frac{2\alpha_f}{\alpha(\alpha+\alpha_f+2)}>0.$$

then one can obtain

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{v^R}{p^R}\phi^2\right)+\frac{1}{2}\frac{\partial}{\partial t}\beta\psi^2+\frac{a_5}{2}\frac{\partial}{\partial t}\left\{\frac{p^R}{v^R}(\frac{v^R}{p^R}\phi-\frac{\alpha+2}{\alpha_fp^R}\chi)^2\right\}+a_6\frac{u_x^R}{p^R}\phi^2+\frac{2(\alpha+2)}{\alpha\alpha_f}\frac{\chi^2}{\tau p^Rv^R}\\ &=-\beta(\phi\psi)_x+a_7u_x^R\phi\zeta+a_8\frac{u_x^R}{p^Rv^R}\phi\chi-\gamma a_5\frac{(\alpha+2)}{\alpha_fv^R}u_x^R\chi\\ &\quad-a_1a_5\frac{(\alpha+2)}{\alpha_fv^R}u_x^R\zeta\chi+a_9\frac{u_x^R}{p^R(v^R)^2}\chi^2,\end{aligned}\tag{3.9}$$

where

$$\begin{aligned}a_6=&-a_3+a_0+a_3a_5-a_2a_5\\ =&-\frac{\alpha+\alpha_f+1}{\alpha+\alpha_f}+\frac{2(\alpha+2)(\alpha+\alpha_f+1)}{\alpha(\alpha+\alpha_f+2)}-\frac{2\alpha_f(\alpha+\alpha_f+1)}{\alpha(\alpha+\alpha_f)(\alpha+\alpha_f+2)}\\ &+\frac{2^2\alpha_f(\alpha+2)(\alpha+\alpha_f+1)}{\alpha^2(\alpha+\alpha_f+2)^2}-\frac{2\alpha_f(\alpha+2)}{\alpha^2(\alpha+\alpha_f+2)}\\ =&\frac{(\alpha+2)(\alpha^2+2\alpha_f^2+3\alpha\alpha_f+\alpha)}{\alpha^2(\alpha+\alpha_f+2)}>0,\\ a_7=&a_1+a_1a_5=\frac{2(\alpha+2)^2(\alpha+\alpha_f)}{\alpha^2(\alpha+\alpha_f+2)^2},\\ a_8=&a_1+a_4a_5-a_2a_5\frac{\alpha+2}{\alpha_f}+2a_3a_5\frac{\alpha+2}{\alpha_f}\\ =&\frac{2(\alpha+2)}{\alpha(\alpha+\alpha_f+2)}+\frac{2^2\alpha_f(\alpha+2)}{\alpha^2(\alpha+\alpha_f+2)^2}-\frac{2(\alpha+2)}{\alpha(\alpha+\alpha_f)}+\frac{2(\alpha+2)^2(\alpha+\alpha_f+1)}{\alpha^2(\alpha+\alpha_f+2)^2}\end{aligned}$$

$$-\frac{2^2(\alpha+2)(\alpha+\alpha_f+1)}{\alpha(\alpha+\alpha_f)(\alpha+\alpha_f+2)}+\frac{2^2(\alpha+2)(\alpha+\alpha_f+1)}{\alpha(\alpha+\alpha_f)(\alpha+\alpha_f+2)}-\frac{2(\alpha+2)^2}{\alpha^2(\alpha+\alpha_f+2)},$$

$$\begin{aligned}a_9=&-a_4a_5\frac{(\alpha+2)}{\alpha_f}-a_3a_5\left(\frac{\alpha+2}{\alpha_f}\right)^2\\=&\frac{2(\alpha+2)^2}{\alpha\alpha_f(\alpha+\alpha_f)}-\frac{2^2(\alpha+2)^2}{\alpha^2(\alpha+\alpha_f+2)^2}-\frac{2(\alpha+2)^2(\alpha+\alpha_f+1)}{\alpha\alpha_f(\alpha+\alpha_f)(\alpha+\alpha_f+2)}.\end{aligned}$$

Thanks to the positivity of a_6, we integrate (3.9) over $\mathbb{R}^1\times[0,t]$, which yields the following inequality

$$\begin{aligned}&||(\phi,\psi,\chi)||^2(t)+\int_0^t\int\left(u_x^R\phi^2+\chi^2\right)(x,t')dxdt'\\\le& C\int_0^t\int\left(|u_x^R\phi\zeta|+|u_x^R\phi\chi|+|u_x^R\zeta\chi|+|u_x^R\chi^2|+|u_x^R\chi|\right)(x,t')dxdt'+||(\phi,\psi,\chi)||^2(0).\end{aligned}\tag{3.10}$$

Now we deal with the terms on the right hand of (3.10). Due to (3.1), Young's inequality implies that

$$\begin{aligned}C\int_0^t\int|u_x^R\phi\zeta|(x,t')dxdt'\le&\frac{1}{8}\int_0^t\int u_x^R\phi^2(x,t')dxdt'+C\int_0^t\int u_x^R\zeta^2(x,t')dxdt'\\\le&\frac{1}{8}\int_0^t\int u_x^R\phi^2(x,t')dxdt'+C.\end{aligned}\tag{3.11}$$

Similarly, one can get

$$C\int_0^t\int|u_x^R\phi\chi|(x,t')dxdt'\le C\delta\int_0^t\int u_x^R\phi^2(x,t')dxdt'+\frac{1}{8}\int_0^t\int\chi^2(x,t')dxdt',\tag{3.12}$$

$$\begin{aligned}C\int_0^t\int|u_x^R\zeta\chi|(x,t')dxdt'\le& C\int_0^t\int u_x^R\zeta^2(x,t')dxdt'+C\delta\int_0^t\int\chi^2(x,t')dxdt'\\\le& C\delta\int_0^t\int\chi^2(x,t')dxdt'+C,\end{aligned}\tag{3.13}$$

$$C\int_0^t\int|u_x^R\chi^2|(x,t')dxdt'\le C\delta\int_0^t\int\chi^2(x,t')dxdt',\tag{3.14}$$

$$\begin{aligned}C\int_0^t\int|u_x^R\chi|(x,t')dxdt'\le& C\int_0^t\int|u_x^R|^2(x,t')dxdt'+\frac{1}{8}\int_0^t\int\chi^2(x,t')dxdt'\\\le& C\delta ln(1+t)+\frac{1}{8}\int_0^t\int\chi^2(x,t')dxdt'\end{aligned}\tag{3.15}$$

by Young's inequality, (3.7) and Lemma 2.1. Therefore, based on (3.10)-(3.15), we have the desired estimate

$$||(\phi,\psi,\chi)||^2(t)+\int_0^t\int\left(u_x^R\phi^2+\chi^2\right)(x,t')dxdt'\le C\delta ln(1+t)+C\tag{3.16}$$

if δ is small enough, which finishes the proof. □

3.3. The estimate for $||(\phi_x,\psi_x,\chi_x)||_{L^2(\mathbb{R}^1)}$. To control the upper bound for $||(\phi,\psi,\chi)||_{L^\infty(\mathbb{R}^1)}$, we shall need to obtain the decay rate for $||(\phi_x,\psi_x,\chi_x)||_{L^2(\mathbb{R}^1)}$. First, we show it is indeed bounded.

LEMMA 3.3. *Assume that the conditions in Theorem 1.1 hold. Then there exists a constant $C>0$ independent of time such that*

$$||(\phi_x,\psi_x,\chi_x)||^2(t)+\int_0^t\int u_x^R(\phi_x^2,\psi_x^2)(x,t')+\chi_x^2(x,t')dxdt'\le C.$$

Proof. First, we take a derivative with respect to x and replace the third equation in (3.6) by a combination of the first and the third ones, then we have

$$\begin{aligned}
\phi_{xt}=&-\beta\left(\frac{p^R}{v^R}\psi_x\right)_x-\frac{2\alpha_f}{\alpha(\alpha+\alpha_f)}\left(\frac{p^R}{v^R}u_x^R\right)_x-a_0\left(\frac{u_x^R}{v^R}\phi\right)_x+a_1\left(\frac{p^R}{v^R}u_x^R\zeta+\frac{u_x^R}{(v^R)^2}\chi\right)_x\\
&-\frac{2}{\tau\alpha}(\frac{\chi}{v^R})_x,\\
\psi_{xt}=&-\phi_{xx},\\
(v_x^R\phi+&v^R\phi_x-a_{10}\chi_x)_t=\gamma(p^Ru_x^R)_x-a_5(u_x^R\phi)_x+a_1(p^Ru_x^R\zeta+\frac{u_x^R}{v^R}\chi)_x+\frac{b_0}{\tau}\chi_x,
\end{aligned}\tag{3.17}$$

where

$$a_{10}=\tfrac{\alpha+2}{\alpha_f}.$$

Testing the equations of (3.17) by $\frac{1}{a_5}\phi_x$, $a_{11}\psi_x$ and $\frac{1}{(v^R)^2}(v_x^R\phi+v^R\phi_x-a_{10}\chi_x)$, respectively, where

$$a_{11}=\tfrac{(\alpha+2)(\alpha+\alpha_f+2)}{2\alpha_f},$$

a straightforward calculation gives

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{a_5}\phi_x^2\right)+\frac{1}{2}\frac{\partial}{\partial t}\left(a_{11}\frac{p^R}{v^R}\psi_x^2\right)+\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{(v^R)^2}(v_x^R\phi+v^R\phi_x-a_{10}\chi_x)^2\right)\\
&\quad+(a_5+a_{12})\frac{u_x^R}{v^R}\phi_x^2+a_{13}\frac{p^Ru_x^R}{v^R}\psi_x^2+\frac{u_x^R}{(v^R)^3}(v_x^R\phi+v^R\phi_x-a_{10}\chi_x)^2\\
&\quad+a_{14}\frac{u_x^R}{(v^R)^3}\chi_x^2+\frac{a_{10}b_0}{\tau(v^R)^2}\chi_x^2\\
&=-a_{11}\left(\frac{p^R}{v^R}\phi_x\psi_x\right)_x+\gamma\frac{p^Ru_x^Rv_x^R\phi_x}{(v^R)^2}-(a_5+a_{12})\frac{u_{xx}^R}{v^R}\phi\phi_x+a_{12}\frac{u_x^Rv_x^R}{(v^R)^2}\phi\phi_x\\
&\quad+(a_1+a_{10})\left(\frac{u_{xx}^R\chi\phi_x}{(v^R)^2}+\frac{u_x^R\chi_x\phi_x}{(v^R)^2}-\frac{u_x^Rv_x^R\chi\phi_x}{(v^R)^3}\right)-a_{10}\frac{u_x^Rv_x^R\phi_x\chi}{(v^R)^3}\\
&\quad+\frac{a_1+a_{10}}{v^R}(p_x^Ru_x^R\zeta\phi_x+p^Ru_{xx}^R\zeta\phi_x+p^Ru_x^R\zeta_x\phi_x)-\frac{a_{10}}{(v^R)^2}p^Ru_x^Rv_x^R\zeta\phi_x\\
&\quad-\frac{b_0}{\tau}\left(\left(\left(\frac{1}{v^R}\right)_x\phi\chi\right)_x+\frac{v_{xx}^R-2(v_x^R)^2}{(v^R)^2}\phi\chi\right)+\gamma\frac{p_x^Ru_x^Rv_x^R\phi+p^Ru_{xx}^Rv_x^R\phi}{(v^R)^2}\\
&\quad-a_5\frac{u_{xx}^Rv_x^R\phi^2+u_x^Rv_x^R\phi\phi_x}{(v^R)^2}+a_1\frac{p_x^Ru_x^Rv_x^R\phi\zeta+p^Ru_{xx}^Rv_x^R\phi\zeta+p^Ru_x^Rv_x^R\phi\zeta_x}{(v^R)^2}\\
&\quad+a_1\left(\frac{u_{xx}^Rv_x^R\phi\chi+u_x^Rv_x^R\phi\chi_x}{(v^R)^3}-\frac{u_x^R(v_x^R)^2\phi\chi}{(v^R)^4}\right)+a_5a_{10}\frac{u_{xx}^R\phi\chi_x+u_x^R\phi_x\chi_x}{(v^R)^2}
\end{aligned}$$

$$-\frac{\gamma a_{10}}{(v^R)^2}(p_x^R u_x^R \chi_x + p^R u_{xx}^R \chi_x) - \frac{a_{14}}{(v^R)^2}(p_x^R u_x^R \chi_x \zeta + p^R u_{xx}^R \chi_x \zeta + p^R u_x^R \chi_x \zeta_x)$$
$$-a_{14}\left(\frac{u_{xx}^R \chi_x \chi}{(v^R)^3} - \frac{u_x^R v_x^R \chi_x \chi}{(v^R)^4}\right), \tag{3.18}$$

where

$$a_{12} = \frac{(\alpha+2)(\alpha+\alpha_f+1)}{\alpha_f},$$
$$a_{13} = a_3 a_{11} = \frac{(\alpha+2)(\alpha+\alpha_f+1)(\alpha+\alpha_f+2)}{2\alpha_f(\alpha+\alpha_f)},$$
$$a_{14} = a_1 a_{10} = a_5 a_{10}^2 = \frac{2(\alpha+2)^2}{\alpha\alpha_f(\alpha+\alpha_f+2)}.$$

Second, we integrate above equality (3.18) over $\mathbb{R}^1$ and find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \frac{1}{a_5}\phi_x^2 dx + \frac{1}{2}\frac{d}{dt}\int a_{11}\frac{p^R}{v^R}\psi_x^2 dx + \frac{1}{2}\frac{d}{dt}\int \frac{1}{(v^R)^2}\left(v_x^R\phi + v^R\phi_x - a_{10}\chi_x\right)^2 dx\\
&+\int (a_5+a_{12})\frac{u_x^R}{v^R}\phi_x^2 dx + \int a_{13}\frac{p^R u_x^R}{v^R}\psi_x^2 dx + \int \frac{u_x^R}{(v^R)^3}\left(v_x^R\phi + v^R\phi_x - a_{10}\chi_x\right)^2 dx\\
&+\int a_{14}\frac{u_x^R}{(v^R)^3}\chi_x^2 dx + \int \frac{a_{10}b_0}{\tau(v^R)^2}\chi_x^2 dx\\
\le & C\int (|u_x^R v_x^R \phi_x| + |u_{xx}^R \phi\phi_x| + |u_x^R v_x^R \phi\phi_x| + |p_x^R u_x^R \phi_x \zeta| + |u_{xx}^R \phi_x \zeta| + |u_x^R \phi_x \zeta_x|\\
&+|u_x^R v_x^R \phi_x \zeta| + |u_{xx}^R \phi_x \chi| + |u_x^R \phi_x \chi_x| + |u_x^R v_x^R \phi_x \chi| + |u_{xx}^R \phi \chi_x| + |v_{xx}^R \phi \chi| + |(v_x^R)^2 \phi \chi|\\
&+|p_x^R u_x^R v_x^R \phi| + |u_{xx}^R v_x^R \phi| + |u_{xx}^R v_x^R \phi^2| + |p_x^R u_x^R v_x^R \zeta\phi| + |u_{xx}^R v_x^R \phi\zeta| + |u_x^R v_x^R \phi\zeta_x|\\
&+|u_{xx}^R v_x^R \phi\chi| + |u_x^R v_x^R \phi\chi_x| + |u_x^R (v_x^R)^2 \phi\chi| + |p_x^R u_x^R \chi_x| + |u_{xx}^R \chi_x|\\
&+|p_x^R u_x^R \chi_x \zeta| + |u_{xx}^R \chi_x \zeta| + |u_x^R \chi_x \zeta_x| + |u_{xx}^R \chi_x \chi| + |u_x^R v_x^R \chi_x \chi|)dx.
\end{aligned} \tag{3.19}$$

As for the terms on the right hand of (3.19) invoking u_{xx}^R or v_{xx}^R, we invoke Lemma 2.1, which in conjunction with Young's inequality implies that

$$\begin{aligned}
&C\int |u_{xx}^R \phi\phi_x| dx \le C\delta^{1/8}(1+t)^{-5/4}\int \phi^2 dx + C\delta^{1/2}\int u_x^R \phi_x^2 dx\\
\le & C\delta^{1/2}\int u_x^R \phi_x^2 dx + C\delta^{9/8}(1+t)^{-5/4} ln(1+t) + C\delta^{1/8}(1+t)^{-5/4}.
\end{aligned} \tag{3.20}$$

