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EXISTENCE OF SOLUTIONS TO AN ANISOTROPIC DEGENERATE CAHN-HILLIARD-TYPE EQUATION*

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Abstract. We prove existence of solutions to an anisotropic Cahn-Hilliard-type equation with degenerate diffusional mobility. In particular, the mobility vanishes at the pure phases, which is typically used to model motion by surface diffusion. The main difficulty of the present existence result is the strong non-linearity given by the fourth-order anisotropic operator. Imposing particular assumptions on the domain and assuming that the strength of the anisotropy is sufficiently small enables to establish appropriate bounds which allow to pass to the limit in the regularized problem. In addition to the existence we show that the absolute value of the corresponding solutions is bounded by 1.

Keywords. Cahn-Hilliard equation; degenerate mobility; anisotropic parabolic equations; existence/boundedness of solutions.

AMS subject classifications. 74Gxx; 74Hxx; 35K55; 35K65; 49Jxx; 82C26.

1. Introduction

In this paper we consider the existence of weak solutions to an anisotropic phase field model, which may be identified as an anisotropic version of the Cahn-Hilliard equation. The corresponding isotropic version, the classical Cahn-Hilliard equation in the form

$$\frac{\partial u}{\partial t} = \operatorname{div}(m(u)\nabla\mu), \tag{1.1a}$$

$$\mu = F'(u) - \epsilon^2\gamma\Delta u, \tag{1.1b}$$

is probably one of the most well-known examples for phase separation and was originally introduced by Cahn and Hilliard to study phase separation of binary fluids [4,5]. In this paper the phase field function u is defined such that $u=1$ denotes the solid phase and $u=-1$ denotes the vapor phase, ϵ is a small parameter that describes the interface width, F is the homogeneous free energy and γ is the surface energy between film and vapor. In order to model motion by surface diffusion, we need to assume that the diffusional mobility $m(u)$ is a non-negative function which is sufficiently strong degenerated at the pure phases, see for example [11]. On the one hand, this constitutes a mathematical difficulty since the a priori estimates, such as are commonly used in existence results, loose their information at points where the mobility degenerates. On the other hand, a degenerate mobility may be beneficial in order to show that solutions which initially take values in the interval $[-1,1]$ will do so for all positive time. Note that this is is not true in general for fourth order parabolic equations without degeneracy since there is no comparison principle available.

Considering present existence results for the Cahn-Hilliard equation with degenerate mobility (1.1), the techniques introduced in the papers by Elliott and Garcke [8], Grün [10] and by Bernis and Friedman [2] have proven to be extremely useful. In each of these papers the general procedure is to replace (1.1) by a family of regularized problems with smooth solutions u_δ, establish particular a priori bounds and show that the approximate solutions u_δ converge to solutions of the original problem as $\delta \to 0$. In [8], for example,

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the degenerate mobility $m(u)$ is approximated by a strictly positive mobility $m_\delta(u)$ which satisfies $m_\delta \to m$, as $\delta \to 0$. The resulting parabolic problem is non-degenerate and provides global and smooth solutions u_δ. With the help of appropriate a priori estimates it is then shown that the integral of u_δ in the region where $|u|>1$ converges to zero as m_δ approaches m, which yields $|u|\leq 1$ in the limit. In fact, it can be shown that solutions to (1.1) with sufficiently strong degenerated mobility preserve the strict inequalities $|u|<1$ for all times $t\geq 0$.

In particular, Elliott and Garcke [8] exploit the dissipation of two particular functionals by solutions to (1.1) which provides the required regularity estimates. The first is the free energy functional

$$\mathcal{E}(u) := \int_\Omega F(u) + \epsilon^2 \frac{\gamma}{2} |\nabla u|^2 \, dx. \tag{1.2}$$

and the second the functional defined by

$$\mathcal{U}(u) := \int_\Omega \Phi(u), \qquad \text{where } \Phi''(u) = \frac{1}{\sqrt{m(u)}}, \tag{1.3}$$

also referred to as entropy functional. In particular, the functional $\mathcal{U}$ has become a key tool in order to provide the bound $|u|\leq 1$.

We note that there is an alternative approach to existence, proposed by Lisini, Matthes and Savaré [13], which exploits the variational structure of (1.1). A major advantage of this newer approach is that essential properties of the solution, such as the bound $|u|\leq 1$, are automatically provided by the construction from so-called *minimizing movements* in the energy landscape, where the terminology *minimizing movement* is due to De Giorgi [9]. Observing that (1.1) is in the shape of a gradient flow for $\mathcal{E}$ with respect to a Wasserstein-like transport metric, weak solutions may be obtained as curves of maximal slope. Unfortunately, the main assumption in [13] is that the mobility is a concave function of u which is not satisfied by the bi-quadratic choice

$$m(u) = (1-u^2)^2, \tag{1.4}$$

which we will apply in the following.

The Cahn-Hilliard equation, even with degenerate mobility, has been studied intensively in the past [1,2,8,10,13,14], but little mathematical analysis has been done for the case where the surface energy is anisotropic, i.e.

$$\frac{\partial u}{\partial t} = \operatorname{div}(m(u)\nabla \mu), \tag{1.5a}$$

$$\mu = F'(u) - \epsilon^2 \operatorname{div}(A(\nabla u)\nabla u), \tag{1.5b}$$

where

$$A(\mathbf{n}) = \begin{bmatrix} \gamma(\theta_{\mathbf{n}})^2 & -\gamma'(\theta_{\mathbf{n}})\gamma(\theta_{\mathbf{n}}) \\ \gamma'(\theta_{\mathbf{n}})\gamma(\theta_{\mathbf{n}}) & \gamma(\theta_{\mathbf{n}})^2 \end{bmatrix} \tag{1.6}$$

and $\theta_{\mathbf{n}}$ denotes the angle between the x-axis and the vector $\mathbf{n}$. The function $\gamma(\theta)$ is given by

$$\gamma(\theta) = 1 + G\cos(n\theta), \tag{1.7}$$

where G is a positive constant and n an integer corresponding to the number of orientations in the symmetry. An existence result for a different model which also includes

(1.6) is provided by Burman and Rappaz [3]. They consider an anisotropic phase field model for the isothermal solidification of a binary alloy due to Warren-Boettinger which in the special case of only one concentration can be identified as an anisotropic version of the well-known Allen-Cahn equation

$$\frac{\partial u}{\partial t} = F'(u) - \epsilon^2 \operatorname{div}(A(\nabla u)\nabla u). \tag{1.8}$$

Burman and Rappaz [3] show that the behavior of the anisotropic second-order operator is strongly depending on the size of G. In particular, for small values of G the anisotropic free energy functional

$$\mathcal{E}(u) := \int_\Omega F(u) + \epsilon^2 \frac{\gamma(\theta_{\nabla u})^2}{2} |\nabla u|^2 \, dx \tag{1.9}$$

is convex with respect to ∇u which implies monotonicity and hemicontinuity of the Eulerian operator. Exploiting the literature, see for instance [16], the existence proof is then essentially based on the theory for monotone operators. Note that the physical interpretation of small values of G is that no corners or sharp edges develop on the surface.

