COMMUN. MATH. SCI.
Vol. 17, No. 7, pp. 1841–1860

GLOBAL STABILITY OF LARGE STEADY-STATES FOR AN ISENTROPIC EULER-MAXWELL SYSTEM IN $\mathbb{R}^3$*

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Abstract. This paper concerns the global existence and stability of smooth solutions near large steady-states for an isentropic Euler-Maxwell system in $\mathbb{R}^3$. This system describes the dynamics of electrons in magnetized plasmas where the ion density is a given smooth function with a positive lower bound, but without any restriction on the size. We establish the well-posedness of large steady-state solutions with zero velocity in $\mathbb{R}^3$. It is achieved through a study for a semilinear elliptic equation by using variational methods. For the initial data close to the steady-state solutions, we solve the stability problem by means of classical energy estimates and an anti-symmetric matrix technique together with an induction argument on the order of the derivatives of solutions with respect to the time and space variables.

Keywords. Euler-Maxwell system; global stability; large steady-state solution; energy estimate.

AMS subject classifications. 35B40; 35Q60; 35Q35.

1. Introduction and main results We study a global stability problem for an isentropic Euler-Maxwell system in $\mathbb{R}^3$. The system is a hydrodynamic model in plasma physics to describe the dynamics of electrons. Let n, $u=(u_1,u_2,u_3)^T$, E and B be the density, the velocity of the electrons, the electric field and magnetic field of the magnetized plasma, respectively. The system satisfied by these variables reads (see [2,4])

$$\begin{cases} \partial_t n+\operatorname{div}(nu)=0, \\ \partial_t(nu)+\operatorname{div}(nu\otimes u)+\nabla p(n)=-nu-n(E+u\times B), \\ \partial_t E-\nabla\times B=nu, \qquad \operatorname{div}E=b(x)-n, \\ \partial_t B+\nabla\times E=0, \qquad \operatorname{div}B=0, \end{cases} \tag{1.1}$$

for the time $t>0$ and the position $x\in\mathbb{R}^3$, where $b(x)$ and p stand for the ion density and the pressure function, respectively.

We assume that p is sufficiently smooth and strictly increasing on $(0,+\infty)$. This covers the usual case for ideal gas $p(n)=An^\gamma$, where $A>0$ and $\gamma\geq 1$ are constants. The system is supplemented by the following initial condition:

$$t=0: \quad (n,u,E,B)=\big(n_0(x),u_0(x),E_0(x),B_0(x)\big), \quad x\in\mathbb{R}^3. \tag{1.2}$$

Throughout this paper, we assume that

$$\operatorname{div}E_0=b(x)-n_0, \quad \operatorname{div}B_0=0.$$

Then, in (1.1) the constraint equations hold for all positive time:

$$\operatorname{div}E=b(x)-n, \quad \operatorname{div}B=0, \quad \forall t>0.$$

*Received: November 15, 2018; Accepted (in revised form): May 9, 2019. Communicated by Feimin Huang.

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For smooth solutions in any non-vacuum field, the momentum equation in (1.1) can be written as

$$\partial_t u + u\cdot\nabla u + \nabla h(n) = -u - E - u\times B, \tag{1.3}$$

where h is the enthalpy function defined by

$$h'(n) = \frac{p'(n)}{n}.$$

Now consider steady-state solutions $(\bar n, \bar u, \bar E, \bar B)$ only depending on x, with zero velocity $\bar u = 0$ and $\bar n > 0$. By (1.1), we have

$$\begin{cases} \nabla h(\bar n) = -\bar E, \\ \nabla\times\bar E = 0, \quad \mathrm{div}\bar E = b(x) - \bar n, \\ \nabla\times\bar B = 0, \quad \mathrm{div}\bar B = 0. \end{cases} \tag{1.4}$$

In (1.4), the equations for $\bar B$ imply that $\bar B$ is a constant vector, and the first equation together with $\mathrm{div}\bar E = b(x) - \bar n$ implies that $\bar n$ satisfies a second-order elliptic equation

$$-\Delta h(\bar n) + \bar n = b(x), \quad \text{in } \mathbb{R}^3. \tag{1.5}$$

Moreover, there exists a potential function $\bar\phi$ such that

$$\bar E = -\nabla\bar\phi, \quad \bar\phi = h(\bar n). \tag{1.6}$$

To precisely show our stability results, we introduce some notations. For any integer $s\in\mathbb{N}$, we denote by H^s, L^2 and L^∞ the usual Sobolev spaces $H^s(\mathbb{R}^3)$, $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$, and by $\|\cdot\|_s$, $\|\cdot\|$ and $\|\cdot\|_\infty$ the corresponding norms, respectively. For a multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^3$, we denote

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}}, \quad \text{with } |\alpha| = \alpha_1+\alpha_2+\alpha_3.$$

For any given time $T>0$, let $B_{s,T}$ be the Banach space defined by

$$B_{s,T} = \bigcap_{k=0}^{s} C^k([0,T];H^{s-k}),$$

equipped with the norm

$$\|v\|_{B_{s,T}} = \max_{0\le t\le T} |||v(t,\cdot)|||_s, \quad \forall v\in B_{s,T},$$

where

$$|||v(t,\cdot)|||_s = \Big(\sum_{|\alpha|+k\le s} \|\partial_t^k\partial_x^\alpha v(t,\cdot)\|^2\Big)^{\frac12}.$$

System (1.1) for variables (n,u,E,B) is symmetrizable hyperbolic when $n>0$. The local existence and uniqueness of smooth solutions was established by Lax [18] and Kato [16] (see also [22]). It can be stated as follows. Let $s\ge 3$ be an integer and $(n_0-\bar n, u_0, E_0-\bar E, B_0-\bar B)\in H^s$ satisfying $n_0\ge \text{const.}>0$. There exists $T_*>0$ such that the Cauchy problem (1.1)-(1.2) admits a unique solution (n,u,E,B) on the domain $[0,T_*]\times\mathbb{R}^3$, and

$$n-\bar{n},u,E-\bar{E},B-\bar{B}\in C\big([0,T_*];H^s\big)\cap C^1\big([0,T_*];H^{s-1}\big),\quad n\geq \text{const.}>0.$$

Using (1.1), we have $(n-\bar{n},u,E-\bar{E},B-\bar{B})\in B_{s,T_*}$.

The stability problem is to study the global existence of smooth solutions to problem (1.1)-(1.2) when (n_0,u_0,E_0,B_0) is sufficiently close to $(\bar{n},\bar{u},\bar{E},\bar{B})$. Such a problem has been investigated by many authors when b is a positive constant [5, 25, 29] or is a small perturbation of a constant [21]. Furthermore, various decay estimates of the solutions were obtained through the study on the pointwise behavior of the solutions to the linearized system, see [5,21,26,27,30]. We also refer to [1,15,28] for stability results for Euler-Poisson systems or to [11,13] for global existence for isentropic Euler-Maxwell systems without the velocity dissipation term but with generalized irrotationality constrains.

When b is large, only fewer related results can be found in the literature. A pioneering work on the stability was carried out for an isentropic Euler-Poisson system [14], by using an anti-symmetric matrix technique and the energy estimates for the divergence and the curl of the velocity. In [24], the last author of the present paper further employed an induction argument on the order of the derivatives of solutions in energy estimates and solved this problem for both the isentropic Euler-Poisson system and Euler-Maxwell system. These results were extended to the two-fluid isentropic systems [9] and to non-isentropic systems with temperature diffusion term [10] or without this term [19,20]. All these results are valid for the system considered in bounded domains with appropriate boundary conditions or for periodic solutions.

In this paper, we consider the stability to problem (1.1)-(1.2) for large b in the whole space. The first obstacle is to solve the stationary Equation (1.5) in the whole space. Here, we use variational methods to study this problem in any dimension $d\geq 1$:

$$-\Delta h(\bar{n})+\bar{n}=b(x),\quad \text{in } \mathbb{R}^d. \tag{1.7}$$

In a bounded domain $\Omega\subset\mathbb{R}^d$ with periodic, Dirichlet or Neumann boundary conditions, it is easy to prove that (1.7) admits a unique solution (see [6, 14]) due to the compact embedding $H^1(\Omega)\hookrightarrow L^2(\Omega)$, which is no longer valid in the whole space. Here, we recover compactness by fine estimates to establish the well-posedness of smooth solutions of (1.7) in $\mathbb{R}^d$. This result is shown in the following Theorem but what we state is in the sense of weak solutions of (1.7), which are defined by

$$\int_{\mathbb{R}^d}\big[\nabla h(n)\cdot\nabla g+(n-b)g\big]dx=0,\quad \forall\ g\in C_0^\infty(\mathbb{R}^d).$$

THEOREM 1.1. *Let $q\geq 1$ be an integer. Suppose that*
(i) $b\in L^\infty(\mathbb{R}^d)$ and $b(x)\geq$ const.>0, a.e. $x\in\mathbb{R}^d$,
(ii) $\nabla b\in H^{q-1}(\mathbb{R}^d)$,
(iii) $h\in C^q(0,+\infty)$ and $h'(n)>0$ for all $n>0$.
Then (1.7) admits a unique weak solution n satisfying $n-b\in H^q(\mathbb{R}^d)$ and

$$0<b_1\overset{def}{=}\operatorname*{ess\,inf}_{y\in\mathbb{R}^d}b(y)\leq n(x)\leq b_2\overset{def}{=}\operatorname*{ess\,sup}_{y\in\mathbb{R}^d}b(y)<+\infty,\quad a.e.\ x\in\mathbb{R}^d. \tag{1.8}$$

Moreover, let $q>\frac{d}{2}$ and $r=q-1-[\frac{d}{2}]\geq 0$. Then, $n\in W^{r,\infty}(\mathbb{R}^d)\cap C^r(\mathbb{R}^d)$. In particular, n is a classical solution of (1.7) when $q>2+\frac{d}{2}$.

