COMMUN. MATH. SCI.
Vol. 17, No. 7, pp. 2005–2034

PERIODIC SOLUTIONS TO NONLINEAR EULER–BERNOULLI BEAM EQUATIONS*

BOCHAO CHEN†, YIXIAN GAO‡, AND YONG LI§

Abstract. Bending vibrations of thin beams and plates can be described by nonlinear Euler–Bernoulli beam equation with x-dependent coefficients. In this paper we demonstrate the existence of families of time-periodic solutions to such a model by virtue of a Lyapunov–Schmidt reduction together with a Nash–Moser method. This result holds for all parameters (ϵ,ω) in a Cantor set with asymptotically full measure as $\epsilon\to 0$.

Keywords. Euler–Bernoulli beam equations; Variable coefficients; Periodic solutions; Nash–Moser iteration.

AMS subject classifications. 35B10; 35L75; 58C15.

1. Introduction

In this article we are concerned with the one-dimensional nonlinear Euler–Bernoulli beam equation

$$\rho(x)u_{tt}+(p(x)u_{xx})_{xx}=\epsilon f(\omega t,x,u)+\epsilon g(\omega t,x),\quad x\in[0,\pi],t\in\mathbb{R} \tag{1.1}$$

with respect to the pinned-pinned boundary conditions

$$u(t,0)=u(t,\pi)=u_{xx}(t,0)=u_{xx}(t,\pi)=0, \tag{1.2}$$

where the coefficients ρ,p are positive, the parameter ϵ is small enough, the force terms $g(\cdot,x),f(\cdot,x,u)$ are 2π-periodic, with $\partial_u^\ell f(\cdot,\cdot,0)=0,\ell\leq 2$. Clearly, $u=0$ is not the solution of Equation (1.1) if $g\neq 0$.

The above model is used to describe bending vibrations of thin beams and plates and reflects the relationship between the applied load and the beam's deflection as well, see [40]. The curve $u(\cdot,x)$ stands for the deflection of the beam at some position x in the vertical direction. The coefficients ρ,p are the density of the beam and the flexural rigidity, respectively. And the terms f,g are the so-called distributed loads depending on x, or u, or x,u, or other variables. Moreover, derivatives of the deflection u have physical significance: u_x is the slope of the beam; $-pu_{xx}$ is the bending moment of the beam and $-(pu_{xx})_x$ is the shear force of the beam.

The free vibrations of non-uniform beams have attracted many investigators since Bernoulli and Euler derived the governing differential equation in the 18th century. Many researchers focused on the study of the spectral problems for the following Euler–Bernoulli operator

$$\mathcal{E}u:=\frac{1}{\rho}(pu'')''+Vu\quad\text{with } V=V(x), \tag{1.3}$$

*Received: April 29, 2018; Accepted (in revised form): July 1, 2019. Communicated by Shi Jin. The research of BC was supported in part by NSFC Grant 11901232 and China Postdoctoral Science Foundation Funded Project 2019M651191. The research of YG was supported in part by NSFC grant 11871140, 11671071, JJKH20180006KJ, JLSTDP 20190201154JC and FRFCU2412019BJ005. The research of YL was supported in part by NSFC grant 11571065.

†College of Mathematics, Jilin University, Changchun 130012, P.R. China (chenbc758@163.com).

‡School of Mathematics and Statistics, Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun, Jilin 130024, P.R. China (gaoyx643@nenu.edu.cn).

§College of Mathematics, Jilin University and School of Mathematics and Statistics, Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun, Jilin 130024, P.R. China (yongli@nenu.edu.cn).

see [1, 2, 24, 35–37] and references therein. In addition, under different boundary conditions, Elishakoff et al. obtained harmonic form solutions, i.e., $u(t,x)=\psi(x)\sin\omega t$, of Equation (1.1) with $f,g=0$ for the first time, recall [20, 21]. In [25], under linear boundary feedback control, Guo considered the Riesz basis property and the stability of the linear one with boundary conditions

$$\begin{cases} u(t,0)=u_x(t,0)=u_{xx}(t,\pi)=0, \\ (p(x)u_{xx})_x(t,\pi)=\mu u_t(t,\pi), \end{cases}$$

where $\mu\geq 0$ is a constant feedback. Despite many studies on the linear model above, the nonlinear problems are equally interesting and challenging. The main focus of the present article is to derive periodic solutions of Equation (1.1). To the best of our knowledge our article establishes the first mathematical analysis for such a model. There will be two main challenges to face in this work. One is that the forced terms have only Sobolev regularity. This leads to the fact that the Green functions will exhibit only a polynomial decay off the diagonal, and not exponential (or subexponential). To overcome this one, we will exploit the interpolation/tame estimates. The other is the so-called "small divisors problem" caused by resonance. If we let $t\to t/\omega$, then Equation (1.1) is equivalent to

$$\omega^2\rho(x)u_{tt}+(p(x)u_{xx})_{xx}=\epsilon f(t,x,u)+\epsilon g(t,x). \tag{1.4}$$

In fact, the spectrum of the operator

$$\mathcal{E}_\omega u:=\omega^2 u_{tt}+\tfrac{1}{\rho}(pu_{xx})_{xx}$$

has the following form

$$-\omega^2 l^2+\lambda_j=-\omega^2 l^2+j^4+\mathrm{a}j^2+\mathrm{b}+O(1/j),\quad l\in\mathbb{Z},\ j\to+\infty.$$

In general, the operator $\mathcal{E}_\omega$ cannot map a functional space into itself, but only into a large one with less regularity. There are two main approaches to deal with the "small divisors problem". One is the infinite-dimensional KAM (Kolmogorov–Arnold–Moser) theory to Hamiltonian PDEs, refer to Kuksin [32], Wayne [39], and recent results [6, 22, 26]. The other more direct bifurcation approach was established by Craig and Wayne [17] and improved by Bourgain [11, 12] based on both a Lyapunov–Schmidt reduction and a Nash–Moser iteration, and recent results [8–10]. Meanwhile, we need to give the asymptotic formulae of the eigenvalues for an Euler–Bernoulli beam problem as well. Actually, the asymptotic property of the eigenvalues for fourth-order operators on the unit interval are less investigated than for second-order ones, see [13, 34].

The existence problem of periodic or quasi-periodic solutions for PDEs has received considerable attention in the last twenty years. Up to now, there have been a number of works devoted to such a problem for nonlinear beam equation with constant coefficients. In [27, 33], Mckenna et al. studied the beam equation which models a suspension bridge and showed that it admits multiple periodic solutions when a parameter exceeds a certain eigenvalue. Subsequently, many researchers use the infinite KAM theorem to handle this kind of problem. We refer the readers to [14, 23, 38] for the existence and stability of small-amplitude quasi-periodic solutions of one-dimensional beam equation with boundary conditions (1.2). For further reference regarding the linearly stable or unstable small-amplitude quasi-periodic solutions of the d-dimensional nonlinear beam equation

$$u_{tt}+\Delta^2 u+mu+\partial_u f(x,u)=0, f(x,u)=u^4+O(u^5),\quad x\in\mathbb{T}^d, t\in\mathbb{R},$$

we may refer to [19]. Notice that the above proof can be carried out in the analytic case. For differential one, the readers can also consult [15] for the application of the Nash–Moser iteration scheme in studying the existence of quasi-periodic solutions of nonlinear beam equation

$$u_{tt}+\Delta^2 u+V(x)u=\epsilon f(\omega t,x,u),\quad x\in\boldsymbol{M},t\in\mathbb{R},$$

where $\boldsymbol{M}$ is any compact Lie group or homogenous manifold with regard to a compact Lie group. Finally, we will introduce some related results about the wave equation with variable coefficients. The problem of looking for periodic solutions to such one was considered by Barbu and Pavel for the first time in [4,5]. Recently, under the general boundary conditions, or periodic or anti-periodic boundary conditions, or Dirichlet boundary conditions, Ji and Li gave a series of results as well, see [28–30]. The above results have to require that the time period is a rational multiple of the length of the spatial interval. If it is an irrational multiple of the length of the spatial interval, then the existence of periodic solutions for the forced vibrations of a nonhomogeneous string was obtained in [3] and [16] by means of the Nash–Moser iteration.

1.1. Main result. The goal of the present article is to look for periodic solutions, with the common period 2π, to Equation (1.4). To do so, we need to introduce our notation and assumption more precisely. Let ρ,p be positive coefficients given by

$$\rho(x)=e^{4\int_0^x\alpha(z)\mathrm{d}z}>0,\quad p(x)=p(0)e^{4\int_0^x\beta(z)\mathrm{d}z}>0 \tag{1.5}$$

with $\alpha(0)+\beta(0)=\alpha(\pi)+\beta(\pi)=0$. Without loss of generality, we assume the following normalization

$$\tfrac{1}{\pi}\int_0^\pi(\rho/p)^{\frac14}\,\mathrm{d}x=1.$$

Let us introduce both the Liouville substitution

$$x=\psi(\xi)\Longleftrightarrow\xi=\phi(x)\quad\text{with }\phi(x):=\int_0^x\zeta(s)\mathrm{d}s,\ \zeta=(\rho/p)^{\frac14},$$

and the unitary Barcilon–Gottlieb transformation

$$\begin{aligned}\mathcal{T}:L^2((0,\pi),\rho(x)\mathrm{d}x)&\longrightarrow L^2((0,\pi),\mathrm{d}\xi)\\ u(x)&\longmapsto \mathrm{y}(\xi)=(\mathcal{T}u)(\psi(\xi))=q(\psi(\xi))u(\psi(\xi)),\quad \xi\in[0,\pi],\end{aligned}$$

where $q=p^{\frac18}\rho^{\frac38}>0$. From (1.5) and (1.2), [1, Lemma 5.1] has showed that both $\mathcal{E}$ (recall (1.3)) and $\mathcal{E}'$, with

$$\mathcal{E}'\mathrm{y}:=\mathrm{y}_{\xi\xi\xi\xi}+2(p_1\mathrm{y}_\xi)_\xi+p_2\mathrm{y},$$

are unitarily equivalent, that is $\mathcal{E}=\mathcal{T}^{-1}\mathcal{E}'\mathcal{T}$. Remark that p_1,p_2 are seen in (5.4)–(5.5) of [1]. Therefore we can directly apply [1, cf. Theorem 1.2, Proposition 6.2] to give the asymptotic formulae of the eigenvalues for the operator $\mathcal{E}$.

For all $s\geq 0$, define the following Sobolev spaces H^s of real-valued functions by

$$\begin{aligned}H^s:=\big\{u:\mathbb{T}\longrightarrow H_p^2((0,\pi);\mathbb{R}),u(t,x)=\textstyle\sum_{l\in\mathbb{Z}}u_l(x)e^{ilt},u_l\in H_p^2((0,\pi);\mathbb{C}),\\ u_{-l}=u_l^*,\|u\|_s<+\infty\big\},\end{aligned}$$

where u_l^* is the complex conjugate of u_l, $\|u\|_s^2:=\sum_{l\in\mathbb{Z}}\|u_l\|_{H^2}^2(1+l^{2s})$ and

$$H_p^2((0,\pi);\mathbb{C}):=\big\{u\in H^2((0,\pi);\mathbb{C}):u(t,0)=u(t,\pi)=u_{xx}(t,0)=u_{xx}(t,\pi)=0\big\}.$$

If $s>1/2$, then $H^s \subset L^\infty(\mathbb{T};H^2_p(0,\pi))$, that is

$$\|u\|_{L^\infty(\mathbb{T};H^2_p(0,\pi))} \le C(s)\|u\|_s, \quad \forall u \in H^s. \tag{1.6}$$

Moreover, one has that for all $u_1,u_2 \in H^s$,

$$\|u_1u_2\|_s \le C(s)\|u_1\|_s\|u_2\|_s.$$

Throughout this paper, our purpose is to look for solutions to Equation (1.4) in H^s with respect to $(t,x)\in \mathbb{T}\times[0,\pi]$ and $f\in\mathcal{C}_k$, where

$$\mathcal{C}_k := \left\{ f\in C(\mathbb{T}\times[0,\pi]\times\mathbb{R};\mathbb{R}) : (t,u)\longmapsto f(t,\cdot,u) \text{ is in } C^k(\mathbb{T}\times\mathbb{R};H^2(0,\pi))\right\}, \quad k\in\mathbb{N}^+.$$

Observe that if $f(t,x,u)=\sum_{l\in\mathbb{Z}} f_l(x,u)e^{\mathrm{i}lt}$, then $f_{-l}=f_l^*$ and $u\longmapsto f_l(\cdot,u)$ belongs to $C^k(\mathbb{R};H^2((0,\pi);\mathbb{C}))$. Moreover, due to the continuous embedding of $H^2(0,\pi)$ into $C^1([0,\pi];\mathbb{R})$, it follows that for all $f\in\mathcal{C}_k$,

$$\partial_x^{\mathrm{p}}\partial_t^i\partial_u^\ell f\in C(\mathbb{T}\times[0,\pi]\times\mathbb{R};\mathbb{R}), \quad 0\le \mathrm{p}\le 1, 0\le i,\ell\le k.$$

In addition, we may decompose the space H^s as the direct sum of V and $W\cap H^s$, where

$$V := H^2_p(0,\pi), \quad W := \left\{ w=\sum_{l\neq 0} w_l(x)e^{\mathrm{i}lt}\in H^0\right\}.$$

Obviously, for every $u\in H^s$, we can write $u=v+w$, where $v\in V$ and $w\in W\cap H^s$. Denote by $\Pi_V: H^s\longrightarrow V$, $\Pi_W: H^s\longrightarrow W$ the corresponding projection operators, respectively. Then, via the Lyapunov–Schmidt reduction, Equation (1.4) is equivalent to the bifurcation equation (Q) and the range equation (P), i.e.,

$$\begin{cases} (pv'')''=\epsilon\Pi_V F(v+w)+\epsilon\Pi_V g & (Q),\\ L_\omega w=\epsilon\Pi_W F(v+w)+\epsilon\Pi_W g & (P),\end{cases} \tag{1.7}$$

where

$$L_\omega w := \omega^2\rho(x)w_{tt}+(p(x)w_{xx})_{xx}, \quad F: u\longmapsto f(t,x,u). \tag{1.8}$$

Since $f(t,x,u)=f_0(x,u)+\sum_{l\neq0} f_l(x,u)e^{\mathrm{i}lt}$, if $w=0$, then

$$\Pi_V F(v)=\Pi_V f(t,x,v(x))=f_0(x,v(x)).$$

Therefore we simplify the (Q)-equation as

$$(pv'')''=\epsilon f_0(x,v)+\epsilon g_0(x) \quad \text{as } w\to 0.$$

This is called the infinite-dimensional "zeroth-order bifurcation equation", recall [3]. Let us make the following assumption.

Assumption 1.1. *There exists a constant* $\epsilon_0\in(0,1)$ *small enough such that for all* $\epsilon\in[0,\epsilon_0]$, *the system*

$$\begin{cases} (p(x)v''(x))''=\epsilon f_0(x,v)+\epsilon g_0(x),\\ v(0)=v(\pi)=v''(0)=v''(\pi)=0\end{cases} \tag{1.9}$$

admits a nondegenerate solution $\hat{v} \in H_p^2(0,\pi)$*, i.e., the linearized equation*

$$(ph'')'' = \epsilon f_0'(\hat{v})h \tag{1.10}$$

possesses only the trivial solution $h=0$ *in* $H_p^2(0,\pi)$.

Observe that Equation (1.10) with $\epsilon=0$ possesses only the trivial solution $h=0$ in $H_p^2(0,\pi)$. Hence $\hat{v}=0$ is the nondegenerate solution of Equation (1.9) with $\epsilon=0$. In view of the implicit function theorem, there exists a constant $\epsilon_0 \in (0,1)$ small enough such that for all $\epsilon \in [0,\epsilon_0]$, Assumption 1.1 is satisfied.

Moreover, define the set

$$A_\gamma := \left\{ (\epsilon,\omega) \in (\epsilon_1,\epsilon_2) \times (\gamma,+\infty) : \frac{\epsilon}{\omega} \le \delta_6 \gamma^5, |\omega l - \bar{\mu}_j| > \frac{\gamma}{l^\tau}, \forall l = 1,\cdots,N_0, \forall j \ge 1 \right\},$$

where $\delta_6 > 0$ shall be fixed in the next theorem, $N_0 > 1$ and $\bar{\lambda}_j = \bar{\mu}_j^2, j \ge 1$ are the eigenvalues of Euler–Bernoulli beam problem

$$\begin{cases} (p(x)y'')'' = \lambda \rho(x) y, \\ y(0) = y(\pi) = y''(0) = y''(\pi) = 0. \end{cases} \tag{1.11}$$

Let us state our main theorem as follows.

THEOREM 1.1. *Let* p,ρ *satisfy* (1.5)*, with* $\alpha,\beta \in H^4(0,\pi)$*,* f *be in* $\mathcal{C}_k$*, with* $\partial_u^\ell f(\cdot,\cdot,0) = 0, \ell \le 2$*, and* $t \longmapsto g(t,\cdot)$ *belong to* $C^k(\mathbb{T};H^2(0,\pi))$*. Fix* $\tau \in (1,2), \gamma \in (0,1), \epsilon_0 \in (0,1)$ *and*

$$\kappa = 6\tau + 4\sigma + 2 \quad \text{with } \sigma = \tau(\tau-1)/(2-\tau). \tag{1.12}$$

If Assumption 1.1 holds for some $\hat{\epsilon} \in [0,\epsilon_0]$*, for* $\frac{\epsilon}{\gamma^5 \omega} \le \delta_6$ *small enough and* $k \ge s + \kappa + 3$*, then there is some constant* $K > 0$ *depending on* $\alpha,\beta,f,g,\epsilon_0,\hat{v},\gamma,\gamma_0,\tau,s$*, a neighborhood* (ϵ_1,ϵ_2) *of* $\hat{\epsilon}$*,* $r \in (0,1)$*, a map* $\tilde{w} \in C^1(A_\gamma; W \cap H^s)$ *with*

$$\|\tilde{w}\|_s \le \tfrac{K\epsilon}{\gamma\omega} < r, \quad \|\partial_\omega \tilde{w}\|_s \le \tfrac{K\epsilon}{\gamma^5\omega}, \quad \|\partial_\epsilon \tilde{w}\|_s \le \tfrac{K}{\gamma^5\omega}, \tag{1.13}$$

and a C^2 *map* $v(\epsilon,\tilde{w})$ *on* $(\epsilon_1,\epsilon_2) \times \{\tilde{w} \in W \cap H^s : \|\tilde{w}\|_s < r\}$*, with values in* V*, satisfying*

$$\|v(\epsilon,\tilde{w}) - v(\epsilon,0)\|_{H^2} \le \tfrac{K\epsilon}{\gamma\omega}, \quad \|v(\epsilon,0) - \hat{v}\|_{H^2} \le K|\epsilon - \hat{\epsilon}|, \tag{1.14}$$

such that for all $(\epsilon,\omega) \in B_\gamma \subset A_\gamma$*,*

$$\tilde{u}(\epsilon,\omega) := v(\epsilon,\tilde{w}(\epsilon,\omega)) + \tilde{w}(\epsilon,\omega) \in V \oplus (W \cap H^s)$$

is a solution of Equation (1.4) *and satisfies*

$$\tilde{u}(t,\cdot) \in H^6(0,\pi) \cap H_p^2(0,\pi), \quad \forall t \in \mathbb{R}.$$

Moreover, the Lebesgue measures of the set B_γ *and its section* $B_\gamma(\epsilon)$ *satisfy*

$$\text{meas}(B_\gamma(\epsilon) \cap (\omega',\omega'')) \ge (1 - K\gamma)(\omega'' - \omega'), \quad \text{meas}(B_\gamma \cap \Omega) \ge (1 - K\gamma)\text{meas}(\Omega),$$

where $\Omega := (\epsilon',\epsilon'') \times (\omega',\omega'')$ *stands for a rectangle contained in* $(\epsilon_1,\epsilon_2) \times (2\gamma,+\infty)$.