The same idea can be used for the estimates of $C\int |u_{xx}^R \phi \chi_x| dx$ and $C\int |u_{xx}^R \chi_x \chi| dx$, and

$$\begin{aligned}
C\int |u_{xx}^R \phi_x \zeta| dx \le & C\delta^{1/8}(1+t)^{-5/4}\int \zeta^2 dx + C\delta^{1/2}\int u_x^R \phi_x^2 dx\\
\le & C\delta^{1/2}\int u_x^R \phi_x^2 dx + C\delta^{1/8}(1+t)^{-5/4}.
\end{aligned} \tag{3.21}$$

Similarly,

$$\begin{aligned}
&C\int |u_{xx}^R \phi_x \chi| dx \le C\delta^{1/2}\int u_x^R \phi_x^2 dx + C\delta^{1/8}(1+t)^{-5/4}\int \chi^2 dx\\
\le & C\delta^{1/2}\int u_x^R \phi_x^2 dx + C\delta^{9/8}(1+t)^{-5/4} ln(1+t) + C\delta^{1/8}(1+t)^{-5/4},
\end{aligned} \tag{3.22}$$

$$\begin{aligned}C\int|v_{xx}^R\phi\chi|dx\leq& C\delta^{1/8}(1+t)^{-5/4}\int(\phi^2+\chi^2)dx\\ &\leq C\delta^{9/8}(1+t)^{-5/4}ln(1+t)+C\delta^{1/8}(1+t)^{-5/4},\end{aligned}\tag{3.23}$$

$$\begin{aligned}&C\int|u_{xx}^Rv_x^R\phi|dx\leq C\delta^{1/8}(1+t)^{-5/4}\int\phi^2dx+C\int|u_{xx}^R(v_x^R)^2|dx\\ &\leq C\delta^{9/8}(1+t)^{-5/4}ln(1+t)+C\delta^{1/8}(1+t)^{-5/4}+C\delta^{9/8}(1+t)^{-9/4},\end{aligned}\tag{3.24}$$

$$\begin{aligned}C\int|u_{xx}^Rv_x^R\phi^2|dx\leq& C\delta^{1/8}(1+t)^{-5/4}\int|v_x^R\phi^2|dx\\ &\leq C\delta^{9/8}(1+t)^{-9/4}ln(1+t)+C\delta^{1/8}(1+t)^{-5/4},\end{aligned}\tag{3.25}$$

$$\begin{aligned}C\int|u_{xx}^Rv_x^R\phi\zeta|dx\leq& C\int|u_{xx}^Rv_x^R\phi^2|dx+C\int|u_{xx}^Rv_x^R\zeta^2|dx\\ &\leq C\delta^{9/8}(1+t)^{-9/4}ln(1+t)+C\delta^{1/8}(1+t)^{-9/4},\end{aligned}\tag{3.26}$$

$$\begin{aligned}C\int|u_{xx}^Rv_x^R\phi\chi|dx\leq& C\int|u_{xx}^Rv_x^R\phi^2|dx+C\int|u_{xx}^Rv_x^R\chi^2|dx\\ &\leq C\delta^{9/8}(1+t)^{-9/4}ln(1+t)+C\delta^{1/8}(1+t)^{-9/4},\end{aligned}\tag{3.27}$$

$$\begin{aligned}C\int|u_{xx}^R\chi_x|dx\leq& C\int|u_{xx}^R|^2dx+\int\frac{a_{10}b_0}{8\tau(v^R)^2}\chi_x^2dx\\ &\leq\int\frac{a_{10}b_0}{8\tau(v^R)^2}\chi_x^2dx+C\delta^{1/8}(1+t)^{-9/4},\end{aligned}\tag{3.28}$$

$$\begin{aligned}C\int|u_{xx}^R\chi_x\zeta|dx\leq& C\int|u_{xx}^R\zeta^2|dx+C\int|u_{xx}^R\chi_x^2|dx\\ &\leq C\delta^{1/2}\int u_x^R\chi_x^2dx+C\delta^{1/8}(1+t)^{-5/4}.\end{aligned}\tag{3.29}$$

Next, we deal with the rest terms on the right side of (3.19), which include p_x^R, u_x^R and v_x^R. We begin with

$$\begin{aligned}C\int|u_x^Rv_x^R\phi_x|dx\leq& C\int|u_x^R(v_x^R)^2|dx+\frac{a_5}{2}\int\frac{u_x^R}{v^R}\phi_x^2dx\\ &\leq\frac{a_5}{2}\int\frac{u_x^R}{v^R}\phi_x^2dx+C\delta(1+t)^{-2},\end{aligned}\tag{3.30}$$

and

$$\begin{aligned}C\int|u_x^Rv_x^R\phi\phi_x|dx\leq& C\int|u_x^Rv_x^R\phi^2|dx+C\int|u_x^Rv_x^R\phi_x^2|dx\\ &\leq C(1+t)^{-2}\int\phi^2dx+C\delta\int u_x^R\phi_x^2dx\end{aligned}$$

$$\le C\delta\int u_x^R\phi_x^2dx+C\delta(1+t)^{-2}ln(1+t)+C(1+t)^{-2} \tag{3.31}$$

due to Lemma 2.1 and Young's inequality. Moreover, $\int|u_x^Rv_x^R\phi_x\chi|dx$, $\int|u_x^Rv_x^R\phi\chi_x|dx$ and $\int|u_x^Rv_x^R\chi\chi_x|dx$ can be estimated like (3.31). Furthermore,

$$\begin{aligned}C\int|p_x^Ru_x^R\phi_x\zeta|dx&\le C\int|p_x^Ru_x^R\phi_x^2|dx+C\int|p_x^Ru_x^R\zeta^2|dx\\&\le C\delta\int u_x^R\phi_x^2dx+C(1+t)^{-2}\end{aligned} \tag{3.32}$$

and the same bound for $\int|u_x^Rv_x^R\phi_x\zeta|dx$ and $\int|p_x^Ru_x^R\chi_x\zeta|dx$ can be also obtained. As for other terms,

$$\begin{aligned}C\int|u_x^R\phi_x\chi_x|dx&\le C\int|u_x^R\phi_x|^2dx+\int\frac{a_{10}b_0}{8\tau(v^R)^2}\chi_x^2dx\\&\le C\delta\int u_x^R\phi_x^2dx+\int\frac{a_{10}b_0}{8\tau(v^R)^2}\chi_x^2dx,\end{aligned} \tag{3.33}$$

$$\begin{aligned}C\int|(v_x^R)^2\phi\chi|dx&\le C\int|v_x^R|^2(\phi^2+\chi^2)dx\\&\le C\delta(1+t)^{-2}ln(1+t)+C(1+t)^{-2},\end{aligned} \tag{3.34}$$

$$\begin{aligned}C\int|p_x^Ru_x^Rv_x^R\phi|dx&\le C\int|u_x^R\phi|^2dx+C\int|p_x^Rv_x^R|^2dx\\&\le C\delta(1+t)^{-2}ln(1+t)+C(1+t)^{-2}+C(1+t)^{-2}\int|v_x^R|^2dx\\&\le C\delta(1+t)^{-2}ln(1+t)+C(1+t)^{-2}+C\delta(1+t)^{-3},\end{aligned} \tag{3.35}$$

$$\begin{aligned}C\int|p_x^Ru_x^Rv_x^R\phi\zeta|dx&\le C\int|u_x^R\phi|^2dx+C\int|p_x^Rv_x^R\zeta|^2dx\\&\le C\delta(1+t)^{-2}ln(1+t)+C(1+t)^{-2}+C(1+t)^{-4}\int\zeta^2dx\\&\le C\delta(1+t)^{-2}ln(1+t)+C(1+t)^{-2},\end{aligned} \tag{3.36}$$

$$\begin{aligned}C\int|u_x^Rv_x^R\phi\zeta_x|dx&\le C\int|v_x^R\phi|^2dx+C\int|u_x^R\zeta_x|^2dx\\&\le C\delta(1+t)^{-2}ln(1+t)+C(1+t)^{-2}+C(1+t)^{-2}\int\zeta_x^2dx\\&\le C\delta(1+t)^{-2}ln(1+t)+C(1+t)^{-2},\end{aligned} \tag{3.37}$$

$$\begin{aligned}C\int|u_x^R(v_x^R)^2\phi\chi|dx&\le C\int|u_x^R(v_x^R)^2|(\phi^2+\chi^2)dx\\&\le C\delta(1+t)^{-3}ln(1+t)+C(1+t)^{-3},\end{aligned} \tag{3.38}$$

$$\begin{aligned} C\int|p_x^Ru_x^R\chi_x|dx &\le C\int(p_x^R)^2u_x^Rdx+C\int u_x^R\chi_x^2dx \\ &\le C\delta(1+t)^{-2}+C\delta\int\chi_x^2dx, \end{aligned} \tag{3.39}$$

$$\begin{aligned} C\int|u_x^R\zeta_x\chi_x|dx &\le C\int(u_x^R)^2\zeta_x^2dx+\int\frac{a_{10}b_0}{8\tau(v^R)^2}\chi_x^2dx \\ &\le C(1+t)^{-2}+\int\frac{a_{10}b_0}{8\tau(v^R)^2}\chi_x^2dx. \end{aligned} \tag{3.40}$$

After estimating the right hand terms in (3.19) except for $\int|u_x^R\phi_x\zeta_x|dx$, then we have the following inequality

$$\begin{aligned} &\frac{d}{dt}\int\phi_x^2dx+\frac{d}{dt}\int\frac{p^R}{v^R}\psi_x^2dx+\frac{d}{dt}\int\frac{1}{(v^R)^2}(v_x^R\phi+v^R\phi_x-a_{10}\chi_x)^2dx \\ &+\int\frac{u_x^R}{v^R}\phi_x^2dx+\int\frac{p^Ru_x^R}{v^R}\psi_x^2dx+\int\frac{u_x^R}{(v^R)^3}(v_x^R\phi+v^R\phi_x-a_{10}\chi_x)^2dx \\ &+\int\frac{u_x^R}{(v^R)^3}\chi_x^2dx+\int\frac{a_{10}b_0}{\tau(v^R)^2}\chi_x^2dx \\ \le& C\int|u_x^R\phi_x\zeta_x|dx+C\delta(1+t)^{-5/4}ln(1+t)+C\delta^{1/8}(1+t)^{-5/4}+C(1+t)^{-2} \end{aligned} \tag{3.41}$$

if δ is small enough. Since Lemma 3.1 warrants the bound on the time evolution of $||\zeta_x||$, then

$$\begin{aligned} &\int\phi_x^2dx+\int\psi_x^2dx+\int\frac{1}{(v^R)^2}(v_x^R\phi+v^R\phi_x-a_{10}\chi_x)^2dx \\ &+\int_0^t\int u_x^R(\phi_x^2+\psi_x^2)(x,t')dxdt'+\int_0^t\int u_x^R(v_x^R\phi+v^R\phi_x-a_{10}\chi_x)^2(x,t')dxdt' \\ &+\int_0^t\int u_x^R\chi_x^2(x,t')dxdt'+\int_0^t\int\chi_x^2(x,t')dxdt' \\ \le& C\delta\int_0^t(1+t')^{-5/4}ln(1+t')+(1+t')^{-5/4}dt'+C\int_0^t\int|u_x^R\phi_x\zeta_x|(x,t')dxdt'+C \\ \le& \frac{1}{8}\int_0^t\int u_x^R\phi_x^2(x,t')dxdt'+C\int_0^t\int u_x^R\zeta_x^2(x,t')dxdt'+C \\ \le& \frac{1}{8}\int_0^t\int u_x^R\phi_x^2(x,t')dxdt'+C \end{aligned} \tag{3.42}$$

holds by integrating (3.41) over $[0,t]$. Therefore we have the following inequalities

$$||\phi_x||^2(t)+||\psi_x||^2(t)+\int\frac{1}{(v^R)^2}(v_x^R\phi+v^R\phi_x-a_{10}\chi_x)^2(t)dx\le C. \tag{3.43}$$

Observing

$$\begin{aligned} \int(v_x^R)^2\phi^2dx &\le C\delta(1+t)^{-1}\int\phi^2dx \\ &\le C\delta^2(1+t)^{-2}ln(1+t)+C\delta(1+t)^{-1} \\ &\le C \end{aligned}$$

holds, and

$$\int (v_x^R\phi+v^R\phi_x)^2dx \le 2\int (v_x^R\phi)^2dx+2\int (v^R\phi_x)^2dx,$$
$$\le C$$

due to (3.43), hence we find that

$$\int \chi_x^2(t)dx \le C.$$

Summing up above inequalities, the desired results are obtained. □

We next show the decay of $||(\phi_x,\psi_x,\chi_x)(t)||^2$ respect to time based on our previous estimates.

LEMMA 3.4. *Under the same assumptions in Theorem 1.1, there exists a constant $C>0$ independent of time such that*

$$||(\phi_x,\psi_x,\chi_x)||^2(t) \le C\delta(1+t)^{-1/4}ln(1+t)+C(1+t)^{-1/4}.$$

Proof. We first estimate $\int_0^t\int(\phi_x^2,\psi_x^2)dxds$. Applying ∂_x to the third equation in (3.6) and multiplying it by $\frac{\alpha}{\alpha_f}\psi_x$ we obtain

$$\psi_x^2=-\frac{\alpha}{\alpha_f}\left(\frac{1}{p^R}\psi_x\chi\right)_t+\frac{\alpha\gamma}{\alpha_f p^R v^R}u_x^R\psi_x\chi-\frac{\alpha}{\alpha_f}\left(\left(\frac{1}{p^R}\phi_x\chi\right)_x+\frac{p_x^R\phi_x\chi}{(p^R)^2}-\frac{\phi_x\chi_x}{p^R}\right)$$
$$-u_x^R\psi_x-\frac{u_x^R\phi\psi_x}{p^R}-\frac{\alpha+\alpha_f}{\tau\alpha_f p^R}\psi_x\chi.$$

Integrating the above inequality over $\mathbb{R}^1\times[0,t]$, we have

$$\begin{aligned}\int_0^t\int \psi_x^2(x,t')dxdt' \le & C(||(\chi,\psi_x)||^2(t)+||(\chi,\psi_x)(0)||^2)\\ & +C\int_0^t\int|\phi_x\chi_x|(x,t')dxdt'+C\delta ln(1+t)\\ \le & C\int_0^t\int|\phi_x\chi_x|(x,t')dxdt'+C\delta ln(1+t)+C. \end{aligned}\tag{3.44}$$

The following inequalities

$$\begin{aligned}C\int_0^t\int u_x^R\psi_x\chi(x,t')dxdt' &\le C\int_0^t\int u_x^R\psi_x^2(x,t')dxdt'+C\int_0^t\int \chi^2(x,t')dxdt'\\ &\le C\delta ln(1+t)+C,\end{aligned}\tag{3.45}$$

$$\begin{aligned}C\int_0^t\int p_x^R\phi_x\chi(x,t')dxdt' &\le C\int_0^t\int u_x^R\phi_x^2(x,t')dxdt'+\int_0^t\int \chi^2(x,t')dxdt'\\ &\le C\delta ln(1+t)+C,\end{aligned}$$

$$\begin{aligned}C\int_0^t\int u_x^R\psi_x(x,t')dxdt' &\le C\int_0^t\int (u_x^R)^2(x,t')dxdt'+\frac{1}{8}\int_0^t\int \psi_x^2(x,t')dxdt'\\ &\le \frac{1}{8}\int_0^t\int \psi_x^2(x,t')dxdt'+C\delta ln(1+t),\end{aligned}$$

$$\begin{aligned} C\int_0^t\int u_x^R\phi\psi_x(x,t')dxdt' &\le C\int_0^t\int u_x^R\phi^2(x,t')dxdt'+C\int_0^t\int u_x^R\psi_x^2(x,t')dxdt' \\ &\le C\delta ln(1+t)+C \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} C\int_0^t\int \psi_x\chi(x,t')dxdt' &\le C\int_0^t\int \chi^2(x,t')dxdt'+\frac{1}{8}\int_0^t\int \psi_x^2(x,t')dxdt' \\ &\le \frac{1}{8}\int_0^t\int \psi_x^2(x,t')dxdt'+C\delta ln(1+t)+C \end{aligned} \tag{3.47}$$

hold by Lemma 2.1, Lemma 3.2, Lemma 3.3 and Cauchy's inequality. Similarly, we apply ∂_x to the second equation in (3.6) and test it by ϕ_x to obtain

$$\begin{aligned} \phi_x^2 = &-(\phi_x\psi)_t+(\phi_t\psi)_x+\beta\frac{p^R}{v^R}\psi_x^2+\frac{2\alpha_f}{\alpha(\alpha+\alpha_f)}\frac{p^R}{v^R}u_x^R\psi_x+\frac{a_0}{v^R}u_x^R\phi\psi_x \\ &-\frac{a_1}{v^R}p^Ru_x^R\zeta\psi_x-\frac{a_1}{(v^R)^2}u_x^R\psi_x\chi+\frac{2}{\tau\alpha v^R}\chi\psi_x. \end{aligned}$$

Integrating above equality over $\mathbb{R}^1\times[0,t]$ to get

$$\begin{aligned} \int_0^t\int \phi_x^2(x,t')dxdt' \le& C(||(\phi_x,\psi)||^2(t)+||(\phi_x,\psi)||^2(0)) \\ &+C\int_0^t\int \psi_x^2(x,t')dxdt'+C\delta ln(1+t) \\ \le& C\delta ln(1+t)+C, \end{aligned} \tag{3.48}$$

since

$$\begin{aligned} C\int_0^t\int |u_x^R\psi_x\zeta|(x,t')dxdt' &\le C\int_0^t\int u_x^R\zeta^2(x,t')dxdt'+C\int_0^t\int u_x^R\psi_x^2(x,t')dxdt' \\ &\le C, \end{aligned}$$

exists, then

$$\begin{aligned} C\int_0^t\int \psi_x^2(x,t')dxdt' &\le C\int_0^t\int |\phi_x\chi_x|(x,t')dxdt'+C\delta ln(1+t)+C \\ &\le \frac{1}{8}\int_0^t\int \phi_x^2(x,t')dxdt'+C\int_0^t\int \chi_x^2(x,t')dxdt'+C\delta ln(1+t)+C \\ &\le \frac{1}{8}\int_0^t\int \phi_x^2(x,t')dxdt'+C\delta ln(1+t)+C, \end{aligned}$$

where we also have used the estimates (3.44)-(3.47), Lemmas 2.1, 3.1, 3.2, 3.3 and Cauchy's inequality. At the same time, we can see

$$\int_0^t\int \psi_x^2(x,t')dxdt' \le C\delta ln(1+t)+C \tag{3.49}$$

holds by (3.44) and (3.48).