Remark 1.1. In [15], Hsiao et al. obtained a result on the existence and uniqueness of solutions $n\in H^4(\mathbb{R}^d)+b$ under the assumptions:
(a) $\lim_{|x|\to+\infty} b(x)>0$ and $b(x)>0$ for $x\in\mathbb{R}^d$,
(b) $b\in C^4(\mathbb{R}^d)$ and $\nabla b\in H^3(\mathbb{R}^d)$.
However, the proof of this result is not given explicitly. Obviously, assumptions (a)-(b) correspond to a particular case of (i)-(ii) with $q=4$. Moreover, the result of Theorem 1.1 is valid for all $q\ge 1$ without condition $b\in C^4(\mathbb{R}^d)$ in (b).

Once $\bar n$ is given by Theorem 1.1, together with (1.6), we obtain a steady-state solution $(\bar n,0,\bar E,\bar B)$ of system (1.1). Then the stability result can be stated as follows.

Theorem 1.2. *Let $s\ge 3$ be an integer and the assumptions of Theorem 1.1 hold with $q\ge s+3$ (and $d=3$). Assume $(n_0-\bar n,u_0,E_0-\bar E,B_0-\bar B)\in H^s$ with $(\bar n,0,\bar E,\bar B)$ being the steady-state solution of (1.1). Then there exist constants $\delta>0$ and $C>0$ such that if*

$$\|(n_0-\bar n,u_0,E_0-\bar E,B_0-\bar B)\|_s\le\delta, \tag{1.9}$$

problem (1.1)-(1.2) admits a unique global smooth solution (n,u,E,B), and for any $t\ge 0$,

$$\begin{aligned}
&|||(n(t,\cdot)-\bar n,u(t,\cdot),E(t)-\bar E,B(t)-\bar B)|||_s^2\\
&\quad+\int_0^t\big(|||(n(\tau,\cdot)-\bar n,u(\tau,\cdot))|||_s^2+|||E(\tau,\cdot)-\bar E|||_{s-1}^2\\
&\quad+|||\partial_\tau B(\tau,\cdot)|||_{s-2}^2+|||\nabla B(\tau,\cdot)|||_{s-2}^2\big)d\tau\\
&\le C\|(n_0-\bar n,u_0,E_0-\bar E,B_0-\bar B)\|_s^2.
\end{aligned}\tag{1.10}$$

Furthermore, we have

$$\lim_{t\to+\infty}|||(n(t)-\bar n,u(t),E(t)-\bar E)|||_{s-1}=0, \tag{1.11}$$

$$\lim_{t\to+\infty}\big(|||\partial_t B(t)|||_{s-2}+|||\nabla B(t)|||_{s-2}\big)=0. \tag{1.12}$$

This paper is organized as follows. We focus on the proof of Theorem 1.1 in the next section. In Section 3, we first establish energy estimates together with dissipation estimates, and then give the proof of Theorem 1.2.

2. Proof of Theorem 1.1

In this section, we give a detailed proof of Theorem 1.1 on the existence and uniqueness of the solution to (1.7). Throughout this section when we say a solution of an equation, it means a weak solution of the equation. For simplifying the notation, for any integer $s\in\mathbb{Z}$, the spaces $H^s(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ are still denoted by H^s, L^2 and L^∞, respectively.

Since h is a strictly increasing function on $(0,+\infty)$, we denote by f the inverse function of h. Clearly, f is also strictly increasing. By using the potential function $\phi=h(n)$ defined in (1.6), (1.7) can be written as

$$-\Delta\phi+f(\phi)=b,\quad \text{in } \mathbb{R}^d.$$

Let's define

$$\phi_b(x)=h(b(x))\quad\text{and}\quad \widetilde{\phi}=\phi-\phi_b.$$

Then

$$-\Delta\widetilde{\phi}+f(\widetilde{\phi}+\phi_b)-b=\Delta\phi_b.$$

Noting that $b = f(\phi_b)$, we further obtain

$$-\Delta\widetilde{\phi} + a(\widetilde{\phi},\phi_b)\widetilde{\phi} = \ell, \quad \text{in } \mathbb{R}^d,$$

where

$$a(\sigma,\phi_b) = \int_0^1 f'(\phi_b + \theta\sigma)d\theta \quad \text{and} \quad \ell = \Delta\phi_b.$$

Since ϕ_b is a given function, for the sake of simplicity, in what follows $a(\sigma,\phi_b)$ is denoted by $a(\sigma)$. From the assumptions in Theorem 1.1, for each σ such that $h(b_1) \le \sigma + \phi_b \le h(b_2)$, we have

$$a_1 \le a(\sigma) \le a_2, \text{ in } \mathbb{R}^d, \tag{2.1}$$

where a_1 and a_2 are two given constants independent of σ.

Let n be a solution of (1.7) satisfying (1.8). For $q = 1$, we have

$$n - b \in H^1 \Longleftrightarrow \widetilde{\phi} \in H^1,$$

and (1.8) is equivalent to

$$h(b_1) - \phi_b(x) \le \widetilde{\phi}(x) \le h(b_2) - \phi_b(x), \text{ a.e. } x \in \mathbb{R}^d.$$

Therefore, we only need to search for a solution $\widetilde{\phi}$ in H^1 to the equation

$$-\Delta\widetilde{\phi} + a(\widetilde{\phi})\widetilde{\phi} = \ell \in H^{-1}, \tag{2.2}$$

where H^{-1} is the dual space of H^1.

Thus, we define

$$\mathcal{C} = \left\{\sigma \middle| \sigma \in H^1, h(b_1) - \phi_b(x) \le \sigma(x) \le h(b_2) - \phi_b(x), \text{ a.e. } x \in \mathbb{R}^d\right\}.$$

It is easy to see $\mathcal{C}$ is not empty, closed and convex, and the linear equation

$$-\Delta v + a(\sigma)v = \ell \in H^{-1} \tag{2.3}$$

has a unique solution $v_\sigma \in H^1$. More precisely, we have the following result.

LEMMA 2.1. *For each $\sigma \in \mathcal{C}$, the Equation (2.3) has a unique solution $v_\sigma \in \mathcal{C}$. Moreover,*

(i) we have the following variational character for v_σ,

$$I_\sigma(v_\sigma) = \inf_{v \in H^1} I_\sigma(v),$$

where

$$I_\sigma(v) = \frac{1}{2}\int_{\mathbb{R}^d} \left(|\nabla v|^2 + a(\sigma)v^2\right)dx - <\ell, v>_{H^{-1},H^1}, \quad v \in H^1;$$

(ii) we have the estimate:

$$\|v_\sigma\|_{H^1} \le C\|\ell\|_{H^{-1}};$$

(iii) if $\ell \in L^2$, then $v_\sigma \in H^2$ and

$$\|v_\sigma\|_{H^2} \le C\|\ell\|_{L^2},$$

where C is a positive constant independent of σ.

Proof. Let $\sigma \in \mathcal{C}$. It is easy to see that $I_\sigma \in C^\infty(H^1)$. Let c_σ be the infimum of the functional I_σ on H^1:

$$c_\sigma = \inf_{v \in H^1} I_\sigma(v).$$

In H^1 we define a new inner product as

$$(u,v)_\sigma = \int_{\mathbb{R}^d} \big(\nabla u \cdot \nabla v + a(\sigma) uv\big)\, dx.$$

According to (2.1), $\|\cdot\|_\sigma$ defined by

$$\|v\|_\sigma = (v,v)_\sigma^{1/2}$$

is a norm of H^1, and it is uniformly equivalent to the standard norm $\|\cdot\|_{H^1}$ with respect to σ. By the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} I_\sigma(v) &= \frac{1}{2}\int_{\mathbb{R}^d} \big(|\nabla v|^2 + a(\sigma) v^2\big)\, dx - <\ell, v>_{H^{-1},H^1} \\ &\ge \frac{1}{2}\|v\|_\sigma^2 - \|v\|_{H^1}\|\ell\|_{H^{-1}} \\ &\ge \frac{1}{4}\|v\|_\sigma^2 - C_1\|\ell\|_{H^{-1}}^2, \end{aligned} \tag{2.4}$$

where C_1 is a positive constant independent of σ. Thus, $c_\sigma > -\infty$.