1.2. Plan of the paper. The rest of the paper is organized as follows. In Subsection 2.1, under Assumption 1.1, we solve the bifurcation equation by the classical implicit function theorem. Given the non-resonance conditions, the goal of Subsection 2.2 is to solve the range equation including initialization, iteration and measure estimates. Section 3 is devoted to investigating invertibility of the linearized operator. Finally, we list the proof of some related results for the sake of completeness in the Appendix.

2. Proof of the main result

The object of this section is to complete the proof of the main result.

2.1. Solutions of the bifurcation equation.

We first solve the bifurcation equation via the classical implicit function theorem.

LEMMA 2.1. *Provided that Assumption 1.1 holds for some* $\hat{\epsilon}\in[0,\epsilon_0]$, *there exists a neighborhood* (ϵ_1,ϵ_2) *of* $\hat{\epsilon}$, $r\in(0,1)$ *and a* C^2 *map*

$$v:(\epsilon_1,\epsilon_2)\times\{w\in W\cap H^s:\|w\|_s<r\}\longrightarrow H^2_p(0,\pi),\quad (\epsilon,w)\longmapsto v(\epsilon,w),$$

such that $v(\epsilon,w)$ *solves the* (Q)*-equation satisfying* $v(\hat{\epsilon},0)=\hat{v}$ *and for some constant* $C>0$,

$$\|v(\epsilon,w)-v(\epsilon,0)\|_{H^2}\leq C\|w\|_s,\quad \|v(\epsilon,0)-\hat{v}\|_{H^2}\leq C|\epsilon-\hat{\epsilon}|. \tag{2.1}$$

Proof. In view of Assumption 1.1, the linearized operator $h\longmapsto(ph'')''-\hat{\epsilon}f_0'(\hat{v})h$ is invertible on V. Since $t\longmapsto g(t,\cdot)$ is in $C^k\left(\mathbb{T};H^2(0,\pi)\right)$, if $f\in\mathcal{C}_k$, with $\partial_u^\ell f(\cdot,\cdot,0)=0,\ell\leq 2$, then using Lemma 4.5 yields that

$$(\epsilon,w,v)\longmapsto(pv'')''-\epsilon\Pi_V F(v+w)-\epsilon\Pi_V g$$

is a C^2 map. Consequently, by the implicit function theorem, there is a C^2-path $(\epsilon,w)\longmapsto v(\epsilon,w)$ satisfying (2.1). □

2.2. Solutions of the range equation.

In this subsection, we will apply a Nash–Moser method to solve the range equation, that is

$$L_\omega w=\epsilon\Pi_W\mathcal{F}(\epsilon,w)+\epsilon\Pi_W g,\quad \mathcal{F}(\epsilon,w):=F(v(\epsilon,w)+w).$$

Let $W=W_N\oplus W_N^\perp$, where

$$W_N:=\left\{w\in W:w=\textstyle\sum_{1\leq|l|\leq N}w_l(x)e^{\mathrm{i}lt}\right\},\quad W_N^\perp:=\left\{w\in W:w=\textstyle\sum_{|l|>N}w_l(x)e^{\mathrm{i}lt}\right\}.$$

Then corresponding projection operators $\mathrm{P}_N:W\longrightarrow W_N$, $\mathrm{P}_N^\perp:W\longrightarrow W_N^\perp$ satisfy

(P1) $\|\mathrm{P}_N w\|_{s+\vartheta}\leq N^\vartheta\|w\|_s,\quad \forall w\in W\cap H^s,\forall s,\vartheta\geq 0.$

(P2) $\|\mathrm{P}_N^\perp w\|_s\leq N^{-\vartheta}\|w\|_{s+\vartheta},\quad \forall w\in W\cap H^{s+\vartheta},\forall s,\vartheta\geq 0.$

Denote by $\mathrm{D}_w\mathcal{F}$ the Fréchet derivative of $\mathcal{F}$ with respect to w. If $f\in\mathcal{C}_k$, with $\partial_u^\ell f(\cdot,\cdot,0)=0,\ell\leq 2$, then for $k\geq s'+3$ with $s'\geq s>1/2$, the following (U1)–(U3) hold.

(U1) (Regularity.) $\mathcal{F}$ is a C^2 map and $\mathcal{F},\mathrm{D}_w\mathcal{F},\mathrm{D}_w^2\mathcal{F}$ are bounded on $\{\|w\|_s\leq 1\}$;

(U2) (Tame.) $\mathcal{F}:W\cap H^{s'}\longrightarrow H^{s'}$, $\mathrm{D}_w\mathcal{F}\in\mathcal{L}(W\cap H^{s'};H^{s'})$, $\mathrm{D}_w^2\mathcal{F}\in\mathcal{L}((W\cap H^{s'})\times(W\cap H^{s'});H^{s'})$ and $\forall w,h,\mathrm{h}\in W\cap H^{s'}$ with $\|w\|_s\leq 1$,

$$\|\mathcal{F}(\epsilon,w)\|_{s'}\leq C(s')(1+\|w\|_{s'}),\quad \|\mathrm{D}_w\mathcal{F}(\epsilon,w)h\|_{s'}\leq C(s')(\|w\|_{s'}\|h\|_s+\|h\|_{s'}),$$
$$\|\mathrm{D}_w^2\mathcal{F}(\epsilon,w)[h,\mathrm{h}]\|_{s'}\leq C(s')(\|w\|_{s'}\|h\|_s\|\mathrm{h}\|_s+\|h\|_{s'}\|\mathrm{h}\|_s+\|h\|_s\|\mathrm{h}\|_{s'});$$

(U3) (Taylor Tame.) $\forall s\leq s'\leq k-3$, $\forall w,h\in W\cap H^{s'}$ with $\|w\|_s\leq 1$ and $\|h\|_s\leq 1$,

$$\|\mathcal{F}(\epsilon,w+h)-\mathcal{F}(\epsilon,w)-\mathrm{D}_w\mathcal{F}(\epsilon,w)[h]\|_s\leq C\|h\|_s^2,$$

$$\|\mathcal{F}(\epsilon,w+h)-\mathcal{F}(\epsilon,w)-\mathrm{D}_w\mathcal{F}(\epsilon,w)[h]\|_{s'}\leq C(s')(\|w\|_{s'}\|h\|_s^2+\|h\|_s\|h\|_{s'}).$$

Remark that (U1)–(U3) follow from Lemmata 4.4–4.5 and Lemma 2.1. Let us define the linearized operator $\mathcal{L}_N(\epsilon,\omega,w)$ as

$$\mathcal{L}_N(\epsilon,\omega,w)[h]:=-L_\omega h+\epsilon \mathrm{P}_N\Pi_W\mathrm{D}_w\mathcal{F}(\epsilon,w)[h],\quad \forall h\in W_N, \tag{2.2}$$

where L_ω is given by (1.8). Denote by $\lambda_j(\epsilon,w)=\mu_j^2(\epsilon,w),j\in\mathbb{N}^+$ the eigenvalues of Euler–Bernoulli beam problem

$$\begin{cases}(p(x)y'')''-\epsilon\Pi_V f'(t,x,v(\epsilon,w(t,x))+w(t,x))y=\lambda\rho(x)y,\\ y(0)=y(\pi)=y''(0)=y''(\pi)=0,\end{cases} \tag{2.3}$$

where

$$\mu_j(\epsilon,w)=\begin{cases}\mathrm{i}\sqrt{-\lambda_j(\epsilon,w)} & \text{if } \lambda_j(\epsilon,w)<0,\\ \sqrt{\lambda_j(\epsilon,w)} & \text{if } \lambda_j(\epsilon,w)>0.\end{cases} \tag{2.4}$$

For fixed $\gamma\in(0,1),\tau\in(1,2)$, define the set $\Delta_N^{\gamma;\tau}(w)$ by

$$\Delta_N^{\gamma;\tau}(w):=\Big\{(\epsilon,\omega)\in(\epsilon_1,\epsilon_2)\times(\gamma,+\infty):|\omega l-\mu_j(\epsilon,w)|>\tfrac{\gamma}{l^\tau},$$
$$|\omega l-j|>\tfrac{\gamma}{l^\tau},\ \forall l=1,2,\cdots,N,j\geq 1\Big\}. \tag{2.5}$$

Note that the non-resonance conditions in (2.5) are trivially satisfied if $\lambda_j(\epsilon,w)<0$.

LEMMA 2.2. *Let $(\epsilon,\omega)\in\Delta_N^{\gamma;\tau}(w)$ for fixed $\gamma\in(0,1),\tau\in(1,2)$. There exist $K,K(s')>0$ such that if*

$$\|w\|_{s+\sigma}\leq 1, \tag{2.6}$$

for $\frac{\epsilon}{\gamma^3\omega}\leq\delta\leq\frac{c}{L}$ (see Lemma 3.5) small enough, then $\mathcal{L}_N(\epsilon,\omega,w)$ is invertible with

$$\big\|\mathcal{L}_N^{-1}(\epsilon,\omega,w)h\big\|_s\leq\tfrac{K}{\gamma\omega}N^{\tau-1}\|h\|_s,\quad\forall s>1/2,$$
$$\big\|\mathcal{L}_N^{-1}(\epsilon,\omega,w)h\big\|_{s'}\leq\tfrac{K(s')}{\gamma\omega}N^{\tau-1}(\|h\|_{s'}+\|w\|_{s'+\sigma}\|h\|_s),\quad\forall s'\geq s>1/2.$$

Moreover, let the symbol $\lfloor\cdot\rfloor$ stand for the integer part. We set

$$N_n:=\lfloor e^{d2^n}\rfloor\quad\text{with } d=\ln N_0, \tag{2.7}$$

where $N_0>1$ will be fixed in Lemma 2.4. Denote by A_0 the open set

$$A_0:=\Big\{(\epsilon,\omega)\in(\epsilon_1,\epsilon_2)\times(\gamma,+\infty):|\omega l-\bar\mu_j|>\frac{\gamma}{l^\tau},\ \forall l=1,\cdots,N_0,j\geq 1\Big\}, \tag{2.8}$$

where $\bar\lambda_j=\bar\mu_j^2,j\geq 1$ are the eigenvalues of Euler–Bernoulli beam problem (1.11). We now give the following inductive theorem.

THEOREM 2.1. *Let r be given by Lemma 2.1. For $\frac{\epsilon}{\gamma^3\omega}\leq\delta_4$ (see Lemma 2.6) small enough, there exists a sequence of subsets $(\epsilon,\omega)\in A_n\subseteq A_{n-1}\subseteq\cdots\subseteq A_1\subseteq A_0$, where*

$$A_n:=\big\{(\epsilon,\omega)\in A_{n-1}:(\epsilon,\omega)\in\Delta_{N_n}^{\gamma;\tau}(w_{n-1})\big\},$$

and a sequence $w_n(\epsilon,\omega)\in W_{N_n}$ *satisfying*

$(\mathbf{S1})_{n\geq 0}$ $\|w_n\|_{s+\sigma}\leq 1$, $\|\partial_\omega w_n\|_s\leq \frac{K_1\epsilon}{\gamma^2\omega}$ *and* $\|\partial_\epsilon w_n\|_s\leq \frac{K_1}{\gamma\omega}$;

$(\mathbf{S2})_{n\geq 1}$ *For all* $1\leq k\leq n$, *one has* $\|w_k-w_{k-1}\|_s\leq \frac{K_2\epsilon}{\gamma\omega}N_k^{-\sigma-1}$ *with* $\frac{K_2\epsilon}{\gamma\omega}<r$, $\|\partial_\omega(w_k-w_{k-1})\|_s\leq \frac{K_3\epsilon}{\gamma^2\omega}N_k^{-1}$ *and* $\|\partial_\epsilon(w_k-w_{k-1})\|_s\leq \frac{K_3}{\gamma\omega}N_k^{-1}$;

$(\mathbf{S3})_{n\geq 0}$ *If* $(\epsilon,\omega)\in A_n$, *then* $w_n(\epsilon,\omega)$ *is a solution of*

$$L_\omega w-\epsilon \mathrm{P}_{N_n}\Pi_W\mathcal{F}(\epsilon,w)=0; \qquad (P_{N_n})$$

$(\mathbf{S4})_{n\geq 1}$ *If we set* $B_k:=1+\|w_k\|_{s+\kappa}$, $B_k':=1+\|\partial_\omega w_k\|_{s+\kappa}$ *and* $B_k'':=1+\|\partial_\epsilon w_k\|_{s+\kappa}$, *then there exist constants* $\bar K=\bar K(N_0), C_i=C_i(d,\tau,\sigma), i=1,2,3$ *such that for all* $1\leq k\leq n$,

$$B_k\leq C_1\bar K N_{k+1}^{\tau-1+\sigma},\ B_k'\leq C_2\bar K\gamma^{-1}N_{k+1}^{3\tau+2\sigma-1},\ B_k''\leq C_3\bar K(\gamma\omega)^{-1}N_{k+1}^{3\tau+2\sigma-1},$$

such that for all $(\epsilon,\omega)\in\cap_{n\geq 0}A_n$, *the sequence* $\{w_n=w_n(\epsilon,\omega)\}_{n\geq 0}$ *converges uniformly in* s*-norm to a map* $w=w_\infty\in C^1\left(\cap_{n\geq 0}A_n\cap\left\{(\epsilon,\omega):\epsilon/\omega\leq\delta_4\gamma^3\right\};W\cap H^s\right)$.

2.2.1. Initialization. First, we need to verity that $(\mathrm{S1})_0,(\mathrm{S3})_0$ hold.

LEMMA 2.3. *Given* $(\epsilon,\omega)\in A_0$, *the operator* $\frac{1}{\rho}L_\omega$ *is invertible with, for* $\tilde K>0$,

$$\|(\tfrac{1}{\rho}L_\omega)^{-1}h\|_s\leq \frac{\tilde K N_0^{\tau-1}}{\gamma\omega}\|h\|_s,\quad \forall s\geq 0,\forall h\in W_{N_0}.$$

Proof. Clearly, the eigenvalues of $\frac{1}{\rho}L_\omega$ on W_{N_0} are

$$-\omega^2l^2+\bar\lambda_j,\quad \forall 1\leq|l|\leq N_0,\forall j\geq 1.$$

For all $(\epsilon,\omega)\in A_0$, we obtain that for all $1\leq|l|\leq N_0, j\geq 1$,

$$|\omega^2l^2-\bar\lambda_j|=|\omega l-\bar\mu_j||\omega l+\bar\mu_j|>\frac{\gamma\omega}{l^{\tau-1}}.$$

Then the operator $\frac{1}{\rho}L_\omega$ is invertible on W_{N_0} satisfying the above estimate. Hence we complete the proof of the lemma. ∎

REMARK 2.1. In the proof of Lemma 2.3, we apply an equivalent scalar product $(\cdot,\cdot)$ on $H_p^2(0,\pi)$, where $(y,z):=\int_0^\pi py''z''+\rho yz\mathrm{d}x$.

By applying Lemma 2.3, solving (P_{N_0}) is equivalent to the fixed point problem $w=\mathcal{U}_0(w)$, where

$$\mathcal{U}_0:W_{N_0}\longrightarrow W_{N_0},\quad w\longmapsto \epsilon\left(\tfrac{1}{\rho}L_\omega\right)^{-1}\tfrac{1}{\rho}\mathrm{P}_{N_0}\Pi_W\mathcal{F}(\epsilon,w).$$

LEMMA 2.4. *Let* $(\epsilon,\omega)\in A_0$ *and* r *be given in Lemma 2.1. For* $\frac{\epsilon}{\gamma\omega}\leq\delta_1N_0^{1-\tau}\leq\delta$, *the map* $\mathcal{U}_0$ *is a contraction in*

$$\mathcal{B}(0,\rho_0):=\{w\in W_{N_0}:\|w\|_s\leq\rho_0\},\quad \rho_0:=\frac{\epsilon K_1N_0^{\tau-1}}{\gamma\omega}\ \textit{with}\ \frac{\epsilon K_1N_0^{\tau}}{\gamma\omega}<r.$$

Proof. It follows from Lemma 2.3 and (U1) that for $\frac{\epsilon}{\gamma\omega}N_0^{\tau-1}\leq\delta_1$ small enough,

$$\|\mathcal{U}_0(w)\|_s\leq\frac{\epsilon\tilde KN_0^{\tau-1}}{\gamma\omega}\|1/\rho\|_{H^2}\|\mathrm{P}_{N_0}\Pi_W\mathcal{F}(\epsilon,w)\|_s\leq\frac{\epsilon}{\gamma\omega}K_1N_0^{\tau-1},$$

$$\|\mathrm{D}_w\mathcal{U}_0(w)\|_s = \|\epsilon(\tfrac{1}{\rho}L_\omega)^{-1}\tfrac{1}{\rho}\mathrm{P}_{N_0}\Pi_W\mathrm{D}_w\mathcal{F}(\epsilon,w)\|_s \le \tfrac{\epsilon}{\gamma\omega}K_1N_0^{\tau-1} \le \tfrac{1}{2}. \tag{2.9}$$

Thus the map $\mathcal{U}_0$ is a contraction in $\mathcal{B}(0,\rho_0)$. □

Denote by w_0 the unique solution of equation (P_{N_0}) in $\mathcal{B}(0,\rho_0)$. For $\frac{\epsilon}{\gamma\omega}N_0^{\tau-1}\le\delta_1$ small enough, by using (P1) and Lemma 2.4, one has

$$\|w_0\|_{s+\sigma}\le 1, \quad \|w_0\|_{s+\kappa} = \|\epsilon(\tfrac{1}{\rho}L_\omega)^{-1}\tfrac{1}{\rho}\mathrm{P}_{N_0}\Pi_W\mathcal{F}(\epsilon,w_0)\|_{s+\kappa}\le\bar{K}. \tag{2.10}$$

Moreover, let us define

$$\mathscr{U}_0(\epsilon,\omega,w) := w - \epsilon(\tfrac{1}{\rho}L_\omega)^{-1}\tfrac{1}{\rho}\mathrm{P}_{N_0}\Pi_W\mathcal{F}(\epsilon,w).$$

It is obvious that $\mathscr{U}_0(\epsilon,\omega,w_0)=0$. By virtue of formula (2.9), for $\frac{\epsilon}{\gamma\omega}N_0^{\tau-1}\le\delta_1$ small enough, we obtain that

$$\mathrm{D}_w\mathscr{U}_0(\epsilon,\omega,w_0) = \mathrm{Id} - \epsilon(\tfrac{1}{\rho}L_\omega)^{-1}\tfrac{1}{\rho}\mathrm{P}_{N_0}\Pi_W\mathrm{D}_w\mathcal{F}(\epsilon,w_0)$$

is invertible. Then the implicit function theorem implies that $w_0\in C^1(A_0;W_{N_0})$. By taking the derivatives of $\mathscr{U}_0(\epsilon,\omega,w_0)=0$ with respect to ω,ϵ, one has

$$\begin{aligned}\partial_\omega w_0 =& \epsilon(\mathrm{Id}-\mathrm{D}_w\mathcal{U}_0(w_0))^{-1}\partial_\omega(\tfrac{1}{\rho}L_\omega)^{-1}\tfrac{1}{\rho}\mathrm{P}_{N_0}\Pi_W\mathcal{F}(\epsilon,w_0),\\ \partial_\epsilon w_0 =& (\mathrm{Id}-\mathrm{D}_w\mathcal{U}_0(w_0))^{-1}(\tfrac{1}{\rho}L_\omega)^{-1}\tfrac{1}{\rho}(\mathrm{P}_{N_0}\Pi_W\mathcal{F}(\epsilon,w_0)+\epsilon\mathrm{P}_{N_0}\Pi_W\partial_\epsilon\mathcal{F}(\epsilon,w_0)).\end{aligned}$$

In addition, taking the derivative of the identity $(\frac{1}{\rho}L_\omega)(\frac{1}{\rho}L_\omega)^{-1}\mathfrak{w}=\mathfrak{w}$ with respect to ω yields that

$$\partial_\omega(\tfrac{1}{\rho}L_\omega)^{-1}\mathfrak{w} = -(\tfrac{1}{\rho}L_\omega)^{-1}(\tfrac{2\omega}{\rho}\partial_{tt})(\tfrac{1}{\rho}L_\omega)^{-1}\mathfrak{w}.$$

Then, in view of (2.9), (P1) and Lemma 2.3, we derive

$$\|\partial_\omega w_0\|_s\le\tfrac{K_1\epsilon}{\gamma^2\omega}, \quad \|\partial_\epsilon w_0\|_s\le\tfrac{K_1}{\gamma\omega}.$$

Combining these estimates with (P1) gives that for $\frac{\epsilon}{\gamma\omega}N_0^{\tau-1}\le\delta_1$ small enough,

$$\|\partial_\omega w_0\|_{s+\kappa}\le\bar{K}\gamma^{-1}, \quad \|\partial_\epsilon w_0\|_{s+\kappa}\le\bar{K}(\gamma\omega)^{-1}. \tag{2.11}$$

Consequently, we have $(\mathrm{S1})_0,(\mathrm{S3})_0$.