Second, we show $||(\phi_x,\psi_x,\chi_x)(t)||$ actually decays in time. For this purpose, we introduce E_1 as follows

$$E_1=\int\left(\phi_x^2+\frac{p^R}{v^R}\psi_x^2+\frac{1}{(v^R)^2}(v_x^R\phi+v^R\phi_x-a_{10}\chi_x)^2\right)(x,t')dx. \tag{3.50}$$

Therefore we have

$$\begin{aligned}\int_0^t((1+t')E_1)_{t'}\,dt'=&\int_0^t(1+t')E_{1t'}+E_1dt'\\ \le& C\int_0^t(\delta(1+t')^{-1/4}ln(1+t')+\delta^{1/8}(1+t')^{-1/4}+(1+t')^{-1/4})dt'\\ &+C\int_0^t(1+t')\int|u_x^R\phi_x\zeta_x|(x,t')dxdt'+\int_0^tE_1dt' \end{aligned}\tag{3.51}$$

due to (3.41). The second term on the right side of (3.51)

$$\begin{aligned}&\int_0^t\int(1+t')|u_x^R\phi_x\zeta_x|(x,t')dxdt'\\ \le&\int_0^t(1+t')\int(u_x^R)^{1/2}\phi_x^2(x,t')dxdt'+\int_0^t(1+t')\int(u_x^R)^{3/2}\zeta_x^2(x,t')dxdt'\\ \le& C(1+t)^{1/2}\int_0^t\int\phi_x^2(x,t')dxdt'+C\int_0^t(1+t')^{-1/2}\int\zeta_x^2(x,t')dxdt'\\ \le& C\delta(1+t)^{1/2}ln(1+t)+C(1+t)^{1/2}\end{aligned}\tag{3.52}$$

would hold because of (3.7) and (3.48). Moreover, we can estimate the third term as

$$\begin{aligned}\int_0^tE_1dt'=&\int_0^t\int\phi_x^2+\frac{p^R}{v^R}\psi_x^2+\left(\frac{1}{(v^R)^2}(v_x^R\phi+v^R\phi_x-a_{10}\chi_x)^2\right)(x,t')dxdt'\\ \le& C\int_0^t\int\left(\phi_x^2+\psi_x^2+(v_x^R\phi)^2+\chi_x^2\right)(x,t')dxdt'\\ \le& C\delta ln(1+t)+C.\end{aligned}\tag{3.53}$$

Based on above inequalities (3.51)-(3.53), the following estimate

$$E_1\le C\delta(1+t)^{-1/4}ln(1+t)+C(1+t)^{-1/4}$$

hold, and finally

$$\int(\phi_x^2,\psi_x^2,\chi_x^2)(t)dx\le C\delta(1+t)^{-1/4}ln(1+t)+C(1+t)^{-1/4}.$$

Together with above estimates, we have finished the proof of Lemma 3.4. □

Proof. (**Proof of Theorem 1.1.**) The following inequality follows from Lemmas 3.2 and 3.4 and Sobolev's inequality,

$$\begin{aligned}||(\phi,\psi,\chi)||^2_{L^\infty(\mathbb{R}^1)}\le& C||(\phi,\psi,\chi)||||(\phi_x,\psi_x,\chi_x)||\\ \le& C(C\delta ln(1+t)+C)^{1/2}(C\delta(1+t)^{-1/4}ln(1+t)+C(1+t)^{-1/4})^{1/2}\\ \le& C\delta(1+t)^{-1/8}ln(1+t)+C(1+t)^{-1/8}\end{aligned}$$

that is,

$$\begin{aligned}||(\phi,\psi,\chi)||_{L^\infty(\mathbb{R}^1)} &\le C\delta^{1/2}(1+t)^{-1/16}(ln(1+t))^{1/2}+C(1+t)^{-1/16}\\ &\le C\delta^{1/2}(1+t)^{-1/16}ln(1+t)+C(1+t)^{-1/16},\end{aligned}$$

which tends to zero as time approaches to infinity. Combining with Lemma 2.1, the desired results are obtained. □

4. Proof of Theorem 1.2

This section is devoted to studying the life span of the solution to the nonlinear system (1.1). We may decompose (v,p,u,χ,s) as

$$(v,p,u,\chi,s)(x,t)=(\varphi+v^R,\phi_1+p^R,\psi_1+u^R,\chi,\zeta_1+\bar{s}).$$

Here we set

$$N^2(T)=\sup_{0\le t\le T}||(v-v^R,p-p^R,u-u^R,\chi,s-\bar{s})||^2_{H^2(\mathbb{R}^1)}, \tag{4.1}$$

in particular

$$N^2(0)=||(v_0-v_0^R,p_0-p_0^R,u_0-u_0^R,\chi_0,s_0-\bar{s})||^2_{H^2(\mathbb{R}^1)}. \tag{4.2}$$

In order to prove Theorem 1.2, it suffices to prove the following proposition.

Proposition 4.1. *Assume that the smooth 1-rarefaction wave (v^R,u^R,p^R,s^R) is given by (1.11) with $\epsilon=\delta^{1/2}$ and let (v,u,p,q) be a solution of the initial value problem (1.1) and (1.16) in the time interval $t\in[0,T]$. Then there exist positive constants δ_1 and C, independent of T, such that if*

$$N(T)\le\delta_1,\ \delta\le\delta_1 \text{ and } T\le\frac{1}{8C}\delta^{-3/2}, \tag{4.3}$$

then

$$N^2(T)\le CN^2(0)+C\delta\ln(1+\delta^{-3/2}). \tag{4.4}$$

4.1. The estimate for $||(\varphi,\phi_1,\psi_1,\chi,\zeta_1)||_{L^2(\mathbb{R}^1)}$.

Lemma 4.1. *Assume that the conditions in Proposition 4.1 hold. Then there exists a constant $C>0$ independent of T such that*

$$||(\varphi,\phi_1,\psi_1,\chi,\zeta_1)||^2(t)+\int_0^t\int\chi^2(x,t')dxdt'\le N^2(0)+C\left(\delta ln(1+t)+\delta^{3/2}TN(T)^2\right).$$

Proof. Let

$$w_1=(v,u,E,q)^T,\ w_1^R=(v^R,u^R,E^R,q^R)^T,$$

where $E=e_1+q+\frac{u^2}{2}$. Therefore, we can rewrite (1.1) as

$$w_{1t}+f(w_1)_x=r(w_1)$$

with

$$f(w_1)=(-u,p,pu,0)^T,r(w)=(0,0,0,\tfrac{\chi}{\tau})^T.$$

Let

$$\eta(w_1,w_1^R)=-s(w_1)+s(w_1^R)+\nabla s(w_1^R)(w_1-w_1^R). \tag{4.5}$$

A simple calculation yields

$$\begin{aligned}\eta_t(w_1,w_1^R)+\frac{\chi^2}{\tau pvq}=&-\left(\frac{\phi_1\psi_1}{p^Rv^R}\right)_x+\frac{u_x^R}{p^R(v^R)^2}(\varphi\phi_1+\frac{\psi_1^2}{\alpha+\alpha_f}-\frac{2\chi}{\alpha+\alpha_f})\\&-\frac{p_x^R\phi_1\psi_1}{v^R(p^R)^2}-\frac{v_x^R\phi_1\psi_1}{p^R(v^R)^2}\end{aligned} \tag{4.6}$$

by (3.1) and

$$\nabla s(w_1)=\left(\frac{1}{v},-\frac{u}{pv},\frac{1}{pv},\frac{\alpha_f}{2}(\frac{1}{q}-\frac{1}{Q})\right).$$

Then we have

$$\begin{aligned}\int\eta(t)dx+\int_0^t\int\chi^2(x,t')dxdt'\le\int\eta(0)dx+C\int_0^t\int(|u_x^R\varphi\phi_1|+|u_x^R\psi_1^2|+|p_x^R\phi_1\psi_1|\\+|v_x^R\phi_1\psi_1|+|u_x^R\chi|)(x,t')dxdt'.\end{aligned}$$

Now we deal with the term on the right hand of the above inequality. Thanks to Lemma 2.1, Young's inequality implies that

$$\begin{aligned}C\int_0^t\int|u_x^R\varphi\phi_1|(x,t')dxdt'\le&C\int_0^t\int u_x^R(\varphi^2+\phi_1^2)(x,t')dxdt'\\\le&CN(T)^2\int_0^t||u_x^R||_{L^\infty}dt'\\\le&C\delta^{3/2}TN(T)^2.\end{aligned} \tag{4.7}$$

Moreover, one can use the same idea to estimate $\int_0^t\int|u_x^R\psi_1^2|,\int_0^t\int|p_x^R\phi_1\psi_1|$ and $\int_0^t\int|v_x^R\phi_1\psi_1|$. As for the last term, we have

$$\begin{aligned}C\int_0^t\int|u_x^R\chi|(x,t')dxdt'\le&C\int_0^t\int(u_x^R)^2(x,t')dxdt'+\frac{1}{8}\int_0^t\int\chi^2(x,t')dxdt'\\\le&C\delta ln(1+t)+\frac{1}{8}\int_0^t\int\chi^2(x,t')dxdt'.\end{aligned} \tag{4.8}$$

Hence, combining the above inequalities together, one obtains the following estimate:

$$\begin{aligned}&||(\varphi,\phi_1,\psi_1,\chi)||^2(t)+\int_0^t\int\chi^2(x,t')dxdt'\\\le&||(\varphi,\phi_1,\psi_1,\chi)||^2(0)+C\left(\delta ln(1+t)+N(T)^2\delta^{3/2}T\right),\end{aligned} \tag{4.9}$$

where we have used the fact that $s(w)$ is strictly concave, which implies that

$$c_1|w_1-w_1^R|^2\le\eta\le c_2|w_1-w_1^R|^2$$

for some positive constants c_1 and c_2 provided w_1 is in a neighborhood of w_1^R. By the fourth equation of (3.1) and $s^R=\bar{s}$, the following inequality holds

$$\begin{aligned}||\zeta_1||^2(t) &\le ||\zeta_1||^2(0)+CN(T)\int_0^t\int \chi^2(x,t')dxdt' \\ &\le ||\zeta_1||^2(0)+CN(T)\left(\delta ln(1+t)+N(T)^2\delta^{3/2}T\right).\end{aligned} \tag{4.10}$$

Putting (4.9) and (4.10) together, we complete the proof of Lemma 4.1. ◻

In order to obtain the estimates on derivatives, we will take the equivalent form (3.1) to (1.1) in the following proof. We take a difference between systems (3.1) and (3.2), to get

$$\begin{cases}\phi_{1t}+\beta\left(\frac{p^R}{v}u_x^R+\frac{p^R}{v}\psi_{1x}+\frac{\phi_1}{v}u_x^R+\frac{\phi_1}{v}\psi_{1x}\right)-\gamma\frac{p^R}{v^R}u_x^R=-\frac{2\chi}{\tau\alpha v},\\ \psi_{1t}+\phi_{1x}=0,\\ \chi_t+\frac{\alpha_f}{\alpha}(p^Ru_x^R+p^R\psi_{1x}+\phi_1u_x^R+\phi_1\psi_{1x})=-\frac{\alpha+\alpha_f}{\alpha}\frac{\chi}{\tau},\\ \zeta_{1t}=\frac{2\chi^2}{\alpha_f\tau T_1T_2}.\end{cases} \tag{4.11}$$

4.2. The estimate for $||(\phi_{1x},\psi_{1x},\chi_x,\zeta_{1x})||_{L^2(\mathbb{R}^1)}$.

LEMMA 4.2. *Assume that the conditions in Proposition 4.1 hold. Then there exists a constant $C>0$ independent of T such that, for $0\le t\le T$,*

$$\begin{aligned}&||(\phi_{1x},\psi_{1x},\chi_x,\zeta_{1x})||^2(t)+\int_0^t\int(\phi_{1x}^2+\psi_x^2+\chi_x^2)(x,t')dxdt' \\ &\le C\left(N^2(0)+\delta ln(1+t)+\delta^{1/8}N^2(T)+\delta^{3/2}TN(T)^2+N(T)^3\right).\end{aligned}$$

Proof. Differentiate (4.11) with respect to x to get

$$\begin{aligned}&\phi_{1xt}=-\beta\left(\frac{p}{v}\psi_{1x}\right)_x-\beta\left(\frac{p^R}{v}u_x^R\right)_x-\beta\left(\frac{\phi_1}{v}u_x^R\right)_x+\gamma\left(\frac{p^R}{v^R}u_x^R\right)_x-\frac{2}{\tau\alpha}\left(\frac{\chi}{v}\right)_x,\\ &\psi_{1xt}=-\phi_{1xx},\\ &\left(\phi_{1x}-a_{10}\frac{\chi_x}{v}\right)_t=\beta\frac{v_x}{v^2}pu_x+\gamma\left(\frac{p^R}{v^R}u_x^R\right)_x+\frac{b_0}{v\tau}\chi_x-\frac{2}{\alpha\tau}\left(\frac{1}{v}\right)_x\chi+a_{10}\frac{u_x^R+\psi_{1x}}{v^2}\chi_x,\\ &\zeta_{1xt}=\left(\frac{2\chi^2}{\alpha_f\tau T_1T_2}\right)_x.\end{aligned} \tag{4.12}$$

Multiplying the first to fourth equations by ϕ_{1x}, $\beta\frac{p}{v}\psi_{1x}$, $a_5\left(\phi_{1x}-a_{10}\frac{\chi_x}{v}\right)$ and ζ_{1x}, respectively, we have