Let $\{v_m\}_m \subset H^1$ be a minimizing sequence of I_σ. From (2.4), $\{v_m\}_m$ is bounded in H^1. Therefore, up to a subsequence, we may assume

$$v_m \rightharpoonup v_\sigma \quad \text{weakly in } H^1, \quad \text{with } v_\sigma \in H^1.$$

According to the well-known Ekeland variational principle [7], we have

$$I'_\sigma(v_m) \to 0, \text{ as } m \to \infty,$$

where

$$I'_\sigma(v)(g) = \int_{\mathbb{R}^d} \big(\nabla v \cdot \nabla g + a(\sigma) v g\big)\, dx - <\ell, g>_{H^{-1},H^1}, \quad g \in H^1.$$

By the definition of weak convergence, it is easy to see that v_σ is a solution of (2.3). The uniqueness of solutions is obvious for the linear equation. Moreover, we check easily that $h(b_1) - \phi_b$ and $h(b_2) - \phi_b$ are a subsolution and supersolution of (2.3). It follows from the maximum principle that

$$h(b_1) - \phi_b \le v_\sigma \le h(b_2) - \phi_b, \quad \text{in } \mathbb{R}^d.$$

Therefore, $v_\sigma \in \mathcal{C}$ is the unique solution of (2.3). On the other hand, the norm is weakly lower-semi continuous in a Banach space, i.e.,

$$\|v_\sigma\|_\sigma \le \liminf_m \|v_m\|_\sigma.$$

By the definition of the weak convergence, we have

$$\lim_m < \ell, v_m >_{H^{-1},H^1} = < \ell, v_\sigma >_{H^{-1},H^1}.$$

Hence,

$$c_\sigma \le I_\sigma(v_\sigma) \le \liminf_m I_\sigma(v_m) = c_\sigma.$$

This shows the conclusion (i). Finally, multiplying (2.3) by v_σ and integrating it on $\mathbb{R}^d$, we obtain (ii) by using the Cauchy-Schwarz inequality. The conclusion (iii) is purely a regularity result in [3,8,12]. □

Thanks to (ii) of Lemma 2.1, there exists a constant $M > 0$ such that

$$\|v_\sigma\|_{H^1} \le M, \text{ for all } \sigma \in \mathcal{C}.$$

Set

$$B_M = \{\sigma \big| \sigma \in H^1, \|\sigma\|_{H^1} \le M\} \quad \text{and} \quad \widetilde{\mathcal{C}} = \mathcal{C} \cap B_M.$$

According to Lemma 2.1, the mapping

$$T : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$$
$$\sigma \mapsto v_\sigma$$

is well-defined.

Proof. **(Proof of Theorem 1.1.)** It suffices to consider the case where $q = 1$. The results of Theorem 1.1 for $q \ge 2$ follow from the regularity of solutions (see [3,8,12]) and the Sobolev embedding theorems.

We first consider the uniqueness of solutions to (1.7). Let n_1 and n_2 be two solutions of (1.7) satisfying $n_1 - b \in H^1$, $n_2 - b \in H^1$ and (1.8). We have

$$n_1 - n_2 \in H^1, \quad h(n_1) - h(n_2) \in H^1$$

and

$$-\Delta(h(n_1) - h(n_2)) + (n_1 - n_2) = 0.$$

Taking $h(n_1) - h(n_2)$ as a test function in the above equation, it yields

$$\int_{\mathbb{R}^d} |\nabla(h(n_1) - h(n_2))|^2 dx + \int_{\mathbb{R}^d} (h(n_1) - h(n_2))(n_1 - n_2) dx = 0.$$

Thus, $n_1 = n_2$, since h is a strictly increasing function.

Now we consider the existence of solutions to (1.7). Obviously, the solution of (2.2) is a fixed point of the mapping T in $\widetilde{\mathcal{C}}$. It is easy to see that $\widetilde{\mathcal{C}}$ is a nonempty, bounded, closed and convex subset of H^1. By the Schauder fixed point theorem, it remains to show that $T(\widetilde{\mathcal{C}})$ is precompact in H^1 and T is continuous.

(1) $T(\widetilde{\mathcal{C}})$ is precompact. Let $\{\sigma_m\}_m$ be a sequence in $\widetilde{\mathcal{C}}$. Then $\{\sigma_m\}_m$ is bounded in H^1. We need to show, up to a subsequence (not re-labelled), that $v_{\sigma_m} \to v_\sigma$ in H^1.

Lemma 2.1 (ii) implies that the sequence $\{v_{\sigma_m}\}_m$ is bounded in H^1, with

$$\|v_{\sigma_m}\|_{H^1} \le C \|\ell\|_{H^{-1}}.$$

As above, up to a subsequence, we may assume

$$v_{\sigma_m} \rightharpoonup v_\sigma \text{ weakly in } H^1, \quad \text{with } v_\sigma \in H^1.$$

It follows from the Rellich compact embedding theorem, up to a subsequence, that

$$v_{\sigma_m} \to v_\sigma \text{ strongly in } L^2_{loc}(\mathbb{R}^d).$$

By the equations of v_{σ_m} and v_σ we have

$$-\Delta(v_{\sigma_m} - v_\sigma) + a(\sigma_m)(v_{\sigma_m} - v_\sigma) + (a(\sigma_m) - a(\sigma))v_\sigma = 0.$$

Multiplying this equation by $v_{\sigma_m} - v_\sigma$ and integrating it on $\mathbb{R}^d$ yields

$$\int_{\mathbb{R}^d} |\nabla(v_{\sigma_m} - v_\sigma)|^2 dx + \int_{\mathbb{R}^d} a(\sigma_m)(v_{\sigma_m} - v_\sigma)^2 dx = -\int_{\mathbb{R}^d} \big(a(\sigma_m) - a(\sigma)\big) v_\sigma (v_{\sigma_m} - v_\sigma) dx.$$

On one hand, since $a(\sigma_m) \geq a_0$, there exists a constant $\delta > 0$ such that

$$\int_{\mathbb{R}^d} |\nabla(v_{\sigma_m} - v_\sigma)|^2 dx + \int_{\mathbb{R}^d} a(\sigma_m)(v_{\sigma_m} - v_\sigma)^2 dx \geq \delta \|v_{\sigma_m} - v_\sigma\|^2_{H^1}.$$

On the other hand, we will show that

$$\lim_m \int_{\mathbb{R}^d} \big(a(\sigma_m) - a(\sigma)\big) v_\sigma (v_{\sigma_m} - v_\sigma) dx = 0. \tag{2.5}$$

These last two relations imply that $v_{\sigma_m} \to v_\sigma$ in H^1.

To prove (2.5), let B_R be the ball of center zero and radius $R > 0$. Since σ, σ_m, $v_\sigma \in \widetilde{\mathcal{C}}$, we have σ, $v_\sigma \in L^2 \cap L^\infty$ and the sequence $\{\sigma_m\}_m$ is bounded in L^∞. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big|\big(a(\sigma_m) - a(\sigma)\big) v_\sigma (v_{\sigma_m} - v_\sigma)\big| dx \\
&= \int_{B_R} \big|\big(a(\sigma_m) - a(\sigma)\big) v_\sigma (v_{\sigma_m} - v_\sigma)\big| dx + \int_{B_R^c} \big|\big(a(\sigma_m) - a(\sigma)\big) v_\sigma (v_{\sigma_m} - v_\sigma)\big| dx \\
&\leq C \|v_\sigma\|_{L^2(B_R)} \|v_{\sigma_m} - v_\sigma\|_{L^2(B_R)} + C \|v_\sigma\|_{L^2(B_R^c)} \|v_{\sigma_m} - v_\sigma\|_{L^2},
\end{aligned}$$

where $B_R^c = \mathbb{R}^d \setminus B_R$ and $C > 0$ is a constant. Hence, by integrable property of v_σ, for any $\varepsilon > 0$, there exists $R > 0$ such that

$$C \|v_\sigma\|_{L^2(B_R^c)} \|v_{\sigma_m} - v_\sigma\|_{L^2} < \frac{\varepsilon}{2}.$$

Then, the strong convergence of $\{v_{\sigma_m}\}_m$ to v_σ in $L^2_{loc}(\mathbb{R}^d)$ implies that there exists an integer $N > 0$ such that for $m > N$, we have

$$C \|v_\sigma\|_{L^2(B_R)} \|v_{\sigma_m} - v_\sigma\|_{L^2(B_R)} < \frac{\varepsilon}{2}.$$

This proves (2.5).