In this paper we consider the anisotropic Cahn-Hilliard-type Equation (1.5) on a rectangular open subset $\Omega \subset \mathbb{R}^2$ with boundary conditions

$$\mathbf{n}_\Omega \cdot \nabla u = 0, \tag{1.10a}$$
$$m(u)\mathbf{n}_\Omega \cdot \nabla \mu = 0, \tag{1.10b}$$

on $\partial\Omega$, where $\mathbf{n}_\Omega$ is the unit outward pointing normal vector onto Ω. Note that $A(\mathbf{n})$ is the anisotropy matrix defined by (1.6). We apply the homogeneous free energy

$$F(u) = \frac{1}{2}(1-u^2)^2 \tag{1.11}$$

and biquadratic diffusional mobility (1.4). As shown in a previous paper [6], this combination recovers motion by pure surface diffusion in the sharp interface limit, i.e. when $\epsilon \to \infty$ in Equation (1.5). We assume that γ is a smooth 2π-periodic function and exploit the fact that in two space dimensions θ can be written in terms of the arctangent function

$$\theta = \arctan \frac{u_y}{u_x}. \tag{1.12}$$

Moreover we will require the interface energy to be only weakly anisotropic, i.e.

$$\gamma(\theta) + \gamma''(\theta) > 0, \tag{1.13}$$

for all $\theta \in [-\pi, \pi]$, to avoid ill-posedness of the resulting evolution equations. To be more precise, if $\gamma^2|\nabla u|^2$ is not convex then the term ∇u may be backwards diffusive for some initial data [7, 17] and in the two-dimensional case, which we consider here, this corresponds to the case if and only if $\gamma(\theta) + \gamma''(\theta) \le 0$, which is referred to as strongly anisotropic. Since Equation (1.5) is of fourth order, we additionally need some higher order bounds on $\operatorname{div}(A(\nabla u)\nabla u)$. This requires the assumption that G is sufficiently small such that at least (2.1) holds true. The second assumption is that the considered function u satisfies

$$\int_\Omega u_{xy}^2 - u_{xx}u_{yy} = 0. \tag{1.14}$$

Note that phase field functions u which are constant on $\partial\Omega$ naturally fulfill (1.14) according to partial integration. In particular the eigenfunctions of the Laplace operator on a rectangular domain Ω with Neumann boundary conditions obviously satisfy (1.14), which will be applied in the main proof.

The energy of the system is then given by (1.9) and in order to derive appropriate energy estimates similar as in the proof by Elliott and Garcke [8] we introduce the function

$$\Phi:(-1,1)\to\mathbb{R}_0^+,$$

where $\mathbb{R}_0^+$ denotes the set of non-negative real numbers, and Φ is defined by

$$\Phi''(u)=\frac{1}{\sqrt{m(u)}},\quad \Phi'(0)=0,\quad \text{and } \Phi(0)=0.$$

The following theorem states the existence of a weak solution to the anisotropic Cahn-Hilliard equation with doubly degenerated mobility on an arbitrary interval $[0,T]$, for some $T\in\mathbb{R}^+$.

Theorem 1.1. *Suppose that (1.14) holds true and that G is sufficiently small, according to Lemma 2.3. Let $u_0\in H^1(\Omega)$ with $|u_0|\leq 1$ a.e. and*

$$\int_\Omega (F(u_0)+\Phi(u_0))\leq C,\qquad C\in\mathbb{R}^+.$$

Then there exists a pair of functions (u,μ) such that
(1) $u\in L^\infty(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))$,
(2) $u_t\in L^2(0,T;(H^1(\Omega))')$,
(3) $u(0)=u_0$,
(4) $m(u)\nabla\mu\in\left[L^2(\Omega_T)\right]^2$
which satisfies (1.5) in the following weak sense:

$$\int_0^T \langle \xi(t),u_t(t)\rangle_{H^1,(H^1)'}=-\int_{\Omega_T} m(u)\nabla\mu\cdot\nabla\xi \tag{1.15}$$

for all $\xi\in L^2(0,T;H^1(\Omega))$ and

$$\int_\Omega \mu\phi=\int_\Omega F'(u)\phi+\int_\Omega \epsilon^2 A(\nabla u)\nabla u\cdot\nabla\phi \tag{1.16}$$

for all $\phi\in H^1(\Omega)$ which fulfill $\mathbf{n}_\Omega\nabla\phi=0$ on $\partial\Omega\times(0,T)$ and almost all $t\in[0,T]$.

Motivated by [3], we exploit the properties of the anisotropy operator (1.6), but since our equation is of fourth order, we additionally need some higher order bounds on $\operatorname{div}(A(\nabla u)\nabla u)$. All necessary properties are collected in Section 2.1. In Section 2.2 we then present the proof of Theorem 1.1. The proof of the existence theorem is divided in two main steps. The first step is to consider the regularized problem, i.e. with a mobility which is bounded away from zero by a small parameter δ, and apply a Galerkin-approximation for this auxiliary problem. We prove the existence of solutions to the approximate problem and derive appropriate a priori bounds, which are in particular independent of the regularization parameter δ. All the results, which correspond to the approximate problem are given in Subsection 2.2.1, for a better overview. The second step of the proof, given in Subsection 2.2.2, is to pass to the limit in the approximate problem and deduce existence of solutions to the degenerate problem.

2. Existence of solutions to the anisotropic Cahn-Hilliard equation with degenerate mobility (1.5)

The main difficulty in the present existence proof, compared to the result in reference [8] or [10], resides in the strongly non-linear fourth-order operator. Motivated by [3], we will exploit the fact that the impact of the anisotropy depends on the size of G and that for small values of G the energy functional (1.9) stays convex with respect to ∇u. This implies monotonicity and hemicontinuity of $u \mapsto \langle (A(\nabla u)\nabla u), \nabla \cdot \rangle$, which will be very useful in situations where we have to identify limits of approximate problems. Furthermore, since the differential Equation (1.5) is of fourth order we will additionally need some higher order bounds on $\operatorname{div}(A(\nabla u)\nabla u)$. These are in particular necessary in order to recover the energy estimates (or a priori estimates) as posed in [8] for the anisotropic case.

In the following section we collect all the crucial properties of the anisotropic operator. Note that the former may also be found in [3].

2.1. Properties of the anisotropic operator.

NOTATION 2.1. *Throughout this and the following sections of this chapter we assume that Ω is an open, bounded domain in $\mathbb{R}^2$, with a Lipschitz boundary $\partial\Omega$. The $L^2(\Omega)$-scalar product will be denoted by $(\cdot,\cdot)$ and $\Omega_T = \Omega \times (0,T)$ will denote the space-time domain for some $T > 0$. For brevity we write H^1 instead of $H^1(\Omega)$ in the indices of corresponding norms or scalar products. We omit the differential "dx" at the end of an integral in order to save space. Furthermore, unless otherwise stated, $C > 0$ denotes a constant.*

The results of this section refer to the particular representation (1.7) of the anisotropic surface energy and the corresponding matrix representation (1.6) of the anisotropy in the partial differential equation. We recall that G represents the strength of the anisotropy and n corresponds to symmetry type.

First of all we repeat the key lemma from Section 4 in the paper by Burman and Rappaz [3].

LEMMA 2.1. *If*

$$G < \frac{1}{n^2 - 1}, \tag{2.1}$$

then

(1) the functional

$$\hat{E}(\mathbf{v}) := \int_\Omega \frac{\gamma(\theta_{\mathbf{v}})^2}{2} |\mathbf{v}|^2$$

is strictly convex in $\mathbf{v}$, $\forall \mathbf{v} \in [L^2(\Omega)]^2$.

(2) the Gateaux derivative of the potential

$$\tilde{E}(u) = \int_\Omega \frac{\gamma(\theta_{\nabla u})^2}{2} |\nabla u|^2$$

exists for each $u \in H^1(\Omega)$ and is given by

$$\tilde{E}'(u)v = \int_\Omega A(\nabla u)\nabla u \cdot \nabla v$$

(3) the anisotropic operator satisfies the following upper and lower bounds

$$(1-G)^2|u|_{H^1}^2 \leq \int_\Omega A(\nabla u)|\nabla u|^2 \; dx \leq (1+G)^2|u|_{H^1}^2.$$

Proof. See Section 4 in [3]. ∎

Properties (1) and (2) turn out to be useful in order to prove the following lemma.