2.2.2. Iteration. The next thing to suppose is that there have been solutions $w_k\in W_{N_k}$ of (P_{N_k}) satisfying $(\mathrm{S1})_k$–$(\mathrm{S4})_k$ for all $k\le n$.

Another step is to seek a solution $w_{n+1}\in W_{N_{n+1}}$ satisfying $(\mathrm{S1})_{n+1}$–$(\mathrm{S4})_{n+1}$ of

$$L_\omega w - \epsilon\mathrm{P}_{N_{n+1}}\Pi_W\mathcal{F}(\epsilon,w)=0. \tag{$P_{N_{n+1}}$}$$

According to Equation (P_{N_n}), if we denote

$$w_{n+1}=w_n+h, \quad h\in W_{N_{n+1}},$$

then

$$\begin{aligned}L_\omega(w_n+h)-\epsilon\mathrm{P}_{N_{n+1}}\Pi_W\mathcal{F}(\epsilon,w_n+h) =& L_\omega h + L_\omega w_n - \epsilon\mathrm{P}_{N_{n+1}}\Pi_W\mathcal{F}(\epsilon,w_n+h)\\ =& -\mathcal{L}_{N_{n+1}}(\epsilon,\omega,w_n)h + R_n(h) + r_n,\end{aligned}$$

where

$$R_n(h) := -\epsilon \mathrm{P}_{N_{n+1}}(\Pi_W \mathcal{F}(\epsilon, w_n + h) - \Pi_W \mathcal{F}(\epsilon, w_n) - \Pi_W \mathrm{D}_w \mathcal{F}(\epsilon, w_n)[h]),$$
$$r_n := \epsilon \mathrm{P}_{N_n} \Pi_W \mathcal{F}(\epsilon, w_n) - \epsilon \mathrm{P}_{N_{n+1}} \Pi_W \mathcal{F}(\epsilon, w_n) = -\epsilon \mathrm{P}_{N_n}^{\perp} \mathrm{P}_{N_{n+1}} \Pi_W \mathcal{F}(\epsilon, w_n).$$

Since $(\epsilon,\omega) \in A_{n+1} \subseteq A_n$ and $\frac{\epsilon}{\gamma\omega} \leq \frac{\epsilon}{\gamma^3\omega} \leq \delta_1 N_0^{1-\tau} \leq \delta$, by means of $(\mathrm{S1})_n$ (i.e., (2.6) holds), using Lemma 2.2 yields that the operator $\mathcal{L}_{N_{n+1}}(\epsilon,\omega,w_n)$ is invertible with

$$\|\mathcal{L}_{N_{n+1}}^{-1}(\epsilon,\omega,w_n)h\|_s \leq \tfrac{K}{\gamma\omega} N_{n+1}^{\tau-1} \|h\|_s, \quad \forall s > 1/2, \tag{2.12}$$
$$\|\mathcal{L}_{N_{n+1}}^{-1}(\epsilon,\omega,w_n)h\|_{s'} \leq \tfrac{K(s')}{\gamma\omega} N_{n+1}^{\tau-1} (\|h\|_{s'} + \|w_n\|_{s'+\sigma} \|h\|_s), \quad \forall s' \geq s > 1/2. \tag{2.13}$$

Define a map

$$\mathcal{U}_{n+1} : W_{N_{n+1}} \longrightarrow W_{N_{n+1}}, \quad h \longmapsto \mathcal{L}_{N_{n+1}}^{-1}(\epsilon,\omega,w_n)(R_n(h) + r_n).$$

Then solving $(P_{N_{n+1}})$ is reduced to finding the fixed point of $h = \mathcal{U}_{n+1}(h)$.

LEMMA 2.5. *Let $(\epsilon,\omega) \in A_{n+1}$ and r be as seen in Lemma 2.1. If $\frac{\epsilon}{\gamma^3\omega} \leq \delta_2 \leq \delta_1 N_0^{1-\tau}$, then there exists $K_2 > 0$ such that the map $\mathcal{U}_{n+1}$ is a contraction in*

$$\mathcal{B}(0,\rho_{n+1}) := \left\{ h \in W_{N_{n+1}} : \|h\|_s \leq \rho_{n+1} \right\}, \quad \rho_{n+1} := \tfrac{\epsilon K_2}{\gamma\omega} N_{n+1}^{-\sigma-1} \ \textit{with}\ \tfrac{\epsilon K_2}{\gamma\omega} < r. \tag{2.14}$$

Moreover, the unique fixed point $h_{n+1}(\epsilon,\omega)$ of the map $\mathcal{U}_{n+1}$ satisfies

$$\|h_{n+1}\|_s \leq \tfrac{\epsilon}{\gamma\omega} K_2 N_{n+1}^{\tau-1} N_n^{-\kappa} B_n. \tag{2.15}$$

Proof. In view of (P2), (U2)–(U3), it follows that

$$\|R_n(h)\|_s \leq \epsilon C \|h\|_s^2, \quad \|r_n\|_s \leq \epsilon C(\kappa) N_n^{-\kappa} B_n,$$

where B_n is as seen in $(\mathrm{S4})_n$. Based on the above estimates and (2.12), we get

$$\begin{aligned} \|\mathcal{U}_{n+1}(h)\|_s &\leq \frac{\epsilon K'}{\gamma\omega} N_{n+1}^{\tau-1} \|h\|_s^2 + \frac{\epsilon K'}{\gamma\omega} N_{n+1}^{\tau-1} N_n^{-\kappa} B_n \\ &\leq \frac{\epsilon K'}{\gamma\omega} N_{n+1}^{\tau-1} \rho_{n+1}^2 + \frac{\epsilon K'}{\gamma\omega} N_{n+1}^{\tau-1} N_n^{-\kappa} B_n. \end{aligned} \tag{2.16}$$

Obviously, it can be seen from the fact $\tau \in (1,2)$ that $\sigma > \tau - 1$. Exploring the definition of ρ_{n+1} together with $(\mathrm{S4})_n$, for $\frac{\epsilon}{\gamma^3\omega} \leq \delta_2$ small enough, one has

$$\frac{\epsilon K'}{\gamma\omega} N_{n+1}^{\tau-1} \rho_{n+1} \leq \frac{1}{2}, \quad \frac{\epsilon K'}{\gamma\omega} N_{n+1}^{\tau-1} N_n^{-\kappa} B_n \overset{(1.12)}{\leq} \frac{\rho_{n+1}}{2}, \tag{2.17}$$

which leads to $\|\mathcal{U}_{n+1}(h)\|_s \leq \rho_{n+1}$. Moreover, by taking the derivative of $\mathcal{U}_{n+1}$ with respect to h, we obtain that $\mathrm{D}_h \mathcal{U}_{n+1}(h)[\mathfrak{w}]$ is equal to

$$-\epsilon \mathcal{L}_{N_{n+1}}^{-1}(\epsilon,\omega,w_n) \mathrm{P}_{N_{n+1}} (\Pi_W \mathrm{D}_w \mathcal{F}(\epsilon, w_n + h) - \Pi_W \mathrm{D}_w \mathcal{F}(\epsilon, w_n)) \mathfrak{w}. \tag{2.18}$$

For $\frac{\epsilon}{\gamma^3\omega} \leq \delta_2$ small enough, by virtue of (U1)–(U2) and $(\mathrm{S1})_n$, it follows that

$$\|\mathrm{D}_h \mathcal{U}_{n+1}(h)[\mathfrak{w}]\|_s \overset{(2.12)}{\leq} \frac{\epsilon K'}{\gamma\omega} N_{n+1}^{\tau-1} \|h\|_s \|\mathfrak{w}\|_s \leq \frac{\epsilon K'}{\gamma\omega} N_{n+1}^{\tau-1} \rho_{n+1} \|\mathfrak{w}\|_s \overset{(2.17)}{\leq} \frac{1}{2} \|\mathfrak{w}\|_s. \tag{2.19}$$

Hence $\mathcal{U}_{n+1}$ is a contraction in $\mathcal{B}(0,\rho_{n+1})$.

Denote by $h_{n+1}(\epsilon,\omega)$ the unique fixed point of $\mathcal{U}_{n+1}$. Using (2.14), (2.16)–(2.17) yields that

$$\|h_{n+1}\|_s \le \frac{1}{2}\|h_{n+1}\|_s + \frac{\epsilon K'}{\gamma\omega} N_{n+1}^{\tau-1} N_n^{-\kappa} B_n.$$

This carries out (2.15). Thus the proof is completed. □

If we set $h_0 = w_0$, for $\frac{\epsilon}{\gamma^3\omega} \le \delta_3 \le \delta_2$ small enough, Lemmata 2.4–2.5 show that

$$\|w_{n+1}\|_{s+\sigma} \le \sum_{i=0}^{n+1} \|h_i\|_{s+\sigma} \overset{(\mathrm{P1})}{\le} \sum_{i=0}^{n+1} N_i^\sigma \|h_i\|_s \le \sum_{i=1}^{n+1} N_i^\sigma \frac{\epsilon K_2}{\gamma\omega} N_i^{-\sigma-1} + N_0^\sigma \frac{\epsilon K_1}{\gamma\omega} N_0^{\tau-1} \le 1.$$

In the following, we will investigate the derivatives of h_{n+1} with respect to ω,ϵ.

Lemma 2.6. *Provided $(\epsilon,\omega)\in A_{n+1}$, if $\frac{\epsilon}{\gamma^3\omega} \le \delta_4 \le \delta_3$, then the unique fixed point $h_{n+1} \in C^1(A_{n+1} \cap \{(\epsilon,\omega) : \epsilon/\omega \le \delta_4\gamma^3\}; W_{N_{n+1}})$ satisfying that for some constant $K_3 > 0$,*

$$\|\partial_\omega h_{n+1}\|_s \le \tfrac{K_3\epsilon}{\gamma^2\omega} N_{n+1}^{-1}, \quad \|\partial_\epsilon h_{n+1}\|_s \le \tfrac{K_3}{\gamma\omega} N_{n+1}^{-1}.$$

Proof. Let us define the map $\mathscr{U}_{n+1}$ as

$$\mathscr{U}_{n+1}(\epsilon,\omega,h) := -L_\omega(w_n+h) + \epsilon \mathrm{P}_{N_{n+1}} \Pi_W \mathcal{F}(\epsilon, w_n+h).$$

By Lemma 2.5, it is easy to see that $\mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1}) = 0$. This shows that

$$\mathrm{D}_h \mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1}) = \mathcal{L}_{N_{n+1}}(\epsilon,\omega,w_{n+1}) \overset{(2.18)}{=} \mathcal{L}_{N_{n+1}}(\epsilon,\omega,w_n)(\mathrm{Id} - \mathrm{D}_h \mathcal{U}(h_{n+1})). \tag{2.20}$$

By means of (2.19), the operator $\mathcal{L}_{N_{n+1}}(\epsilon,\omega,w_{n+1})$ is invertible with

$$\|\mathcal{L}_{N_{n+1}}^{-1}(\epsilon,\omega,w_{n+1})\|_s \le \|(\mathrm{Id} - \mathrm{D}_h\mathcal{U}(h_{n+1}))^{-1} \mathcal{L}_{N_{n+1}}^{-1}(\epsilon,\omega,w_n)\|_s \overset{(2.12)}{\le} \frac{2K}{\gamma\omega} N_{n+1}^{\tau-1}. \tag{2.21}$$

Then using the implicit function theorem yields that $h_{n+1} \in C^1(A_{n+1} \cap \{(\epsilon,\omega) : \epsilon/\omega \le \delta_4\gamma^3\}; W_{N_{n+1}})$. Hence it can be obtained that

$$\partial_{\omega,\epsilon} \mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1}) + \mathrm{D}_h \mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1}) \partial_{\omega,\epsilon} h_{n+1} = 0.$$

Consequently, combining $w_{n+1} = w_n + h_{n+1}$ with Equation (P_{N_n}) gives that

$$\partial_{\omega,\epsilon} h_{n+1} = -\mathcal{L}_{N_{n+1}}^{-1}(\epsilon,\omega,w_{n+1}) \partial_{\omega,\epsilon} \mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1}), \tag{2.22}$$

where

$$\begin{aligned}
\partial_\omega \mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1}) = &-2\omega\rho(x)(h_{n+1})_{tt} + \epsilon \mathrm{P}_{N_n}^\perp \mathrm{P}_{N_{n+1}} \Pi_W \mathrm{D}_w \mathcal{F}(\epsilon,w_n) \partial_\omega w_n \\
&+ \epsilon \mathrm{P}_{N_{n+1}} (\Pi_W \mathrm{D}_w \mathcal{F}(\epsilon, w_n+h_{n+1}) - \Pi_W \mathrm{D}_w \mathcal{F}(\epsilon,w_n)) \partial_\omega w_n, \quad (2.23)\\
\partial_\epsilon \mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1}) = &\mathrm{P}_{N_n}^\perp \mathrm{P}_{N_{n+1}} \mathcal{F}(\epsilon,w_n) + \mathrm{P}_{N_{n+1}} (\Pi_W \mathcal{F}(\epsilon,w_n+h_{n+1}) - \Pi_W \mathcal{F}(\epsilon,w_n)) \\
&+ \epsilon \mathrm{P}_{N_n}^\perp \mathrm{P}_{N_{n+1}} \Pi_W \partial_\epsilon \mathcal{F}(\epsilon,w_n) \\
&+ \epsilon \mathrm{P}_{N_{n+1}} (\Pi_W \partial_\epsilon \mathcal{F}(\epsilon, w_n+h_{n+1}) - \Pi_W \partial_\epsilon \mathcal{F}(\epsilon,w_n)). \quad (2.24)
\end{aligned}$$

According to Lemma 2.1 and Lemma 4.5, we can see

$$\|\Pi_W\partial_\epsilon\mathcal{F}(\epsilon,w_n+h_{n+1})-\Pi_W\partial_\epsilon\mathcal{F}(\epsilon,w_n)\|_s\leq C(1+\|\partial_\epsilon w_n\|_s)\|h_{n+1}\|_s$$

and

$$\|\Pi_W\partial_\epsilon\mathcal{F}(\epsilon,w_n)\|_{s+\kappa}\leq C(\kappa)\|w_n\|_{s+\kappa}(1+\|\partial_\epsilon w_n\|_s)+C(\kappa)(1+\|\partial_\epsilon w_n\|_{s+\kappa}),\tag{2.25}$$

$$\begin{aligned}\|\Pi_W\partial_\epsilon\mathcal{F}(\epsilon,w_n+h_{n+1})-\Pi_W\partial_\epsilon\mathcal{F}(\epsilon,w_n)\|_{s+\kappa}&\leq C(\kappa)\|w_n\|_{s+\kappa}(1+\|\partial_\epsilon w_n\|_s)\|h_{n+1}\|_s\\&+C(\kappa)(1+\|\partial_\epsilon w_n\|_s)\|h_{n+1}\|_{s+\kappa}\\&+C(\kappa)(1+\|\partial_\epsilon w_n\|_{s+\kappa})\|h_{n+1}\|_s.\end{aligned}\tag{2.26}$$

If $\frac{\epsilon}{\gamma^3\omega}\leq\delta_4$ is small enough, then it follows from (P1), (U1)–(U2) and $(\mathrm{S1})_n$ that

$$\|\partial_\omega\mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1})\|_s\overset{(2.15)}{\leq}\epsilon K'\gamma^{-1}N_{n+1}^{\tau+1}N_n^{-\kappa}B_n+\epsilon K'N_n^{-\kappa}B_n',\tag{2.27}$$

$$\|\partial_\epsilon\mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1})\|_s\overset{(2.15)}{\leq}K'N_{n+1}^{\tau-1}N_n^{-\kappa}B_n+\epsilon K'N_n^{-\kappa}B_n'',\tag{2.28}$$

where B_n,B_n',B_n'' are given by $(\mathrm{S4})_n$. Based on the above estimates, by means of (2.21)–(2.22), (1.12) and $(\mathrm{S4})_n$, we have $\|\partial_\omega h_{n+1}\|_s\leq\frac{K_3\epsilon}{\gamma^2\omega}N_{n+1}^{-1}$, $\|\partial_\epsilon h_{n+1}\|_s\leq\frac{K_3}{\gamma\omega}N_{n+1}^{-1}$. This ends the proof of the lemma. ∎

As a consequence, we complete the proof of $(\mathrm{S1})_{n+1}$–$(\mathrm{S3})_{n+1}$. Our next task is devoted to establishing the upper bounds of $h_{n+1},\partial_{\omega,\epsilon}h_{n+1}$ in $(s+\kappa)$-norm.