(2) T is continuous. Let $\sigma_m \in \widetilde{\mathcal{C}}$ such that $\sigma_m \to \sigma$ in H^1. It is clear that $\sigma \in \widetilde{\mathcal{C}}$. Similar to the proof in **(1)**, it is quite easy to prove $v_{\sigma_m} \to v_\sigma$ in H^1. Thus, T is continuous. This ends the proof of Theorem 1.1. ∎

3. Energy estimates and proof of Theorem 1.2

3.1. Preliminaries. Let $(\bar{n},0,\bar{E},\bar{B})$ be the steady-state solution given by Theorem 1.1. Let $T>0$ and (n,u,E,B) be the smooth solution of problem (1.1)-(1.2) defined in the time interval $[0,T]$. We denote

$$N=n-\bar{n},\quad F=E-\bar{E},\quad G=B-\bar{B},$$

$$U=\begin{pmatrix} N\\ u\end{pmatrix},\quad W=\begin{pmatrix} N\\ u\\ F\\ G\end{pmatrix},\quad W_0=\begin{pmatrix} n_0-\bar{n}\\ u_0\\ E_0-\bar{E}\\ B_0-\bar{B}\end{pmatrix}, \tag{3.1}$$

and

$$W_T=\sup_{t\in[0,T]}|||W(t,\cdot)|||_s.$$

In what follows, we assume W_T is sufficiently small which implies that

$$\frac{\bar{n}}{2}\le n\le\frac{3\bar{n}}{2},\quad |u|\le\frac{1}{2}. \tag{3.2}$$

Let $C>0$ be a generic constant independent of any time. For all $t\in[0,T]$, we want to establish the energy estimate (1.10), i.e.

$$|||W(t,\cdot)|||_s^2+\int_0^t\left(|||U(\tau,\cdot)|||_s^2+|||F(\tau,\cdot)|||_{s-1}^2+|||\partial_\tau G(\tau,\cdot)|||_{s-2}^2+|||\nabla G(\tau,\cdot)|||_{s-2}^2\right)d\tau$$
$$\le C\|W_0\|_s^2. \tag{3.3}$$

According to [23], this estimate implies the global existence result of Theorem 1.2.

To prove (3.3), we need two lemmas below on calculus inequalities, similar to the version in H^s (see [17,22]). Lemma 3.1 can be found in [16,31] and Lemma 3.2 is proved in [24].

Lemma 3.1. *Let $s\ge 3$ be an integer and $\alpha\in\mathbb{N}^3$ with $1\le|\alpha|\le s$. Then*

$$\|\partial_x^\alpha(uv)-u\partial_x^\alpha v\|\le C\|\nabla u\|_{s-1}\|v\|_{|\alpha|-1},\quad \forall u,v\in H^s,$$

and

$$\|\partial_x^\alpha(uv)\|\le C\|u\|_s\|v\|_{|\alpha|}.$$

Lemma 3.2. *Suppose f is a smooth function and $u,v\in B_{s,T}$ with $s\ge 3$. Let $k,\ l\in\mathbb{N}$ and $\alpha,\ \beta\in\mathbb{N}^3$.*

i) If $k+|\alpha|\le s$, then

$$\|\partial_x^\alpha(u\partial_t^k\nabla v)-u\partial_x^\alpha\partial_t^k\nabla v\|\le C|||u|||_s|||v|||_s.$$

ii) If

$$k+|\alpha|\le s,\quad l+|\beta|\le s,\quad k+l\le s,\quad |\alpha|+|\beta|\le s,\quad k+l+|\alpha|+|\beta|\le s+1,$$

then

$$\|(\partial_t^k \partial_x^\alpha u)(\partial_t^l \partial_x^\beta v)\| \le C|||u|||_s|||v|||_s.$$

Furthermore, if $l+|\alpha|\ge 1$ *and* $k+|\beta|\ge 1$, *then*

$$\|(\partial_t^k \partial_x^\alpha u)(\partial_t^l \partial_x^\beta v)\| \le C|||\partial u|||_{s-1}|||\partial v|||_{s-1},$$

where ∂ *denotes any first order derivative with respect to* t *or* x. *Hence,*

$$|||uv|||_s \le C|||u|||_s|||v|||_s.$$

iii) If $1\le k$ *and* $k+|\alpha|\le s$, *then*

$$\|\partial_t^k \partial_x^\alpha f(v)\| \le C|||\partial_t v|||_{s-1}.$$

Finally, for $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $\beta=(\beta_1,\beta_2,\beta_3)$, $\beta\le\alpha$ stands for $\beta_j\le\alpha_j$ for all $j=1,\cdots,3$, and $\beta<\alpha$ stands for $\beta\le\alpha$ and $\beta\ne\alpha$. We recall the Leibniz formulas

$$\partial_x^\alpha(uv)=\sum_{\gamma\le\alpha} m_{\alpha\gamma}\partial_x^{\alpha-\gamma}u\partial_x^\gamma v, \quad \forall\alpha\in\mathbb{N}^3,$$

$$\partial_t^k(uv)=\sum_{l=0}^{k} m'_{kl}\partial_t^{k-l}u\partial_t^l v, \quad \forall k\in\mathbb{N},$$

where $m_{\alpha\gamma}$ and m'_{kl} are positive constants. These formulas will be used in the next subsections.

3.2. Energy estimates with dissipation estimates of u. Substituting

$$n=N+\bar{n}, \quad u, \quad E=F+\bar{E}, \quad B=G+\bar{B}$$

into (1.1) and (1.3), yields the system satisfied by W:

$$\begin{cases} \partial_t N+u\cdot\nabla N+N\mathrm{div}u+u\cdot\nabla\bar{n}=0,\\ \partial_t u+u\cdot\nabla u+\nabla(h(n)-h(\bar{n}))=-F-u-u\times(\bar{B}+G),\\ \partial_t F-\nabla\times G=nu, \quad \mathrm{div}F=-N,\\ \partial_t G+\nabla\times F=0, \quad \mathrm{div}G=0, \end{cases} \tag{3.4}$$

where

$$\nabla(h(n)-h(\bar{n}))=h'(n)\nabla N+\nabla h'(\bar{n})N+r(\bar{n},N),$$

with

$$r(\bar{n},N)=(h'(n)-h'(\bar{n})-h''(\bar{n})N)\nabla\bar{n}=O(N^2).$$

From (1.2) and (3.1), the initial condition for (3.4) is

$$t=0: \quad W=W_0, \quad \text{in } \mathbb{R}^3.$$

We write the Euler equations in (3.4) for U in the form

$$\partial_t U+\sum_{j=1}^{d} A_j(n,u)\partial_{x_j} U+L(x)U+M(W)=f,$$

where

$$A_j(n,u)=\begin{pmatrix} u_j & ne_j^T \\ h'(n)e_j & u_j\mathbf{I}_3 \end{pmatrix},\quad j=1,2,3,$$

$$L(x)=\begin{pmatrix} 0 & (\nabla\bar{n})^T \\ \nabla h'(\bar{n}) & 0 \end{pmatrix},$$

$$M(W)=\begin{pmatrix} 0 \\ F+u+u\times\bar{B} \end{pmatrix},$$

$$f=-\begin{pmatrix} 0 \\ u\times G+r \end{pmatrix}.$$

Let us introduce the matrix

$$A_0(n)=\begin{pmatrix} h'(n) & 0 \\ 0 & n\mathbf{I}_3 \end{pmatrix},\quad \widetilde{A}_j(n,u)=A_0(n)A_j(n,u),$$

and

$$B(n,u,x)=\sum_{j=1}^{3}\partial_{x_j}\widetilde{A}_j(n,u)-2A_0(n)L(x).$$

It is easy to see that $A_0(n)$ is symmetric and positive definite when $n>0$, and $\widetilde{A}_j(n,u)$ is symmetric. Furthermore,

$$B(n,u,x)=\begin{pmatrix} \operatorname{div}(h'(n)u) & (\nabla p'(n)-2h'(n)\nabla\bar{n})^T \\ (\nabla p'(n)-2n\nabla h'(\bar{n})) & \operatorname{div}(nu)\mathbf{I}_3 \end{pmatrix},$$

and

$$B(n,u,x)\big|_{(n,u)=(\bar{n},0)}=\begin{pmatrix} 0 & (\nabla(p'(\bar{n})-2h(\bar{n})))^T \\ -\nabla(p'(\bar{n})-2h(\bar{n})) & 0 \end{pmatrix},$$

which is an anti-symmetric matrix.