LEMMA 2.2. *The mapping*

$$u \in H^1(\Omega) \mapsto \langle A(\nabla u)\nabla u, \nabla \cdot \; \rangle_{(H^1)',H^1} \in (H^1(\Omega))'$$

is monotone and hemicontinuous.

Proof. From Lemma 4.8 in [16], we know that $u \in H^1(\Omega) \mapsto \langle A(\nabla u)\nabla u, \nabla \cdot \; \rangle \in (H^1(\Omega))'$ is monotone and radially continuous in the sense of Definition 2.3 in [15]. Using Lemma 2.16 in [15] then gives hemicontinuity as well. ∎

The following lemma states the particular bounds on $\operatorname{div}(A(\nabla u)\nabla u)$.

LEMMA 2.3. *Let $u \in C^2(\Omega)$ and assume that (1.14) is satisfied. Then there exists $0 < G_0 \leq 1/(n^2-1)$ such that for all $G \leq G_0$ there exists a constant $C(n,G) > 0$, only depending on n and G, such that*

$$0 \leq \int_\Omega (\operatorname{div}(A(\nabla u)\nabla u))^2 \leq C(n,G) \int_\Omega \operatorname{div}(A(\nabla u)\nabla u)\Delta u.$$

Proof. Exploiting the particular representation of θ, i.e. (1.12), we have

$$\theta_x = \frac{u_{yx}u_x - u_y u_{xx}}{|\nabla u|^2}, \qquad \theta_y = \frac{u_{yy}u_x - u_y u_{xy}}{|\nabla u|^2}.$$

On the one hand, we obtain

$$\nabla\theta \cdot \begin{pmatrix} -u_y \\ u_x \end{pmatrix} = \frac{1}{|\nabla u|^2}\left(-2u_{yx}u_x u_y + u_y^2 u_{xx} + u_{yy}u_x^2\right),$$

where "$\cdot$" denotes the standard Euclidean scalar product of two vectors and we exploited the fact that $u_{xy} = u_{yx}$.

On the other hand we have

$$\begin{aligned} |\nabla\theta|^2|\nabla u|^2 &= \frac{1}{|\nabla u|^2}\left((u_{yx}u_x - u_y u_{xx})^2 + (u_{yy}u_x - u_y u_{xy})^2\right) \\ &= \frac{1}{|\nabla u|^2}\left(-2u_{xy}u_x u_y \Delta u + u_{xy}^2|\nabla u|^2 + u_y^2 u_{xx}^2 + u_x^2 u_{yy}^2\right) \\ &= \frac{1}{|\nabla u|^2}\left(-2u_{yx}u_x u_y + u_y^2 u_{xx} + u_{yy}u_x^2\right)\Delta u + u_{xy}^2 - u_{xx}u_{yy}, \end{aligned}$$

which together reveals the relation

$$|\nabla\theta|^2|\nabla u|^2 = \left(\nabla\theta \cdot \begin{pmatrix} -u_y \\ u_x \end{pmatrix}\right)\Delta u + u_{xy}^2 - u_{xx}u_{yy}.$$

Moreover, denoting the angle between $\nabla\theta$ and ∇u by α, we have

$$\begin{aligned} (\nabla\theta \cdot \nabla u) &= \cos(\alpha) \; |\nabla\theta||\nabla u|, \\ \left(\nabla\theta \cdot \begin{pmatrix} -u_y \\ u_x \end{pmatrix}\right) &= \cos(\frac{\pi}{2} - \alpha) \; |\nabla\theta||\nabla u| = \sin(\alpha) \; |\nabla\theta||\nabla u|, \end{aligned} \tag{2.2}$$

which gives

$$(|\nabla\theta||\nabla u|)^2 = \sin(\alpha)\Delta u\ |\nabla\theta||\nabla u| + u_{xy}^2 - u_{xx}u_{yy}$$

and consequently

$$(|\nabla\theta||\nabla u|)_{1,2} = \frac{1}{2}\sin(\alpha)\Delta u \pm \sqrt{\frac{1}{4}\sin^2(\alpha)(\Delta u)^2 + u_{xy}^2 - u_{xx}u_{yy}}.$$

Observing that $|\nabla\theta||\nabla u|$ is positive and real and $x \in \mathbb{R}^2 \to |x|$ is a surjective mapping we may conclude that only

$$|\nabla\theta||\nabla u| = \frac{1}{2}\sin(\alpha)\Delta u + \sqrt{\frac{1}{4}\sin^2(\alpha)(\Delta u)^2 + u_{xy}^2 - u_{xx}u_{yy}} \tag{2.3}$$

is a reasonable solution.

Consider now $\operatorname{div}(A(\nabla u)\nabla u)$ and apply the representation (2.2)

$$\begin{aligned}\operatorname{div}(A(\nabla u)\nabla u) &= \operatorname{div}(\gamma^2\nabla u) + \operatorname{div}\left(\gamma\gamma'\begin{pmatrix}-u_y\\ u_x\end{pmatrix}\right)\\ &= \gamma^2\Delta u + 2\gamma\gamma'(\nabla\theta\cdot\nabla u) + ((\gamma')^2 + \gamma\gamma'')\left(\nabla\theta\cdot\begin{pmatrix}-u_y\\ u_x\end{pmatrix}\right)\\ &= \gamma^2\Delta u + \left(\underbrace{2\gamma\gamma'\cos(\alpha) + ((\gamma')^2 + \gamma\gamma'')\sin(\alpha)}_{=:c_1(\alpha,\theta,G)}\right)|\nabla\theta||\nabla u|\end{aligned}$$

where $c_1(\alpha,\theta,G)$ is uniformly bounded and satisfies

$$\begin{aligned}\gamma^2 + c_1(\alpha,\theta,G)\sin\alpha &= \gamma^2 + 2\gamma\gamma'\cos(\alpha)\sin(\alpha) + ((\gamma')^2 + \gamma\gamma'')\sin^2(\alpha)\\ &= \underbrace{(\gamma\cos(\alpha) + \gamma'\sin(\alpha))^2}_{\geq 0} + (\gamma(\underbrace{\gamma + \gamma''}_{>0}))\underbrace{\sin^2(\alpha)}_{\geq 0}.\end{aligned} \tag{2.4}$$

Consequently we have

$$\gamma^2 + \frac{c_1}{2}(\alpha,\theta,G)\sin\alpha \geq \frac{\gamma^2}{2} > 0, \tag{2.5}$$

which we keep in mind for the following estimates.