Lemma 2.7. *Let $(\epsilon,\omega)\in A_{n+1}$. If $\frac{\epsilon}{\gamma^3\omega}\leq\delta_4$, then the first term in $(\mathrm{S4})_{n+1}$ holds.*

Proof. First of all, we can claim that for $\frac{\epsilon}{\gamma^3\omega}\leq\delta_4$ small enough,

$$B_{n+1}\leq(1+N_{n+1}^{\tau-1+\sigma})B_n.\tag{2.29}$$

Due to (2.7), it is clear that $N_{n+1}^2\leq e^{d2^{n+2}}<N_{n+2}+1\leq 2N_{n+2}$. Combining this with (2.29) shows that

$$\begin{aligned}B_{n+1}&\leq B_0\prod_{k=1}^{n+1}(1+N_k^{\tau-1+\sigma})\leq B_0\prod_{k=1}^{n+1}(1+e^{d2^k(\tau-1+\sigma)})\\&\leq\prod_{k=1}^{+\infty}(1+e^{-d2^k(\tau-1+\sigma)})B_0e^{d2^{n+2}(\tau-1+\sigma)}\\&\leq 2^{\tau-1+\sigma}\prod_{k=1}^{+\infty}(1+e^{-d2^k(\tau-1+\sigma)})B_0N_{n+2}^{\tau-1+\sigma}.\end{aligned}$$

This together with (2.10) gives the first term in $(\mathrm{S4})_{n+1}$.

Our goal is now to prove (2.29). Observe that

$$B_{n+1}\leq 1+\|w_n\|_{s+\kappa}+\|h_{n+1}\|_{s+\kappa}=B_n+\|h_{n+1}\|_{s+\kappa}.\tag{2.30}$$

Then we just need to establish the upper bound of $\|h_{n+1}\|_{s+\kappa}$. It follows from Lemma 2.5 and (U2)–(U3) that

$$\|r_n\|_s\leq\epsilon C,\quad\|R_n(h_{n+1})\|_s\leq\epsilon C\rho_{n+1}^2,\quad\|r_n\|_{s+\kappa}\leq\epsilon C(\kappa)B_n,$$

$$\|R_n(h_{n+1})\|_{s+\kappa}\leq \epsilon C(\kappa)(\rho_{n+1}^2 B_n+\rho_{n+1}\|h_{n+1}\|_{s+\kappa}).$$

Hence using the equality $h_{n+1}=\mathcal{L}_{N_{n+1}}^{-1}(\epsilon,\omega,w_n)(R_n(h_{n+1})+r_n)$ yields that

$$\|h_{n+1}\|_{s+\kappa}\overset{(\mathrm{P1}),(2.13)}{\leq}\frac{\epsilon K'}{\gamma\omega}N_{n+1}^{\tau-1+\sigma}B_n+\frac{\epsilon K'}{\gamma\omega}N_{n+1}^{\tau-1}\rho_{n+1}\|h_{n+1}\|_{s+\kappa}.$$

For this, owing to (2.17), one has that for $\frac{\epsilon}{\gamma\omega}\leq\frac{\epsilon}{\gamma^3\omega}\leq\delta_4$ small enough,

$$\|h_{n+1}\|_{s+\kappa}\leq\frac{2\epsilon K'}{\gamma\omega}N_{n+1}^{\tau-1+\sigma}B_n\leq N_{n+1}^{\tau-1+\sigma}B_n. \tag{2.31}$$

Obviously, formula (2.29) follows directly from (2.30)–(2.31). ∎

Let us study the operator $\mathcal{L}_{N_{n+1}}^{-1}(\epsilon,\omega,w_{n+1})$ (recall (2.20)) in $(s+\kappa)$-norm.

LEMMA 2.8. *Given $(\epsilon,\omega)\in A_{n+1}$, for $\frac{\epsilon}{\gamma^3\omega}\leq\delta_4$ small enough, there is $K_4>0$ such that for all $\mathfrak{w}\in W_{N_{n+1}}$,*

$$\|\mathcal{L}_{N_{n+1}}^{-1}(\epsilon,\omega,w_{n+1})\mathfrak{w}\|_{s+\kappa}\leq\frac{K_4}{\gamma\omega}N_{n+1}^{\tau-1}\|\mathfrak{w}\|_{s+\kappa}+\frac{K_4}{\gamma\omega}N_{n+1}^{2\tau-2}(\|w_n\|_{s+\kappa+\sigma}+\|h_{n+1}\|_{s+\kappa})\|\mathfrak{w}\|_s.$$

Proof. Let $\mathfrak{L}(h_{n+1}):=(\mathrm{Id}-\mathrm{D}_h\mathcal{U}_{n+1}(h_{n+1}))^{-1}\mathfrak{w}$. Observe that

$$\mathfrak{L}(h_{n+1})=\mathfrak{w}+\mathrm{D}_h\mathcal{U}_{n+1}(h_{n+1})\mathfrak{L}(h_{n+1}),\quad \|\mathfrak{L}(h_{n+1})\|_s\overset{(2.19)}{\leq}2\|\mathfrak{w}\|_s.$$

In view of (2.18), (2.13) and (U2), we can get

$$\|\mathrm{D}_h\mathcal{U}_{n+1}(h_{n+1})\|_{s+\kappa}\leq\frac{\epsilon K'}{\gamma\omega}N_{n+1}^{\tau-1}(\|w_n\|_{s+\kappa+\sigma}\|h_{n+1}\|_s+\|h_{n+1}\|_{s+\kappa}).$$

It follows from (2.19) that

$$\begin{aligned}\|\mathfrak{L}(h_{n+1})\|_{s+\kappa}\leq&\|\mathfrak{w}\|_{s+\kappa}+\frac{2\epsilon K'}{\gamma\omega}N_{n+1}^{\tau-1}(\|w_n\|_{s+\kappa+\sigma}\|h_{n+1}\|_s+\|h_{n+1}\|_{s+\kappa})\|\mathfrak{w}\|_s\\&+\frac{1}{2}\|\mathfrak{L}(h_{n+1})\|_{s+\kappa}.\end{aligned}$$

Thus it can be seen that for $\frac{\epsilon}{\gamma\omega}\leq\frac{\epsilon}{\gamma^3\omega}\leq\delta_4$ small enough,

$$\|\mathfrak{L}(h_{n+1})\|_{s+\kappa}\leq 2\|\mathfrak{w}\|_{s+\kappa}+\frac{4\epsilon K'}{\gamma\omega}N_{n+1}^{\tau-1}(\|w_n\|_{s+\kappa+\sigma}\|h_{n+1}\|_s+\|h_{n+1}\|_{s+\kappa})\|\mathfrak{w}\|_s.$$

As a consequence, using (2.12)–(2.13) can give the conclusion of the lemma. ∎

LEMMA 2.9. *For $(\epsilon,\omega)\in A_{n+1}$ and $\frac{\epsilon}{\gamma^3\omega}\leq\delta_4$, the last two terms in* $(\mathrm{S4})_{n+1}$ *hold.*

Proof. We first claim that for $\frac{\epsilon}{\gamma^3\omega}\leq\delta_4$ small enough,

$$B'_{n+1}\leq(1+N_{n+1}^{\tau-1})B'_n+\frac{K'}{\gamma}N_{n+1}^{2\tau+\sigma}B_n,\quad B''_{n+1}\leq(1+N_{n+1}^{\tau-1})B''_n+\frac{K'}{\gamma\omega}N_{n+1}^{2\tau+\sigma}B_n. \tag{2.32}$$

Our next purpose is to study the upper bound of B'_{n+1}. Let $\alpha_1:=\tau-1$, $\alpha_2:=2\tau+\sigma$, $\alpha_3:=\tau-1+\sigma$. The first formula in (2.32) leads to

$$B'_{n+1}\leq\mathcal{S}_1+\mathcal{S}_2,\quad \mathcal{S}_1=B'_0\textstyle\prod_{k=1}^{n+1}(1+N_k^{\alpha_1}),\quad \mathcal{S}_2=\sum_{k=1}^{n+1}\mathcal{S}_{2,k},$$

where $\mathcal{S}_{2,1}=\frac{K'}{\gamma}N_{n+1}^{\alpha_2}B_n$ and

$$\mathcal{S}_{2,k}=\frac{K'}{\gamma}\left(\textstyle\prod_{j=2}^{k}\left(1+N_{n+1-(j-2)}^{\alpha_1}\right)\right)N_{n+1-(k-1)}^{\alpha_2}B_{n+1-k},\quad 2\leq k\leq n+1.$$

An argument similar to the one used in the proof of the upper bound on B_{n+1} shows that

$$\mathcal{S}_1 \le C(d,\tau,\sigma)B_0' N_{n+2}^{\alpha_1}.$$

On the other hand, according to the first term in $(\mathrm{S4})_n$, it follows that

$$\mathcal{S}_{2,1} \le \frac{K''}{\gamma} B_0 e^{(\alpha_2+\alpha_3)d2^{n+1}} \le \frac{C_1'}{\gamma} B_0 N_{n+2}^{\alpha_2+\alpha_3}.$$

In addition, one has

$$\begin{aligned}\sum_{k=2}^{n+1}\mathcal{S}_{2,k} &\le K''\gamma^{-1}B_0\sum_{k=2}^{n+1} e^{\alpha_1 d(2^{n+2}-2^{n+3-k})} e^{\alpha_2 d2^{n+2-k}} e^{\alpha_3 d2^{n+2-k}}\\ &\le K''\gamma^{-1}B_0 e^{\alpha_1 d2^{n+2}}\sum_{k=2}^{n+1} e^{(-\alpha_1+\alpha_2+\alpha_3)d2^{n+3-k}}\\ &\le K''\gamma^{-1}B_0 e^{(\alpha_2+\alpha_3)d2^{n+2}} \le C_1'\gamma^{-1}B_0 N_{n+2}^{\alpha_2+\alpha_3}.\end{aligned}$$

Hence, because of (2.10)–(2.11), we can get the upper bound of B_{n+1}'. The upper bound of B_{n+1}'' can be proved by the method analogous to that used above.

It remains to verify (2.32). It is straightforward that

$$B_{n+1}' \le 1+\|\partial_\omega w_n\|_{s+\kappa}+\|\partial_\omega h_{n+1}\|_{s+\kappa},\quad B_{n+1}'' \le 1+\|\partial_\epsilon w_n\|_{s+\kappa}+\|\partial_\epsilon h_{n+1}\|_{s+\kappa}. \tag{2.33}$$

Then we just investigate the upper bounds of $\partial_{\omega,\epsilon}h_{n+1}$ in $(s+\kappa)$-norm. Due to formula (2.22) and Lemma 2.8, we can obtain

$$\begin{aligned}\|\partial_{\omega,\epsilon}h_{n+1}\|_{s+\kappa} \le& \frac{K_4}{\gamma\omega} N_{n+1}^{\tau-1}\|\partial_{\omega,\epsilon}\mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1})\|_{s+\kappa}\\ &+\frac{K_4}{\gamma\omega}N_{n+1}^{2\tau-2}(\|w_n\|_{s+\kappa+\sigma}+\|h_{n+1}\|_{s+\kappa})\|\partial_{\omega,\epsilon}\mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1})\|_s.\end{aligned}$$

Moreover, applying (U1)–(U2), $(\mathrm{S1})_n$ and (2.31) gives that for $\frac{\epsilon}{\gamma^3\omega}\le\delta_4$ small enough,

$$\begin{aligned}\|\partial_\omega\mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1})\|_{s+\kappa} &\overset{(2.23)}{\le} C'(\kappa)\omega N_{n+1}^{\tau+1+\sigma}B_n+\epsilon C'(\kappa)B_n',\\ \|\partial_\epsilon\mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1})\|_{s+\kappa} &\overset{(2.24)\text{–}(2.26)}{\le} C'(\kappa)N_{n+1}^{\tau-1+\sigma}B_n+\epsilon C'(\kappa)B_n''.\end{aligned}$$

On the other hand, due to (2.27)–(2.28), (1.12) and $(\mathrm{S4})_n$, it follows that

$$\|\partial_\omega\mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1})\|_s\le\epsilon C'\gamma^{-1},\quad \|\partial_\epsilon\mathscr{U}_{n+1}(\epsilon,\omega,h_{n+1})\|_s\le C'.$$

The combination of the above estimates establishes that for $\frac{\epsilon}{\gamma^3\omega}\le\delta_4$ small enough,

$$\begin{aligned}\|\partial_\omega h_{n+1}\|_{s+\kappa} &\le \frac{K'}{\gamma}N_{n+1}^{2\tau+\sigma}B_n+\frac{\epsilon K'}{\gamma\omega}N_{n+1}^{\tau-1}B_n',\\ \|\partial_\epsilon h_{n+1}\|_{s+\kappa} &\le \frac{K'}{\gamma\omega}N_{n+1}^{2\tau+\sigma}B_n+\frac{\epsilon K'}{\gamma\omega}N_{n+1}^{\tau-1}B_n''.\end{aligned}$$

Hence combining these with (2.33) shows that (2.32) holds. The proof of the lemma is now completed. ∎

2.2.3. Whitney extension. Finally, we need to look for a set of parameters ϵ,ω unrelated to the iteration step n. Let us define

$$\begin{aligned}\hat{A}_n &:= \Big\{(\epsilon,\omega)\in A_n : \mathrm{dist}((\epsilon,\omega),\partial A_n) > \tfrac{\gamma_0\gamma^4}{N_n^{\tau+1}}\Big\},\\ \tilde{A}_n &:= \Big\{(\epsilon,\omega)\in A_n : \mathrm{dist}((\epsilon,\omega),\partial A_n) > \tfrac{2\gamma_0\gamma^4}{N_n^{\tau+1}}\Big\} \subset \hat{A}_n. \end{aligned}\tag{2.34}$$

Note that γ_0 will be given in Lemma 2.10. Exploiting the characteristic function of the set $\hat{A}_n$, there exists a C^∞ cut-off function $\varphi_n : A_0 \longrightarrow [0,1]$ satisfying

$$0\le \varphi_n(\epsilon,\omega)\le 1,\ \mathrm{supp}(\varphi_n)\subseteq A_n,\ \varphi_n(\epsilon,\omega)=1 \text{ if } (\epsilon,\omega)\in\tilde{A}_n,\ |\partial_{\omega,\epsilon}\varphi_n|\le C\tfrac{N_n^{\tau+1}}{\gamma_0\gamma^4}. \tag{2.35}$$

Then, for $(\epsilon,\omega)\in A_0$, we can define

$$\tilde{h}_n(\epsilon,\omega) := \begin{cases}\varphi_n(\epsilon,\omega)h_n(\epsilon,\omega) & \text{if} \quad (\omega,\epsilon)\in A_n,\\ 0 & \text{if} \quad (\omega,\epsilon)\notin A_n.\end{cases}$$

It is clear that

$$\tilde{h}_n \in C^1(A_0\cap\{(\epsilon,\omega):\epsilon/\omega\le\delta_4\gamma^3\}; W_{N_n}).$$

According to Theorem 2.1, Lemma 2.4 and (2.35), it follows that

$$\begin{aligned}&\|\tilde{h}_0\|_s \le \tfrac{\tilde{C}\epsilon}{\gamma\omega}N_0^{-1}, \quad \|\partial_\omega\tilde{h}_0\|_s\le\tfrac{\tilde{C}(\gamma_0)\epsilon}{\gamma^5\omega}, \quad \|\partial_\epsilon\tilde{h}_0\|_s\le\tfrac{\tilde{C}(\gamma_0)}{\gamma^5\omega},\\ &\|\tilde{h}_n\|_s \le \tfrac{\tilde{C}\epsilon}{\gamma\omega}N_n^{-\sigma-1}, \quad \|\partial_\omega\tilde{h}_n\|_s\le\tfrac{\tilde{C}(\gamma_0)\epsilon}{\gamma^5\omega}N_n^{-1}, \quad \|\partial_\epsilon\tilde{h}_n\|_s\le\tfrac{\tilde{C}(\gamma_0)}{\gamma^5\omega}N_n^{-1}, \quad \forall n\in\mathbb{N}^+,\end{aligned}$$

where $\frac{\tilde{C}\epsilon}{\gamma\omega}<r$. Obviously, the function $\tilde{w}_n=\sum_{k=0}^n\tilde{h}_k$ is an extension of w_n and if $(\epsilon,\omega)\in\tilde{A}_n\cap\{(\epsilon,\omega):\epsilon/\omega\le\delta_4\gamma^3\}$, then $\tilde{w}_n(\epsilon,\omega)=w_n(\epsilon,\omega)$. Hence we can obtain

$$\tilde{w}=\tilde{w}_\infty\in C^1(A_0\cap\{(\epsilon,\omega):\epsilon/\omega\le\delta_4\gamma^3\}; W\cap H^s)$$

satisfying

$$\|\tilde{w}\|_s\le\tfrac{K\epsilon}{\gamma\omega}<r, \quad \|\partial_\omega\tilde{w}\|_s\le\tfrac{K(\gamma_0)\epsilon}{\gamma^5\omega}, \quad \|\partial_\epsilon\tilde{w}\|_s\le\tfrac{K(\gamma_0)}{\gamma^5\omega}. \tag{2.36}$$

Moreover, using (2.15), (1.12) and $(S4)_n$ yields that for all $n\in\mathbb{N}^+$,

$$\|\tilde{w}-\tilde{w}_n\|_s \le \frac{\tilde{C}\epsilon}{\gamma\omega}\sum_{k\ge n+1}e^{-(\tau+\sigma+2)d2^k}\le\frac{\tilde{C}\epsilon}{\gamma\omega}e^{-(\tau+\sigma+2)d2^n}\le\frac{\tilde{C}\epsilon}{\gamma\omega}N_{n+1}^{-\frac{\tau+\sigma+2}{2}}. \tag{2.37}$$

Denote by $\lambda_j(\epsilon,\tilde{w})=\mu_j^2(\epsilon,\tilde{w}), j\in\mathbb{N}^+$ the eigenvalues of Euler–Bernoulli beam problem

$$\begin{cases}(py'')''-\epsilon\Pi_V f'(v(\epsilon,\tilde{w})+\tilde{w})y=\lambda\rho y,\\ y(0)=y(\pi)=y''(0)=y''(\pi)=0.\end{cases}$$

Let us define the set B_γ as

$$\begin{aligned}B_\gamma := \big\{(\epsilon,\omega)\in(\epsilon_1,\epsilon_2)\times(2\gamma,+\infty) : \tfrac{\epsilon}{\omega}\le\delta_6\gamma^5, |\omega l-\bar{\mu}_j|>\tfrac{2\gamma}{l^\tau}, \forall l=1,\cdots,N_0, \forall j\ge1,\\ |\omega l-j|>\tfrac{2\gamma}{l^\tau}, |\omega l-\mu_j(\epsilon,\tilde{w})|>\tfrac{2\gamma}{l^\tau}, \forall l\ge1, \forall j\ge1\big\}.\end{aligned}\tag{2.38}$$

LEMMA 2.10. *For* $\frac{\epsilon}{\gamma^2\omega} \le \frac{\epsilon}{\gamma^3\omega} \le \delta_5 \le \delta_4$ *small enough, there is some* $\gamma_0 > 0$ *such that*

$$B_\gamma \subseteq \tilde{A}_n \subset A_n, \quad \forall n \in \mathbb{N}.$$

In order to prove Lemma 2.10, let us introduce the following fact.