Let $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}^3$ with $k+|\alpha|\le s$. Applying $\partial_t^k\partial_x^\alpha$ to (3.4), we get

$$\partial_t U_{k,\alpha}+\sum_{j=1}^{d} A_j(n,u)\partial_{x_j} U_{k,\alpha}+L(x)U_{k,\alpha}+\partial_t^k\partial_x^\alpha M=\partial_t^k\partial_x^\alpha f+g^{k,\alpha}, \tag{3.5}$$

supplemented by the Maxwell equation

$$\begin{cases} \partial_t F_{k,\alpha} - \nabla\times G_{k,\alpha} = (nu)_{k,\alpha}, \quad \mathrm{div} F_{k,\alpha} = -N_{k,\alpha}, \\ \partial_t G_{k,\alpha} + \nabla\times F_{k,\alpha} = 0, \quad \mathrm{div} G_{k,\alpha} = 0, \end{cases} \tag{3.6}$$

where

$$g^{k,\alpha} = \sum_{j=1}^{d}\left[A_j(n,u)\partial_{x_j}U_{k,\alpha} - \partial_t^k\partial_x^\alpha(A_j(n,u)\partial_{x_j}U)\right] + L(x)U_{k,\alpha} - \partial_t^k\partial_x^\alpha(L(x)U). \tag{3.7}$$

When $|\alpha|\geq 1$, we have the following estimate.

LEMMA 3.3. *For all $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}^3$ with $1\leq|\alpha|$ and $k+|\alpha|\leq s$, we have*

$$\begin{aligned} &\frac{d}{dt}\Big(\langle A_0U_{k,\alpha},U_{k,\alpha}\rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2\Big) + C_0\|u_{k,\alpha}\|^2 \\ &\leq C\big(\|\partial_t^k u\|^2_{|\alpha|-1} + \|\partial_t^k F\|^2_{|\alpha|-1} + \|\partial_t^k N\|^2_{|\alpha|}\big) + C|||U|||_s^2|||W|||_s, \end{aligned} \tag{3.8}$$

here, and hereafter, $<\cdot,\cdot>$ denotes the inner product of L^2, C_0 is a positive constant.

Proof. Taking the inner product of (3.5) with $A_0(n)U_{k,\alpha}$ in L^2 yields a classical energy equality

$$\begin{aligned} \frac{d}{dt}\langle A_0(n)U_{k,\alpha},U_{k,\alpha}\rangle = &\ \langle \partial_t A_0(n)U_{k,\alpha},U_{k,\alpha}\rangle + \langle B(n,u,x)U_{k,\alpha},U_{k,\alpha}\rangle \\ &-2\langle A_0\partial_t^k\partial_x^\alpha M,U_{k,\alpha}\rangle + 2\langle A_0 g^{k,\alpha},U_{k,\alpha}\rangle \\ &+2\langle A_0\partial_t^k\partial_x^\alpha f,U_{k,\alpha}\rangle \\ = &\ I_1+I_2+I_3+I_4+I_5, \end{aligned} \tag{3.9}$$

with the natural correspondence for I_1, I_2, I_3, I_4 and I_5. In what follows, we control each term on the right-hand side of (3.9).

Estimate of I_1. Using classical Sobolev embedding theorem together with the first equation in (1.1), we have

$$\|\partial_t n\|_\infty \leq C\|(u,\nabla u)\|_\infty \leq C|||U|||_s.$$

Hence,

$$|I_1| = \Big|\langle \partial_t A_0(n)U_{k,\alpha},U_{k,\alpha}\rangle\Big| \leq C\|\partial_t n\|_\infty\|U_{k,\alpha}\|^2 \leq C|||U|||_s^2|||W|||_s. \tag{3.10}$$

Estimate of I_2. Noting that the matrix $B(n,u,x)$ is anti-symmetric at point $(n,u)=(\bar n,0)$. It follows easily that

$$|I_2| = \big|\langle B(n,u,x)U_{k,\alpha},U_{k,\alpha}\rangle\big| \leq C|||U|||_s^2|||W|||_s. \tag{3.11}$$

Estimate of I_3. We write I_3 as

$$I_3 = -2\langle A_0\partial_t^k\partial_x^\alpha M,U_{k,\alpha}\rangle = -2\langle nu_{k,\alpha},u_{k,\alpha}\rangle - 2\langle F_{k,\alpha},nu_{k,\alpha}\rangle. \tag{3.12}$$

Estimate of I_4. By (3.7), we write $g^{k,\alpha}$ as

$$g^{k,\alpha} = g_1^{k,\alpha} + g_2^{k,\alpha},$$

where

$$g_1^{k,\alpha} = \sum_{j=1}^{d} \left[A_j(n,u)\partial_{x_j} U_{k,\alpha} - \partial_t^k \partial_x^\alpha (A_j(n,u)\partial_{x_j} U) \right],$$

$$g_2^{k,\alpha} = L(x)U_{k,\alpha} - \partial_t^k \partial_x^\alpha (L(x)U).$$

A straightforward calculation yields

$$\begin{aligned} \langle A_0 g_1^{k,\alpha}, U_{k,\alpha} \rangle = & \langle u\cdot\nabla N_{k,\alpha} - \partial_t^k \partial_x^\alpha (u\cdot\nabla N), h'(n)N_{k,\alpha} \rangle \\ & + \langle u\cdot\nabla u_{k,\alpha} - \partial_t^k \partial_x^\alpha (u\cdot\nabla u), n u_{k,\alpha} \rangle \\ & + \langle n(\mathrm{div} u)_{k,\alpha} - \partial_t^k \partial_x^\alpha (n \mathrm{div} u), h'(n)N_{k,\alpha} \rangle \\ & + \langle h'(n)\nabla(N_{k,\alpha}) - \partial_t^k \partial_x^\alpha (h'(n)\nabla N), n u_{k,\alpha} \rangle. \end{aligned} \tag{3.13}$$

For the first term on the right-hand side of (3.13), the Leibniz formula gives

$$u\cdot\nabla N_{k,\alpha} - \partial_t^k \partial_x^\alpha (u\cdot\nabla N) = \sum_{\substack{0\le l\le k,\beta\le\alpha \\ l+|\beta|\ge 1}} m'_{kl} m_{\alpha\beta} \partial_t^l \partial_x^\beta u \cdot \partial_t^{k-l} \partial_x^{\alpha-\beta} \nabla N.$$

Applying Lemma 3.2, we get

$$\| u\cdot\nabla N_{k,\alpha} - \partial_t^k \partial_x^\alpha (u\cdot\nabla N) \| \le C|||u|||_s |||N|||_s.$$

Then,

$$\left| \langle u\cdot\nabla N_{k,\alpha} - \partial_t^k \partial_x^\alpha (u\cdot\nabla N), h'(n)N_{k,\alpha} \rangle \right| \le C|||N|||_s^2 |||u|||_s. \tag{3.14}$$

In the same way, we get

$$\left| \langle u\cdot\nabla u_{k,\alpha} - \partial_t^k \partial_x^\alpha (u\cdot\nabla u), n u_{k,\alpha} \rangle \right| \le C|||u|||_s^3. \tag{3.15}$$

We write the third term as

$$\begin{aligned} \langle n(\mathrm{div} u)_{k,\alpha} - \partial_t^k \partial_x^\alpha (n\mathrm{div} u), h'(n)N_{k,\alpha} \rangle = & \langle \bar{n}(\mathrm{div} u)_{k,\alpha} - \partial_t^k \partial_x^\alpha (\bar{n}\mathrm{div} u), h'(n)N_{k,\alpha} \rangle \\ & + \langle N(\mathrm{div} u)_{k,\alpha} - \partial_t^k \partial_x^\alpha (N\mathrm{div} u), h'(n)N_{k,\alpha} \rangle. \end{aligned}$$

It is easy to see that

$$\left| \langle N(\mathrm{div} u)_{k,\alpha} - \partial_t^k \partial_x^\alpha (N\mathrm{div} u), h'(n)N_{k,\alpha} \rangle \right| \le C|||N|||_s^2 |||u|||_s,$$

and the Leibniz formula yields

$$\begin{aligned} \bar{n}(\mathrm{div} u)_{k,\alpha} - \partial_t^k \partial_x^\alpha (\bar{n}\mathrm{div} u) = & \bar{n} \partial_x^\alpha \partial_t^k (\mathrm{div} u) - \partial_x^\alpha (\bar{n} \partial_t^k \mathrm{div} u) \\ = & - \sum_{0<\beta\le\alpha} m_{\alpha\beta} \partial_x^\beta \bar{n} \partial_t^k (\mathrm{div} \partial_x^{\alpha-\beta} u). \end{aligned}$$