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\phi_{1x}^2+\frac{1}{2}\frac{\partial}{\partial t}\left(\beta\frac{p}{v}\psi_{1x}^2\right)+\frac{a_5}{2}\frac{\partial}{\partial t}\left(\phi_{1x}-a_{10}\frac{\chi_x}{v}\right)^2+\frac{1}{2}\frac{\partial}{\partial t}\zeta_{1x}^2+\beta\frac{u_x^R}{v}\phi_{1x}^2\\ &\quad+a_{15}\frac{pu_x^R}{v^2}\psi_{1x}^2+a_{14}\frac{u_x^R}{v^3}\chi_x^2+\frac{2a_{10}}{\tau\alpha v^2}\chi_x^2\\ &=-\beta\left(\frac{p}{v}\phi_{1x}\psi_{1x}\right)_x-\beta\left(\frac{p^Ru_x^R}{v}-\frac{p^Ru_x^R}{v^R}\right)_x\phi_{1x}-\beta\frac{u_{xx}^R}{v}\phi_1\phi_{1x}\\ &\quad-\beta\left(\frac{1}{v}\right)_xu_x^R\phi_1\phi_{1x}-\frac{2}{\alpha\tau}\left(\frac{1}{v}\right)_x\chi\phi_{1x}-a_{15}\frac{p\psi_{1x}^3}{v^2}-\beta\frac{\chi\psi_{1x}^2}{\alpha\tau v^2}+a_5\beta\frac{pv_xu_x\phi_{1x}}{v^2}\end{aligned}$$

$$
\begin{aligned}
&-a_5\frac{2}{\alpha\tau}\left(\frac{1}{v}\right)_x\chi\phi_{1x}+a_5a_{10}\frac{u_x^R}{v^2}\phi_{1x}\chi_x+a_5a_{10}\frac{\phi_{1x}\psi_{1x}\chi_x}{v^2}-\frac{a_5a_{10}\beta p}{v^3}v_xu_x\chi_x\\
&-\gamma a_5a_{10}\left(\frac{p^Ru_x^R}{v^R}\right)_x\frac{\chi_x}{v}+\frac{2a_5a_{10}}{\alpha\tau v}\left(\frac{1}{v}\right)_x\chi\chi_x-a_5a_{10}^2\frac{\psi_{1x}\chi_x^2}{v^3}\\
&+\frac{2}{\alpha_f}\zeta_{1x}\left[\frac{2\chi\chi_x}{\tau T_1T_2}-\frac{\chi^2T_{1x}}{\tau T_1^2T_2}-\frac{\chi^2T_{2x}}{\tau T_2^2T_1}\right]\\
=:&-\beta\left(\frac{p}{v}\phi_{1x}\psi_{1x}\right)_x+\sum_{i=1}^{15}K_i,
\end{aligned}
\tag{4.13}
$$

where

$$
a_{15}=\frac{(\alpha+1)(\alpha+2)}{\alpha^2},
$$

and β a_3, a_5, a_{10}, a_{14} and a_{15} are defined in Section 3. Now we estimate K_i, $1\le i\le 15$ one by one by invoking Lemma 2.1 and Young's inequality. For K_1,

$$
\begin{aligned}
K_1&\le C\left(|p_x^Ru_x^R\varphi\phi_{1x}|+|u_{xx}^R\varphi\phi_{1x}|+|u_x^Rv_x\varphi\phi_{1x}|+|u_x^R\varphi_x\phi_{1x}|+|u_x^Rv_x^R\varphi\phi_{1x}|\right)\\
&\le C\left(|p_x^Ru_x^R\varphi\phi_{1x}|+|u_{xx}^R\varphi\phi_{1x}|+|u_x^Rv_x^R\varphi\phi_{1x}|+|u_x^R\varphi_x\phi_{1x}|+|u_x^R\varphi_x\varphi\phi_{1x}|\right).
\end{aligned}
\tag{4.14}
$$

Since

$$
\begin{aligned}
&C\int_0^t\int|p_x^Ru_x^R\varphi\phi_{1x}|(x,t')dxdt'\\
&\le C\int_0^t||p_x^R||_{L^\infty}^2\int\varphi^2(x,t')dxdt'+C\int_0^t||u_x^R||_{L^\infty}^2\int\phi_{1x}^2(x,t')dxdt'\\
&\le C\delta N(T)^2\int_0^t(1+t)^{-4/3}dt'\\
&\le C\delta N(T)^2.
\end{aligned}
\tag{4.15}
$$

Similarly, $C\int_0^t\int|u_x^Rv_x^R\varphi\phi_{1x}|$ can be controlled by the same procedure as (4.15). Moreover, one can obtain

$$
\begin{aligned}
&C\int_0^t\int|u_{xx}^R\varphi\phi_{1x}|(x,t')dxdt'\\
&\le C\int_0^t||u_{xx}^R||_{L^\infty}\int(\varphi^2+\phi_{1x}^2)(x,t')dxdt'\\
&\le C\delta^{1/8}N(T)^2\int_0^t(1+t)^{-5/4}dt'\\
&\le C\delta^{1/8}N(T)^2.
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
&C\int_0^t\int|u_x^R\varphi_x\phi_{1x}|(x,t')dxdt'\\
\le&C\int_0^t||u_x^R||_{L^\infty}\int\varphi^2(x,t')dxdt'+C\int_0^t\int u_x^R\phi_{1x}^2(x,t')dxdt'\\
\le&C\delta^{3/2}N(T)^2T+C\delta\int_0^t\int\phi_{1x}^2(x,t')dxdt',
\end{aligned}
\tag{4.17}
$$

and

$$C\int_0^t\int|u_x^R\varphi_x\varphi\phi_{1x}|(x,t')dxdt'\le CN(T)\int_0^t||u_x^R||_{L^\infty}\int(\varphi^2+\phi_{1x}^2)(x,t')dxdt'$$
$$\le C\delta^{3/2}N(T)^3T. \tag{4.18}$$

Combine (4.15) and (4.18) together to get

$$C\int_0^t\int K_1(x,t')dxdt'\le C(\delta^{1/8}N(T)^2+\delta^{3/2}N(T)^2T)+C\delta\int_0^t\int\phi_{1x}^2(x,t')dxdt', \tag{4.19}$$

due the smallness of δ and $N(T)$. For K_2, one can take the same approach as (4.16) to get

$$C\int_0^t\int K_2(x,t')dxdt'\le C\int_0^t\int|u_{xx}^R\phi_1\phi_{1x}|(x,t')dxdt'$$
$$\le C\delta^{1/8}N(T)^2. \tag{4.20}$$

For K_3, we have

$$K_3\le C|u_x^Rv_x\phi_1\phi_{1x}|\le C\left(|u_x^Rv_x^R\phi_1\phi_{1x}|+|u_x^R\varphi_x\phi_{1x}\phi_1|\right), \tag{4.21}$$

where $C\int_0^t\int|u_x^Rv_x^R\phi_1\phi_{1x}|$ can be estimated in the same way as (4.15) and by (4.17), we have

$$C\int_0^t\int|u_x^R\varphi_x\phi_{1x}\phi_1|(x,t')dxdt'\le CN(T)\int_0^t\int|u_x^R\varphi_x\phi_{1x}|(x,t')dxdt'$$
$$\le C\delta^{3/2}N(T)^3T+C\delta N(T)\int_0^t\int\phi_{1x}^2(x,t')dxdt'. \tag{4.22}$$

Therefore, one can get

$$C\int_0^t\int K_3(x,t')dxdt'\le C(\delta N(T)^2+\delta^{3/2}N(T)^2T)+C\delta N(T)\int_0^t\int\phi_{1x}^2(x,t')dxdt'. \tag{4.23}$$

For K_4 and K_8, they can be controlled by

$$C(K_4,K_8)\le C|v_x\phi_{1x}\chi|\le C\left(|v_x^R\phi_{1x}\chi|+|\varphi_x\phi_{1x}\chi|\right). \tag{4.24}$$

Then the following inequalities are achieved

$$\int_0^t\int(K_4,K_8)(x,t')dxdt'$$
$$\le C\delta N(T)^2+\frac{1}{16}\int_0^t\int\chi^2(x,t')dxdt'+CN(T)\int_0^t\int(\phi_{1x}^2+\chi^2)(x,t')dxdt', \tag{4.25}$$

by

$$C\int_0^t\int|v_x^R\phi_{1x}\chi|(x,t')dxdt'\le C\int_0^t|u_x^R|_{L^\infty}^2\int\phi_{1x}^2(x,t')dxdt'+\frac{1}{16}\int_0^t\int\chi^2(x,t')dxdt'$$

$$\leq C\delta N(T)^2 \int_0^t (1+t')^{-4/3} dt' + \frac{1}{16}\int_0^t \int \chi^2(x,t')dxdt'$$
$$\leq C\delta N(T)^2 + \frac{1}{16}\int_0^t \int \chi^2(x,t')dxdt', \tag{4.26}$$

and

$$C\int_0^t \int |\varphi_x \phi_{1x}\chi|(x,t')dxdt' \leq CN(T)\int_0^t \int (\phi_{1x}^2+\chi^2)(x,t')dxdt'. \tag{4.27}$$

For K_5, we have

$$C\int_0^t \int K_5(x,t')dxdt' \leq C\int_0^t \int |\psi_{1x}^3|(x,t')dxdt' \leq CN(T)\int_0^t \int \psi_{1x}^2(x,t')dxdt'. \tag{4.28}$$

Similarly, the following estimate holds for K_6

$$C\int_0^t \int K_6(x,t')dxdt' \leq C\int_0^t \int |\psi_{1x}^2\chi|(x,t')dxdt' \leq CN(T)\int_0^t \int \psi_{1x}^2(x,t')dxdt'. \tag{4.29}$$

While for K_7, one has

$$K_7 \leq C\left(|u_x^R\varphi_x\phi_{1x}| + |v_x^R\phi_{1x}\psi_{1x}| + |u_x^R v_x^R\phi_{1x}| + |\varphi_x\phi_{1x}\psi_{1x}|\right). \tag{4.30}$$

Since $C\int_0^t\int |u_x^R\varphi_x\phi_{1x}|$ and $C\int_0^t\int |v_x^R\phi_{1x}\psi_{1x}|$ could be estimated similar to (4.17), and the following two inequalities

$$C\int_0^t \int |u_x^R v_x^R \phi_{1x}|(x,t')dxdt' \leq C\int_0^t \int ((v_x^R)^2 + (u_x^R)^2\phi_{1x}^2)(x,t')dxdt'$$
$$\leq C\delta ln(1+t) + C\delta\int_0^t \int \phi_{1x}^2(x,t')dxdt', \tag{4.31}$$

and

$$C\int_0^t \int |\varphi_x\phi_{1x}\psi_{1x}|(x,t')dxdt' \leq CN(T)\int_0^t \int (\phi_{1x}^2+\psi_{1x}^2)(x,t')dxdt'. \tag{4.32}$$

Hence,

$$C\int_0^t \int K_7(x,t')dxdt' \leq C\left(\delta ln(1+t) + \delta^{3/2}TN(T)^2\right)$$
$$+ C(\delta+N(T))\int_0^t \int (\phi_{1x}^2+\psi_{1x}^2)(x,t')dxdt'. \tag{4.33}$$

One can give a similar estimate as (4.26) for K_9,

$$C\int_0^t \int K_9(x,t')dxdt' \leq C\int_0^t \int |u_x^R\phi_{1x}\chi_x|(x,t')dxdt'$$
$$\leq C\delta N(T)^2 + \frac{a_{10}}{16\alpha\tau}\int_0^t \int \frac{\chi_x^2}{v^2}(x,t')dxdt'. \tag{4.34}$$

For K_{10}, we have

$$C\int_0^t\int K_{10}(x,t')dxdt' \le C\int_0^t\int |\phi_{1x}\psi_{1x}\chi_x|(x,t')dxdt' \\ \le CN(T)\int_0^t\int (\psi_x^2+\chi^2)(x,t')dxdt'. \tag{4.35}$$

For K_{11}, we can use the same idea as K_7 to have the following estimate

$$C\int_0^t\int K_{11}(x,t')dxdt' \le C\left(\delta ln(1+t)+\delta^{3/2}TN(T)^2\right) \\ +C(\delta+N(T))\int_0^t\int (\chi_x^2+\psi_{1x}^2)(x,t')dxdt'. \tag{4.36}$$

For K_{12}, we have

$$K_{12}\le C\left(|p_x^R u_x^R\chi_x|+|u_{xx}^R\chi_x|+|u_x^R v_x^R\chi_x|\right). \tag{4.37}$$

Due to the estimates

$$C\int_0^t\int |u_x^R(p_x^R,v_x^R)\chi_x|(x,t')dxdt' \le C\int_0^t\int \left((p_x^R,v_x^R)^2+(u_x^R)^2\chi_x^2\right)(x,t')dxdt' \\ \le C\delta ln(1+t)+C\delta\int_0^t\int \chi_x^2(x,t')dxdt', \tag{4.38}$$

and

$$C\int_0^t\int |u_{xx}^R\chi_x|(x,t')dxdt' \le C\delta\int_0^t\int (u_x^R)^2(x,t')dxdt'+\frac{a_{10}}{16\alpha\tau}\int_0^t\int \frac{\chi_x^2}{v^2}(x,t')dxdt' \\ \le C\delta ln(1+t)+\frac{a_{10}}{16\alpha\tau}\int_0^t\int \frac{\chi_x^2}{v^2}(x,t')dxdt', \tag{4.39}$$

one has

$$C\int_0^t\int K_{12}(x,t')dxdt' \le C\delta ln(1+t)+(C\delta+\frac{a_{10}}{16\alpha\tau})\int_0^t\int \frac{\chi_x^2}{v^2}(x,t')dxdt'. \tag{4.40}$$

For K_{13}, we have

$$C\int_0^t\int K_{13}(x,t')dxdt' \le \int_0^t\int (|v_x^R\chi_x\chi|+|\varphi_x\chi_x\chi|)(x,t')dxdt' \\ \le C(\delta+N(T))\int_0^t\int (\chi_x^2+\chi^2)(x,t')dxdt'. \tag{4.41}$$

For K_{14}, one has

$$C\int_0^t\int K_{14}(x,t')dxdt' \le \int_0^t\int |\psi_{1x}\chi_x^2|(x,t')dxdt' \\ \le CN(T)\int_0^t\int \chi_x^2(x,t')dxdt'. \tag{4.42}$$

K_{15} can be bounded by

$$K_{15}\le C\left(|\zeta_{1x}^R\chi_x\chi|+|T_{1x}\zeta_{1x}^R\chi^2|+|T_{1x}\zeta_{1x}^R\chi^2|\right). \tag{4.43}$$

Because

$$T_1=pv,\ T_2-T_1=-\tfrac{2}{\alpha_f}\chi,$$

then

$$|T_{1x}|\le C(|p^R_x|+|v^R_x|+|\phi_{1x}|+|\varphi_x|),\ |T_{2x}|\le C(|p^R_x|+|v^R_x|+|\phi_{1x}|+|\varphi_x|+|\chi|)$$

hold, which implies

$$|T_{1x}|\le C(\delta+N(T)),\ |T_{2x}|\le C(\delta+N(T)) \tag{4.44}$$

by virtue of Lemma 2.1 and Sobolev's inequality. For K_{15}, we have

$$\begin{aligned}
C\int_0^t\int K_{15}(x,t')dxdt' \le& \int_0^t\int (|\zeta_{1x}\chi_x\chi|+|(T_{1x},T_{2x})\zeta_{1x}\chi^2|)(x,t')dxdt'\\
\le& CN(T)\int_0^t\int (\chi_x^2+\chi^2)(x,t')dxdt'\\
&+C(\delta+N(T))N(T)\int_0^t\int \chi^2(x,t')dxdt'\\
\le& CN(T)\int_0^t\int (\chi_x^2+\chi^2)(x,t')dxdt',
\end{aligned} \tag{4.45}$$

where we have used the smallness of δ and $N(T)$.