LEMMA 2.11. *For all* $(\epsilon, w), (\bar{\epsilon}, \bar{w}) \in (\epsilon_1, \epsilon_2) \times \{W \cap H^s : \|w\|_s < r\}$, *the eigenvalues* $\lambda_j(\epsilon, w)$ *of* (2.3) *satisfy that for some constant* $\nu > 0$,

$$|\lambda_j(\epsilon, w) - \lambda_j(\bar{\epsilon}, \bar{w})| \le \nu(|\epsilon - \bar{\epsilon}| + \|w - \bar{w}\|_s), \quad j \in \mathbb{N}^+. \tag{2.39}$$

Proof. Let $\psi_j(\mathrm{g})$ denote the eigenfunctions with respect to the eigenvalues $\lambda_j(\mathrm{g})$, where

$$\mathrm{g}(\cdot) := -\epsilon \Pi_V f'(t, \cdot, v(\epsilon, w(t, \cdot)) + w(t, \cdot)) \in H_p^2(0, \pi) \subset C^1([0, \pi]; \mathbb{R}).$$

Since the coefficients in problem (2.3) satisfy the assumptions of [31, Theorem 4.4], it follows that

$$\mathrm{D}_\mathrm{g} \lambda_j(\mathrm{g})[h] = -\int_0^\pi (\psi_j(\mathrm{g}))^2 h \mathrm{d}x.$$

Notice that $\int_0^\pi (\psi_j(\mathrm{g} + \mathfrak{v}(\bar{\mathrm{g}} - \mathrm{g})))^2 \rho \mathrm{d}x = 1$ (see (3.6)). Then applying Lemma 4.5 and Lemma 2.1 yields that

$$\begin{aligned} |\lambda_j(\mathrm{g}) - \lambda_j(\bar{\mathrm{g}})| &= \left| \int_0^1 \int_0^\pi (\psi_j(\mathrm{g} + \mathfrak{v}(\bar{\mathrm{g}} - \mathrm{g})))^2 (\mathrm{g} - \bar{\mathrm{g}}) \mathrm{d}x \mathrm{d}\mathfrak{v} \right| \\ &\le \max_{\mathfrak{v} \in [0,1]} \left| \int_0^\pi (\psi_j(\mathrm{g} + \mathfrak{v}(\bar{\mathrm{g}} - \mathrm{g})))^2 (\mathrm{g} - \bar{\mathrm{g}}) \mathrm{d}x \right| \\ &\le \|(\mathrm{g} - \bar{\mathrm{g}})/\rho\|_{L^\infty(0,\pi)} \max_{\mathfrak{v} \in [0,1]} \left| \int_0^\pi (\psi_j(\mathrm{g} + \mathfrak{v}(\bar{\mathrm{g}} - \mathrm{g})))^2 \rho \mathrm{d}x \right| \\ &\le C \|(\mathrm{g} - \bar{\mathrm{g}})/\rho\|_{H^2(0,\pi)} \le \nu(|\epsilon - \bar{\epsilon}| + \|w - \bar{w}\|_s). \end{aligned}$$

This ends the proof. □

Observe that the non-degeneracy of $\hat{v} = v(\hat{\epsilon}, 0)$ means that $\lambda_j(\hat{\epsilon}, 0) \ne 0$. Then it follows from Lemma 2.11 that

$$\nu_0 := \inf\{|\lambda_j(\epsilon, \omega)| : \ j \ge 1, \epsilon \in [\epsilon_1, \epsilon_2], \|w\|_s \le r\} > 0.$$

If necessary, here we may take that $|\epsilon_2 - \epsilon_1|$ and r are smaller than the ones in Lemma 2.1. Moreover, for the sake of brevity, we denote

$$\mu_{j,n}^2(\epsilon, \omega) = \lambda_{j,n}(\epsilon, \omega) := \lambda_j(\epsilon, w_n(\epsilon, \omega)), \quad \tilde{\mu}_j^2(\epsilon, \omega) = \tilde{\lambda}_j(\epsilon, \omega) := \lambda_j(\epsilon, \tilde{w}(\epsilon, \omega)).$$

We are now turning to the proof of Lemma 2.10.

Proof. **(Proof of Lemma 2.10.)** Clearly, the definition of $\tilde{A}_n$ (recall (2.34)) shows that $\tilde{A}_n \subset A_n, \forall n \in \mathbb{N}$. Our next goal is to verify

(F1): If $\frac{\epsilon}{\gamma^2\omega} \le \delta_5$ is small enough, then there exists $\gamma_0 > 0$ such that for all $(\epsilon, \omega) \in B_\gamma$,

$$\mathcal{B}((\epsilon, \omega), \tfrac{2\gamma_0\gamma^4}{N_n^{\tau+1}}) \subseteq A_n, \quad \forall n \in \mathbb{N}.$$

This implies that for all $n\in\mathbb{N}$, if $(\epsilon,\omega)\in B_\gamma$, then (ϵ,ω) can belong to $\tilde{A}_n$. Let us check the fact (F1) by induction.

For all $(\bar\epsilon,\bar\omega)\in\mathcal{B}((\epsilon,\omega),\frac{2\gamma_0\gamma^4}{N_0^{\tau+1}})$, when we take $\gamma_0\le\frac{1}{2}$, it can be seen that

$$|\bar\omega l-\bar\mu_j|\ge|\omega l-\bar\mu_j|-|\omega-\bar\omega|l>\frac{2\gamma}{l^\tau}-\frac{2\gamma_0\gamma^4}{N_0^{\tau+1}}l\ge\frac{\gamma}{l^\tau}+\frac{\gamma}{N_0^\tau}-\frac{2\gamma_0\gamma^4}{N_0^\tau}\ge\frac{\gamma}{l^\tau},\quad l=1,\cdots,N_0.$$

This gives rise to $(\bar\epsilon,\bar\omega)\in A_0$.

If we assume that $\mathcal{B}((\epsilon,\omega),\frac{2\gamma_0\gamma^4}{N_n^{\tau+1}})\subseteq A_n$, then $(\epsilon,\omega)\in\tilde{A}_n$. As a result, it can be obtained that $\tilde{w}_n(\epsilon,\omega)=w_n(\epsilon,\omega)$.

It remains to show that the fact (F1) holds at $(n+1)$-th step. For all $(\bar\epsilon,\bar\omega)\in\mathcal{B}((\epsilon,\omega),\frac{2\gamma_0\gamma^4}{N_{n+1}^{\tau+1}})$, if $\gamma_0\le\frac{1}{2}$, then

$$|\bar\omega l-j|\ge|\omega l-j|-|\omega-\bar\omega|l>\frac{2\gamma}{l^\tau}-\frac{2\gamma_0\gamma^4}{N_{N_{n+1}}^{\tau+1}}l\ge\frac{\gamma}{l^\tau}+\frac{\gamma}{N_{n+1}^\tau}-\frac{2\gamma_0\gamma^4}{N_{n+1}^\tau}\ge\frac{\gamma}{l^\tau},\quad 1\le l\le N_{n+1}.$$

Moreover, it follows from (2.39), $(\mathrm{S1})_n$ and (2.37) that

$$\begin{aligned}|\mu_{j,n}(\bar\epsilon,\bar\omega)-\tilde\mu_j(\epsilon,\omega)|&=\frac{|\lambda_{j,n}(\bar\epsilon,\bar\omega)-\tilde\lambda_j(\epsilon,\omega)|}{|\mu_{j,n}(\bar\epsilon,\bar\omega)|+|\tilde\mu_j(\epsilon,\omega)|}\le\frac{1}{\sqrt{\nu_0}}|\lambda_{j,n}(\bar\epsilon,\bar\omega)-\tilde\lambda_j(\epsilon,\omega)|\\&\le\frac{\nu}{\sqrt{\nu_0}}(|\bar\epsilon-\epsilon|+\|w_n(\bar\epsilon,\bar\omega)-\tilde{w}(\epsilon,\omega)\|_s)\\&\le\frac{\nu}{\sqrt{\nu_0}}|\bar\epsilon-\epsilon|+\frac{\nu}{\sqrt{\nu_0}}\|w_n(\bar\epsilon,\bar\omega)-w_n(\bar\epsilon,\omega)\|_s\\&\quad+\frac{\nu}{\sqrt{\nu_0}}\|w_n(\bar\epsilon,\omega)-w_n(\epsilon,\omega)\|_s+\frac{\nu}{\sqrt{\nu_0}}\|\tilde{w}_n(\epsilon,\omega)-\tilde{w}(\epsilon,\omega)\|_s\\&\le\frac{\nu}{\sqrt{\nu_0}}\left(\frac{2\gamma_0\gamma^4}{N_{n+1}^{\tau+1}}+\frac{2K_1}{\gamma^2\omega}\frac{2\gamma_0\gamma^4}{N_{n+1}^{\tau+1}}+\frac{\tilde{C}\epsilon}{\gamma\omega}\frac{1}{N_{n+1}^{(\tau+\sigma+2)/2}}\right).\end{aligned}$$

Using (1.12) gives that $\frac{\tau+\sigma+2}{2}\ge\tau+\frac{1}{2}$. Since $\omega>\gamma$, for $\gamma_0,\frac{\epsilon}{\gamma^2\omega}$ small enough, one has

$$|\mu_{j,n}(\bar\epsilon,\bar\omega)-\tilde\mu_j(\epsilon,\omega)|\le\frac{\gamma}{2l^\tau}.$$

Consequently, for all $(\bar\epsilon,\bar\omega)\in\mathcal{B}((\epsilon,\omega),\frac{2\gamma_0\gamma^4}{N_{n+1}^{\tau+1}})$, we can obtain that for $\gamma_0,\frac{\epsilon}{\gamma^2\omega}$ small enough,

$$\begin{aligned}|\bar\omega l-\mu_{j,n}(\bar\epsilon,\bar\omega)|&\ge|\omega l-\tilde\mu_j(\epsilon,\omega)|-|\omega-\bar\omega|l-|\mu_{j,n}(\bar\epsilon,\bar\omega)-\tilde\mu_j(\epsilon,\omega)|\\&>\frac{2\gamma}{l^\tau}-\frac{2\gamma_0\gamma^4}{N_{n+1}^{\tau+1}}l-\frac{\gamma}{2l^\tau}\ge\frac{\gamma}{l^\tau},\quad l=1,\cdots,N_{n+1}.\end{aligned}$$

We have thus proved the lemma. □

Let $\Omega:=(\epsilon',\epsilon'')\times(\omega',\omega'')$ denote a rectangle contained in $(\epsilon_1,\epsilon_2)\times(2\gamma,+\infty)$ and set

$$\begin{aligned}\nu_1&:=\inf\{|\mu_{j+1}(\epsilon,\omega)-\mu_j(\epsilon,\omega)|:j\ge1,\epsilon\in[\epsilon_1,\epsilon_2],\|w\|_s\le r\}>0,\\\nu_2&:=\inf\{|\mu_{j+1}(\epsilon,\omega)-\mu_j(\epsilon,\omega)|:j\ge1,(\epsilon,\omega)\in B_\gamma\}.\end{aligned}\tag{2.40}$$

The proof of the fact $\nu_1>0$ will be given later. Moreover, without loss of generality, we assume that $\omega''-\omega'\ge1$.

LEMMA 2.12. *For fixed $\epsilon \in (\epsilon', \epsilon'')$, if $\frac{\epsilon}{\gamma^5 \omega} \le \delta_6 \le \delta_5$ small enough, then the measure estimate on $B_\gamma(\epsilon)$ satisfies that for some constant $\mathcal{Q} > 0$,*

$$\mathrm{meas}(B_\gamma(\epsilon) \cap (\omega', \omega'')) \ge (1 - \mathcal{Q}\gamma)(\omega'' - \omega'), \tag{2.41}$$

where $B_\gamma(\epsilon) := \{\omega : (\epsilon, \omega) \in B_\gamma\}$. Furthermore,

$$\mathrm{meas}(B_\gamma \cap \Omega) \ge (1 - \mathcal{Q}\gamma)\mathrm{meas}(\Omega) = (1 - \mathcal{Q}\gamma)(\omega'' - \omega')(\epsilon'' - \epsilon').$$

Proof. Denote by $(B_\gamma(\epsilon))^c$ the complementary set of $B_\gamma(\epsilon)$. By the definition of B_γ (recall (2.38)), it is evident that

$$(B_\gamma(\epsilon))^c \subseteq \mathfrak{R}^1(\epsilon) \cup \mathfrak{R}^2 \cup \mathfrak{R}^3,$$

where

$$\mathfrak{R}^1(\epsilon) = \bigcup_{l \ge 1, j \ge 1} \mathfrak{R}^1_{l,j}(\epsilon), \quad \mathfrak{R}^1_{l,j}(\epsilon) := \left\{ \omega \in (\omega', \omega'') : |\omega l - \tilde{\mu}_j(\epsilon, \omega)| \le \frac{2\gamma}{l^\tau} \right\},$$

$$\mathfrak{R}^2 = \bigcup_{l \ge 1, j \ge 1} \mathfrak{R}^2_{l,j}, \quad \mathfrak{R}^2_{l,j} := \left\{ \omega \in (\omega', \omega'') : |\omega l - \bar{\mu}_j| \le \frac{2\gamma}{l^\tau} \right\},$$

$$\mathfrak{R}^3 = \bigcup_{l \ge 1, j \ge 1} \mathfrak{R}^3_{l,j}, \quad \mathfrak{R}^3_{l,j} := \left\{ \omega \in (\omega', \omega'') : |\omega l - j| \le \frac{2\gamma}{l^\tau} \right\}.$$

We first consider the Lebesgue measure of the set $\mathfrak{R}^1(\epsilon)$. Since $B_\gamma \subseteq \tilde{A}_n$ (see Lemma 2.10), one has $\tilde{w}(\epsilon, \omega) = w(\epsilon, \omega)$. Then $\tilde{\lambda}_j(\epsilon, \omega) = \lambda_j(\epsilon, \omega)$ on B_γ. Hence formula (2.36) implies that $\nu_2 \ge \nu_1 > 0$. Using (2.39), (2.36) and the definition of ν_0 yields that

$$\begin{aligned} |\tilde{\mu}_j(\epsilon, \omega_1) - \tilde{\mu}_j(\epsilon, \omega_2)| &= \frac{|\tilde{\lambda}_j(\epsilon, \omega_1) - \tilde{\lambda}_j(\epsilon, \omega_2)|}{|\tilde{\mu}_j(\epsilon, \omega_1)| + |\tilde{\mu}_j(\epsilon, \omega_2)|} \le \frac{1}{\sqrt{\nu_0}} |\tilde{\lambda}_j(\epsilon, \omega_1) - \tilde{\lambda}_j(\epsilon, \omega_2)| \\ &\le \frac{\nu}{\sqrt{\nu_0}} \|\tilde{w}(\epsilon, \omega_1) - \tilde{w}(\epsilon, \omega_2)\|_s \le \frac{\epsilon \nu K(\gamma_0)}{\sqrt{\nu_0}\gamma^5 \omega} |\omega_1 - \omega_2|. \end{aligned}$$

This gives rise to $|\partial_\omega \tilde{\mu}_j(\epsilon, \omega)| \le \frac{\epsilon \nu K(\gamma_0)}{\sqrt{\nu_0}\gamma^5 \omega}$. If we set $a(\omega) := \omega l - \tilde{\mu}_j(\epsilon, \omega)$, for $\frac{\epsilon}{\gamma^5 \omega} \le \delta_6$ small enough, then

$$\partial_\omega a(\omega) = l - \partial_\omega \tilde{\mu}_j(\epsilon, \omega) \ge l/2.$$

Combining this with the definition of $\mathfrak{R}^1_{l,j}(\epsilon)$ gives that

$$\mathrm{meas}(\mathfrak{R}^1_{l,j}(\epsilon)) \le \frac{|a(\omega_1) - a(\omega_2)|}{\inf |\partial_\omega a(\omega)|} \le \frac{8\gamma}{l^{\tau+1}}.$$

If $\mathfrak{R}^1_{l,j}(\epsilon) \neq \emptyset$, then we also have that for fixed l,

$$\omega' l - \frac{2\gamma}{l^\tau} \le \tilde{\mu}_j(\epsilon, \omega) \le \omega'' l + \frac{2\gamma}{l^\tau}.$$

Moreover, it follows that $\sharp j \le \frac{1}{\nu_1}(l(\omega'' - \omega') + \frac{4\gamma}{l^\tau}) + 1$, where $\sharp j$ denotes the number of j. Therefore,

$$\mathrm{meas}(\mathfrak{R}^1(\epsilon)) \le \textstyle\sum_{l=1}^{+\infty} \frac{8\gamma}{l^{\tau+1}} \left(\frac{1}{\nu_1}(l(\omega'' - \omega') + \frac{4\gamma}{l^\tau}) + 1 \right)$$

$$\leq \sum_{l=1}^{+\infty} \frac{8\gamma}{l^{\tau+1}} \mathcal{Q}'' l(\omega''-\omega') \leq \mathcal{Q}'\gamma(\omega''-\omega').$$

An argument similar to the one used above can establish the upper bounds of $\text{meas}(\mathfrak{R}^2)$ and $\text{meas}(\mathfrak{R}^3)$. Thus formula (2.41) follows.

In addition, it can be seen that

$$\text{meas}(B_\gamma \cap \Omega) = \int_{\epsilon'}^{\epsilon''} \text{meas}(B_\gamma(\epsilon) \cap (\omega', \omega''))\text{d}\epsilon \geq (1-\mathcal{Q}\gamma)\text{meas}(\Omega).$$

Thus the proof is completed. □

Theorem 1.1 follows from Lemma 2.1, Lemma 2.12 and Theorem 2.1.

Proof. **(Proof of Theorem 1.1.)** By means of Theorem 2.1 and the steps of Whitney extension, the function $\tilde{w}(\epsilon,\omega)$, with $\tilde{w} \in C^1(A_\gamma; W \cap H^s)$, can solve the range equation (P) in (1.7). For $\frac{\epsilon}{\gamma^5 \omega} \leq \delta_6$ small enough, according to the fact $\|\tilde{w}\|_s < r$ (recall (2.36)), Lemma 2.1 presents that $v(\epsilon, \tilde{w})$ solves the bifurcation equation (Q) in (1.7). As a consequence, it follows that

$$\tilde{u}(\epsilon,\omega) = v(\epsilon, \tilde{w}(\epsilon,\omega)) + \tilde{w}(\epsilon,\omega) \in H_p^2(0,\pi) \oplus (W \cap H^s)$$

is a solution of Equation (1.4). Meanwhile, formulae (1.13)–(1.14) can be obtained by (2.1) and (2.36).