Since $q \ge s+3$, by Theorem 1.1, we have $\bar{n} \in W^{s+1,\infty}(\mathbb{R}^3)$. Hence,

$$\left| \langle \bar{n}(\mathrm{div} u)_{k,\alpha} - \partial_t^k \partial_x^\alpha (\bar{n}\mathrm{div} u), h'(n)N_{k,\alpha} \rangle \right|$$

$$\leq \frac{\varepsilon}{6}\|u_{k,\alpha}\|^2 + C\|\partial_t^k u\|_{|\alpha|-1}^2 + C\|\partial_t^k N\|_{|\alpha|}^2. \tag{3.16}$$

By the Leibniz formula, we write the last term on the right-hand side of (3.13) as

$$\begin{aligned}&h'(n)\nabla N_{k,\alpha} - \partial_t^k\partial_x^\alpha(h'(n)\nabla N)\\ =&h'(n)\partial_x^\alpha(\partial_t^k\nabla N) - \partial_x^\alpha(h'(n)\partial_t^k\nabla N) - \sum_{\substack{0<l\leq k\\ 0\leq\beta\leq\alpha}} m'_{kl}m_{\alpha\beta}\partial_x^\beta\partial_t^l(h'(n))\partial_x^{\alpha-\beta}\partial_t^{k-l}(\nabla N).\end{aligned}$$

Applying Lemma 3.2, we have

$$\begin{aligned}\|h'(n)\partial_x^\alpha(\partial_t^k\nabla N) - \partial_x^\alpha(h'(n)\partial_t^k\nabla N)\| &\leq C\|\nabla h'(n)\|_{s-1}\|\partial_t^k N\|_{|\alpha|}\\ &\leq C\|\partial_t^k N\|_{|\alpha|} + C|||N|||_s^2\end{aligned}$$

and

$$\|m_{kl}m_{\alpha\beta}\partial_x^\beta\partial_t^l(h'(n))\partial_x^{\alpha-\beta}\partial_t^{k-l}(\nabla N)\| \leq C|||\partial_t h'(n)|||_{s-1}|||\nabla N|||_{s-1} \leq C|||N|||_s^2.$$

Hence,

$$\begin{aligned}&\left|\left\langle h'(n)\nabla(N_{k,\alpha}) - \partial_t^k\partial_x^\alpha(h'(n)\nabla N), nu_{k,\alpha}\right\rangle\right|\\ \leq&\frac{\varepsilon}{6}\|u_{k,\alpha}\|^2 + C\|\partial_t^k N\|_{|\alpha|}^2 + C|||U|||_s^2|||W|||_s.\end{aligned} \tag{3.17}$$

The combination of (3.14)-(3.17) yields

$$\left|2\left\langle A_0 g_1^{k,\alpha}, U_{k,\alpha}\right\rangle\right| \leq \frac{2\varepsilon}{3}\|u_{k,\alpha}\|^2 + C\|\partial_t^k u\|_{|\alpha|-1}^2 + C\|\partial_t^k N\|_{|\alpha|}^2 + C|||U|||_s^2|||W|||_s. \tag{3.18}$$

For $2\left\langle A_0 g_2^{k,\alpha}, U_{k,\alpha}\right\rangle$, the Leibniz formula yields

$$g_2^{k,\alpha} = L(x)U_{k,\alpha} - \partial_t^k\partial_x^\alpha(L(x)U) = -\sum_{0<\beta\leq\alpha} m_{\alpha\beta}\partial_x^\beta(L(x))\partial_x^{\alpha-\beta}(\partial_t^k U).$$

Noticing the expression of $L(x)$ and $\bar{n}\in W^{s+1,\infty}(\mathbb{R}^3)$, we have $L\in W^{s,\infty}(\mathbb{R}^3)$. Then,

$$\left|2\left\langle A_0 g_2^{k,\alpha}, U_{k,\alpha}\right\rangle\right| \leq \frac{\varepsilon}{3}\|u_{k,\alpha}\|^2 + C\|\partial_t^k N\|_{|\alpha|}^2. \tag{3.19}$$

Thus, from (3.18) and (3.19), we obtain

$$\begin{aligned}|I_4| =& \left|2\left\langle A_0 g^{k,\alpha}, U_{k,\alpha}\right\rangle\right|\\ \leq& \varepsilon\|u_{k,\alpha}\|^2 + C\|\partial_t^k u\|_{|\alpha|-1}^2 + C\|\partial_t^k N\|_{|\alpha|}^2 + C|||U|||_s^2|||W|||_s.\end{aligned} \tag{3.20}$$

Estimate of I_5. Obviously,

$$|I_5| = \left|2\left\langle A_0\partial_t^k\partial_x^\alpha f, U_{k,\alpha}\right\rangle\right| \leq C|||U|||_s^2|||W|||_s. \tag{3.21}$$

Hence, by (3.10)-(3.12), (3.20)-(3.21), we obtain

$$\frac{d}{dt}\left(\left\langle A_0 U_{k,\alpha}, U_{k,\alpha}\right\rangle\right) + 2\left\langle nu_{k,\alpha}, u_{k,\alpha}\right\rangle$$

$$\leq \varepsilon\|u_{k,\alpha}\|^2 + C\|\partial_t^k u\|_{|\alpha|-1}^2 + C\|\partial_t^k N\|_{|\alpha|}^2 + C|||U|||_s^2|||W|||_s + 2\langle F_{k,\alpha}, -nu_{k,\alpha}\rangle. \tag{3.22}$$

Next, by Maxwell equations (3.6), we get

$$\frac{d}{dt}\Big(\|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2\Big) - 2\langle (nu)_{k,\alpha}, F_{k,\alpha}\rangle = 0. \tag{3.23}$$

The combination of (3.22) and (3.23) yields

$$\begin{aligned} &\frac{d}{dt}\Big(\langle A_0 U_{k,\alpha}, U_{k,\alpha}\rangle + \|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2\Big) + 2\langle nu_{k,\alpha}, u_{k,\alpha}\rangle \\ &\leq \varepsilon\|u_{k,\alpha}\|^2 + C\|\partial_t^k u\|_{|\alpha|-1}^2 + C\|\partial_t^k N\|_{|\alpha|}^2 + 2\langle F_{k,\alpha}, (nu)_{k,\alpha} - nu_{k,\alpha}\rangle + C|||U|||_s^2|||W|||_s. \end{aligned} \tag{3.24}$$

For the term $2\langle F_{k,\alpha}, (nu)_{k,\alpha} - nu_{k,\alpha}\rangle$ on the right-hand side of (3.24), noticing $n = \bar{n} + N$ and $|\alpha| \geq 1$, we can use an integration by parts and Lemmas 3.1-3.2 to get

$$\begin{aligned} \Big|2\langle F_{k,\alpha}, (nu)_{k,\alpha} - nu_{k,\alpha}\rangle\Big| &\leq \Big|2\langle F_{k,\alpha}, (\bar{n}u)_{k,\alpha} - \bar{n}u_{k,\alpha}\rangle\Big| + \Big|2\langle F_{k,\alpha}, (Nu)_{k,\alpha} - Nu_{k,\alpha}\rangle\Big| \\ &\leq \Big|2\langle F_{k,\alpha_1}, \partial_x\big((\bar{n}u)_{k,\alpha} - \bar{n}u_{k,\alpha}\big)\rangle\Big| + C|||U|||_s^2|||W|||_s, \end{aligned} \tag{3.25}$$

where $\alpha_1 \in \mathbb{N}^3$ with $|\alpha_1| = |\alpha| - 1$. The Leibniz formula implies that the highest order derivatives of u and $\bar{n}$ in $\partial_x(\bar{n}u_{k,\alpha} - (\bar{n}u)_{k,\alpha})$ are $k + |\alpha|$ and $1 + |\alpha|$, respectively. Noticing $\bar{n} \in W^{s+1,\infty}(\mathbb{R}^3)$, we get

$$\begin{aligned} 2\Big|\langle F_{k,\alpha_1}, \partial_x(\bar{n}u_{k,\alpha} - (\bar{n}u)_{k,\alpha})\rangle\Big| &\leq C\|\partial_t^k F\|_{|\alpha|-1}(\|u_{k,\alpha}\| + \|\partial_t^k u\|_{|\alpha|-1}) \\ &\leq \varepsilon\|u_{k,\alpha}\|^2 + C\|\partial_t^k u\|_{|\alpha|-1}^2 + C\|\partial_t^k F\|_{|\alpha|-1}^2. \end{aligned} \tag{3.26}$$

These inequalities together with (3.24) yield

$$\begin{aligned} &\frac{d}{dt}\Big(\langle A_0 U_{k,\alpha}, U_{k,\alpha}\rangle + (\|F_{k,\alpha}\|^2 + \|G_{k,\alpha}\|^2)\Big) + 2\langle nu_{k,\alpha}, u_{k,\alpha}\rangle \\ &\leq 2\varepsilon\|u_{k,\alpha}\|^2 + C\|\partial_t^k u\|_{|\alpha|-1}^2 + C\|\partial_t^k F\|_{|\alpha|-1}^2 + C\|\partial_t^k N\|_{|\alpha|}^2 + C|||U|||_s^2|||W|||_s. \end{aligned} \tag{3.27}$$

Let $\varepsilon > 0$ be small enough. Due to (3.2), there is a constant $C_0 > 0$ such that

$$2n - 2\varepsilon \geq C_0.$$

Thus, this inequality together with (3.27) implies (3.8). ∎

For $\alpha = 0$, we have a simpler estimate, which also implies the classical L^2 energy estimate when $k = 0$.