Putting the estimates for $\int_0^t\int K_i(x,t')dxdt'$, $1\le i\le 15$ together, we arrive at

$$\begin{aligned}
&||(\phi_{1x},\psi_{1x},\chi_x,\zeta_{1x})||^2(t)+\int_0^t\int \left(u^R_x(\phi_{1x}^2,\psi_x^2,\chi_x^2)+\chi_x^2\right)(x,t')dxdt'\\
\le& N^2(0)+C\left(\delta ln(1+t)+\delta^{1/8}N^2(T)+\delta^{3/2}TN(T)^2\right)\\
&+C(\delta+N(T))\int_0^t\int (\phi_{1x}^2+\psi_{1x}^2)(x,t')dxdt',
\end{aligned} \tag{4.46}$$

by Lemma 4.1 and the smallness of δ and $N(T)$. Therefore, we are only left with $\int_0^t\int(\phi_{1x}^2,\psi_{1x}^2)$ to estimate. After a simple calculation, ψ_{1x}^2 can be expressed as

$$\begin{aligned}
\psi_{1x}^2=&-\frac{\alpha}{\alpha_f}\left(\frac{1}{p^R}\psi_{1x}\chi\right)_t+\frac{\alpha\gamma}{\alpha_f p^R v^R}u^R_x\psi_{1x}\chi-\frac{\alpha}{\alpha_f}\left(\left(\frac{1}{p^R}\phi_{1x}\chi\right)_x+\frac{p^R_x\phi_{1x}\chi}{(p^R)^2}-\frac{\phi_{1x}\chi_x}{p^R}\right)\\
&-u^R_x\psi_{1x}-\frac{u^R_x\phi_1\psi_{1x}}{p^R}-\frac{\phi_1\psi_{1x}^2}{p^R}-\frac{\alpha+\alpha_f}{\tau\alpha_f p^R}\psi_{1x}\chi.
\end{aligned}$$

For $\int_0^t\int|u^R_x\psi_{1x}\chi|(x,t')dxdt'$ and $\int_0^t\int|p^R_x\phi_{1x}\chi|(x,t')dxdt'$, we can take the same idea as (4.26) to have

$$C\int_0^t\int|u^R_x\psi_{1x}\chi|(x,t')dxdt'\le C\delta N(T)^2+\frac{1}{16}\int_0^t\int\chi^2(x,t')dxdt', \tag{4.47}$$

and

$$C\int_0^t\int|p^R_x\phi_{1x}\chi|(x,t')dxdt'\le C\delta N(T)^2+\frac{1}{16}\int_0^t\int\chi^2(x,t')dxdt'. \tag{4.48}$$

Moreover, one can obtain

$$C\int_0^t\int|u^R_x\psi_{1x}|(x,t')dxdt'\le C\delta ln(1+t)+\frac{1}{16}\int_0^t\int\psi_{1x}^2(x,t')dxdt', \tag{4.49}$$

$$C\int_0^t\int |u_x^R\psi_{1x}\phi_1|(x,t')dxdt' \leq \int_0^t ||u_x^R||_{L^\infty}^2 \int \phi_1^2(x,t')dxdt + \frac{1}{16}\int_0^t\int \psi_{1x}^2(x,t')dxdt'$$
$$\leq C\delta N(T)^2 + \frac{1}{16}\int_0^t\int \psi_{1x}^2(x,t')dxdt', \tag{4.50}$$

$$C\int_0^t\int |\phi_1\psi_{1x}^2|(x,t')dxdt' \leq CN(T)\int_0^t\int \psi_{1x}^2(x,t')dxdt' \tag{4.51}$$

and

$$C\int_0^t\int |\psi_{1x}\chi|(x,t')dxdt' \leq C\int_0^t\int \chi^2(x,t')dxdt + \frac{1}{16}\int_0^t\int \psi_{1x}^2(x,t')dxdt'. \tag{4.52}$$

Combine (4.49)-(4.52) and Lemma 4.1 together to obtain

$$\int_0^t\int \psi_{1x}^2(x,t')dxdt' \leq C(N(0)^2 + N(T)^2 + \delta ln(1+t) + \delta N(T)^2 + \delta^{3/2}TN(T)^2)$$
$$+ C\int_0^t\int |\phi_{1x}\chi_x|(x,t')dxdt. \tag{4.53}$$

A straightforward computation gives

$$\phi_{1x}^2 = -(\phi_{1x}\psi_1)_t + (\phi_{1t}\psi_1)_x + \beta\frac{p^R}{v}\psi_{1x}^2 + \beta\frac{p^R}{v}u_x^R\psi_{1x} + \beta\frac{\phi_1}{v}u_x^R\psi_{1x}$$
$$+ \beta\frac{\phi_1}{v}\psi_{1x}^2 - \gamma\frac{p^R}{v^R}u_x^R\psi_{1x} + \frac{2}{\tau\alpha v}\chi\psi_{1x}.$$

Then

$$\int_0^t\int \phi_{1x}^2(x,t')dxdt' \leq C(N(0)^2 + N(T)^2 + \delta ln(1+t) + \delta N(T)^2 + \delta^{3/2}TN(T)^2)$$
$$+ C\int_0^t\int \psi_{1x}^2(x,t')dxdt$$
$$\leq C(N(0)^2 + N(T)^2 + \delta ln(1+t) + \delta N(T)^2 + \delta^{3/2}TN(T)^2)$$
$$+ C\int_0^t\int \chi_x^2(x,t')dxdt + \frac{1}{8}\int_0^t\int \phi_{1x}^2(x,t')dxdt'$$
$$\leq C(N(0)^2 + N(T)^2 + \delta ln(1+t) + \delta^{1/8}N(T)^2 + \delta^{3/2}TN(T)^2)$$
$$+ C(\delta + N(T))\int_0^t\int \psi_{1x}^2(x,t')dxdt' \tag{4.54}$$

holds, where we have used the inequalities (4.46), (4.48)-(4.52). On the other hand, we have

$$\int_0^t\int \psi_{1x}^2(x,t')dxdt' \leq C(N(0)^2 + N(T)^2 + \delta ln(1+t) + \delta^{1/8}N(T)^2 + \delta^{3/2}TN(T)^2) \tag{4.55}$$

by inequalities (4.46), (4.53) and (4.54). It is easy to see that

$$\int_0^t\int (\phi_{1x}^2, \psi_{1x}^2)(x,t')dxdt' \leq C(N(0)^2 + N(T)^2 + \delta ln(1+t) + \delta^{1/8}N(T)^2 + \delta^{3/2}TN(T)^2) \tag{4.56}$$

from (4.54) and (4.55). Then we obtain

$$\begin{aligned}&||(\phi_{1x},\psi_{1x},\chi_x,\zeta_{1x})||^2(t)+\int_0^t\int(\phi_{1x}^2,\psi_x^2,\chi_x^2)(x,t')dxdt'\\&\le C\left(N^2(0)+\delta ln(1+t)+\delta^{1/8}N^2(T)+\delta^{3/2}TN(T)^2+N(T)^3\right),\end{aligned}\tag{4.57}$$

where we have used the smallness of δ and $N(T)$. Therefore, the desired estimate is obtained. ∎

4.3. The estimate for $||(\phi_{1xx},\psi_{1xx},\chi_{xx},\zeta_{1xx})||_{L^2(\mathbb{R}^1)}$.

LEMMA 4.3. *Assume that the conditions in Proposition 4.1 hold. Then there exists a constant $C>0$ independent of T such that*

$$\begin{aligned}&||(\phi_{1xx},\psi_{1xx},\chi_{xx},\zeta_{1xx})||^2(t)+\int_0^t\int\left(\phi_{1xx}^2+\psi_{1xx}^2+\chi_{xx}^2\right)(x,t')dxdt'\\&\le C\left(N^2(0)+\delta ln(1+t)+\delta^{1/8}N^2(T)+\delta^{3/2}TN(T)^2+N(T)^3\right),\end{aligned}$$

for $0\le t\le T$.

Proof. Differentiate (4.11) with respect to x to get

$$\begin{aligned}&\phi_{1xxt}=-\beta\left(\frac{p}{v}\psi_{1x}\right)_{xx}-\beta\left(\frac{p^R}{v}u_x^R\right)_{xx}-\beta\left(\frac{\phi_1}{v}u_x^R\right)_{xx}+\gamma\left(\frac{p^R}{v^R}u_x^R\right)_{xx}-\frac{2}{\tau\alpha}\left(\frac{\chi}{v}\right)_{xx},\\&\psi_{1xxt}=-\phi_{1xxx},\\&\left(\phi_{1xx}-a_{10}\frac{\chi_{xx}}{v}\right)_t=-\beta\left[\left(\frac{pu_x}{v}\right)_{xx}-\frac{1}{v}(pu_x)_{xx}\right]+\gamma\left(\frac{p^R}{v^R}u_x^R\right)_{xx}-\frac{2}{\alpha\tau}\left(\frac{\chi}{v}\right)_{xx}\\&\qquad+\frac{a_{10}(\alpha+\alpha_f)}{\alpha\tau v}\chi_{xx}+a_{10}\frac{u_x^R+\psi_{1x}}{v^2}\chi_{xx},\\&\zeta_{1xxt}=\left(\frac{2\chi^2}{\alpha_f\tau T_1T_2}\right)_{xx}.\end{aligned}\tag{4.58}$$

Multiplying the first to fourth equations by ϕ_{1xx}, $\beta\frac{p}{v}\psi_{1xx}$, $a_5\left(\phi_{1xx}-a_{10}\frac{\chi_{xx}}{v}\right)$ and ζ_{1xx}, respectively, we have

$$\begin{aligned}&\frac12\frac{\partial}{\partial t}\phi_{1xx}^2+\frac12\frac{\partial}{\partial t}\left(\beta\frac{p}{v}\psi_{1xx}^2\right)+\frac{a_5}{2}\frac{\partial}{\partial t}\left(\phi_{1xx}-a_{10}\frac{\chi_{xx}}{v}\right)^2+\frac12\frac{\partial}{\partial t}\zeta_{1xx}^2+\beta\frac{u_x^R}{v}\phi_{1xx}^2\\&\quad+a_{15}\frac{pu_x^R}{v^2}\psi_{1xx}^2+a_{14}\frac{u_x^R}{v^3}\chi_{xx}^2+\frac{2a_{10}}{\tau\alpha v^2}\chi_{xx}^2\\&=-\beta\left(\frac{p}{v}\phi_{1xx}\psi_{1xx}\right)_x-\beta\left(\frac{p}{v}\right)_x\phi_{1xx}\psi_{1xx}-\beta\left(\frac{p}{v}\right)_{xx}\psi_{1x}\phi_{1xx}-2\beta\frac{u_{xx}^R}{v}\phi_{1x}\phi_{1xx}\\&\quad+2\beta\frac{v_x}{v^2}u_x^R\phi_{1x}\phi_{1xx}-\beta\frac{u_{xxx}^R}{v}\phi_1\phi_{1xx}+2\beta\frac{v_x}{v^2}u_{xx}^R\phi_1\phi_{1xx}+\beta\left(\frac{v_x}{v^2}\right)_xu_x^R\phi_1\phi_{1xx}\\&\quad-\beta\left(\frac{p^Ru_x^R}{v}-\frac{p^Ru_x^R}{v^R}\right)_{xx}\phi_{1xx}+\frac{2}{\alpha\tau}\left(v_{xx}\chi-2\frac{v_x^2\chi}{v}+2v_x\chi_x\right)\frac{\phi_{1xx}}{v^2}\\&\quad-a_{15}\frac{p}{v^2}\psi_{1xx}^2\psi_{1x}-\beta\frac{\chi}{\tau\alpha v^2}\psi_{1xx}^2-a_5\beta\left[\left(\frac{p}{v}u_x\right)_{xx}-\frac{1}{v}(pu_x)_{xx}\right]\phi_{1xx}\\&\quad-\frac{2a_5}{\alpha\tau}\left(\frac{1}{v}\right)_{xx}\chi\phi_{1xx}-\frac{4a_5}{\alpha\tau}\left(\frac{1}{v}\right)_x\chi_x\phi_{1xx}+a_5a_{10}\frac{u_x^R+\psi_{1x}}{v^2}\phi_{1xx}\chi_{xx}\end{aligned}$$

$$
\begin{aligned}
&+a_5a_{10}\beta\left[\left(\frac{p}{v}u_x\right)_{xx}-\frac{1}{v}(pu_x)_{xx}\right]\frac{\chi_{xx}}{v}-\gamma a_5a_{10}\left(\frac{p^Ru_x^R}{v^R}\right)_{xx}\frac{\chi_{xx}}{v}\\
&+\frac{2a_5a_{10}}{\alpha\tau v}\left(\frac{1}{v}\right)_{xx}\chi\chi_{xx}+\frac{4a_5a_{10}}{\alpha\tau v}\left(\frac{1}{v}\right)_x\chi_x\chi_{xx}-a_5a_{10}^2\frac{\psi_{1x}\chi_{xx}^2}{v^3}\\
&+\frac{2}{\alpha_f\tau}\zeta_{1xx}\left[\frac{2(\chi_x+\chi\chi_{xx})}{T_1T_2}+4\left(\frac{1}{T_1T_2}\right)_x\chi\chi_x+\left(\frac{1}{T_1T_2}\right)_{xx}\chi^2\right]\\
=&-\beta\left(\frac{p}{v}\phi_{1xx}\psi_{1xx}\right)_x+\sum_{i=1}^{21}R_i.
\end{aligned}\tag{4.59}
$$

In order to get the estimate for $||(\phi_{1xx},\psi_{1xx},\chi_{xx},\zeta_{1xx})||_{L^2(\mathbb{R}^1)}$, we firstly deal with the right-hand side terms in sequence. For R_1, one have

$$
R_1\le C\left(|p_x^R\phi_{1xx}\psi_{1xx}|+|v_x^R\phi_{1xx}\psi_{1xx}|+|\phi_{1x}\phi_{1xx}\psi_{1xx}|+|\varphi_x\phi_{1xx}\psi_{1xx}|\right).\tag{4.60}
$$

It is easy to get the following inequality by Lemma 2.1 and Young's inequality

$$
C\int_0^t\int|(p_x^R,v_x^R)\phi_{1xx}\psi_{1xx}|(x,t')dxdt'\le C\delta\int_0^t\int(\phi_{1xx}^2+\psi_{1xx}^2)(x,t')dxdt'.\tag{4.61}
$$

And

$$
C\int_0^t\int|(\phi_{1x},\varphi_x)\phi_{1xx}\psi_{1xx}|(x,t')dxdt'\le CN(T)\int_0^t\int(\phi_{1xx}^2+\psi_{1xx}^2)(x,t')dxdt'.\tag{4.62}
$$

Hence

$$
C\int_0^t\int R_1(x,t')dxdt'\le C(\delta+N(T))\int_0^t\int(\phi_{1xx}^2+\psi_{1xx}^2)(x,t')dxdt'.\tag{4.63}
$$

R_2 can have a bound as

$$
\begin{aligned}
R_2\le C(&|p_x^R\varphi_x\phi_{1xx}\psi_{1x}|+|v_x^R\phi_{1x}\psi_{1x}\phi_{1xx}|+|p_x^Rv_x^R\psi_x\phi_{1xx}|+|\varphi_x\phi_{1x}\psi_{1x}\phi_{1xx}|\\
&+|v_{xx}^R\phi_{1xx}\psi_{1x}|+|\varphi_{xx}\phi_{1xx}\psi_{1x}|+|(v_x^R)^2\phi_{1xx}\psi_{1x}|+|\varphi_x^2\phi_{1xx}\psi_{1x}|\\
&+|p_{xx}^R\phi_{1xx}\psi_{1x}|+|\phi_{1xx}^2\psi_{1x}|).
\end{aligned}\tag{4.64}
$$

For R_2, one can obtain

$$
C\int_0^t\int R_2(x,t')dxdt'\le C\delta^{1/8}N(T)^2+CN(T)\int_0^t\int(\phi_{1xx}^2+\psi_{1xx}^2+\psi_{1x}^2)(x,t')dxdt',\tag{4.65}
$$

where the smallness of δ and $N(T)$ and the following estimates have been used,

$$
C\int_0^t\int|p_x^R\varphi_x\phi_{1xx}\psi_{1x}|(x,t')dxdt'\le C\delta N(T)\int_0^t\int(\phi_{1xx}^2+\psi_{1x}^2)(x,t')dxdt',\tag{4.66}
$$

$$
C\int_0^t\int|v_x^R\phi_{1x}\psi_{1x}\phi_{1xx}|(x,t')dxdt'\le C\delta N(T)\int_0^t\int(\phi_{1xx}^2+\psi_{1x}^2)(x,t')dxdt',\tag{4.67}
$$

$$
C\int_0^t\int|(p_x^R,v_x^R)v_x^R\phi_{1xx}\psi_{1x}|(x,t')dxdt'\le C\int_0^t||(p_x^R,v_x^R)||_{L^\infty}^2\int(\phi_{1xx}^2+\psi_{1x}^2)(x,t')dxdt'
$$

$$\leq C\delta N(T)^2 \int_0^t (1+t')^{-4/3} dt', \\ \leq C\delta N(T)^2, \tag{4.68}$$

$$\begin{aligned} C\int_0^t \int |(p^R_{xx}, v^R_{xx})\phi_{1xx}\psi_{1x}|(x,t')dxdt' &\leq C\int_0^t ||(p^R_{xx}, v^R_{xx})||_{L^\infty} \int (\phi^2_{1xx} + \psi^2_{1x})(x,t')dxdt' \\ &\leq CN(T)^2\delta^{1/8} \int_0^t (1+t')^{-5/4} dt' \\ &\leq C\delta^{1/8} N(T)^2, \end{aligned} \tag{4.69}$$

$$C\int_0^t \int |(\phi_{1x}, \varphi_x)\varphi_x\psi_{1x}\phi_{1xx}|(x,t')dxdt' \leq CN(T)^2 \int_0^t \int (\phi^2_{1xx} + \psi^2_{1x})(x,t')dxdt', \tag{4.70}$$

$$\begin{aligned} & C\int_0^t \int |\psi_{1x}\varphi_{xx}\phi_{1xx}|(x,t')dxdt' \\ \leq & \frac{C}{N(T)} \int_0^t \int (\psi^2_{1x}\varphi^2_{1xx})(x,t')dxdt' + CN(T)\int_0^t \int \phi^2_{1xx}(x,t')dxdt' \\ \leq & \frac{C}{N(T)} \int_0^t |\psi_{1x}|^2_{L^\infty} \int \varphi^2_{1xx}(x,t')dxdt' + CN(T)\int_0^t \int \phi^2_{1xx}(x,t')dxdt' \\ \leq & CN(T)\int_0^t \int (\psi^2_{1x} + \psi^2_{1xx} + \phi^2_{1xx})(x,t')dxdt' \end{aligned} \tag{4.71}$$

$$C\int_0^t \int |\phi^2_{1xx}\psi_{1x}|(x,t')dxdt' \leq CN(T)\int_0^t \int \phi^2_{1xx}(x,t')dxdt'. \tag{4.72}$$