In addition, since the function $\tilde{u}$ solves $-(p(x)u_{xx})_{xx} = \epsilon f(t,x,u) - \omega^2 \rho(x) u_{tt}$, we obtain

$$-(p\tilde{u}_{xx})_{xx} \in H^2(0,\pi), \quad \forall t \in \mathbb{R}.$$

Due to (1.5), if $\alpha, \beta \in H^4(0,\pi)$, then $\rho, p \in H^5(0,\pi)$. Hence it can be seen that $\tilde{u}(t,\cdot) \in H^6(0,\pi) \cap H_p^2(0,\pi) \subset C^5[0,\pi]$ for all $t \in \mathbb{R}$. □

3. Invertibility of linearized operator

The object of this section is to show the invertibility of the linearized operator $\mathcal{L}_N(\epsilon,\omega,w)$ (recall (2.2)). More precisely, we will complete the proof of Lemma 2.2. We rewrite $\mathcal{L}_N(\epsilon,\omega,w)$ as

$$\mathcal{L}_N(\epsilon,\omega,w)[h] = \mathfrak{L}_1(\epsilon,\omega,w)[h] + \mathfrak{L}_2(\epsilon,w)[h], \quad \forall h \in W_N,$$

where

$$\mathfrak{L}_1(\epsilon,\omega,w)[h] := -L_\omega h + \epsilon \text{P}_N \Pi_W f'(t,x,v(\epsilon,\omega,w)+w)h,$$
$$\mathfrak{L}_2(\epsilon,w)[h] := \epsilon \text{P}_N \Pi_W f'(t,x,v(\epsilon,w)+w)\text{D}_w v(\epsilon,w)[h].$$

Let $b(t,x) := f'(t,x,v(\epsilon,\omega,w(t,x)) + w(t,x))$. If $\|w\|_{s+\sigma} \leq 1$, then it follows from (4.11) and Lemma 2.1 that

$$\|b\|_s \leq \|b\|_{s+\sigma} \leq C, \quad \forall s > 1/2, \tag{3.1}$$
$$\|b\|_{s'} \leq C(s')(1+\|w\|_{s'}), \quad \forall s' \geq s > 1/2. \tag{3.2}$$

By decomposing $b(t,x) = \sum_{k\in\mathbb{Z}} b_k(x) e^{\text{i}kt}, h(t,x) = \sum_{1\leq|l|\leq N} h_l(x) e^{\text{i}lt}$, we can write the operator $\mathfrak{L}_1(\epsilon,\omega,w)$ as

$$\mathfrak{L}_1(\epsilon,\omega,w)[h] = \sum_{1\leq|l|\leq N} \left(\omega^2 l^2 \rho h_l - (p(h_l)'')''\right) e^{\text{i}lt} + \epsilon \text{P}_N \Pi_W \sum_{k\in\mathbb{Z}, 1\leq|l|\leq N} b_{k-l} h_l e^{\text{i}kt}$$

$$=\rho\mathfrak{L}_{1,\mathrm{D}}[h]-\rho\mathfrak{L}_{1,\mathrm{ND}}[h],$$

where $b_0(x)=\Pi_V f'(t,x,v(\epsilon,w)+w)$ and

$$\mathfrak{L}_{1,\mathrm{D}}[h]=\sum_{1\le|l|\le N}\left(\omega^2l^2h_l-\frac{1}{\rho}(ph_l'')''+\epsilon\frac{b_0}{\rho}h_l\right)e^{\mathrm{i}lt},$$
$$\mathfrak{L}_{1,\mathrm{ND}}[h]=-\frac{\epsilon}{\rho}\sum_{1\le|l|,|k|\le N,l\ne k}b_{k-l}h_le^{\mathrm{i}kt}.$$

By virtue of [1, cf. Theorem 1.2, Proposition 6.2], we first give the asymptotic formulae of the eigenvalues for problem (2.3).

Lemma 3.1. *Let $\zeta=(\rho/p)^{\frac{1}{4}}$. Denote by $\lambda_j(\epsilon,w)$ and $\psi_j(\epsilon,w)$ the eigenvalues and the eigenfunctions of problem (2.3), respectively. One has*

$$\lambda_1(\epsilon,w)<\lambda_2(\epsilon,w)<\cdots<\lambda_j(\epsilon,w)<\cdots, \tag{3.3}$$

with $\lambda_j(\epsilon,w)\to+\infty$ as $j\to+\infty$, and for all $\epsilon\in(\epsilon_1,\epsilon_2)$, $w\in\{W\cap H^s:\|w\|_s<r\}$,

$$\lambda_j(\epsilon,w)=j^4+2j^2\upsilon_0+\upsilon_1(\epsilon,w)-\varrho_j(\epsilon,w)+\frac{o(1)}{j}\quad \text{as } j\to+\infty. \tag{3.4}$$

Note that

$$\begin{aligned}&\upsilon_0=\mathfrak{d}(\pi)-\mathfrak{d}(0)+\tfrac{1}{\pi}\textstyle\int_0^\pi\frac{\mathfrak{x}(x)}{\zeta(x)}\mathrm{d}x,\\&\upsilon_1(\epsilon,w)=\Lambda(\pi)-\Lambda(0)+\tfrac{1}{\pi}\textstyle\int_0^\pi\Gamma(x)\zeta(x)\mathrm{d}x+\frac{\upsilon_0^2}{2}-\frac{1}{\pi}\int_0^\pi\frac{\epsilon}{\rho(x)}\Pi_Vf'(v(\epsilon,w)+w)(x)\zeta(x)\mathrm{d}x,\\&\varrho_j(\epsilon,w)=\tfrac{1}{\pi}\textstyle\int_0^\pi\left(-\frac{\epsilon}{\rho(x)}\Pi_Vf'(v(\epsilon,w)+w)(x)\zeta(x)+\frac{\alpha_{xxx}(x)-\beta_{xxx}(x)}{4\zeta^3(x)}\right)\cos\left(2j\int_0^x\zeta(z)\mathrm{d}z\right)\mathrm{d}x,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}&\Lambda=\tfrac{1}{\zeta^3}\left(\tfrac{2\chi^3}{3}-\tfrac{\eta_-^3}{2}-2\eta\eta_+-(\chi-\eta_-)\eta_-\chi+(\chi-\eta_-)_x\chi-(\alpha\eta_-)_x-\tfrac{(\eta_-)_{xx}}{4}\right),\\&\Gamma=\tfrac{1}{8\zeta^4}((\eta_-)_x-\eta_-^2-2\mathfrak{x})^2-\tfrac{1}{\zeta^4}((\eta_+)_x-2\eta)^2,\\&\mathfrak{d}=\tfrac{3\alpha+5\beta}{2\zeta},\quad \mathfrak{x}=\tfrac{5\alpha^2+5\beta^2+6\alpha\beta}{4}\ge\tfrac{\alpha^2+\beta^2}{2}\ge0,\\&\chi=\tfrac{\alpha+3\beta}{2},\quad \eta=\eta_+\eta_-,\quad \eta_\pm=\beta\pm\alpha.\end{aligned}$$

Moreover, the eigenfunctions $\psi_j(\epsilon,w)$ form an orthogonal basis of $L^2(0,\pi)$ with the scalar product

$$(y,z)_{L^2_\rho}:=\int_0^\pi\rho yz\mathrm{d}x. \tag{3.6}$$

For $\Theta>0$ large enough, define a scalar product $(\cdot,\cdot)_{\epsilon,w}$ on $H^2_p(0,\pi)$ by

$$(y,z)_{\epsilon,w}:=\int_0^\pi py''z''-\epsilon\Pi_Vf'(v(\epsilon,\omega,w)+w)yz+\Theta\rho yz\mathrm{d}x$$

satisfying that for all $y\in H^2_p(0,\pi)$,

$$L_1\|y\|_{H^2}\le\|y\|_{\epsilon,w}\le L_2\|y\|_{H^2},\quad L_1,L_2>0. \tag{3.7}$$

In addition, the eigenfunctions $\psi_j(\epsilon,w)$ are an orthogonal basis of $H_p^2(0,\pi)$ with respect to the scalar product $(\cdot,\cdot)_{\epsilon,w}$ as well, and for $y=\sum_{j\geq 1}\hat{y}_j\psi_j(\epsilon,w)$,

$$\|y\|_{L^2_\rho}^2=\sum_{j\geq 1}(\hat{y}_j)^2, \quad \|y\|_{\epsilon,w}^2=\sum_{j\geq 1}(\lambda_j(\epsilon,w)+\Theta)(\hat{y}_j)^2, \tag{3.8}$$

where $\lambda_j(\epsilon,w)+\Theta>0$.

Proof. Let us verify formulae (3.7)–(3.8). By using the Poincaré inequality, we get $\|y'\|_{L^2(0,\pi)}\leq C\|y''\|_{L^2(0,\pi)}$ if $y\in H_p^2(0,\pi)$. Hence it can be obtained that (3.7) holds. Moreover, observe that

$$(p\psi_j''(\epsilon,w))''-\epsilon\Pi_V f'(t,x,v(\epsilon,w)+w)\psi_j(\epsilon,w)+\Theta\rho\psi_j(\epsilon,w)=(\lambda_j(\epsilon,w)+\Theta)\rho\psi_j(\epsilon,w).$$

Multiplying the above equality by $\psi_{j'}(\epsilon,w)$ and integrating by parts yield that

$$(\psi_j,\psi_{j'})_{\epsilon,w}=\delta_{j,j'}(\lambda_j(\epsilon,w)+\Theta).$$

Therefore we arrive at (3.8). □

According to (3.7)–(3.8), it can be seen that for $w=\sum_{|l|\geq 1,j\geq 1}\hat{w}_{l,j}\psi_j(\epsilon,w)e^{\mathrm{i}lt}$,

$$L_1^2\|w\|_s^2\leq \sum_{|l|\geq 1,j\geq 1}(\lambda_j(\epsilon,w)+\Theta)(\hat{w}_{l,j})^2(1+l^{2s})\leq L_2^2\|w\|_s^2. \tag{3.9}$$

This means that we have sought the equivalent norm of the s-norm restricted to $W\cap H^s$. Moreover, it follows from Lemma 3.1 that

$$\omega^2l^2h_l-\frac{1}{\rho}(ph_l'')''+\epsilon\frac{b_0}{\rho}h_l=\sum_{j\geq 1}(\omega^2l^2-\lambda_j(\epsilon,w))\hat{h}_{l,j}\psi_j(\epsilon,w).$$

This means that $\mathfrak{L}_{1,\mathrm{D}}$ is a diagonal operator on W_N. Let us define the operator

$$|\mathfrak{L}_{1,\mathrm{D}}|^{\frac{1}{2}}h=\sum_{1\leq|l|\leq N,j\geq 1}|\omega^2l^2-\lambda_j(\epsilon,w)|^{\frac{1}{2}}\hat{h}_{l,j}\psi_j(\epsilon,w)e^{\mathrm{i}lt}, \quad \forall h\in W_N.$$

For all $1\leq|l|\leq N,\ j\geq 1$, if $\omega^2l^2-\lambda_j(\epsilon,w)\neq 0$, then its inverse operator is

$$|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}h:=\sum_{1\leq|l|\leq N,j\geq 1}\frac{1}{|\omega^2l^2-\lambda_j(\epsilon,w)|^{\frac{1}{2}}}\hat{h}_{l,j}\psi_j(\epsilon,w)e^{\mathrm{i}lt}.$$

Hence we can rewrite $\mathcal{L}_N(\epsilon,\omega,w)$ as

$$\mathcal{L}_N(\epsilon,\omega,w)=\rho|\mathfrak{L}_{1,\mathrm{D}}|^{\frac{1}{2}}(|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}\mathfrak{L}_{1,\mathrm{D}}|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}-R_1-R_2)|\mathfrak{L}_{1,\mathrm{D}}|^{\frac{1}{2}},$$

where

$$R_1=|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}\mathfrak{L}_{1,\mathrm{ND}}|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}, \quad R_2=-|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}\left(\tfrac{1}{\rho}\mathfrak{L}_2\right)|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}. \tag{3.10}$$

Note that for all $h\in W_N$,

$$(|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}\mathfrak{L}_{1,\mathrm{D}}|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}})h=\sum_{1\leq|l|\leq N,j\geq 1}\mathrm{sign}(\omega^2l^2-\lambda_j(\epsilon,w))\hat{h}_{l,j}\psi_j(\epsilon,w)e^{\mathrm{i}lt}.$$

Then it is invertible with, for all $s\geq 0$,

$$\|(|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}\mathfrak{L}_{1,\mathrm{D}}|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}})^{-1}h\|_s \overset{(3.9)}{\leq} \tfrac{L_2}{L_1}\|h\|_s. \tag{3.11}$$

Therefore $\mathcal{L}_N(\epsilon,\omega,w)$ may be reduced to

$$\mathcal{L}_N(\epsilon,\omega,w)=\rho|\mathfrak{L}_{1,\mathrm{D}}|^{\frac{1}{2}}(|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}\mathfrak{L}_{1,\mathrm{D}}|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}})(\mathrm{Id}-\mathcal{R})|\mathfrak{L}_{1,\mathrm{D}}|^{\frac{1}{2}}, \tag{3.12}$$

where $\mathcal{R}=\mathcal{R}_1+\mathcal{R}_2$ with

$$\mathcal{R}_1=(|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}\mathfrak{L}_{1,\mathrm{D}}|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}})^{-1}R_1,\quad \mathcal{R}_2=(|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}\mathfrak{L}_{1,\mathrm{D}}|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}})^{-1}R_2.$$

In order to investigate the invertibility of $\mathrm{Id}-\mathcal{R}$, we need to impose some non-resonance conditions. For fixed $\tau\in(1,2)$, we assume

$$|\omega l-\mu_j(\epsilon,w)|>\frac{\gamma}{l^\tau},\quad \forall 1\leq l\leq N,\ \forall j\geq 1. \tag{3.13}$$

Then it follows from the definition of $\mu_j(\epsilon,w)$ (recall (2.4)) that

$$|\omega^2l^2-\lambda_j(\epsilon,w)|=|\omega l-\mu_j(\epsilon,w)||\omega l+\mu_j(\epsilon,w)|>\frac{\gamma\omega}{l^{\tau-1}},\quad \forall 1\leq l\leq N,\ \forall j\geq 1. \tag{3.14}$$

Moreover, we set

$$\omega_l:=\min_{j\geq 1}|\omega^2l^2-\lambda_j(\epsilon,w)|=|\omega^2l^2-\lambda_{j^*}(\epsilon,w)|,\quad \forall 1\leq |l|\leq N. \tag{3.15}$$

It is evident that $\omega_l=\omega_{-l}, 1\leq l\leq N$.

Lemma 3.2. *Provided (3.13), the operator $|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}$ is invertible with, for all $h\in W_N$,*

$$\||\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}h\|_s\leq \tfrac{\sqrt{2}L_2}{\sqrt{\gamma\omega}L_1}\|h\|_{s+\frac{\tau-1}{2}},\quad \forall s\geq 0, \tag{3.16}$$

$$\||\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}h\|_s\leq \tfrac{\sqrt{2}L_2}{\sqrt{\gamma\omega}L_1}N^{\frac{\tau-1}{2}}\|h\|_s,\quad \forall s\geq 0. \tag{3.17}$$

Proof. Since $|l|^{\tau-1}(1+l^{2s})<2(1+|l|^{2s+\tau-1})$ for $l\geq 1$, using (3.9), (3.14)–(3.15) yields that

$$\begin{aligned}\||\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}h\|_s^2 &\leq \frac{1}{L_1^2}\sum_{1\leq |l|\leq N, j\geq 1}\frac{\lambda_j(\epsilon,w)+\Theta}{\omega_l}(\hat{h}_{l,j})^2(1+l^{2s})\\ &\leq \frac{2}{\gamma\omega L_1^2}\sum_{1\leq |l|\leq N, j\geq 1}(\lambda_j(\epsilon,w)+\Theta)(\hat{h}^{\pm}_{l,j})^2(1+|l|^{2s+\tau-1})\\ &\leq \frac{2L_2^2}{\gamma\omega L_1^2}\|h\|^2_{s+\frac{\tau-1}{2}}\leq \frac{2L_2^2N^{\tau-1}}{\gamma\omega L_1^2}\|h\|_s^2.\end{aligned}$$

Thus the proof is completed. ∎

In addition, for fixed $\tau\in(1,2)$, we also assume

$$|\omega l-j|>\frac{\gamma}{l^\tau},\quad \forall 1\leq l\leq N, \forall j\geq 1. \tag{3.18}$$

Under the non-resonance conditions (3.13) and (3.18), we have

(F2): Let $\tau \in (1,2), \gamma \in (0,1)$. If $\omega > \gamma$, then there is $\tilde{L} > 0$ such that

$$\omega_l \omega_k \geq \tilde{L}^2 \gamma^6 \omega^2 |l-k|^{-2\sigma}.$$

Based on the above fact, we give the following lemma.