LEMMA 3.4. *For all $k \in \mathbb{N}$ with $k \leq s$, we have*

$$\frac{d}{dt}\Big(\langle A_0 U_{k,0}, U_{k,0}\rangle + \|F_{k,0}\|^2 + \|G_{k,0}\|^2\Big) + C_0\|u_{k,0}\|^2 \leq C|||U|||_s^2|||W|||_s. \tag{3.28}$$

Proof. According to the steps in the proof of Lemma 3.3, we only need to prove

$$2\Big|\langle A_0 g^{k,0}, W_{k,0}\rangle\Big| \leq C|||U|||_s^2|||W|||_s, \tag{3.29}$$

and

$$\Big|2\langle F_{k,0}, (nu)_{k,0} - nu_{k,0}\rangle\Big| \leq C|||U|||_s^2|||W|||_s, \tag{3.30}$$

which correspond to (3.20) and (3.25)-(3.26), respectively. In this case, noting that

$$\bar{n}(\mathrm{div} u)_{k,0} - \partial_t^k(\bar{n}\mathrm{div} u) = 0$$

and

$$h'(n)\nabla(N_{k,0}) - \partial_t^k(h'(n)\nabla N) = -\sum_{0<l\le k} m_{kl}\partial_t^l(h'(n))\partial_t^{k-l}\nabla N.$$

Applying Lemma 3.2, we get

$$\|h'(n)\nabla(N_{k,0}) - \partial_t^k(h'(n)\nabla N)\| \le C|||N|||_s^2.$$

These last two formulas together with related estimates yield (3.29).

Next,

$$\begin{aligned}\Big|2\big\langle F_{k,0},(nu)_{k,0} - nu_{k,0}\big\rangle\Big| &= \Big|2\big\langle F_{k,\alpha},(Nu)_{k,0} - Nu_{k,0}\big\rangle\Big| \\ &\le C|||U|||_s^2|||W|||_s,\end{aligned}$$

which implies (3.30). □

3.3. Dissipation estimates of (N,F)**.** In order to prove Theorem 1.2, we still need to control the terms $\|\partial_t^k F\|_{|\alpha|-1}^2$ and $\|\partial_t^k N\|_{|\alpha|}^2$ appearing on the right-hand side of (3.8). This is achieved in the next lemmas.

LEMMA 3.5. *For all $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}^3$ with $k+|\alpha|\le s$ and $1\le|\alpha|$, we have*

$$\|\partial_t^k N\|_{|\alpha|}^2 \le C\Big(\|\partial_t^k(N,u)\|_{|\alpha|-1}^2 + \|\partial_t^{k+1}u\|_{|\alpha|-1}^2\Big) + C|||U|||_s^2|||W|||_s, \tag{3.31}$$

and

$$\|\partial_t^k F\|_{|\alpha|-1}^2 \le C\Big(\|\partial_t^k(N,u)\|_{|\alpha|-1}^2 + \|\partial_t^{k+1}u\|_{|\alpha|-1}^2\Big) + C|||U|||_s^2|||W|||_s. \tag{3.32}$$

Proof. We write the second equation in (3.4) as

$$\begin{aligned}\nabla(h'(\bar{n})N) &= (h'(n)-h'(\bar{n}))\nabla N - u\times G - u\cdot\nabla u - \partial_t u - u - F - u\times\bar{B} \\ &= R_1 - \partial_t u - u - F - u\times\bar{B},\end{aligned} \tag{3.33}$$

where

$$R_1 = (h'(n)-h'(\bar{n}))\nabla N - u\times G - u\cdot\nabla u.$$

Let $|\beta|\le|\alpha|-1$. Applying $\partial_t^k\partial_x^\beta$ to (3.33), we get

$$\begin{aligned}h'(\bar{n})\partial_x^\beta\nabla\partial_t^k N = {}& h'(\bar{n})\partial_x^\beta\nabla\partial_t^k N - \partial_x^\beta(h'(\bar{n})\nabla\partial_t^k N) \\ &+ \partial_t^k\partial_x^\beta R_1 - u_{k+1,\beta} - u_{k,\beta} - F_{k,\beta} - u_{k,\beta}\times\bar{B}.\end{aligned}$$

Taking the inner product with $(\nabla N)_{k,\beta}$ in L^2, we obtain

$$\begin{aligned}\Big\langle h'(\bar{n})\partial_x^\beta\nabla\partial_t^k N, \partial_x^\beta\nabla\partial_t^k N\Big\rangle = {}& -\big\langle F_{k,\beta},(\nabla N)_{k,\beta}\big\rangle - \big\langle u_{k+1,\beta} + u_{k,\beta} + u_{k,\beta}\times\bar{B},(\nabla N)_{k,\beta}\big\rangle \\ &- \big\langle h'(\bar{n})\partial_x^\beta\nabla\partial_t^k N - \partial_x^\beta(h'(\bar{n})\nabla\partial_t^k N),(\nabla N)_{k,\beta}\big\rangle \\ &+ \big\langle \partial_t^k\partial_x^\beta R_1,(\nabla N)_{k,\beta}\big\rangle.\end{aligned} \tag{3.34}$$

Using the compatibility condition $\operatorname{div}F = -N$, we have

$$-\big\langle F_{k,\beta},(\nabla N)_{k,\beta}\big\rangle = -\big\|N_{k,\beta}\big\|^2.$$

Applying the Young inequality yields

$$\Big|\big\langle u_{k,\beta}+u_{k+1,\beta}+u_{k,\beta}\times\bar{B},(\nabla N)_{k,\beta}\big\rangle\Big| \le C\big(\|\partial_t^k u\|^2_{|\alpha|-1}+\|\partial_t^{k+1}u\|^2_{|\alpha|-1}\big)+\varepsilon\|(\nabla N)_{k,\beta}\|^2.$$

For $\big\langle h'(\bar{n})\partial_x^\beta\nabla\partial_t^k N-\partial_x^\beta(h'(\bar{n})\nabla\partial_t^k N),(\nabla N)_{k,\beta}\big\rangle$, notice that the highest order derivatives of N and $\bar{n}$ in $h'(\bar{n})\partial_x^\beta\nabla\partial_t^k N-\partial_x^\beta(h'(\bar{n})\nabla\partial_t^k N)$ are $k+|\beta|$ and $|\beta|$, respectively. Due to $\bar{n}\in W^{s+1,\infty}(\mathbb{R}^3)$, the Young inequality yields

$$\Big|\big\langle h'(\bar{n})\partial_x^\beta\nabla\partial_t^k N-\partial_x^\beta(h'(\bar{n})\nabla\partial_t^k N),(\nabla N)_{k,\beta}\big\rangle\Big| \le C\|\partial_t^k N\|^2_{|\alpha|-1}+\varepsilon\|(\nabla N)_{k,\beta}\|^2.$$

By Lemmas 3.1-3.2, we easily get

$$\Big|\big\langle \partial_t^k\partial_x^\beta R_1,(\nabla N)_{k,\beta}\big\rangle\Big| \le C|||U|||_s^2|||W|||_s.$$

Adding the inequalities above by taking $\varepsilon>0$ small enough such that $\varepsilon\le\frac{h'(n)}{3}$, from (3.34) we get

$$\|(\nabla N)_{k,\beta}\|^2+\|N_{k,\beta}\|^2 \le C\Big(\|\partial_t^k(N,u)\|^2_{|\alpha|-1}+\|\partial_t^{k+1}u\|^2_{|\alpha|-1}\Big)+C|||U|||_s^2|||W|||_s.$$

Summing these inequalities for all indexes $|\beta|\le|\alpha|-1$, we get (3.31). Finally, from

$$F=-\nabla(h'(\bar{n})N)+R_1-\partial_t u-u-F-u\times\bar{B},$$

we get (3.32). □

We give a dissipation estimate of $\|\partial_t^s N\|$ and a refined estimate of (3.31) for $\|\partial_t^k N\|_1$ with $k\in\mathbb{N}$ and $k\le s-1$.