For R_3, one can obtain

$$C\int_0^t \int R_3(x,t')dxdt' \leq C\int_0^t \int |u^R_{xx}\phi_{1xx}\phi_{1x}|(x,t')dxdt' \leq C\delta^{1/8}N(T)^2 \tag{4.73}$$

by the same way as (4.69). For R_4, we have

$$\begin{aligned} C\int_0^t \int R_4(x,t')dxdt' &\leq C\int_0^t \int (|u^R_x v^R_x \phi_{1xx}\phi_{1x}| + |u^R_x \varphi_x \phi_{1xx}\phi_{1x}|)(x,t')dxdt' \\ &\leq C\delta N(T)^2 + C\delta N(T)\int_0^t \int (\phi^2_{1xx} + \phi^2_{1x})(x,t')dxdt', \end{aligned} \tag{4.74}$$

which follows the same methods as (4.67) and (4.68), respectively. $\int_0^t \int R_5$ can have an upper bound as

$$\begin{aligned} C\int_0^t \int R_5(x,t')dxdt' &\leq C\int_0^t \int (|u^R_{xxx}\phi_{1xx}\phi_1|(x,t')dxdt' \\ &\leq C\delta \int_0^t ||u^R_x||^2_{L^\infty} \int \phi^2_1(x,t')dxdt' + C\delta\int_0^t \int \phi^2_{1xx}(x,t')dxdt' \\ &\leq C\delta N(T)^2 + C\delta \int_0^t \int \phi^2_{1xx}(x,t')dxdt'. \end{aligned} \tag{4.75}$$

Since the following inequality for R_6 exists

$$R_6 \le C(|v_x^R u_{xx}^R \phi_{1xx}\phi_1| + |\varphi_x u_{xx}^R \phi_{1xx}\phi_1|), \tag{4.76}$$

$$\begin{aligned} &C\int_0^t \int |v_x^R u_{xx}^R \phi_{1xx}\phi_1|(x,t')dxdt' \\ \le& C\int_0^t ||u_{xx}^R||_{L^\infty}^2 \int \phi_1^2(x,t')dxdt' + C\delta \int_0^t \int u_x^R \phi_{1xx}^2(x,t')dxdt' \\ \le& C\delta N(T)^2 + C\delta^2 \int_0^t \int \phi_{1xx}^2(x,t')dxdt', \end{aligned} \tag{4.77}$$

and

$$\begin{aligned} &C\int_0^t \int |\varphi_x u_{xx}^R \phi_{1xx}\phi_1|(x,t')dxdt' \\ \le& CN(T)\int_0^t ||u_{xx}^R||_{L^\infty}^2 \int \varphi_x^2(x,t')dxdt' + CN(T)\int_0^t \int \phi_{1xx}^2(x,t')dxdt' \\ \le& C\delta N(T)^3 \int_0^t (1+t')^{-4/3} + CN(T)\int_0^t \int \phi_{1xx}^2(x,t')dxdt' \\ \le& C\delta N(T)^2 + CN(T)\int_0^t \int \phi_{1xx}^2(x,t')dxdt', \end{aligned} \tag{4.78}$$

which implies

$$C\int_0^t \int R_6(x,t')dxdt' \le C\delta N(T)^2 + C(\delta + N(T))\int_0^t \int \phi_{1xx}^2(x,t')dxdt'. \tag{4.79}$$

A calculation gives us

$$R_7 \le C(|u_x^R v_{xx}^R \phi_{1xx}\phi_1| + |u_x^R \varphi_{xx}\phi_{1xx}\phi_1| + |u_x^R (v_x^R)^2 \phi_{1xx}\phi_1| + |u_x^R \varphi_x^2 \phi_{1xx}\phi_1|), \tag{4.80}$$

and

$$C\int_0^t \int |u_x^R v_{xx}^R \phi_{1xx}\phi_1|(x,t')dxdt' \le C\delta N(T)^2 + C\delta^2 \int_0^t \int \phi_{1xx}^2(x,t')dxdt', \tag{4.81}$$

which follows from (4.77), and

$$\begin{aligned} &C\int_0^t \int |u_x^R \varphi_{xx}\phi_{1xx}\phi_1|(x,t')dxdt' \\ \le& CN(T)\int_0^t ||u_x^R||_{L^\infty}^2 \int \varphi_{xx}^2(x,t')dxdt' + CN(T)\int_0^t \int \phi_{1xx}^2(x,t')dxdt' \\ \le& C\delta N(T)^3 + CN(T)\int_0^t \int \phi_{1xx}^2(x,t')dxdt', \end{aligned} \tag{4.82}$$

$$\begin{aligned} &C\int_0^t \int |u_x^R (v_x^R)^2 \phi_{1xx}\phi_1|(x,t')dxdt' \\ \le& C\delta \int_0^t ||u_x^R||_{L^\infty}^2 \int \phi_1^2(x,t')dxdt' + C\delta \int_0^t \int \phi_{1xx}^2(x,t')dxdt' \end{aligned}$$

$$\leq C\delta^2 N(T)^2 + C\delta \int_0^t \int \phi_{1xx}^2(x,t')dxdt', \tag{4.83}$$

$$\begin{aligned}
&C\int_0^t \int |u_x^R \varphi_x^2 \phi_{1xx}\phi_1|(x,t')dxdt' \\
\leq &CN(T)^2 \int_0^t ||u_x^R||_{L^\infty}^2 \int \phi_1^2(x,t')dxdt' + CN(T)^2 \int_0^t \int \phi_{1xx}^2(x,t')dxdt' \\
\leq &C\delta N(T)^2 + CN(T)^2 \int_0^t \int \phi_{1xx}^2(x,t')dxdt'.
\end{aligned} \tag{4.84}$$

Combining (4.81)-(4.84) together, one can obtain

$$C\int_0^t \int R_7(x,t')dxdt' \leq C\delta N(T)^2 + C(\delta + N(T)) \int_0^t \int \phi_{1xx}^2(x,t')dxdt', \tag{4.85}$$

by using the smallness of δ and $N(T)$. For R_8, one has

$$\begin{aligned}
R_8 \leq C(&|p_{xx}^R u_x^R \phi_{1xx}\varphi| + |p_x^R u_{xx}^R \phi_{1xx}\varphi| + |p_x^R u_x^R \phi_{1xx}\varphi_x| + |p_x^R u_x^R v_x^R \phi_{1xx}\varphi| \\
&+ |p_x^R u_x^R \varphi_x \varphi \phi_{1xx}| + |u_x^R \varphi_{xx}\phi_{1xx}| + |u_{xx}^R \varphi_x \phi_{1xx}| + |u_x^R \varphi_x^2 \phi_{1xx}| \\
&+ |u_x^R v_x^R \varphi_x \phi_{1xx}| + |u_{xxx}^R \varphi \phi_{1xx}| + |u_{xx}^R v_x^R \varphi \phi_{1xx}| + |u_{xx}^R \varphi_x \phi_{1xx}\varphi| \\
&+ |u_x^R (v_x^R)^2 \phi_{1xx}\varphi| + |u_x^R \varphi_x^2 \phi_{1xx}\varphi| + |u_x^R \varphi_{xx}\phi_{1xx}\varphi| + |u_x^R v_{xx}^R \phi_{1xx}\varphi| \\
&+ |u_x^R v_x^R \varphi_x \phi_{1xx}\varphi|).
\end{aligned} \tag{4.86}$$

$\int_0^t \int |p_{xx}^R u_x^R \phi_{1xx}\varphi|$, $\int_0^t \int |p_x^R u_{xx}^R \phi_{1xx}\varphi|$, $\int_0^t \int |u_{xx}^R v_x^R \varphi \phi_{1xx}|$ and $\int_0^t \int |u_x^R v_{xx}^R \phi_{1xx}\varphi|$ can be controlled using the same method as the following

$$\begin{aligned}
&C\int_0^t \int |p_{xx}^R u_x^R \phi_{1xx}\varphi|(x,t')dxdt' \\
\leq &C\delta \int_0^t ||p_{xx}^R||_{L^\infty}^2 \int \varphi^2(x,t')dxdt' + C\delta \int_0^t \int \phi_{1xx}^2(x,t')dxdt' \\
\leq &C\delta N(T)^2 + C\delta \int_0^t \int \phi_{1xx}^2(x,t')dxdt'.
\end{aligned} \tag{4.87}$$

At the same time, we have

$$C\int_0^t \int |(p_x^R, v_x^R) u_x^R \phi_{1xx}\varphi_x|(x,t')dxdt' \leq C\delta N(T)^2 \tag{4.88}$$

by the same idea as (4.68), which also induces

$$\begin{aligned}
&C\int_0^t \int |(p_x^R, v_x^R) u_x^R \phi_{1xx}\varphi_x \varphi|(x,t')dxdt' \\
\leq &CN(T) \int_0^t \int |(p_x^R, v_x^R) u_x^R \phi_{1xx}\varphi_x|(x,t')dxdt' \\
\leq &C\delta N(T)^3.
\end{aligned} \tag{4.89}$$

Using the similar steps to (4.83), one can get

$$C\int_0^t \int |(p_x^R, v_x^R) u_x^R v_x^R \phi_{1xx}\varphi|(x,t')dxdt' \leq C\delta^2 N(T)^2 + C\delta \int_0^t \int \phi_{1xx}^2(x,t')dxdt'. \tag{4.90}$$

Since

$$\begin{aligned} &C\int_0^t\int |u_x^R\phi_{1xx}\varphi_x^2|(x,t')dxdt' \\ \leq &CN(T)\int_0^t ||u_x^R||_{L^\infty}^2\int \varphi_x^2(x,t')dxdt'+CN(T)\int_0^t\int \phi_{1xx}^2(x,t')dxdt' \\ \leq &C\delta N(T)^2+CN(T)\int_0^t\int \phi_{1xx}^2(x,t')dxdt' \end{aligned} \tag{4.91}$$

establishes, which also implies

$$\begin{aligned} &C\int_0^t\int |u_x^R\phi_{1xx}\varphi_x^2\varphi|(x,t')dxdt' \\ \leq &CN(T)\int_0^t\int |u_x^R\phi_{1xx}\varphi_x^2|(x,t')dxdt' \\ \leq &C\delta N(T)^2+CN(T)\int_0^t\int \phi_{1xx}^2(x,t')dxdt'. \end{aligned} \tag{4.92}$$

One has

$$C\int_0^t\int |u_{xx}^R\phi_{1xx}\varphi_x|(x,t')dxdt'\leq C\delta^{1/8}N(T)^2 \tag{4.93}$$

by the similar idea as (4.69), which tells us

$$\begin{aligned} C\int_0^t\int |u_{xx}^R\phi_{1xx}\varphi_x\varphi|(x,t')dxdt' \leq &CN(T)\int_0^t\int |u_{xx}^R\phi_{1xx}\varphi_x|(x,t')dxdt' \\ \leq &C\delta^{1/8}N(T)^2. \end{aligned} \tag{4.94}$$

The following inequality holds

$$\begin{aligned} C\int_0^t\int |u_x^R\phi_{1xx}\varphi_{xx}|(x,t')dxdt' \leq &C\int_0^t\int |u_x^R(\phi_{1xx}^2+\varphi_{xx}^2)|(x,t')dxdt' \\ \leq &C\int_0^t ||u_x^R||_{L^\infty}\int (\phi_{1xx}^2+\varphi_{xx}^2)(x,t')dxdt' \\ \leq &C\delta^{3/2}TN(T)^2+C\delta\int_0^t\int \phi_{1xx}^2(x,t')dxdt', \end{aligned} \tag{4.95}$$

which implies

$$\begin{aligned} &C\int_0^t\int |u_x^R\phi_{1xx}\varphi_{xx}\varphi|(x,t')dxdt' \\ \leq &CN(T)\int_0^t\int |u_x^R\phi_{1xx}\varphi_{xx}|(x,t')dxdt' \\ \leq &C\delta^{3/2}TN(T)^3+C\delta N(T)\int_0^t\int \phi_{1xx}^2(x,t')dxdt'. \end{aligned} \tag{4.96}$$

The only rest term of R_7 can be controlled by

$$C\int_0^t\int |u_{xxx}^R\phi_{1xx}\varphi|(x,t')dxdt'\leq C\delta N(T)^2+C\delta\int_0^t\int \phi_{1xx}^2(x,t')dxdt', \tag{4.97}$$

due to the similar step as (4.75). Summing up from (4.87) to (4.97)

$$
\begin{aligned}
&C\int_0^t\int R_8(x,t')dxdt'\\
\le&C(\delta^{1/8}N(T)^2+\delta^{3/2}TN(T)^2)+C(\delta+N(T))\int_0^t\int\phi_{1xx}^2(x,t')dxdt'.
\end{aligned}\tag{4.98}
$$

For R_9, it is clear to see

$$
R_9\le C(|v_{xx}^R\chi\phi_{1xx}|+|\varphi_{xx}\chi\phi_{1xx}|+|(v_x^R)^2\chi\phi_{1xx}|+|\varphi_x^2\chi\phi_{1xx}|+|v_x^R\chi_x\phi_{1xx}|+|\varphi_x\chi_x\phi_{1xx}|).\tag{4.99}
$$

Since we have the following inequalities

$$
C\int_0^t\int|v_{xx}^R\chi\phi_{1xx}|(x,t')dxdt'\le C\delta\int_0^t\int(\chi^2+\phi_{1xx}^2)(x,t')dxdt',\tag{4.100}
$$

$$
\begin{aligned}
&C\int_0^t\int|\varphi_{xx}\chi\phi_{1xx}|(x,t')dxdt'\\
\le&\frac{C}{N(T)}\int_0^t||\chi||_{L^\infty}^2\int\varphi_{xx}^2(x,t')dxdt'+CN(T)\int_0^t\int\phi_{1xx}^2(x,t')dxdt'\\
\le&CN(T)\int(\chi^2+\chi_x^2+\phi_{1xx}^2)(x,t')dxdt',
\end{aligned}\tag{4.101}
$$

$$
C\int_0^t\int|(v_x^R)^2\chi\phi_{1xx}|(x,t')dxdt'\le C\delta\int_0^t\int(\chi^2+\phi_{1xx}^2)(x,t')dxdt',\tag{4.102}
$$

$$
C\int_0^t\int|\varphi_x^2\chi\phi_{1xx}|(x,t')dxdt'\le CN(T)\int_0^t\int(\chi^2+\phi_{1xx}^2)(x,t')dxdt',\tag{4.103}
$$

$$
C\int_0^t\int|v_x^R\chi_x\phi_{1xx}|(x,t')dxdt'\le C\delta\int_0^t\int(\chi_x^2+\phi_{1xx}^2)(x,t')dxdt',\tag{4.104}
$$

$$
C\int_0^t\int|\varphi_x\chi_x\phi_{1xx}|(x,t')dxdt'\le CN(T)\int_0^t\int(\chi_x^2+\phi_{1xx}^2)(x,t')dxdt',\tag{4.105}
$$

then

$$
C\int_0^t\int R_9(x,t')dxdt'\le C(\delta+N(T))\int_0^t\int(\chi^2+\chi_x^2+\phi_{1xx}^2)(x,t')dxdt'.\tag{4.106}
$$