LEMMA 3.3. *Given (3.13) and (3.18), if $\|w\|_{s+\sigma} \leq 1$, then there exists $L > 0$ such that for all $s' \geq s > 1/2$,*

$$\|\mathcal{R}_1 h\|_{s'} \leq \tfrac{\epsilon L}{2\gamma^3 \omega} \left(\|h\|_{s'} + \|w\|_{s'+\sigma} \|h\|_s\right), \quad \forall h \in W_N. \tag{3.19}$$

Proof. By formula (3.10) and the definitions of $\mathfrak{L}_{1,\mathrm{ND}}, |\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}}$, it can be seen that

$$\begin{aligned}
R_1 h &= |\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}} \mathfrak{L}_{1,\mathrm{ND}} \left(\sum_{1 \leq |l| \leq N, j \geq 1} \frac{\hat{h}_{l,j}}{\sqrt{|\omega^2 l^2 - \lambda_j(\epsilon, w)|}} \psi_j(\epsilon, w) e^{\mathrm{i}lt} \right) \\
&= -\epsilon |\mathfrak{L}_{1,\mathrm{D}}|^{-\frac{1}{2}} \left(\sum_{\substack{1 \leq |l|, |k| \leq N \\ l \neq k, j \geq 1}} \frac{\hat{h}_{l,j}}{\sqrt{|\omega^2 l^2 - \lambda_j(\epsilon, w)|}} \frac{b_{k-l}}{\rho} \psi_j(\epsilon, w) e^{\mathrm{i}kt} \right) \\
&= -\epsilon \sum_{\substack{1 \leq |l|, |k| \leq N \\ l \neq k, j \geq 1}} \frac{\hat{h}_{l,j}}{\sqrt{|\omega^2 k^2 - \lambda_j(\epsilon, w)|} \sqrt{|\omega^2 l^2 - \lambda_j(\epsilon, w)|}} \frac{b_{k-l}}{\rho} \psi_j(\epsilon, w) e^{\mathrm{i}kt}.
\end{aligned}$$

This carries out

$$(R_1 h)_k = -\epsilon \sum_{\substack{1 \leq |l| \leq N \\ l \neq k, j \geq 1}} \frac{\hat{h}_{l,j}}{\sqrt{|\omega^2 k^2 - \lambda_j(\epsilon, w)|} \sqrt{|\omega^2 l^2 - \lambda_j(\epsilon, w)|}} \frac{b_{k-l}}{\rho} \psi_j(\epsilon, w).$$

Combining this with (3.7)–(3.8) yields that

$$\begin{aligned}
\|(R_1 h)_k\|_{H^2} &\leq \frac{\epsilon L_2}{L_1} \sum_{1 \leq |l| \leq N, l \neq k} \frac{1}{\sqrt{\omega_l \omega_k}} \|b_{k-l}/\rho\|_{H^2} \|h_l\|_{H^2} \\
&\leq \frac{\epsilon L_2}{\gamma^3 \omega \tilde{L} L_1} \sum_{1 \leq |l| \leq N, l \neq k} \|b_{k-l}/\rho\|_{H^2} |k-l|^\sigma \|h_l\|_{H^2}.
\end{aligned} \tag{3.20}$$

If we denote

$$\begin{aligned}
&\Upsilon(x) := \textstyle\sum_{1 \leq |l|, |k| \leq N} \|\frac{b_{k-l}}{\rho}\|_{H^2} |k-l|^\sigma \|h_l\|_{H^2} e^{\mathrm{i}kt}, \\
&\mathrm{p}(x) := \textstyle\sum_{l \in \mathbb{Z}} \|\frac{b_l}{\rho}\|_{H^2} |l|^\sigma e^{\mathrm{i}lt}, \quad \mathrm{q}(x) := \sum_{1 \leq |l| \leq N} \|h_l\|_{H^2} e^{\mathrm{i}lt},
\end{aligned}$$

then $\Upsilon = \mathrm{P}_N(\mathrm{pq})$. Moreover, by (3.2), we deduce that for all $s' \geq s > 1/2$,

$$\|\mathrm{p}\|_{s'} \leq C'(s')(1 + \|w\|_{s'+\sigma}), \quad \|\mathrm{q}\|_{s'} = \|h\|_{s'}.$$

Thus it follows from (3.20) and (4.2) that for $\|w\|_{s+\sigma} \leq 1$,

$$\|R_1 h\|_{s'} \leq \tfrac{\epsilon L_2}{\gamma^3 \omega \tilde{L} L_1} \|\Upsilon\|_{s'} \leq \tfrac{\epsilon L_2 C(s')}{\gamma^3 \omega \tilde{L} L_1} (\|\mathrm{p}\|_{s'} \|\mathrm{q}\|_s + \|\mathrm{p}\|_s \|\mathrm{q}\|_{s'})$$

$$\leq \tfrac{\epsilon L_2 C''(s')}{\gamma^3\omega \tilde{L} L_1}(\|w\|_{s'+\sigma}\|h\|_s+\|h\|_{s'}).$$

Consequently, by the above estimate together with (3.11), we obtain (3.19). ◻

LEMMA 3.4. *Given* (3.13)*, if* $\|w\|_{s+\sigma}\leq 1$*, then for all* $s'\geq s>1/2$*,*

$$\|\mathcal{R}_2 h\|_{s'} \leq \tfrac{\epsilon L}{2\gamma\omega}\left(\|h\|_{s'}+\|w\|_{s'+\sigma}\|h\|_s\right), \quad \forall h\in W_N. \tag{3.21}$$

Proof. Lemma 2.1 shows that

$$\mathrm{D}_w v(\epsilon,w)[|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac12}h]\in H^2_p(0,\pi).$$

Moreover, it can be seen from the fact $\tau\in(1,2)$ that $\sigma>\tau-1$. Thus, by virtue of (3.1)–(3.2), it follows that

$$\begin{aligned}
\|R_2 h\|_{s'} &\overset{(3.16)}{\leq} \tfrac{\epsilon\sqrt{2}L_2}{\sqrt{\gamma\omega}L_1}\|\tfrac{1}{\rho}\|_{H^2}\|b(t,x)\mathrm{D}_w v(\epsilon,w)[|\mathfrak{L}_{1,\mathrm{D}}|^{-\frac12}h]\|_{s'+\frac{\tau-1}{2}}\\
&\overset{(4.2)}{\leq} \tfrac{\epsilon\sqrt{2}L_2}{\sqrt{\gamma\omega}L_1}\|\tfrac{1}{\rho}\|_{H^2}C'(s')(\|b\|_{s'+\sigma}\||\mathfrak{L}_{1,\mathrm{D}}|^{-\frac12}h\|_{s-\frac{\tau-1}{2}}+\|b\|_{s+\sigma}\||\mathfrak{L}_{1,\mathrm{D}}|^{-\frac12}h\|_{s-\frac{\tau-1}{2}})\\
&\overset{(3.16)}{\leq} \tfrac{2\epsilon L_2^2 C''(s')}{\gamma\omega L_1^2}(\|w\|_{s'+\sigma}\|h\|_s+\|h\|_{s'}).
\end{aligned}$$

Consequently, we can infer (3.21) because of the above estimate and (3.11). ◻

LEMMA 3.5. *Provided* (3.13) *and* (3.18)*, if* $\|w\|_{s+\sigma}\leq 1$*, for* $\frac{\epsilon L}{\gamma^3\omega}\leq \mathrm{c}$ *small enough, then* $(\mathrm{Id}-\mathcal{R})$ *is invertible with, for all* $s'\geq s>1/2$*,*

$$\|(\mathrm{Id}-\mathcal{R})^{-1}h\|_{s'}\leq 2(\|h\|_{s'}+\|w\|_{s'+\sigma}\|h\|_s), \quad \forall h\in W_N. \tag{3.22}$$

Proof. It follows from Lemmata 3.3–3.4 that for $\frac{\epsilon L}{\gamma^3\omega}\leq \mathrm{c}$ small enough,

$$\|\mathcal{R}h\|_s\leq \tfrac{\epsilon L}{\gamma^3\omega}\|h\|_s\leq \tfrac12\|h\|_s.$$

Then, according to the Neumann series, the operator $(\mathrm{Id}-\mathcal{R})$ is invertible. In order to prove (3.22), we claim

(F3): If $\|w\|_{s+\sigma}\leq 1$, then for $\ell\in\mathbb{N}^+$,

$$\|\mathcal{R}^\ell h\|_{s'}\leq (\tfrac{\epsilon L}{\gamma^3\omega})^\ell(\|h\|_{s'}+\ell\|w\|_{s'+\sigma}\|h\|_s), \quad \forall h\in W_N. \tag{3.23}$$

Then applying the fact (F3) gives that for $\frac{\epsilon L}{\gamma^3\omega}\leq \mathrm{c}$ small enough,

$$\begin{aligned}
\|(\mathrm{Id}-\mathcal{R})^{-1}h\|_{s'} &= \|(\mathrm{Id}+\textstyle\sum_{\ell\in\mathbb{N}^+}\mathcal{R}^\ell)h\|_{s'}\leq \|h\|_{s'}+\sum_{\ell\in\mathbb{N}^+}\|\mathcal{R}^\ell h\|_{s'}\\
&\leq \|h\|_{s'}+\textstyle\sum_{\ell\in\mathbb{N}^+}(\tfrac{\epsilon L}{\gamma^3\omega})^\ell(\|h\|_{s'}+\ell\|w\|_{s'+\sigma}\|h\|_s)\\
&\leq 2\|h\|_{s'}+2\|w\|_{s'+\sigma}\|h\|_s.
\end{aligned}$$

It remains to show that the fact (F3) holds by a recursive argument. For $\ell=1$, owing to (3.19) and (3.21), it follows that

$$\|\mathcal{R}h\|_{s'}\leq \tfrac{\epsilon L}{\gamma^3\omega}\left(\|h\|_{s'}+\|w\|_{s'+\sigma}\|h\|_s\right).$$

Suppose that formula (3.23) holds at ℓ-step, with $\ell\in\mathbb{N}^+,\ell\geq 2$. Let us check that (3.23) holds at $(\ell+1)$-step. Based on the assumption at ℓ-step, we can get

$$\|\mathcal{R}^{\ell+1}h\|_{s'}=\|\mathcal{R}^\ell(\mathcal{R}h)\|_{s'}\leq (\tfrac{\epsilon L}{\gamma^3\omega})^\ell(\|\mathcal{R}h\|_{s'}+\ell\|w\|_{s'+\sigma}\|\mathcal{R}h\|_s)$$

$$\leq (\tfrac{\epsilon L}{\gamma^3\omega})^{\ell}\left(\tfrac{\epsilon L}{\gamma^3\omega}\|h\|_{s'}+\left(\tfrac{\epsilon L\ell}{\gamma^3\omega}+\tfrac{\epsilon L}{\gamma^3\omega}\right)\|w\|_{s'+\sigma}\|h\|_s\right)$$
$$\leq (\tfrac{\epsilon L}{\gamma^3\omega})^{\ell+1}(\|h\|_{s'}+(\ell+1)\|w\|_{s'+\sigma}\|h\|_s).$$

Thus we complete the proof of the lemma. □

As a result, Lemma 2.1 follows from (3.11)–(3.12), (3.17) and (3.22).

Now, we will give the proof of the fact (F2). It is clear that $\rho,\zeta\in H^5(0,\pi)$ because of $\alpha,\beta\in H^4(0,\pi)$. If we denote

$$\Xi(\epsilon,w):=-\tfrac{\epsilon}{\rho}\Pi_V f'(v(\epsilon,w)+w)\zeta+\tfrac{\alpha_{xxx}-\beta_{xxx}}{4\zeta^3},$$

then $\Xi(\epsilon,w)\in H^1(0,\pi)$. Hence integrating by parts yields that

$$|\varrho_j(\epsilon,w)|=\left|\tfrac{1}{\pi}\int_0^\pi \Xi(\epsilon,w)(x)\cos\left(2j\int_0^x\zeta(z)\mathrm{d}z\right)\mathrm{d}x\right|\leq \tfrac{C\|\Xi/\zeta\|_{H^1}}{j},$$

where $\varrho_j(\epsilon,w)$ is defined in (3.5). For $\Theta>0$ large enough, it follows from (3.4) that

$$\lambda_j(\epsilon,w)=j^4+2j^2\upsilon_0+\upsilon_1(\epsilon,w)+\tfrac{r(\epsilon,w)}{j},\quad |r(\epsilon,w)|\leq\Theta,\quad \text{as } j\to+\infty. \tag{3.24}$$

By Taylor expansion and (3.24), there is $J_0>\max\{2|\upsilon_0|,1\}>0$ large enough such that

$$\left|\mu_j(\epsilon,w)-(j^2+\upsilon_0)\right|\leq\tfrac{\Theta}{j^2},\quad \forall j>J_0. \tag{3.25}$$

In addition, from (3.3) and the definition of j^*, if $\omega^2l^2-\lambda_{J_0+1}(\epsilon,w)>0$, then $j^*\geq J_0+1$. Thus there exists $J_1:=J_1(J_0)>0$ such that for every $l>J_1/\omega$,

$$j^*\geq\Theta_0\sqrt{\omega l}. \tag{3.26}$$

Proof. **(Proof of** (F2)**.)** Let $l,k\geq1$ with $l\neq k$. Denote

$$\omega_l=|\omega^2l^2-\lambda_{j^*}(\epsilon,w)|,\quad \omega_k=|\omega^2k^2-\lambda_{i^*}(\epsilon,w)|,\quad \varsigma=(2-\tau)/\tau\in(0,1).$$

Then we consider the following two cases.

Case 1: $2|k-l|>(\max\{k,l\})^\varsigma$. It follows from (3.14) that

$$\omega_l\omega_k\geq\frac{(\gamma\omega)^2}{(kl)^{\tau-1}}\geq\frac{(\gamma\omega)^2}{(\max\{k,l\})^{2(\tau-1)}}\geq\frac{(\gamma\omega)^2}{2^{2(\tau-1)/\varsigma}|k-l|^{2(\tau-1)/\varsigma}}.$$

Case 2: $0<2|k-l|\leq(\max\{k,l\})^\varsigma$. Clearly, either $k>l$ or $l>k$ follows. By the fact $\varsigma\in(0,1)$, in the first case $2l>k$ and in the latter $2k>l$, namely, $k/2<l<2k$.

(i) If $\lambda_{j^*}(\epsilon,w)<0,\lambda_{i^*}(\epsilon,w)<0$, then $\omega_l\geq\omega^2l^2,\omega_k\geq\omega^2k^2$, which leads to

$$\omega_l\omega_k\geq\omega^4\geq\gamma^2\omega^2.$$

(ii) If either $\lambda_{j^*}(\epsilon,w)<0$ or $\lambda_{i^*}(\epsilon,w)<0$, then in the first case

$$\omega_l\omega_k\overset{(3.14)}{\geq}\omega^2l^2\frac{\gamma\omega}{k^{\tau-1}}\geq2^{1-\tau}\gamma\omega^3\geq2^{1-\tau}\gamma^2\omega^2$$

and in the latter

$$\omega_l\omega_k\overset{(3.14)}{\geq}\frac{\gamma\omega}{l^{\tau-1}}\omega^2k^2\geq2^{1-\tau}\gamma\omega^3\geq2^{1-\tau}\gamma^2\omega^2.$$

(iii) Let us study the case $\lambda_{j^*}(\epsilon,w)>0,\lambda_{i^*}(\epsilon,w)>0$. Let

$$k_\star := \max\left\{\tfrac{2J_1}{\omega},\left(\tfrac{6\Theta}{\Theta_0^2\gamma\omega}\right)^{\frac{1}{1-\varsigma\tau}}\right\}.$$

Suppose that $\max\{k,l\}=k>k_\star$. According to (3.18), (3.25)–(3.26), it follows that

$$\begin{aligned}|(\omega l-\mu_{j^*}(\epsilon,w))-(\omega k-\mu_{i^*}(\epsilon,w))| &\geq \frac{\gamma}{|l-k|^\tau}-\frac{\Theta}{(j^*)^2}-\frac{\Theta}{(i^*)^2}\geq\frac{2^\tau\gamma}{k^{\varsigma\tau}}-\frac{\Theta}{\Theta_0^2\omega l}-\frac{\Theta}{\Theta_0^2\omega k}\\ &\geq \frac{\gamma}{2k^{\varsigma\tau}}+\frac{\gamma}{k^{\varsigma\tau}}+\frac{\gamma}{2k^{\varsigma\tau}}-\frac{3\Theta}{\Theta_0^2\omega k}\geq\frac{1}{2}\left(\frac{\gamma}{k^{\varsigma\tau}}+\frac{\gamma}{l^{\varsigma\tau}}\right).\end{aligned}$$

This shows that either $|\omega k-\mu_{i^*}(\epsilon,w)|\geq\frac{\gamma}{2k^{\varsigma\tau}}$ or $|\omega l-\mu_{j^*}(\epsilon,w)|\geq\frac{\gamma}{2l^{\varsigma\tau}}$. The same conclusion is reached if $\max\{k,l\}=l>k_\star$. For brevity, we just consider the case $|\omega k-\mu_{i^*}(\epsilon,w)|\geq\frac{\gamma}{2k^{\varsigma\tau}}$. Observe that

$$\omega_k=|\omega^2k^2-\lambda_{i^*}(\epsilon,w)|=|\omega k-\mu_{i^*}(\epsilon,w)||\omega k+\mu_{i^*}(\epsilon,w)|\geq\tfrac{\gamma\omega}{2}k^{1-\varsigma\tau}.$$

This arrives at

$$\omega_l\omega_k\geq\frac{\gamma\omega}{l^{\tau-1}}\frac{\gamma\omega}{2}k^{1-\varsigma\tau}\geq\frac{(\gamma\omega)^2}{2^\tau}k^{2-\tau-\varsigma\tau}=\frac{(\gamma\omega)^2}{2^\tau}$$

if we take $\varsigma=(2-\tau)/\tau$.

On the other hand, we consider the case $\max\{j,k\}\leq k_\star$. Since $\omega>\gamma$, we can obtain that for $k_\star=\frac{2J_1}{\omega}$,

$$\omega_j\omega_k\geq\frac{(\gamma\omega)^2}{(jk)^{\tau-1}}\geq\frac{(\gamma\omega)^2}{(k_\star)^{2(\tau-1)}}=\frac{(\gamma\omega)^2}{(2J_1/\omega)^{2(\tau-1)}}\geq\frac{\gamma^4\omega^2}{(2J_1)^{2(\tau-1)}}.$$

For $k_\star=\left(\frac{6\Theta}{\Theta_0^2\gamma\omega}\right)^{\frac{1}{1-\varsigma\tau}}$, it can be seen that

$$\omega_l\omega_k\geq\frac{(\gamma\omega)^2}{(kl)^{\tau-1}}\geq\frac{(\gamma\omega)^2}{(k_\star)^{2(\tau-1)}}=\gamma^2\omega^2\left(\frac{\Theta_0^2\gamma\omega}{6\Theta}\right)^2\geq\frac{\Theta_0^4}{(6\Theta)^2}\gamma^6\omega^2.$$

In addition, note that $\omega_l=\omega_{-l},\omega_k=\omega_{-k}$. The remainder of the lemma may be proved in the similar way as above. Thus the proof of (F2) is now completed. ∎

Finally, let us complete the proof of formula (2.40).

Proof. **(Proof of formula (2.40).)** In view of (3.25), it follows that

$$\begin{aligned}&\inf_{j>\max\{J_0,2\Theta^{\frac{1}{2}}\}}|\mu_{j+1}(\epsilon,w)-\mu_j(\epsilon,w)|\\ &\geq 1-|\mu_{j+1}(\epsilon,w)-((j+1)^2+\upsilon_0)|-|\mu_j(\epsilon,w)-(j^2+\upsilon_0)|\geq 1-\tfrac{2\Theta}{j^2}>\tfrac{1}{2}\end{aligned}$$

uniformly in $\epsilon\in[\epsilon_1,\epsilon_2]$, $w\in\mathcal{B}(0,r)$. In the proof of Lemma 2.11, one has $|\lambda_j(\mathrm{g})-\lambda_j(\bar{\mathrm{g}})|\leq C\|(\mathrm{g}-\bar{\mathrm{g}})/\rho\|_{H^2(0,\pi)}$, where $\mathrm{g}\in H^2(0,\pi)\subset L^\infty(0,\pi)$. Then it can be seen from (3.3) that for $1\leq j\leq\max\{J_0,2\Theta^{\frac{1}{2}}\}$, the following

$$\Xi_j:=\min_{\epsilon\in[\epsilon_1,\epsilon_2],w\in\mathcal{B}(0,r)}|\mu_{j+1}(\epsilon,w)-\mu_j(\epsilon,w)|$$

can be attained. Thus we complete the proof. ∎

Appendix. In this appendix, we will supplement some Lemmata used in the proof of Theorem 1.1.