Exploiting (2.3) and applying short forms, i.e. c_1 for $c_1(\alpha,\theta,G)$ and c_2 for $c_2(\alpha,\theta,G)$, we then have

$$\operatorname{div}(A(\nabla u)\nabla u) = \left(\gamma^2 + \frac{c_1}{2}\sin(\alpha)\right)\Delta u + c_1\sqrt{\frac{1}{4}\sin^2(\alpha)(\Delta u)^2 + u_{xy}^2 - u_{xx}u_{yy}}. \tag{2.6}$$

Multiplying (2.6) with $\Delta u \in H^1(\Omega)$ and integrating over Ω we have

$$\begin{aligned}\int_\Omega \operatorname{div}(A(\nabla u)\nabla u)\Delta u &= \int_\Omega \left(\gamma^2 + \frac{c_1}{2}\sin(\alpha)\right)(\Delta u)^2\\ &\quad + \int_\Omega c_1\Delta u\sqrt{\frac{1}{4}\sin^2(\alpha)(\Delta u)^2 + u_{xy}^2 - u_{xx}u_{yy}}\end{aligned}$$

$$\geq \int_\Omega \left(\gamma^2 + \frac{c_1}{2}\sin(\alpha)\right)(\Delta u)^2 - \int_\Omega |c_1 \Delta u| \sqrt{\frac{1}{4}\sin^2(\alpha)(\Delta u)^2 + u_{xy}^2 - u_{xx}u_{yy}}. \tag{2.7}$$

Concerning the last integral, we may deduce by applying Young's inequality with $\epsilon_Y > 0$

$$\begin{aligned}
&\int_\Omega |c_1 \Delta u| \sqrt{\frac{1}{4}\sin^2(\alpha)(\Delta u)^2 + u_{xy}^2 - u_{xx}u_{yy}} \\
\leq &\epsilon_Y \int_\Omega |c_1 \Delta u|^2 + \frac{1}{4\epsilon_Y}\int_\Omega \frac{1}{4}\sin^2(\alpha)(\Delta u)^2 + u_{xy}^2 - u_{xx}u_{yy} \\
\leq &\left(\epsilon_Y C_1 + \frac{1}{8\epsilon_Y}\right)\int_\Omega |\Delta u|^2,
\end{aligned} \tag{2.8}$$

where $C_1 := \max_{\alpha,\theta,G} c_1^2$ and we exploited the fact that $u_{xy}^2 - u_{xx}u_{yy}$ has zero mean value.

Introducing the function

$$Y(\epsilon_Y) := \epsilon_Y C_1 + \frac{1}{8\epsilon_Y},$$

and calculating the derivative with respect to ϵ_Y

$$Y'(\epsilon_Y) := C_1 - \frac{1}{8\epsilon_Y^2},$$

reveals that Y has a minimum at $1/\sqrt{8C_1}$ and

$$Y\left(\frac{1}{\sqrt{8C_1}}\right) = \frac{1}{2}\sqrt{C_1} =: \epsilon_G > 0,$$

so that we can choose at least $\epsilon_Y = \epsilon_G$ in (2.8). Then, considering C_1 and exploiting the particular representation of γ, i.e. (1.7), we have

$$\begin{aligned}
\sqrt{C_1} &= \max_{\alpha,\theta,G}\left(2\gamma\gamma'\cos(\alpha) + ((\gamma')^2 + \gamma\gamma'')\sin(\alpha)\right) \\
&\leq |2\gamma\gamma'| + (\gamma')^2 + |\gamma\gamma''| \\
&\leq Gn((2+n) + Gn),
\end{aligned}$$

which basically reveals that C_1 tends to zero for sufficiently small G. On the other hand, (2.5) implies boundedness from below of $\gamma^2 + \frac{c_1}{2}\sin(\alpha)$ by a positive constant. Now, going back to (2.7), we are in the position to deduce that for G sufficiently small we may choose ϵ_G such that

$$0 < \epsilon_G \leq \left(\gamma^2 + \frac{c_1}{2}\sin(\alpha)\right)$$

for all α, θ and consequently

$$\int_\Omega \operatorname{div}(A(\nabla u)\nabla u)\Delta u \geq \int_\Omega \left(\gamma^2 + \frac{c_1}{2}\sin(\alpha)\right)(\Delta u)^2 - \epsilon_G \int_\Omega (\Delta u)^2 \geq 0. \tag{2.9}$$

We now consider the right-hand side of the inequality in Lemma 2.3. Multiplying (2.6) with $\operatorname{div}(A(\nabla u)\nabla u)$ and integrating over Ω we obtain

$$\begin{aligned}
0 &\leq \int_\Omega (\operatorname{div}(A(\nabla u)\nabla u))^2 \\
&= \int_\Omega \Big(\gamma^2 + \frac{c_1}{2}\sin(\alpha)\Big) \operatorname{div}(A(\nabla u)\nabla u)\Delta u \\
&\quad + \int_\Omega \operatorname{div}(A(\nabla u)\nabla u) c_1 \sqrt{\frac{1}{4}\sin^2(\alpha)(\Delta u)^2 + u_{xy}^2 - u_{xx}u_{yy}} \\
&\leq \int_\Omega \Big(\gamma^2 + \frac{c_1}{2}\sin(\alpha)\Big) \operatorname{div}(A(\nabla u)\nabla u)\Delta u \\
&\quad + \epsilon_Y \int_\Omega |\operatorname{div}(A(\nabla u)\nabla u)c_1|^2 + \frac{1}{4\epsilon_Y}\int_\Omega \frac{1}{4}\sin^2(\alpha)(\Delta u)^2, \qquad (2.10)
\end{aligned}$$

where we again exploited the fact that $u_{xy}^2 - u_{xx}u_{yy}$ has zero mean value. Choosing $\epsilon_Y = 1/(C_1+1)$ and observing that from (2.9) we know that there exists a constant $C>0$ such that

$$\int_\Omega |\Delta u|^2 \leq C \int_\Omega \operatorname{div}(A(\nabla u)\nabla u)\Delta u,$$

we obtain from (2.10)

$$\begin{aligned}
0 &\leq \int_\Omega (\operatorname{div}(A(\nabla u)\nabla u))^2 \\
&\leq \int_\Omega C \operatorname{div}(A(\nabla u)\nabla u)\Delta u + \frac{C_1}{C_1+1}\int_\Omega |\operatorname{div}(A(\nabla u)\nabla u)|^2.
\end{aligned}$$

Note that $C>0$ is now a different constant which we still denote the same to simplify matters. Finally we conclude that

$$\frac{1}{C_1+1}\int_\Omega (\operatorname{div}(A(\nabla u)\nabla u))^2 \leq \int_\Omega C \operatorname{div}(A(\nabla u)\nabla u)\Delta u,$$

which completes the proof. $\square$

Remark 2.1. In the Appendix it is shown that G can be at least as big as $1/5\ G_0$ in order to satisfy Lemma 2.3.

We are now in the position to prove the existence result, i.e. Theorem 1.1.

2.2. Proof of Theorem 1.1. We now present the proof of Theorem 1.1, which consists of two main steps. The first step is to consider (1.5) with a mobility which is bounded away from zero by a small parameter δ, and apply a Galerkin-approximation for this regularized problem. The existence of solutions to the approximate problem is stated in Lemma 2.4 and appropriate a priori bounds are given in Lemma 2.5. These are in particular independent of the regularization parameter δ and the corresponding proof basically exploits the dissipation of two energy functionals. For a better overview, we collect the results corresponding to the approximate problem, or the first step, respectively, in the following subsection.

The second step of the proof, given in Subsection 2.2.2, is to pass to the limit in the approximate problem and deduce existence of solutions to the degenerate problem.

2.2.1. Step 1: Galerkin approximation of the regularized problem. Consider the anisotropic Cahn-Hilliard Equation (1.5) with the regularized mobility $m_\delta(u)$ defined by

$$m_\delta(u) := \begin{cases} m(-1+\delta) & \text{for} \quad u \leq -1+\delta, \\ m(u) & \text{for} \quad u < 1-\delta, \\ m(1-\delta) & \text{for} \quad u \geq 1-\delta, \end{cases}$$

where $\delta \ll 1$ and define $\Phi_\delta(u)$ such that

$$\Phi_\delta''(u) = \frac{1}{\sqrt{m_\delta(u)}}, \quad \Phi_\delta'(0) = 0, \quad \text{and } \Phi_\delta(0) = 0. \tag{2.11}$$

We point out that $\Phi_\delta(u) = \Phi(u)$ when $|u| \leq 1-\delta$. In a similar way we define $\Psi_\delta(u)$ such that

$$\Psi_\delta''(u) = \frac{1}{m_\delta(u)}, \quad \Psi_\delta'(0) = 0, \quad \text{and } \Psi_\delta(0) = 0, \tag{2.12}$$

which will prove useful in order to derive appropriate bounds for the anisotropy operator.