It is a straightforward computation for R_{10} and R_{11} to get

$$
\begin{aligned}
C\int_0^t\int R_{10}(x,t')dxdt'\le&C\int_0^t\int|\psi_{1x}\psi_{1xx}^2|(x,t')dxdt'\\
\le&CN(T)\int_0^t\int\int\psi_{1xx}^2(x,t')dxdt'
\end{aligned}\tag{4.107}
$$

and

$$C\int_0^t\int R_{11}(x,t')dxdt' \le C\int_0^t\int |\chi\psi_{1xx}^2|(x,t')dxdt'$$
$$\le CN(T)\int_0^t\int\int \psi_{1xx}^2(x,t')dxdt'. \tag{4.108}$$

Next, we estimate $C\int_0^t\int R_{12}$, where

$$\begin{aligned}R_{12} \le & C(|u_x^R\varphi_{xx}\phi_{1xx}|+|u_x^Rv_{xx}^R\phi_{1xx}|+|\psi_{1x}v_{xx}^R\phi_{1xx}|+|\psi_{1x}\varphi_{xx}\phi_{1xx}|\\ &+|(v_x^R)^2u_x^R\phi_{1xx}|+|(v_x^R)^2\psi_{1x}\phi_{1xx}|+|\varphi_x^2\psi_{1x}\phi_{1xx}|+|u_x^R\varphi_x^2\phi_{1xx}|\\ &+|p_x^Ru_x^Rv_x^R\phi_{1xx}|+|p_x^R\psi_{1x}v_x^R\phi_{1xx}|+|u_x^Rv_x^R\phi_{1x}\phi_{1xx}|+|v_x^R\phi_{1x}\psi_{1x}\phi_{1xx}|\\ &+|\varphi_xp_x^Ru_x^R\phi_{1xx}|+|\varphi_xp_x^R\psi_{1x}\phi_{1xx}|+|\varphi_x\phi_{1x}u_x^R\phi_{1xx}|+|\varphi_x\phi_{1x}\psi_{1x}\phi_{1xx}|\\ &+|u_{xx}^Rv_x^R\phi_{1xx}|+|v_x^R\psi_{xx}\phi_{1xx}|+|\varphi_xu_{xx}^R\phi_{1xx}|+|\varphi_x\psi_{1xx}\phi_{1xx}|).\end{aligned} \tag{4.109}$$

Then we have

$$\begin{aligned}&C\int_0^t\int |u_x^Rv_{xx}^R\phi_{1xx}|(x,t')dxdt'\\ \le & C\int_0^t\int u_x^R(v_{xx}^R)^2(x,t')dxdt'+C\int_0^t\int u_x^R\phi_{1xx}^2(x,t')dxdt'\\ \le & C\delta ln(1+t)+C\delta\int_0^t\int \phi_{1xx}^2(x,t')dxdt',\end{aligned} \tag{4.110}$$

$$\begin{aligned}&C\int_0^t\int |u_x^R(v_x^R)^2\phi_{1xx}|(x,t')dxdt'\\ \le & C\int_0^t\int u_x^R(v_x^R)^4(x,t')dxdt'+C\int_0^t\int u_x^R\phi_{1xx}^2(x,t')dxdt'\\ \le & C\delta ln(1+t)+C\delta\int_0^t\int \phi_{1xx}^2(x,t')dxdt',\end{aligned} \tag{4.111}$$

where $C\int_0^t\int |u_{xx}^Rv_x^R\phi_{1xx}|$ and $C\int_0^t\int |p_x^Ru_x^Rv_x^R\phi_{1xx}|$ can be controlled by the similar method to (4.110) and (4.111), respectively.

Then based on above equalities (4.110)-(4.111) with the inequalities (4.61), (4.62), (4.66)-(4.71), (4.74), (4.88), (4.91), (4.93), (4.95), one can obtain

$$\begin{aligned}C\int_0^t\int R_{12}(x,t')dxdt \le & C(\delta ln(1+t)+\delta^{1/8}N(T)^2+\delta^{3/2}TN(T)^2)\\ &+C(\delta+N(T))\int_0^t\int (\phi_{1x}^2+\psi_{1x}^2+\phi_{1xx}^2+\psi_{1xx}^2)(x,t')dxdt'.\end{aligned} \tag{4.112}$$

For R_{13}, it is easy to see

$$R_{13} \le C(|v_{xx}^R\chi\phi_{1xx}|+|(v_x^R)^2\chi\phi_{1xx}|+|\varphi_x^2\chi\phi_{1xx}|+|\varphi_{xx}\chi\phi_{1xx}|), \tag{4.113}$$

which implies

$$C\int_0^t\int R_{13}(x,t')dxdt \le C(\delta+N(T))\int_0^t\int (\chi^2+\chi_x^2+\phi_{1xx}^2)(x,t')dxdt' \tag{4.114}$$

by inequalities (4.100)-(4.103). For R_{14}, we have

$$C\int_0^t\int R_{14}(x,t')dxdt \le C\int_0^t\int(|v_x^R\chi_x\phi_{1xx}|+|\varphi_x\chi_x\phi_{1xx}|)(x,t')dxdt \le C(\delta+N(T))\int_0^t\int(\chi_x^2+\phi_{1xx}^2)(x,t')dxdt' \quad (4.115)$$

which comes from (4.104) and (4.105). It is obvious that

$$C\int_0^t\int R_{15}(x,t')dxdt \le C\int_0^t\int(|u_x^R\chi_{xx}\phi_{1xx}|+|\psi_x\chi_{xx}\phi_{1xx}|)(x,t')dxdt \le C(\delta+N(T))\int_0^t\int(\chi_{xx}^2+\phi_{1xx}^2)(x,t')dxdt' \quad (4.116)$$

holds for R_{15}. Here we can use the same method as $C\int_0^t\int R_{12}$ to deal with $C\int_0^t\int R_{16}$ (we only need to use χ_{xx} to replace ϕ_{1xx}), then we can get

$$C\int_0^t\int R_{16}(x,t')dxdt \le C(\delta ln(1+t)+\delta^{1/8}N(T)^2+\delta^{3/2}TN(T)^2) +C(\delta+N(T))\int_0^t\int(\phi_{1x}^2+\psi_{1x}^2+\psi_{1xx}^2+\chi_{xx}^2)(x,t')dxdt'. \quad (4.117)$$

After a simple calculation, one obtains that

$$R_{17} \le C(|p_{xx}^R u_x^R\chi_{xx}|+|p_x^R u_{xx}^R\chi_{xx}|+|p_x^R u_x^R v_x^R\chi_{xx}|+|u_{xxx}^R\chi_{xx}| +|u_{xx}^R v_x^R\chi_{xx}|+|v_{xx}^R u_x^R\chi_{xx}|+|(v_x^R)^2u_x^R\chi_{xx}|), \quad (4.118)$$

and

$$\begin{aligned}&C\int_0^t\int|p_{xx}^R u_x^R\chi_{xx}|(x,t')dxdt'\\ \le& C\int_0^t\int|u_x^R(p_{xx}^R)^2|(x,t')dxdt'+C\int_0^t\int|u_x^R\chi_{xx}^2|(x,t')dxdt'\\ \le& C\delta ln(1+t)+C\delta\int_0^t\int\chi_{xx}^2(x,t')dxdt',\end{aligned} \quad (4.119)$$

where $C\int_0^t\int|p_x^R u_{xx}^R\chi_{xx}|$, $C\int_0^t\int|u_{xx}^R v_x^R\chi_{xx}|$ and $C\int_0^t\int|v_{xx}^R u_x^R\chi_{xx}|$ can be controlled by the same method as (4.119). Moreover, we have

$$\begin{aligned}&C\int_0^t\int|(p_x^R,v_x^R)u_x^R v_x^R\chi_{xx}|(x,t')dxdt'\\ \le& C\delta\int_0^t\int(u_x^R v_x^R)^2(x,t')dxdt'+C\delta\int_0^t\int\chi_{xx}^2(x,t')dxdt'\\ \le& C\delta ln(1+t)+C\delta\int_0^t\int\chi_{xx}^2(x,t')dxdt',\end{aligned} \quad (4.120)$$

$$C\int_0^t\int|u_{xxx}^R\chi_{xx}|(x,t')dxdt' \le C\delta^2\int_0^t\int(u_x^R)^2(x,t')dxdt'+\frac{a_{10}}{16\alpha\tau}\int_0^t\int\frac{\chi_{xx}^2}{v^2}(x,t')dxdt'$$

$$\leq C\delta^3 ln(1+t)+\frac{a_{10}}{16\alpha\tau}\int_0^t\int\frac{\chi_{xx}^2}{v^2}(x,t')dxdt'. \tag{4.121}$$

Then for R_{17}, the above inequalities give

$$C\int_0^t\int R_{17}(x,t')dxdt\leq C\delta ln(1+t)+(C\delta+\frac{a_{10}}{16\alpha\tau})\int_0^t\int\frac{\chi_{xx}^2}{v^2}(x,t')dxdt'. \tag{4.122}$$

For $C\int_0^t\int R_{18}$, we follow the way to handle $C\int_0^t\int R_{13}$ to get

$$C\int_0^t\int R_{18}(x,t')dxdt\leq C(\delta+N(T))\int_0^t\int(\chi^2+\chi_x^2+\chi_{xx}^2)(x,t')dxdt'. \tag{4.123}$$

It is obvious to obtain

$$\begin{aligned}C\int_0^t\int R_{19}(x,t')dxdt&\leq C\int_0^t\int(|v_x^R\chi_x\chi_{xx}|+|\varphi_x\chi_x\chi_{xx}|)(x,t')dxdt\\&\leq C(\delta+N(T))\int_0^t\int(\chi_x^2+\chi_{xx}^2)(x,t')dxdt',\end{aligned} \tag{4.124}$$

and

$$\begin{aligned}C\int_0^t\int R_{20}(x,t')dxdt&\leq C\int_0^t\int|\psi_{1x}\chi_{xx}^2|)(x,t')dxdt\\&\leq CN(T)\int_0^t\int\chi_{xx}^2(x,t')dxdt'.\end{aligned} \tag{4.125}$$

In order to estimate $C\int_0^t\int R_{21}(x,t')dxdt$, we are first to analysis R_{21},

$$R_{21}=\frac{2}{\alpha_f\tau}\zeta_{1xx}\left[\frac{2(\chi_x^2+\chi\chi_{xx})}{T_1T_2}+4\left(\frac{1}{T_1T_2}\right)_x\chi\chi_x+\left(\frac{1}{T_1T_2}\right)_{xx}\chi^2\right],$$

which implies

$$\begin{aligned}|R_{21}|&\leq C|\zeta_{1xx}|(|\chi_x^2|+|\chi\chi_{xx}|+|T_{1x}\chi\chi_x|+|T_{2x}\chi\chi_x|+|T_{1xx}\chi^2|+|T_{2xx}\chi^2|+|T_{1x}^2\chi^2|+|T_{2x}^2\chi^2|)\\&\leq C(|\zeta_{1xx}\chi^2|+|\zeta_{1xx}\chi_{xx}\chi|+|\zeta_{1xx}\chi_x^2|+|T_{1xx}\zeta_{1xx}\chi^2|+|T_{2xx}\zeta_{1xx}\chi^2|),\end{aligned} \tag{4.126}$$

where we have used Young's inequality and (4.44). Since

$$\begin{aligned}|T_{1xx}|&\leq C(|p_{xx}^R|+|v_{xx}^R|+|\phi_{1xx}|+|\varphi_{xx}|+|(p_x^R)^2|+|(v_x^R)^2|+|\phi_{1x}^2|+|\varphi_x^2|),\\|T_{2xx}|&\leq C(|T_{1xx}|+|\chi_{xx}|)\\&\leq C(|p_{xx}^R|+|v_{xx}^R|+|\chi_{xx}|+|\phi_{1xx}|+|\varphi_{xx}|+|(p_x^R)^2|+|(v_x^R)^2|+|\phi_{1x}^2|+|\varphi_x^2|)\end{aligned}$$

hold, then

$$\begin{aligned}||T_{1xx}||^2&\leq C(\delta^{7/2}+N(T)^2),\\||T_{1xx}||^2&\leq C(\delta^{7/2}+N(T)^2),\end{aligned} \tag{4.127}$$

due to Lemma 2.1 and (4.1). Obviously

$$C\int_0^t\int|\zeta_{1xx}\chi^2|(x,t')dxdt'$$

$$\le \frac{C}{N(T)} \int_0^t ||\chi||_{L^\infty}^2 \int \zeta_{1xx}^2(x,t')dxdt' + CN(T)\int_0^t \int \chi^2(x,t')dxdt'$$
$$\le CN(T)\int_0^t \int (\chi^2+\chi_x^2)(x,t')dxdt', \tag{4.128}$$

$$C\int_0^t \int |\zeta_{1xx}\chi_x^2|(x,t')dxdt'$$
$$\le C\int_0^t ||\chi_x||_{L^\infty}^2 \int \zeta_{1xx}^2(x,t')dxdt' + \frac{a_{10}}{16\alpha\tau}\int_0^t \int \frac{\chi_{xx}^2}{v^2}(x,t')dxdt'$$
$$\le CN^2(T)\int_0^t \int \chi_x^2(x,t')dxdt' + (\frac{a_{10}}{16\alpha\tau} + CN^2(T))\int_0^t \int \frac{\chi_{xx}^2}{v^2}(x,t')dxdt', \tag{4.129}$$

$$C\int_0^t \int |\zeta_{1xx}\chi_{xx}\chi|(x,t')dxdt'$$
$$\le C\int_0^t ||\chi||_{L^\infty}^2 \int \zeta_{1xx}^2(x,t')dxdt' + \frac{a_{10}}{16\alpha\tau}\int_0^t \int \frac{\chi_{xx}^2}{v^2}(x,t')dxdt'$$
$$\le CN^2(T)\int_0^t \int (\chi^2+\chi_x^2)dxdt' + \frac{a_{10}}{16\alpha\tau}\int_0^t \int \frac{\chi_{xx}^2}{v^2}dxdt', \tag{4.130}$$

and

$$C\int_0^t \int |(T_{1xx},T_{2xx})\zeta_{1xx}\chi^2|(x,t')dxdt'$$
$$\le C\int_0^t ||\chi||_{L^\infty}^2 \int (T_{1xx}^2,T_{2xx}^2)(x,t')dxdt' + C\int_0^t ||\chi||_{L^\infty}^2 \int \zeta_{1xx}^2(x,t')dxdt'$$
$$\le C(\delta^{7/2}+N(T)^2)\int_0^t \int (\chi^2+\chi_x^2)(x,t')dxdt', \tag{4.131}$$

where we have used the facts (4.127). Therefore we have

$$C\int_0^t \int R_{21}(x,t')dxdt \le C(\delta+N(T))\int_0^t \int (\chi^2+\chi_x^2)(x,t')dxdt'$$
$$+ (\frac{a_{10}}{16\alpha\tau} + CN^2(T))\int_0^t \int \frac{\chi_{xx}^2}{v^2}(x,t')dxdt'. \tag{4.132}$$

Combining the estimates for $\int_0^t \int R_i(x,t')dxdt'$, $1 \le i \le 21$, the following inequality is obtained

$$||(\phi_{1xx},\psi_{1xx},\chi_{xx},\zeta_{1xx})||^2(t) + \int_0^t \int \left(u_x^R(\phi_{1xx}^2,\psi_{xx}^2,\chi_{xx}^2) + \chi_{xx}^2\right)(x,t')dxdt'$$
$$\le C\left(N^2(0) + \delta ln(1+t) + \delta^{1/8}N^2(T) + \delta^{3/2}TN(T)^2 + N(T)^3\right)$$
$$+ C(\delta+N(T))\int_0^t \int (\phi_{1xx}^2+\psi_{1xx}^2), \tag{4.133}$$

where we also invoke the results of Lemmas 4.1 and 4.2. Therefore, only $\int_0^t \int (\phi_{1xx}^2,\psi_{1xx}^2)$ is left for us to estimate. Here, one can use the same idea as in Subsection 4.2 to estimate