LEMMA 4.1 (Moser–Nirenberg). *Let $s' \geq 0$ and $s > \frac{1}{2}$. One has that for all $u_1, u_2 \in H^{s'} \cap H^s$,*

$$\|u_1 u_2\|_{s'} \leq C(s') \left(\|u_1\|_{L^\infty(\mathbb{T}, H^2(0,\pi))} \|u_2\|_{s'} + \|u_1\|_{s'} \|u_2\|_{L^\infty(\mathbb{T}, H^2(0,\pi))}\right) \quad (4.1)$$
$$\leq C(s') (\|u_1\|_s \|u_2\|_{s'} + \|u_1\|_{s'} \|u_2\|_s). \quad (4.2)$$

LEMMA 4.2 (Logarithmic convexity). *Let $0 \leq \mathrm{a}' \leq \mathrm{a} \leq \mathrm{b} \leq \mathrm{b}'$ satisfy $\mathrm{a} + \mathrm{b} = \mathrm{a}' + \mathrm{b}'$. One has that for all $u_1, u_2 \in H^{\mathrm{b}'}$,*

$$\|u_1\|_{\mathrm{a}} \|u_2\|_{\mathrm{b}} \leq \theta \|u_1\|_{\mathrm{a}'} \|u_2\|_{\mathrm{b}'} + (1-\theta) \|u_2\|_{\mathrm{a}'} \|u_1\|_{\mathrm{b}'}, \quad \theta = \tfrac{\mathrm{b}' - \mathrm{a}}{\mathrm{b}' - \mathrm{a}'}.$$

In particular, for $u \in H^{\mathrm{b}'}$, we have

$$\|u\|_{\mathrm{a}} \|u\|_{\mathrm{b}} \leq \|u\|_{\mathrm{a}'} \|u\|_{\mathrm{b}'}. \quad (4.3)$$

LEMMA 4.3. *Let $\mathscr{C}_k := \left\{ f \in C([0,\pi] \times \mathbb{R}; \mathbb{R}) : u \longmapsto f(\cdot, u) \text{ is in } C^k(\mathbb{R}; H^2(0,\pi)) \right\}$. If we denote $U = \|u\|_{L^\infty(0,\pi)}$, then for $f \in \mathscr{C}_1$, the composition operator $u(x) \longmapsto f(x, u(x))$ is in $C(H^2(0,\pi); H^2(0,\pi))$ with*

$$\|f(x,u)\|_{H^2} \leq C \left(\max_{u \in [-U,U]} \|f(\cdot,u)\|_{H^2} + \max_{u \in [-U,U]} \|\partial_u f(\cdot,u)\|_{H^2} \|u\|_{H^2}\right).$$

Remark that the above ones may be found in [7, cf. Lemmata 2.1–2.3].

LEMMA 4.4. *Let $f \in \mathcal{C}_k$ with $k \geq 1$. For all $s > 1/2$, $0 \leq s' \leq k-1$, the composition operator $u(t,x) \longmapsto f(t,x,u(t,x))$ belongs to $C(H^s \cap H^{s'}; \tilde{H}^{s'})$, where*

$$\tilde{H}^s := \left\{ u : \mathbb{T} \longrightarrow H^2((0,\pi);\mathbb{R}), u(t,x) = \textstyle\sum_{l \in \mathbb{Z}} u_l(x) e^{ilt}, u_l \in H^2((0,\pi);\mathbb{C}), \right.$$
$$\left. u_{-l} = u_l^*, \|u\|_s < +\infty \right\}.$$

Moreover, one has

$$\|f(t,x,u)\|_{s'} \leq C(s', \|u\|_s)(1 + \|u\|_{s'}). \quad (4.4)$$

Proof. If $s' = \ell \in \mathbb{N}$ with $\ell \leq k-1$, by induction, then we derive that

$$\|f(t,x,u)\|_\ell \leq C(\ell, \|u\|_s)(1 + \|u\|_\ell), \quad \forall u \in H^s \cap H^\ell, \quad (4.5)$$

and that

$$f(t,x,u_n) \to f(t,x,u) \quad \text{as } u_n \to u \text{ in } H^s \cap H^\ell. \quad (4.6)$$

We first verify the fact (4.5). For $\ell = 0$ $(k=1)$, in view of (1.6) and Lemma 4.3, it follows that

$$\|f(t,x,u)\|_0 \leq C \max_{t \in \mathbb{T}} \|f(t,\cdot,u(t,\cdot))\|_{H^2(0,\pi)} \leq C'(1 + \max_{t \in \mathbb{T}} \|u(t,\cdot)\|_{H^2(0,\pi)})$$
$$\leq C''(1 + \|u\|_s) =: C(\|u\|_s). \quad (4.7)$$

A similar argument as above can yield that for $k \geq 2$,

$$\|\partial_t f(t,x,u)\|_0 \leq C(\|u\|_s), \quad \max_{t \in \mathbb{T}} \|\partial_u f(t,\cdot,u(t,\cdot))\|_{H^2(0,\pi)} \leq C(\|u\|_s). \quad (4.8)$$

Suppose that (4.5) holds at ℓ-step, with $\ell\in\mathbb{N}^+$. Let us check that it also holds at $(\ell+1)$-step, with $\ell+1\le k-1$. Since $\partial_t f,\partial_u f\in\mathcal{C}_{k-1}$, the assumption at ℓ-step shows that

$$\|\partial_t f(t,x,u)\|_\ell\le C(\ell,\|u\|_s)(1+\|u\|_\ell),\quad \|\partial_u f(t,x,u)\|_\ell\le C(\ell,\|u\|_s)(1+\|u\|_\ell). \tag{4.9}$$

If we set $a(t,x):=f(t,x,u(t,x))$, it is clear that $\partial_t a(t,x)=\sum_{l\in\mathbb{Z}} \mathrm{i}la_l(x)e^{\mathrm{i}lt}$. Therefore,

$$\|a(t,x)\|^2_{\ell+1}=\sum_{l\in\mathbb{Z}}\|a_l\|^2_{H^2}+\sum_{l\in\mathbb{Z}} l^{2\ell}\|\mathrm{i}la_l\|^2_{H^2}\le(\|a\|_0+\|\partial_t a\|_\ell)^2.$$

This leads to

$$\|f(t,x,u)\|_{\ell+1}\le\|f(t,x,u)\|_0+\|\partial_t f(t,x,u)\|_\ell+\|\partial_u f(t,x,u)\partial_t u\|_\ell. \tag{4.10}$$

Hence, because of (4.8), we obtain that for $\ell=1$,

$$\begin{aligned}\|f(t,x,u)\|_1&\le\|f(t,x,u)\|_0+\|\partial_t f(t,x,u)\|_0+C\max_{t\in\mathbb{T}}\|\partial_u f(t,\cdot,u(t,\cdot))\|_{H^2(0,\pi)}\|\partial_t u\|_0\\&\le 2C(\|u\|_s)+C'(\|u\|_s)\|u\|_1\le C(1,\|u\|_s)(1+\|u\|_1),\end{aligned}$$

where $C(1,\|u\|_s):=\max\{2C(\|u\|_s),C'(\|u\|_s)\}$. Observe that

$$s_1<1<s_1+1<2\quad\text{and}\quad s_1<s_1+1<\ell<\ell+1,\ell\ge 2,$$

where $s_1\in(1/2,\min(1,s))$. Combining this with (4.3) gives that

$$\|u\|_\ell\|u\|_{s_1+1}\le\|u\|_{\ell+1}\|u\|_{s_1}\le\|u\|_{\ell+1}\|u\|_s.$$

For this, according to (4.7)–(4.10), (4.1), it follows that

$$\begin{aligned}\|f(t,x,u)\|_{\ell+1}\le&C(\|u\|_s)+C(\ell,\|u\|_s)(1+\|u\|_\ell)+C(\ell)\|\partial_u f(t,x,u)\|_\ell\|\partial_t u\|_{L^\infty(\mathbb{T};H^2(0,\pi))}\\&+C(\ell)\|\partial_u f(t,x,u)\|_{L^\infty(\mathbb{T};H^2(0,\pi))}\|u\|_{\ell+1}\\\le&C(\|u\|_s)+C(\ell,\|u\|_s)(1+\|u\|_\ell)+C(\ell)C(\ell,\|u\|_s)(1+\|u\|_\ell)\|u\|_{s_1+1}\\&+C(\ell)C(\|u\|_s)\|u\|_{\ell+1}\\\le&C(\ell+1,\|u\|_s)(1+\|u\|_{\ell+1}).\end{aligned}$$

Our next task is to check the fact (4.6). By virtue of (1.6), one has

$$\max_{t\in\mathbb{T}}\|u_n(t,\cdot)-u(t,\cdot)\|_{H^2(0,\pi)}\to 0\quad\text{as } u_n\to u\ \text{ in } H^s\cap H^0.$$

Then using the continuity property in Lemma 4.3 and the compactness of $\mathbb{T}$ yields that

$$\|f(t,x,u_n)-f(t,x,u)\|_0\le C\max_{t\in\mathbb{T}}\|f(t,\cdot,u_n(t,\cdot))-f(t,\cdot,u(t,\cdot))\|_{H^2(0,\pi)}\to 0$$

if $u_n\to u$ in $H^s\cap H^0$. If we assume that (4.6) holds at ℓ-step, then it follows from the inequality (4.10) that it also holds at $(\ell+1)$-step, with $\ell+1\le k-1$.

If s' is not an integer, then we shall adopt a similar procedure as in the proof of Lemma A.1 in [18]. ∎

LEMMA 4.5. *Let $0\le s'\le k-3$ with $k\ge 3$. If $f\in\mathcal{C}_k$, with $\partial_u^\ell f(\cdot,\cdot,0)=0,\ell\le 2$, then the map*

$$F:H^s\cap H^{s'}\longrightarrow H^{s'}\subset\tilde{H}^{s'},\quad u\longmapsto f(t,x,u).$$

is C^2 *with respect to* u *and for all* $h\in H^s\cap H^{s'}$,

$$\mathrm{D}F(u)[h]=\partial_u f(t,x,u)h,\quad \mathrm{D}^2F(u)[h,h]=\partial_u^2 f(t,x,u)h^2.$$

Moreover, one has

$$\|\partial_u f(t,x,u)\|_{s'}\le C(s',\|u\|_s)(1+\|u\|_{s'}),\ \|\partial_u^2 f(t,x,u)\|_{s'}\le C(s',\|u\|_s)(1+\|u\|_{s'}).\tag{4.11}$$

Proof. Observe that $\partial_u f,\partial_u^2 f$ are in $\mathcal{C}_{k-1},\mathcal{C}_{k-2}$. Then it follows from Lemma 4.4 that the maps $u\longmapsto\partial_u f(t,x,u)$, $u\longmapsto\partial_u^2 f(t,x,u)$ are continuous and that the estimates in (4.11) are satisfied.

It remains to see that F is C^2 with respect to u. From the continuity property of $u\longmapsto\partial_u f(t,x,u)$, we obtain

$$\begin{aligned}&\|f(t,x,u+h)-f(t,x,u)-\partial_u f(t,x,u)h\|_{s'}=\|h\textstyle\int_0^1(\partial_u f(t,x,u+\mathfrak{v}h)-\partial_u f(t,x,u))\mathrm{d}\mathfrak{v}\|_{s'}\\ \le&C(s')\|h\|_{\max\{s,s'\}}\max_{\mathfrak{v}\in[0,1]}\|\partial_u f(t,x,u+\mathfrak{v}h)-\partial_u f(t,x,u)\|_{\max\{s,s'\}}=o(\|h\|_{\max\{s,s'\}}).\end{aligned}$$

For this, it can be seen that $\mathrm{D}_uF(u)[h]=\partial_u f(t,x,u)h$ for $h\in H^s\cap H^{s'}$ and that $u\longmapsto \mathrm{D}_uF(u)$ is continuous. In addition,

$$\begin{aligned}&\partial_u f(t,x,u+\mathfrak{v}h)h-\partial_u f(t,x,u)h-\partial_u^2 f(t,x,u)h^2\\ =&h^2\textstyle\int_0^1(\partial_u^2 f(t,x,u+\mathfrak{v}h)-\partial_u^2 f(t,x,u))\mathrm{d}\mathfrak{v}.\end{aligned}$$

Proceeding using a similar procedure as above yields that F is twice differentiable with respect to u and that $u\longmapsto\mathrm{D}_u^2F(u)$ is continuous. ∎

REFERENCES

[1] A. Badanin and E. Korotyaev, *Inverse problems and sharp eigenvalue asymptotics for Euler–Bernoulli operators*, Inverse Problems, 31(5):055004, 2015. 1, 1.1, 3

[2] A. Badanin and E.L. Korotyaev, *Resonances for Euler–Bernoulli operator on the half-line*, J. Diff. Eqs., 263(1):534–566, 2017. 1

[3] P. Baldi and M. Berti, *Forced vibrations of a nonhomogeneous string*, SIAM J. Math. Anal., 40(1):382–412, 2008. 1, 1.1

[4] V. Barbu and N.H. Pavel, *Periodic solutions to one-dimensional wave equation with piece-wise constant coefficients*, J. Diff. Eqs., 132(2):319–337, 1996. 1

[5] V. Barbu and N.H. Pavel, *Periodic solutions to nonlinear one-dimensional wave equation with x-dependent coefficients*, Trans. Amer. Math. Soc., 349(5):2035–2048, 1997. 1

[6] M. Berti, L. Biasco, and M. Procesi, *KAM for reversible derivative wave equations*, Arch. Ration. Mech. Anal., 212(3):905–955, 2014. 1

[7] M. Berti and P. Bolle, *Cantor families of periodic solutions of wave equations with C^k nonlinearities*, Nonlinear Diff. Eqs. Appl., 15(1-2):247–276, 2008. 3

[8] M. Berti, P. Bolle, and M. Procesi, *An abstract Nash–Moser theorem with parameters and applications to PDEs*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27(1):377–399, 2010. 1

[9] M. Berti, L. Corsi, and M. Procesi, *An abstract Nash–Moser theorem and quasi-periodic solutions for NLW and NLS on compact Lie groups and homogeneous manifolds*, Comm. Math. Phys., 334(3):1413–1454, 2015. 1

[10] M. Berti and M. Procesi, *Nonlinear wave and Schrödinger equations on compact Lie groups and homogeneous spaces*, Duke Math. J., 159(3):479–538, 2011. 1

[11] J. Bourgain, *Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE*, Inter. Math. Res. Notices, 11:475–497, 1994. 1

[12] J. Bourgain, *Construction of periodic solutions of nonlinear wave equations in higher dimension*, Geom. Funct. Anal., 5(4):629–639, 1995. 1

[13] L.F. Caudill Jr., P.A. Perry, and A.W. Schueller, *Isospectral sets for fourth-order ordinary differential operators*, SIAM J. Math. Anal., 29(4):935–966, 1998. 1

[14] J. Chang, Y. Gao, and Y. Li, *Quasi-periodic solutions of nonlinear beam equation with prescribed frequencies*, J. Math. Phys., 56(5):052701, 2015. 1

[15] B. Chen, Y. Gao, S. Jiang, and Y. Li, *Quasi-periodic solutions to nonlinear beam equations on compact Lie groups with a multiplicative potential*, J. Diff. Eqs., 264(11):6959–6993, 2018. 1

[16] B. Chen, Y. Li, and X. Yang, *Periodic solutions to nonlinear wave equation with x-dependent coefficients under the general boundary conditions*, J. Dynam. Diff. Eqs., 31(1):321–368, 2019. 1

[17] W. Craig and C.E. Wayne, *Newton's method and periodic solutions of nonlinear wave equations*, Comm. Pure Appl. Math., 46(11):1409–1498, 1993. 1

[18] J.-M. Delort, *Periodic solutions of nonlinear Schrödinger equations: a paradifferential approach*, Anal. PDE, 4(5):639–676, 2011. 3

[19] L.H. Eliasson, B. Grébert, and S.B. Kuksin, *KAM for the nonlinear beam equation*, Geom. Funct. Anal., 26(6):1588–1715, 2016. 1

[20] I. Elishakoff and V. Johnson, *Apparently the first closed-form solution of vibrating inhomogeneous beam with a tip mass*, J. Sound Vib., 286(4-5):1057–1066, 2005. 1

[21] I. Elishakoff and D. Pentaras, *Apparently the first closed-form solution of inhomogeneous elastically restrained vibrating beams*, J. Sound Vib., 298(1-2):439–445, 2006. 1

[22] Y. Gao, Y. Li, and J. Zhang, *Invariant tori of nonlinear Schrödinger equation*, J. Diff. Eqs., 246(8):3296–3331, 2009. 1

[23] J. Geng and J. You, *KAM tori of Hamiltonian perturbations of 1D linear beam equations*, J. Math. Anal. Appl., 277(1):104–121, 2003. 1

[24] G.M.L. Gladwell and A. Morassi, *A family of isospectral Euler–Bernoulli beams*, Inverse Problems, 26(3):035006, 12, 2010. 1

[25] B.-Z. Guo, *Riesz basis property and exponential stability of controlled Euler–Bernoulli beam equations with variable coefficients*, SIAM J. Control Optim., 40(6):1905–1923, 2002. 1

[26] E. Haus and M. Procesi, *KAM for beating solutions of the quintic NLS*, Comm. Math. Phys., 354(3):1101–1132, 2017. 1

[27] L.D. Humphreys and P.J. McKenna, *Multiple periodic solutions for a nonlinear suspension bridge equation*, IMA J. Appl. Math., 63(1):37–49, 1999. 1

[28] S. Ji, *Periodic solutions for one dimensional wave equation with bounded nonlinearity*, J. Diff. Eqs., 264(9):5527–5540, 2018. 1

[29] S. Ji and Y. Li, *Time-periodic solutions to the one-dimensional wave equation with periodic or anti-periodic boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A, 137(2):349–371, 2007. 1

[30] S. Ji and Y. Li, *Time periodic solutions to the one-dimensional nonlinear wave equation*, Arch. Ration. Mech. Anal., 199(2):435–451, 2011. 1

[31] Q. Kong, H. Wu, and A. Zettl, *Dependence of eigenvalues on the problem*, Math. Nachr., 188:173–201, 1997. 2.2.3

[32] S.B. Kuksin, *Hamiltonian perturbations of infinite-dimensional linear systems with imaginary spectrum*, Funct. Anal. Its Appl. , 21(3):22–37, 1987. 1

[33] P.J. McKenna and W. Walter, *Nonlinear oscillations in a suspension bridge*, Arch. Ration. Mech. Anal., 98(2):167–177, 1987. 1

[34] M.A. Najmark, *Linear Differential Operators*, Harrap, 1968. 1

[35] V.G. Papanicolaou, *The spectral theory of the vibrating periodic beam*, Comm. Math. Phys., 170(2):359–373, 1995. 1

[36] V.G. Papanicolaou, *The periodic Euler–Bernoulli equation*, Trans. Amer. Math. Soc., 355(9):3727–3759, 2003. 1

[37] V.G. Papanicolaou and D. Kravvaritis, *The Floquet theory of the periodic Euler–Bernoulli equation*, J. Diff. Eqs., 150(1):24–41, 1998. 1

[38] Y. Wang and J. Si, *A result on quasi-periodic solutions of a nonlinear beam equation with a quasi-periodic forcing term*, Z. Angew. Math. Phys., 63(1):189–190, 2012. 1

[39] C.E. Wayne, *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory*, Comm. Math. Phys., 127(3):479–528, 1990. 1

[40] W. Weaver Jr., S.P. Timoshenko, and D.H. Young, *Vibration Problems in Engineering*, John Wiley & Sons Limited, 1990. 1