LEMMA 3.6. *For all $k\in\mathbb{N}$ with $k\le s-1$, we have*

$$\|\partial_t^k N\|_1^2 \le C\big(\|\partial_t^k u\|^2+\|\partial_t^{k+1}u\|^2\big)+C|||U|||_s^2|||W|||_s, \tag{3.35}$$

$$\|\partial_t^s N\|^2 \le C\|\partial_t^{s-1}u\|_1^2+C|||U|||_s^2|||W|||_s. \tag{3.36}$$

Proof. Taking $\beta=0$ in (3.34), we get

$$\Big\langle h'(\bar{n})\partial_t^k\nabla N,\partial_t^k\nabla N\Big\rangle = -\big\langle\partial_t^k F,\partial_t^k\nabla N\big\rangle-\big\langle\partial_t^{k+1}u+\partial_t^k u+\partial_t^k u\times\bar{B},\partial_t^k\nabla N\big\rangle$$
$$+\big\langle\partial_t^k R_1,\partial_t^k\nabla N\big\rangle.$$

For the first term on the right-hand side of the above equality, we have

$$-\big\langle\partial_t^k F,\partial_t^k\nabla N\big\rangle = -\big\langle\partial_t^k N,\partial_t^k N\big\rangle.$$

Similar to the proof of Lemma 3.5, we also have

$$\big|\big\langle\partial_t^{k+1}u+\partial_t^k u+\partial_t^k u\times\bar{B},\partial_t^k\nabla N\big\rangle\big|+\big|\big\langle\partial_t^k R_1,\partial_t^k\nabla N\big\rangle\big|$$
$$\le C\big(\|\partial_t^k u\|^2+\|\partial_t^{k+1}u\|^2\big)+2\varepsilon\|\partial_t^k\nabla N\|^2+C|||U|||_s^2|||W|||_s.$$

From these three relations, we obtain (3.35). Finally, using the first equation in (3.4) and applying Lemma 3.2, we easily get (3.36). ∎

From Lemmas 3.3-3.6, we get the following results.

PROPOSITION 3.1. *For all $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}^3$ with $1\le|\alpha|$ and $k+|\alpha|\le s$, we have*

$$\begin{aligned}&\frac{d}{dt}\sum_{\beta\le\alpha}\Big(\langle A_0U_{k,\beta},U_{k,\beta}\rangle+\|F_{k,\beta}\|^2+\|G_{k,\beta}\|^2\Big)+C_0\|\partial_t^kU\|_{|\alpha|}^2\\ &\le C\big(\|\partial_t^kU\|_{|\alpha|-1}^2+\|\partial_t^{k+1}u\|_{|\alpha|-1}^2\big)+C|||U|||_s^2|||W|||_s.\end{aligned}\tag{3.37}$$

PROPOSITION 3.2. *When W_T is small enough, we have*

$$\frac{d}{dt}\Big(\langle A_0\partial_t^sU,\partial_t^sU\rangle+\|\partial_t^sF\|^2+\|\partial_t^sG\|^2\Big)+C_0\|\partial_t^sU\|^2\le C\|\partial_t^{s-1}U\|_1^2+C|||U|||_s^2|||W|||_s.\tag{3.38}$$

3.4. Proof of Theorem 1.2. We shall use an induction argument on the order of the derivatives of the solution to prove (3.3). First, for any fixed index $k\in\mathbb{N}$ with $k\le s-1$, we carry on the induction on $|\alpha|$ $(1\le|\alpha|\le s-k)$ of space derivatives for (3.37) from $|\alpha|=1$ to $|\alpha|=s-k$. More specially, for $|\alpha|\ge2$, $\|\partial_t^kU\|_{|\alpha|-1}$ on the right-hand side of (3.37) can be controlled by $\|\partial_t^kU\|_{|\alpha|}$ in the preceding step on the left-hand side of (3.37) multiplying an appropriate positive constant. Thus, we get

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\le s-k}a_{k,\alpha}\Big(\langle A_0U_{k,\alpha},U_{k,\alpha}\rangle+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\Big)+\|\partial_t^kU\|_{s-k}^2\\ &\le C\big(\|\partial_t^kU\|^2+\|\partial_t^{k+1}u\|_{s-k-1}^2\big)+C|||U|||_s^2|||W|||_s,\end{aligned}\tag{3.39}$$

where $a_{k,\alpha}>0\,(k\le s-1,1\le|\alpha|\le s-k)$ are constants.

Next, we carry on the induction on k from $k=s$ to $k=0$. The corresponding estimate for $k=s$ is given by (3.38). For $k=s-1$, (3.39) yields

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\le1}a_{s-1,\alpha}\Big(\langle A_0U_{s-1,\alpha},U_{s-1,\alpha}\rangle+\|F_{s-1,\alpha}\|^2+\|G_{s-1,\alpha}\|^2\Big)+\|\partial_t^{s-1}U\|_1^2\\ &\le C\big(\|\partial_t^{s-1}U\|^2+\|\partial_t^su\|^2\big)+C|||U|||_s^2|||W|||_s.\end{aligned}\tag{3.40}$$

Obviously, the term $\|\partial_t^{s-1}U\|^2$ on the right-hand side of (3.38) can be controlled by the same term on the left-hand side of (3.40) multiplying an appropriate constant. Similarly, $\|\partial_t^{k+1}u\|_{s-k-1}^2$ can be also controlled by $\|\partial_t^kU\|_{s-k}^2$ in the preceding step. In this way, by induction on k, we get

$$\begin{aligned}&\frac{d}{dt}\sum_{k+|\alpha|\le s}a_{k,\alpha}\Big(\langle A_0U_{k,\alpha},U_{k,\alpha}\rangle+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\Big)+\sum_{k=0}^{s}\|\partial_t^kU\|_{s-k}^2\\ &\le C\sum_{k=0}^{s-1}\big(\|\partial_t^kU\|^2+\|\partial_t^{k+1}u\|^2\big)+C|||U|||_s^2|||W|||_s,\end{aligned}\tag{3.41}$$

where the positive constants $a_{k,\alpha}$ are possibly amended based on (3.39). Noting the equivalence of $\sum_{k=0}^{s}\|\partial_t^k U\|_{s-k}^2$ and $|||U|||_s^2$, from (3.28), (3.35) and (3.41), we get

$$\frac{d}{dt}\sum_{k+|\alpha|\le s} a_{k,\alpha}\Big(\big\langle A_0U_{k,\alpha},U_{k,\alpha}\big\rangle+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\Big)+2|||U(t,\cdot)|||_s^2$$
$$\le C|||U(t,\cdot)|||_s^2|||W|||_s,$$

where $a_{k,\alpha}>0$ is possibly modified again. Since W_T is sufficiently small, we obtain

$$\frac{d}{dt}\sum_{k+|\alpha|\le s} a_{k,\alpha}\Big(\big\langle A_0U_{k,\alpha},U_{k,\alpha}\big\rangle+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\Big)+|||U(t,\cdot)|||_s^2\le 0.$$

From the equivalence of $|||W|||_s^2$ and

$$\sum_{k+|\alpha|\le s} a_{k,\alpha}\Big(\big\langle A_0U_{k,\alpha},U_{k,\alpha}\big\rangle+\|F_{k,\alpha}\|^2+\|G_{k,\alpha}\|^2\Big),$$

we get

$$|||W(t,\cdot)|||_s^2+\int_0^t |||U(\tau,\cdot)|||_s^2\,d\tau\le C\|W_0\|_s^2,\quad t\in[0,T].$$

Finally, from the second equation and Maxwell equations in (3.4), we get

$$|||F|||_{s-1}^2\le C|||U|||_s^2+C|||U|||_s^2|||W|||_s,$$

$$|||\partial_t G|||_{s-2}^2+|||\nabla G|||_{s-2}^2\le C|||U|||_s^2+C|||U|||_s^2|||W|||_s.$$

These last three estimates give (3.3), i.e. (1.10). This proves the global existence of smooth solution (n,u,E,B) to (1.1)-(1.2).

Finally, from (1.10), for any index $k\in\mathbb{N}$ and $\beta\in\mathbb{N}^3$ with $k+|\beta|\le s-1$, we have

$$\partial_t^k\partial_x^\beta(n-\bar{n},u,E-\bar{E})\in L^2(\mathbb{R}^+;L^2)\cap W^{1,\infty}(\mathbb{R}^+;L^2)$$

In addition, if $1\le k+|\beta|\le s-1$, we have

$$\partial_t^k\partial_x^\beta B\in L^2(\mathbb{R}^+;L^2)\cap W^{1,\infty}(\mathbb{R}^+;L^2).$$

From these two formulas, we get (1.11) and (1.12).

Acknowledgements. The research of this work was supported in part by National Natural Science Foundation of China under grants 11401421, 11601363, 11671295 and Science Foundation of Shanxi Province under grants 201601D021011, 201701D121008.

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