$\int_0^t\int(\phi_{1xx}^2,\psi_{1xx}^2)$. A simple calculation yields

$$\begin{aligned}\psi_{1xx}^2=&-\frac{\alpha}{\alpha_f}\left(\frac{1}{p^R}\psi_{1xx}\chi_x\right)_t+\frac{\alpha\gamma}{\alpha_f p^R v^R}u_x^R\psi_{1xx}\chi_x\\&-\frac{\alpha}{\alpha_f}\left(\left(\frac{1}{p^R}\phi_{1xx}\chi_x\right)_x+\frac{p_x^R\phi_{1xx}\chi_x}{(p^R)^2}-\frac{\phi_{1xx}\chi_{xx}}{p^R}\right)\\&-\frac{(p^Ru_x^R)_x}{p^R}\psi_{1xx}-\frac{(\phi_1u_x^R)_x}{p^R}\psi_{1xx}-\frac{(\phi_1\psi_{1x})_x}{p^R}\psi_{1xx}-\frac{\alpha+\alpha_f}{\tau\alpha_f p^R}\psi_{1xx}\chi_x-\frac{p_x^R\psi_{1x}\psi_{1xx}}{p^R}\\\le&-\frac{\alpha}{\alpha_f}\left(\frac{1}{p^R}\psi_{1xx}\chi_x\right)_t-\frac{\alpha}{\alpha_f}\left(\frac{1}{p^R}\phi_{1xx}\chi_x\right)_x+C(|u_x^R\psi_{1xx}\chi_x|+|\chi_{xx}\phi_{1xx}|\\&+|p_x^R\phi_{1xx}\chi_x|+|p_x^Ru_x^R\psi_{1xx}|+|u_{xx}^R\psi_{1xx}|+|u_x^R\phi_{1x}\psi_{1xx}|+|u_{xx}^R\phi_1\psi_{1xx}|\\&+|\phi_{1x}\psi_{1xx}\psi_{1x}|+|\phi_1\psi_{1xx}^2|+|\chi_x\psi_{1xx}|+|p_x^R\psi_{1x}\psi_{1xx}|).\end{aligned}$$

The following inequalities are set up,

$$C\int_0^t\int|u_x^R\psi_{1xx}\chi_x|(x,t')dxdt'\le C\delta\int_0^t\int(\psi_{1xx}^2+\chi_x^2)(x,t')dxdt',\tag{4.134}$$

$$\begin{aligned}C\int_0^t\int|p_x^R\phi_{1xx}\chi_x|(x,t')dxdt'&\le C\int_0^t||p_x^R||_{L^\infty}^2\int\phi_{1xx}^2(x,t')dxdt'+C\int_0^t\int\chi_x^2(x,t')dxdt'\\&\le C\delta N(T)^2+C\int_0^t\int\chi_x^2(x,t')dxdt',\end{aligned}\tag{4.135}$$

$$\begin{aligned}C\int_0^t\int|p_x^Ru_x^R\psi_{1xx}|(x,t')dxdt'&\le C\int_0^t\int((p_x^R)^2u_x^R+u_x^R\psi_{1xx}^2)(x,t')dxdt'\\&\le C\delta ln(1+t)+C\delta\int_0^t\int\psi_{1xx}^2(x,t')dxdt',\end{aligned}\tag{4.136}$$

$$\begin{aligned}C\int_0^t\int|u_{xx}^R\psi_{1xx}|(x,t')dxdt'&\le C\int_0^t\int(u_{xx}^R)^2(x,t')dxdt'+\frac{1}{16}\int_0^t\int\psi_{1xx}^2(x,t')dxdt'\\&\le C\delta ln(1+t)+\frac{1}{16}\int_0^t\int\psi_{1xx}^2(x,t')dxdt',\end{aligned}\tag{4.137}$$

$$\begin{aligned}C\int_0^t\int|u_{xx}^R\phi_1\psi_{1xx}|(x,t')dxdt'&\le CN(T)\int_0^t\int|u_{xx}^R\psi_{1xx}|(x,t')dxdt'\\&\le C\delta N(T)ln(1+t)+CN(T)\int_0^t\int\psi_{1xx}^2(x,t')dxdt',\end{aligned}\tag{4.138}$$

$$C\int_0^t\int|u_x^R\phi_{1x}\psi_{1xx}|(x,t')dxdt'\le C\delta\int_0^t\int(\phi_{1x}^2+\psi_{1xx}^2)(x,t')dxdt',\tag{4.139}$$

$$C\int_0^t\int|\phi_{1x}\psi_{1xx}\psi_{1x}|(x,t')dxdt'\le CN(T)\int_0^t\int(\phi_{1x}^2+\psi_{1xx}^2)(x,t')dxdt',\tag{4.140}$$

$$C\int_0^t\int|\phi_1\psi_{1xx}^2|(x,t')dxdt'\le CN(T)\int_0^t\int\psi_{1xx}^2(x,t')dxdt',\tag{4.141}$$

$$C\int_0^t\int|\chi_x\psi_{1xx}|(x,t')dxdt' \le C\int_0^t\int\chi_x^2(x,t')dxdt'+\frac{1}{16}\int_0^t\int\psi_{1xx}^2(x,t')dxdt', \quad (4.142)$$

$$C\int_0^t\int|p_x^R\psi_{1x}\psi_{1xx}|(x,t')dxdt' \le C\delta\int_0^t\int(\psi_{1x}^2+\psi_{1xx}^2)(x,t')dxdt', \quad (4.143)$$

which induce that

$$\begin{aligned}&\int_0^t\int\psi_{1xx}^2(x,t')dxdt'\\ &\le C\left(N^2(0)+\delta ln(1+t)+\delta^{1/8}N^2(T)+\delta^{3/2}TN(T)^2+N(T)^3\right)\\ &\quad +C\int_0^t\int|\phi_{1xx}\chi_{xx}|(x,t')dxdt'+C(\delta+N(T))\int_0^t\int\phi_{1xx}^2(x,t')dxdt', \end{aligned} \quad (4.144)$$

by Lemma 4.2, (4.133) and the smallness of δ and $N(T)$. For ϕ_{1xx}^2, one has

$$\begin{aligned}\phi_{1xx}^2=&-(\phi_{1xx}\psi_{1x})_t+(\phi_{1xt}\psi_{1x})_x+\beta\left(\frac{p^R}{v}\psi_{1x}\right)_x\psi_{1xx}+\beta\left(\frac{p^R}{v}u_x^R\right)_x\psi_{1xx}\\ &+\beta\left(\frac{\phi_1}{v}\psi_{1x}\right)_x\psi_{1xx}+\beta\left(\frac{\phi_1}{v}u_x^R\right)_x\psi_{1xx}-\gamma\left(\frac{p^R}{v^R}u_x^R\right)_x\psi_{1xx}\\ &+\frac{2}{\tau\alpha}\left(\frac{\chi}{v}\right)_x\psi_{1xx}\\ \le&-(\phi_{1xx}\psi_{1x})_t+(\phi_{1xt}\psi_{1x})_x+C(|p_x^R\psi_{1x}\psi_{1xx}|+|\psi_{1xx}^2|+|v_x^R\psi_{1x}\psi_{1xx}|\\ &+|\varphi_x\psi_{1x}\psi_{1xx}|+|p_x^Ru_x^R\psi_{1xx}|+|u_{xx}^R\psi_{1xx}|+|v_x^Ru_x^R\psi_{1xx}|+|\varphi_xu_x^R\psi_{1xx}|\\ &+|\phi_{1x}\psi_{1x}\psi_{1xx}|+|\phi_1\psi_{1xx}^2|+|v_x^R\phi_1\psi_{1x}\psi_{1xx}|+|\varphi_x\phi_1\psi_{1x}\psi_{1xx}|\\ &+|\phi_{1x}u_x^R\psi_{1xx}|+|\phi_1u_{xx}^R\psi_{1xx}|+|\phi_1u_x^Rv_x^R\psi_{1xx}|+|\phi_1u_x^R\varphi_x\psi_{1xx}|\\ &+|\chi_x\psi_{1xx}|+|v_x^R\chi\psi_{1xx}|+|\varphi_x\chi\psi_{1xx}|).\end{aligned}$$

We also have the following estimates

$$C\int_0^t\int|v_x^R\psi_{1x}\psi_{1xx}|(x,t')dxdt' \le C\delta\int_0^t\int(\psi_{1x}^2+\psi_{1xx}^2)(x,t')dxdt', \quad (4.145)$$

$$C\int_0^t\int|\varphi_x\psi_{1x}\psi_{1xx}|(x,t')dxdt' \le CN(T)\int_0^t\int(\psi_{1x}^2+\psi_{1xx}^2)(x,t')dxdt', \quad (4.146)$$

$$\begin{aligned}C\int_0^t\int|v_x^Ru_x^R\psi_{1xx}|(x,t')dxdt' &\le C\delta\int_0^t\int((u_x^R)^2+\psi_{1xx}^2)(x,t')dxdt'\\ &\le C\delta ln(1+t)+C\delta\int_0^t\int\psi_{1xx}^2(x,t')dxdt', \end{aligned} \quad (4.147)$$

and

$$C\int_0^t\int|u_x^R\varphi_x\psi_{1xx}|(x,t')dxdt'$$

$$\begin{aligned}
&\leq C\int_0^t\int((u_x^R)^2\varphi_x^2+\psi_{1xx}^2)(x,t')dxdt'\\
&\leq C\int_0^t||u_x^R||_{L^\infty}^2\int\varphi_x^2(x,t')dxdt'+C\int_0^t\int\psi_{1xx}^2(x,t')dxdt'\\
&\leq C\delta N^2(T)+C\int_0^t\int\psi_{1xx}^2(x,t')dxdt',
\end{aligned}\tag{4.148}$$

which implies

$$\begin{aligned}
C\int_0^t\int|v_x^R\phi_1\psi_{1x}\psi_{1xx}|(x,t')dxdt'\leq &CN(T)\int_0^t\int|v_x^R\psi_{1x}\psi_{1xx}|(x,t')dxdt'\\
\leq &C\delta N(T)\int_0^t\int(\psi_{1x}^2+\psi_{1xx}^2)(x,t')dxdt',
\end{aligned}\tag{4.149}$$

$$\begin{aligned}
C\int_0^t\int|\phi_1\varphi_x\psi_{1x}\psi_{1xx}|(x,t')dxdt'\leq &CN(T)\int_0^t\int|\varphi_x\psi_{1x}\psi_{1xx}|(x,t')dxdt'\\
\leq &CN^2(T)\int_0^t\int(\psi_{1x}^2+\psi_{1xx}^2)(x,t')dxdt',
\end{aligned}\tag{4.150}$$

$$\begin{aligned}
&C\int_0^t\int|\phi_1v_x^Ru_x^R\psi_{1xx}|(x,t')dxdt'\\
\leq &CN(T)\int_0^t\int|v_x^Ru_x^R\psi_{1xx}|(x,t')dxdt'\\
\leq &C\delta N(T)\int_0^t\int((u_x^R)^2+\psi_{1xx}^2)(x,t')dxdt'\\
\leq &C\delta N(T)ln(1+t)+C\delta N(T)\int_0^t\int\psi_{1xx}^2(x,t')dxdt',
\end{aligned}\tag{4.151}$$

$$\begin{aligned}
C\int_0^t\int|\phi_1u_x^R\varphi_x\psi_{1xx}|(x,t')dxdt'\leq &CN(T)\int_0^t\int|u_x^R\varphi_x\psi_{1xx}|(x,t')dxdt'\\
\leq &C\delta N^3(T)+CN(T)\int_0^t\int\psi_{1xx}^2(x,t')dxdt',
\end{aligned}\tag{4.152}$$

respectively. Moreover, one can obtain

$$C\int_0^t\int|v_x^R\chi\psi_{1xx}|(x,t')dxdt'\leq C\delta\int_0^t\int(\chi^2+\psi_{1xx}^2)(x,t')dxdt',\tag{4.153}$$

and

$$C\int_0^t\int|\varphi_x\chi\psi_{1xx}|(x,t')dxdt'\leq CN(T)\int_0^t\int(\chi^2+\psi_{1xx}^2)(x,t')dxdt'.\tag{4.154}$$

Putting (4.136)-(4.154) together, then one has

$$\begin{aligned}
\int_0^t\int\phi_{1xx}^2(x,t')dxdt'\leq &C\left(N^2(0)+\delta ln(1+t)+\delta^{1/8}N^2(T)+\delta^{3/2}TN(T)^2+N(T)^3\right)\\
&+C\int_0^t\int\psi_{1xx}^2(x,t')dxdt'
\end{aligned}$$

$$\leq C\left(N^2(0)+\delta ln(1+t)+\delta^{1/8}N^2(T)+\delta^{3/2}TN(T)^2+N(T)^3\right) + C\int_0^t\int|\phi_{1xx}\chi_{xx}|(x,t')dxdt'. \tag{4.155}$$

The term $C\int_0^t\int|\phi_{1xx}\chi_{xx}|(x,t')dxdt'$ can have an upper bound as

$$C\int_0^t\int|\phi_{1xx}\chi_{xx}|(x,t')dxdt' \leq C\int_0^t\int\chi_{xx}^2(x,t')dxdt'+\frac{1}{16}\int_0^t\int\phi_{1xx}^2(x,t')dxdt', \tag{4.156}$$

which, with (4.144) and (4.155), implies

$$\int_0^t\int(\phi_{1xx}^2,\psi_{1xx}^2)(x,t')dxdt' \leq C\left(N^2(0)+\delta ln(1+t)+\delta^{1/8}N^2(T)+\delta^{3/2}TN(T)^2+N(T)^3\right) \tag{4.157}$$

by the smallness of δ and $N(T)$. Combining with (4.133), we finished our proof. □

4.4. The estimate for $||(\varphi_x,\varphi_{xx})||_{L^2(\mathbb{R}^1)}$.

LEMMA 4.4. *Assume that the conditions in Proposition 4.1 hold. Then there exists a constant $C>0$ independent of T such that*

$$||(\varphi_x,\varphi_{xx})||^2(t)\leq C\left(N^2(0)+\delta^{5/2}+\delta^{1/8}N^2(T)+\delta ln(1+t)+\delta^{3/2}TN(T)^2+N(T)^3\right),$$

for $0\leq t\leq T$.

Proof. We follow the idea of [1], and it is easy to see $v=v(p,\chi,s)$ by virtue of (3.4) and (3.5). Hence, one has

$$\begin{aligned}|v_x| &\leq C(|p_x|+|\chi_x|+|s_x|)\\ &\leq C(|p_x^R|+|\chi_x|+|\phi_{1x}|+|\zeta_{1x}|),\\ |v_{xx}| &\leq C(|p_{xx}|+|\chi_{xx}|+|s_{xx}|+|p_x|^2+|\chi_x|^2+|s_x|^2)\\ &\leq C(|p_{xx}^R|+|\chi_{xx}|+|\zeta_{1xx}|+|p_x^R|^2+|\chi_x|^2+|\phi_{1x}|^2+|\zeta_{1x}|^2).\end{aligned}$$

Here invoking the results of Lemmas 2.1, 4.2 and 4.3, the following inequalities are established,

$$\begin{aligned}||\varphi_x||^2 &\leq C(||p_x^R||^2+||v_x^R||^2+||\chi_x||^2+||\phi_{1x}||^2+||\zeta_{1x}||^2)\\ &\leq C(N^2(0)+\delta^{5/2}+\delta^{1/8}N^2(T)+\delta ln(1+t)+\delta^{3/2}TN(T)^2+N(T)^3),\\ ||\varphi_{xx}||^2 &\leq C(||p_{xx}^R||^2+||v_{xx}^R||^2+||\chi_{xx}||^2+||\zeta_{1xx}||^2+||\chi_x||^2+||\phi_{1x}||^2+||\zeta_{1x}||^2)\\ &\leq C(N^2(0)+\delta^{7/2}+\delta^{1/8}N^2(T)+\delta ln(1+t)+\delta^{3/2}TN(T)^2+N(T)^3).\end{aligned} \tag{4.158}$$

Therefore, the desired estimates are obtained if δ and $N(T)$ are small enough. □

Proof. (**Proof of Theorem 1.2.**) The following inequality follows from Lemmas 4.1-4.4:

$$N^2(T)\leq C\left(N^2(0)+\delta^{5/2}+\delta^{1/8}N^2(T)+\delta ln(1+T)+\delta^{3/2}TN(T)^2+N(T)^3\right),$$

that is,

$$N^2(T) \le C\left(N^2(0) + \delta^{5/2} + \delta ln(1+T) + \delta^{3/2} T N(T)^2\right) \tag{4.159}$$

by using the smallness of δ and $N(T)$. If

$$C\delta^{3/2} T N(T)^2 \le \frac{N^2(T)}{8}, \tag{4.160}$$

i.e.,

$$T \le \frac{1}{8C}\delta^{-3/2}. \tag{4.161}$$

In this case,

$$C\delta ln(1+T) \le C\delta ln(1 + \frac{1}{8C}\delta^{-3/2}), \tag{4.162}$$

and

$$\delta^{5/2} \le C\delta ln(1 + \frac{1}{8C}\delta^{-3/2}),$$

for small δ. This finishes the proof of proposition if $\delta \le \delta_1$. Consequently, Theorem 1.2 has been proved. □

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