We observe that $m_\delta \in C(\mathbb{R}, \mathbb{R}^+)$ and there exist $m_1, M_1 > 0$ such that

$$m_1 \leq |m_\delta(u)| \leq M_1$$

for all $u \in \mathbb{R}$.

We now apply a Galerkin approximation to the regularized problem. Let $\{\phi_i\}_{i\in\mathbb{N}}$ be the eigenfunctions of the Laplace operator with Neumann boundary conditions which is an orthogonal basis of $H^1(\Omega)$. We suppose that the ϕ_i are normalized in the $L^2(\Omega)$ scalar product, i.e. $(\phi_i, \phi_j)_{L^2(\Omega)} = \delta_{ij}$ and that without loss of generality the first eigenfunction ϕ_1 corresponds to the eigenvalue $\lambda_1 = 0$, i.e. $\Delta\phi_1 = 0$.

Consider the following Galerkin ansatz for u and μ

$$u^N(t,x) = \sum_{i=1}^N c_i^N(t)\phi_i(x), \quad \mu^N(t,x) = \sum_{i=1}^N d_i^N(t)\phi_i(x) \tag{2.13}$$

$$\int_\Omega \partial_t u^N \phi_j = -\int_\Omega m_\delta(u^N)\nabla\mu^N \cdot \nabla\phi_j \quad \text{for } j = 1,..,N, \tag{2.14}$$

$$\int_\Omega \mu^N \phi_j = \int_\Omega \epsilon^2 A(\nabla u^N)\nabla u^N \cdot \nabla\phi_j + \int_\Omega F'(u^N)\phi_j \quad \text{for } j = 1,..,N, \tag{2.15}$$

$$u^N(0) = \sum_{i=1}^N (u_0, \phi_i)_{L^2(\Omega)}\phi_i, \tag{2.16}$$

which leads to an initial value problem for a system of ordinary differential equations for $(c_1,..,c_N)$

$$\partial_t c_j^N = -\sum_{k=1}^N d_k^N \int_\Omega m_\delta\left(\sum_{i=1}^N c_i^N(t)\phi_i(x)\right)\nabla\phi_k \cdot \nabla\phi_j \tag{2.17}$$

$$d_j^N = \int_\Omega \epsilon^2 A\left(\sum_{i=1}^N c_i^N(t)\nabla\phi_i(x)\right)\sum_{k=1}^N c_k^N(t)\nabla\phi_k(x)\cdot\nabla\phi_j + \int_\Omega F'\left(\sum_{i=1}^N c_i^N(t)\phi_i(x)\right)\phi_j \tag{2.18}$$

$$c_j^N(0)=(u_0,\phi_j)_{L^2(\Omega)} \tag{2.19}$$

which has to hold for $j=1,...,N$.

LEMMA 2.4 (Existence of approximate solutions). *Let $u_0\in H^1(\Omega)$. Then the initial value problem (2.17)-(2.19) admits a global solution $(u_\delta^N,\mu_\delta^N)$ for every $j=1,...,N$, such that*

(1) $u_\delta^N\in L^\infty(0,T;H^1(\Omega))$

(2) $\partial_t u_\delta^N\in L^2(0,T;(H^1(\Omega))')$,

(3) $\mu_\delta^N\in L^2(0,T;H^1(\Omega))$.

Furthermore the solution u_δ^N satisfies

$$\nabla\left(\operatorname{div}\left(A(\nabla u_\delta^N)\nabla u_\delta^N\right)\right)\in\left[L^2(\Omega_T)\right]^2. \tag{2.20}$$

Proof. Recalling that the mapping

$$u\in H^1(\Omega)\mapsto\langle A(\nabla u)\nabla u,\nabla\cdot\ \rangle\in(H^1(\Omega))'$$

is hemicontinuous implies that

$$t\in\mathbb{R}\mapsto\langle A(\nabla(u+tv))\nabla(u+tv),\nabla w\rangle$$

is continuous for all $u,v,w\in H^1(\Omega)$. We then conclude that

$$c_k\mapsto\left\langle A\left(\left(u_{\neq k}^N+c_k\nabla\phi_k\right)\right)\left(u_{\neq k}^N c_i^N(t)\nabla\phi_i(x)+c_k\nabla\phi_k\right),\nabla\phi_j\right\rangle,$$

where

$$u_{\neq k}^N=\sum_{i=1,i\neq k}^N c_i^N(t)\nabla\phi_i(x),$$

is continuous for every c_k, which reveals continuity of the right-hand side of (2.17) and therefore existence of a local weak solution to (2.17)-(2.19) due to the Peano existence theorem. In order to provide that this solution exists globally in time we need to show that the energy stays bounded uniformly in t. For this purpose, consider the time derivative of the energy $\mathcal{E}$

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t)&=\frac{d}{dt}\int_\Omega\left(F(u^N)+\epsilon^2\frac{\gamma(\theta_{\nabla u^N})^2}{2}|\nabla u^N|^2\right)\\&=\int_\Omega F'(u^N)\partial_t u^N+\epsilon^2\left(\gamma\gamma'\theta_t\ |\nabla u|^2+\gamma^2\nabla u^N\nabla u_t^N\right)\\&=\int_\Omega F'(u^N)u_t^N+\epsilon^2\left(\gamma\gamma'\begin{pmatrix}-\partial_y u^N\\ \partial_x u^N\end{pmatrix}+\gamma^2\nabla u^N\right)\cdot\nabla u_t^N\\&=\int_\Omega\mu^N\partial_t u^N\quad=-\int_\Omega m_\delta(u^N)|\nabla\mu^N|^2,\end{aligned}$$

where we exploited the particular representation of θ, i.e. (1.12). Integrating over $[0,t]$ then reveals

$$\int_\Omega\epsilon^2\frac{\gamma(\theta_{\nabla u^N(t)})^2}{2}|\nabla u^N(t)|^2+\int_\Omega F(u^N(t))+\int_{\Omega_t}m_\delta(u^N)|\nabla\mu^N|^2$$

$$= \int_\Omega \epsilon^2 \frac{\gamma(\theta_{\nabla u^N(0)})^2}{2} |\nabla u^N(0)|^2 + \int_\Omega F(u^N(0)) \leq C, \tag{2.21}$$

where C is a constant which is independent of N, δ and t. From (2.14) with $j=1$ we deduce that $\partial_t \int_\Omega u^N = 0$ and since γ is bounded uniformly we obtain from Poincaré's inequality

$$\text{ess sup}_{0<t<T} \|u^N(t)\|_{H^1(\Omega)} \leq C, \tag{2.22}$$

which implies that $c_1^N, ..., c_N^N$ are bounded uniformly and therefore a global solution to (2.17)-(2.19) exists, which we denote by

$$(u_\delta^N, \mu_\delta^N),$$

and $u_\delta^N \in L^\infty(0,T; H^1(\Omega))$.

Let now Π_N denote the projection of $L^2(\Omega)$ onto $span\{\phi_1, ..., \phi_N\}$. Considering an arbitrary function $\phi \in L^2(0,T; H^1(\Omega))$ then reveals

$$\begin{aligned}
\left| \int_{\Omega_T} \partial_t u_\delta^N \phi \right| &= \left| \int_{\Omega_T} \partial_t u_\delta^N \Pi_N \phi \right| \\
&= \left| \int_{\Omega_T} m_\delta(u_\delta^N) \nabla \mu_\delta^N \nabla \Pi_N \phi \right| \\
&= \left(\int_{\Omega_T} \left| m_\delta(u_\delta^N) \nabla \mu_\delta^N \right|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega_T} |\nabla \Pi_N \phi|^2 \right)^{\frac{1}{2}} \\
&\leq C_0 \left(\int_{\Omega_T} m_\delta(u_\delta^N) \left| \nabla \mu_\delta^N \right|^2 \right)^{\frac{1}{2}} \|\nabla \phi\|_{L^2(\Omega_T)} \\
&\leq C_1 \|\nabla \phi\|_{L^2(\Omega_T)}
\end{aligned}$$

where C_0 and C_1 are independent of N, δ and t. Note that we exploited (2.21) for the last inequality. Consequently we have $\partial_t u_\delta^N \in L^2(0,T; (H^1(\Omega))')$.

We now show the boundedness of μ_δ^N. Since

$$\begin{aligned}
&(A(\theta_\xi)\xi)^T \cdot (A(\theta_\xi)\xi) \\
=&\begin{pmatrix} \gamma^2(\theta_\xi)\xi_1 - \gamma'(\theta_\xi)\gamma(\theta_\xi)\xi_2 \\ \gamma'(\theta_\xi)\gamma(\theta_\xi)\xi_1 + \gamma^2(\theta_\xi)\xi_2 \end{pmatrix}^t \cdot \begin{pmatrix} \gamma^2(\theta_\xi)\xi_1 - \gamma'(\theta_\xi)\gamma(\theta_\xi)\xi_2 \\ \gamma'(\theta_\xi)\gamma(\theta_\xi)\xi_1 + \gamma^2(\theta_\xi)\xi_2 \end{pmatrix} \\
=&\gamma^2(\theta_\xi)(\gamma^2(\theta_\xi) + (\gamma'(\theta_\xi))^2)(\xi_1^2 + \xi_2^2) \\
\leq& C(n,G)|\xi|^2
\end{aligned} \tag{2.23}$$

we obtain as a direct consequence

$$\int_\Omega |A(\nabla u_\delta^N) \nabla u_\delta^N|^2 \leq C(n,G) \int_\Omega |\nabla u_\delta^N|^2. \tag{2.24}$$

Due to (2.22), the right-hand side in the last inequality is uniformly bounded and hence $A(\nabla u_\delta^N)\nabla u_\delta^N$ is uniformly bounded in $L^2(\Omega)$.

Then, exploiting (2.24) together with the uniform boundedness of $F'(u_\delta^N(t))$ for $t \in [0,T)$, we first obtain that $\int_\Omega \mu_\delta^N(t) \leq C(\delta)$ and consequently, including (2.21), Poincaré's inequality leads to

$$\|\mu_\delta^N\|_{L^2(0,T;H^1(\Omega))} \leq C(\delta).$$

Note that in order to apply (2.21) in Poincaré's inequality it is necessary to assume that $m_\delta > 0$, so that C in this case is independent of N but not of δ.

In order to show (2.20) we first show $\operatorname{div}\left(A(\nabla u_\delta^N)\nabla u_\delta^N\right) \in L^2(\Omega_T)$. From (2.15) we have

$$\int_\Omega (F'(u_\delta^N) - \mu_\delta^N)\phi_j + \epsilon^2 A(\nabla u_\delta^N)\nabla u_\delta^N \cdot \nabla \phi_j = 0, \tag{2.25}$$

for $j = 1,..N$. Exploiting the projection Π_N we then have

$$\begin{aligned}\left|\int_\Omega \epsilon^2 A(\nabla u_\delta^N)\nabla u_\delta^N \cdot \nabla \Pi_N \psi\right| &\leq \int_\Omega |(\mu_\delta^N - F'(u_\delta^N))\Pi_N \psi| \\ &\leq C\ \|\Pi_N \psi\|_{L^2(\Omega)} \\ &\leq C\ \|\psi\|_{L^2(\Omega)},\end{aligned}$$

which implies that

$$\psi \mapsto \epsilon^2 A(\nabla u_\delta^N)\nabla u_\delta^N \cdot \nabla \Pi_N \psi, \qquad \psi \in C_c^\infty(\Omega) \tag{2.26}$$

is a linear and continuous functional on $C_c^\infty(\Omega)$ with respect to the L^2-norm. Since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, this functional can be extended uniquely to a linear and continuous functional on $L^2(\Omega)$. From the Riesz representation theorem we then obtain existence of a unique function $v \in L^2(\Omega)$, such that v corresponds to the weak divergence of $A(\nabla u_\delta^N)\nabla u_\delta^N$ and consequently

$$\operatorname{div}\left(A(\nabla u_\delta^N)\nabla u_\delta^N\right) \in L^2(\Omega). \tag{2.27}$$

We may now apply the identity $\mu_\delta^N = F'(u_\delta^N) - \epsilon^2 \operatorname{div}\left(A(\nabla u_\delta^N)\nabla u_\delta^N\right)$ and since $\nabla F'(u_\delta^N) = F''(u_\delta^N)\nabla u_\delta^N \in \left[L^2(\Omega_T)\right]^2$ and $\nabla \mu_\delta^N \in \left[L^2(\Omega_T)\right]^2$ we obtain that also $\nabla\left(\operatorname{div}\left(A(\nabla u_\delta^N)\nabla u_\delta^N\right)\right) \in \left[L^2(\Omega_T)\right]^2$. This completes the proof of Lemma 2.4. ◻

Therefore we can apply the weak form

$$\int_0^T \left\langle \Pi_N \zeta, \partial_t u_\delta^N \right\rangle_{H^1,(H^1)'} = -\int_{\Omega_T} m_\delta(u_\delta^N)\nabla\Big(F'(u_\delta^N) - \epsilon^2 \operatorname{div}\left(A(\nabla u_\delta^N)\nabla u_\delta^N\right)\Big) \cdot \nabla \Pi_N \zeta, \tag{2.28}$$

for all $\zeta \in L^2(0,T;H^1(\Omega))$.

In the next step, we prove the essential energy estimates which provide in particular uniform bounds independent of δ.

Lemma 2.5 (Energy estimates). *Suppose that (1.14) holds true and that G is sufficiently small, according to Lemma 2.3. Let $u_0 \in H^1(\Omega)$ with $|u_0| \leq 1$ a.e. and*

$$\int_\Omega (F(u_0) + \Phi(u_0)) \leq C, \qquad C \in \mathbb{R}^+.$$

Then there exists a δ_0 such that for all $0 < \delta \leq \delta_0$ the following estimates hold for the pair of solutions $(u_\delta^N, \mu_\delta^N)$ with a constant C independent of N and δ:

(a) $\quad \operatorname{ess\,sup}_{0<t<T} \int_\Omega \Phi_\delta(u_\delta^N(t)) \leq C$

(b) $\quad \int_{\Omega_T} |\operatorname{div}\left(A(\nabla u_\delta^N)\nabla u_\delta^N\right)|^2 \leq C$

$$(c)\quad \text{ess sup}_{0<t<T}\int_\Omega (|u_\delta^N|-1)_+^2 \le C\delta^2$$

$$(d)\quad \int_{\Omega_T} |\mathbf{J}_\delta^N|^2 \le C, \qquad \textit{where} \qquad \mathbf{J}_\delta^N := m_\delta(u_\delta^N)\nabla\mu_\delta^N.$$

Proof. To prove (a), we consider the function $\Phi_\delta(u_\delta^N)$ defined by (2.11). Since $\Phi_\delta''(u_\delta^N)$ is bounded uniformly in t, we have $\Phi_\delta'(u_\delta^N) \in L^2(0,T;H^1(\Omega))$ and therefore $\Phi_\delta'(u_\delta^N)$ is an admissible test function in (2.28). On the one hand, we have that

$$\int_0^t \langle \Phi_\delta'(u_\delta^N), \partial_t u_\delta^N\rangle_{H^1,(H^1)'} = \int_\Omega \Phi_\delta(u_\delta^N(t)) - \int_\Omega \Phi_\delta(u_\delta^N(0))$$

is true for almost all $t\in[0,T]$. On the other hand, we have

$$\begin{aligned}
&\int_0^t \langle \Phi_\delta'(u_\delta^N), \partial_t u_\delta^N\rangle_{H^1,(H^1)'} = \int_0^t \langle \Pi_N\Phi_\delta'(u_\delta^N), \partial_t u_\delta^N\rangle_{H^1,(H^1)'}\\
&= \int_{\Omega_t} -m_\delta(u_\delta^N)\nabla\mu_\delta^N\cdot\nabla\Pi_N\Phi_\delta'(u_\delta^N)\\
&\le \left(\int_{\Omega_t} m_\delta(u_\delta^N(t))|\nabla\mu_\delta^N|^2\right)^{1/2}\left(\int_{\Omega_t} m_\delta(u_\delta^N(t))|\nabla\Pi_N\Phi_\delta'(u_\delta^N)|^2\right)^{1/2}\\
&\le C_0\left(\int_{\Omega_t} m_\delta(u_\delta^N(t))|\nabla\mu_\delta^N|^2\right)^{1/2}\left(\int_{\Omega_t} m_\delta(u_\delta^N(t))|\nabla\Phi_\delta'(u_\delta^N)|^2\right)^{1/2}\\
&= C_0\left(\int_{\Omega_t} m_\delta(u_\delta^N(t))|\nabla\mu_\delta^N|^2\right)^{1/2}\left(\int_{\Omega_t} m_\delta(u_\delta^N(t))|\Phi_\delta''(u_\delta^N)\nabla u_\delta^N|^2\right)^{1/2}\\
&= C_0\left(\int_{\Omega_t} m_\delta(u_\delta^N(t))|\nabla\mu_\delta^N|^2\right)^{1/2}\left(\int_{\Omega_t} |\nabla u_\delta^N|^2\right)^{1/2},
\end{aligned} \tag{2.29}$$

where the right-hand side is bounded. It follows that there exists a constant C which is independent of δ such that

$$\int_\Omega \Phi_\delta(u_\delta^N(t)) \le C + \int_\Omega \Phi_\delta(u_\delta^N(0)),$$

which proves (a).

Consider now Ψ_δ defined by (2.12). Similar as in (2.29) we obtain

$$\begin{aligned}
&\int_\Omega \Psi_\delta(u_\delta^N(t)) - \int_\Omega \Psi_\delta(u_\delta^N(0)) = \int_{\Omega_t} -m_\delta(u_\delta^N)\nabla\mu_\delta^N\Psi_\delta''(u_\delta^N)\cdot\nabla u_\delta^N\\
&= \int_{\Omega_t} -\epsilon^2\operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N)\Delta u_\delta^N - F''(u_\delta^N)|\nabla u_\delta^N|^2,
\end{aligned}$$

which again implies that there exists a constant C which is independent of δ such that

$$\begin{aligned}
&\int_\Omega \Psi_\delta(u_\delta^N(t)) + \int_{\Omega_t} \epsilon^2\operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N)\Delta u_\delta^N + F''(u_\delta^N)|\nabla u_\delta^N|^2\\
&\le C + \int_\Omega \Psi_\delta(u_\delta^N(0)).
\end{aligned}$$

Realizing that Ψ_δ and F'' are both convex functions which are bounded from below and taking Lemma 2.3 into account, we conclude that there exists another constant, which is independent of δ such that

$$\int_{\Omega_T} |\mathrm{div}(A(\nabla u_\delta^N)\nabla u_\delta^N)|^2 \le C + \int_\Omega \Psi_\delta(u_\delta^N(0)),$$

which proves (b).

We will now use the bound for $\int_\Omega \Phi_\delta(u_\delta^N)$ to derive a bound for $\int_\Omega (|u_\delta^N|-1)_+^2$. If $z>1$ and $\delta<1$, then we have

$$\begin{aligned}\Phi_\delta(z) &= \underbrace{\Phi(1-\delta)}_{\ge 0} + \underbrace{\Phi'(1-\delta)}_{\ge 0}\underbrace{(z-(1-\delta))}_{\ge 0} + \frac{1}{2}\Phi''(1-\delta)(z-(1-\delta))^2 \\ &\ge \frac{1}{2}\Phi''(1-\delta)(z-1)^2 = \frac{1}{2}\frac{1}{\sqrt{m(1-\delta)}}(z-1)^2 \\ &= \frac{1}{2}\frac{1}{1-(1-\delta)^2}(z-1)^2 \ge C^{-1}\delta^{-2}(z-1)^2.\end{aligned}$$

It follows that $(z-1)^2 \le C\delta^2\Phi_\delta(z)$. Similarly we obtain $(-z-1)^2 \le C\delta^2\Phi_\delta(z)$ for $z<-1$. This implies

$$\int_\Omega (|u_\delta^N|-1)_+^2 \le C\delta^2 \int_\Omega \Phi_\delta(u_\delta^N) \le C\delta^2,$$

which proves (c).

Assertion (d) follows easily from the energy estimate (2.21), and this finishes the proof of Lemma 2.5. ∎

2.2.2. Step 2: Convergence of the approximate problem. We are now in the position to prove Theorem 1.1 by passing to the limit in the approximate problem.

Proof. (**Proof of Theorem 1.1.**) Due to Lemma 2.4 the initial value problem (2.17)-(2.19) admits a global solution $(u_\delta^N, \mu_\delta^N)$, satisfying (2.20) and the uniform bounds of Lemma 2.5.

Exploiting the compact embeddings

$$\left\{u \in L^2\left(0,T;H^1(\Omega)\right) | \partial_t u \in L^2\left(0,T;\left(H^1(\Omega)\right)'\right)\right\} \hookrightarrow L^2\left(0,T;L^2(\Omega)\right), \tag{2.30}$$

(see [12], p. 57) and

$$\left\{u \in L^\infty\left(0,T;H^1(\Omega)\right) | \partial_t u \in L^2\left(0,T;\left(H^1(\Omega)\right)'\right)\right\} \hookrightarrow C\left([0,T];L^2(\Omega)\right), \tag{2.31}$$

(see [18], p. 422) we are in the position to deduce that there exist subsequences (which we still denote by u_δ^N) such that

$$\begin{array}{llll} u_\delta^N \overset{*}{\rightharpoonup} u & \text{weak}-* & \text{in} & L^\infty(0,T;H^1(\Omega)), \\ u_\delta^N \to u & \text{strongly} & \text{in} & C([0,T];L^2(\Omega)), \\ \partial_t u_\delta^N \rightharpoonup \partial_t u & \text{weakly} & \text{in} & L^2(0,T;(H^1(\Omega))'), \text{ and} \\ u_\delta^N \to u & \text{strongly} & \text{in} & L^2(0,T;L^p(\Omega)) \text{ and a.e. in } \Omega_T, \end{array}$$

where $p<\infty$.

According to the bounds of Lemma 2.5 together with standard compactness properties, we obtain that there exists a function $\mathbf{J}$ such that

$$\mathbf{J}_\delta^N \rightharpoonup \mathbf{J} \qquad \text{in} \qquad \left[L^2(\Omega_T)\right]^2. \tag{2.32}$$

Moreover by the boundedness of $\operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N)$ in $L^2(\Omega_T)$ we have that

$$A(\nabla u_\delta^N)\nabla u_\delta^N \rightharpoonup \chi \qquad \text{in } \left[H^1(\Omega)\right]^2,$$

for some function $\chi \in \left[H^1(\Omega_T)\right]^2$. At this point we apply Minty's Trick in order to identify χ as $A(\nabla u_\delta^N)\nabla u_\delta^N$. Adding and subtracting elements we obtain, due to the monotonicity property, that

$$\begin{aligned}
&\langle \chi - A(\nabla v)\nabla v, \nabla u - \nabla v\rangle \\
&= \left\langle \chi - A(\nabla u_\delta^N)\nabla u_\delta^N + A(\nabla u_\delta^N)\nabla u_\delta^N - A(\nabla v)\nabla v, \nabla u - \nabla v\right\rangle \\
&= \left\langle \chi - A(\nabla u_\delta^N)\nabla u_\delta^N, \nabla u - \nabla v\right\rangle \\
&\quad + \left\langle A(\nabla u_\delta^N)\nabla u_\delta^N - A(\nabla v)\nabla v, \nabla u - \nabla v\right\rangle \\
&= \left\langle \chi - A(\nabla u_\delta^N)\nabla u_\delta^N, \nabla u - \nabla v\right\rangle \\
&\quad + \left\langle A(\nabla u_\delta^N)\nabla u_\delta^N - A(\nabla v)\nabla v, \nabla u - \nabla u_\delta^N + \nabla u_\delta^N - \nabla v\right\rangle \\
&= \left\langle \chi - A(\nabla u_\delta^N)\nabla u_\delta^N, \nabla u - \nabla v\right\rangle \\
&\quad + \left\langle A(\nabla u_\delta^N)\nabla u_\delta^N - A(\nabla v)\nabla v, \nabla u - \nabla u_\delta^N\right\rangle \\
&\quad + \left\langle A(\nabla u_\delta^N)\nabla u_\delta^N - A(\nabla v)\nabla v, \nabla u_\delta^N - \nabla v\right\rangle \\
&\geq \left\langle \chi - A(\nabla u_\delta^N)\nabla u_\delta^N, \nabla u - \nabla v\right\rangle \\
&\quad + \left\langle A(\nabla u_\delta^N)\nabla u_\delta^N - A(\nabla v)\nabla v, \nabla u - \nabla u_\delta^N\right\rangle,
\end{aligned}$$

for all $v \in H^1(\Omega)$, where the last inequality holds because of the monotonicity property. Taking the limit we observe that the right-hand side goes to zero and hence

$$(\chi - A(\nabla v)\nabla v, \nabla u - \nabla v) \geq 0. \tag{2.33}$$

We are now in the position to apply Minty's Trick (Lemma 2.13 in [15]) and deduce that

$$\chi = A(\nabla u)\nabla u. \tag{2.34}$$

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ the weak convergence $A(\nabla u_\delta^N)\nabla u_\delta^N \rightharpoonup A(\nabla u)\nabla u$ in $\left[H^1(\Omega)\right]^2$ implies

$$A(\nabla u_\delta^N)\nabla u_\delta^N \to A(\nabla u)\nabla u \qquad \text{in } \left[L^2(\Omega)\right]^2. \tag{2.35}$$

Passing to the limit in

$$\int_\Omega (|u_\delta^N| - 1)_+^2 \leq C\delta^2$$

yields $|u| \leq 1$ a.e. in Ω_T.

It remains to show that u fulfills the limit equation. The weak convergence of $\partial_t u_\delta^N$ and $\mathbf{J}_\delta^N$ gives in the limit

$$\int_0^T \langle \xi, \partial_t u \rangle_{H^1,(H^1)'} = \int_{\Omega_T} \mathbf{J} \cdot \nabla \xi,$$

for all $\xi \in L^2(0,T;H^1(\Omega))$. Now we have to identify $\mathbf{J}$. Therefore, we want to pass to the limit in the equation

$$\int_{\Omega_T} \mathbf{J}_\delta^N \cdot \Pi_N \eta = \int_{\Omega_T} m_\delta(u_\delta^N) \nabla(-\epsilon^2 \operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N) + F'(u_\delta^N))\Pi_N \eta, \tag{2.36}$$

where $\eta \in L^2(0,T;H^1(\Omega,\mathbb{R}^n)) \cap L^\infty(\Omega_T,\mathbb{R}^n)$ with $\eta \cdot \mathbf{n}_\Omega = 0$ on $\partial\Omega \times (0,T)$. Note that the projection operator is applied component-wise in this case. Realizing that $\Pi_N \eta \to \eta$ in $L^2(0,T;H^1(\Omega,\mathbb{R}^n))$ and taking (2.32) into account implies that the left-hand side converges to $\int_{\Omega_T} \mathbf{J} \cdot \eta$. Since $\nabla \operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N)$ may not have a limit in $L^2(\Omega_T)$, we integrate the first term on the right-hand side of (2.36) by parts to get

$$\begin{aligned}
&\int_{\Omega_T} m_\delta(u_\delta^N)\nabla(-\epsilon^2 \operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N))\Pi_N \eta \\
&= \int_{\Omega_T} \epsilon^2 \operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N) m_\delta(u_\delta^N) \nabla \Pi_N \eta \\
&\qquad + \int_{\Omega_T} \epsilon^2 \operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N) m_\delta'(u_\delta^N) \nabla u_\delta^N \cdot \Pi_N \eta \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{2.37}$$

Using the fact that for all $z \in \mathbb{R}$

$$|m_\delta(z) - m(z)| \le \sup_{1-\delta \le |y| \le 1} |m(y)| \to 0 \quad \text{as } \delta \to 0,$$

it follows that $m_\delta \to m$ uniformly.

Hence we have

$$m_\delta(u_\delta^N) \to m(u) \quad \text{a.e. in } \Omega_T.$$

Exploiting that $\operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N)$ is uniformly bounded in $L^2(\Omega_T)$, we may deduce that there exists $\rho \in L^2(\Omega_T)$ such that

$$\operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N) \rightharpoonup \rho \qquad \text{in } L^2(\Omega_T). \tag{2.38}$$

From the definition of the weak divergence and the already established convergence (2.35), we then have that for any test function $\Psi \in C_c^\infty(\Omega_T)$

$$\begin{aligned}
\int_{\Omega_T} \operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N)\Psi &= -\int_{\Omega_T} A(\nabla u_\delta^N)\nabla u_\delta^N \cdot \nabla \Psi \\
&\to -\int_{\Omega_T} A(\nabla u)\nabla u \cdot \nabla \Psi \\
&= \int_{\Omega_T} \operatorname{div}(A(\nabla u)\nabla u)\Psi.
\end{aligned} \tag{2.39}$$

Since the weak divergence is unique we immediately obtain

$$\operatorname{div}(A(\nabla u)\nabla u) = \rho. \tag{2.40}$$

Recalling that m_δ is uniformly bounded, we conclude

$$\int_{\Omega_T} \epsilon^2 \operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N) m_\delta(u_\delta^N)\nabla \Pi_N \eta \to \int_{\Omega_T} \epsilon^2 \operatorname{div}(A(\nabla u)\nabla u) m(u)\nabla \eta,$$

as $\delta \to 0$, which equals the convergence of I in (2.37) . Now we pass to the limit in II. As for m, we have $m'_\delta \to m'$ uniformly, which gives

$$m'_\delta(u_\delta^N) \to m'(u) \quad \text{a.e. in } \Omega_T.$$

By using

$$A(\nabla u_\delta^N)\nabla u_\delta^N \to A(\nabla u)\nabla u \quad \text{in } \left[L^2(\Omega_T)\right]^2 \quad \text{and a.e. in } \Omega_T,$$

and the fact that m'_δ is uniformly bounded a generalized version of the Lebesgue convergence theorem yields

$$m'_\delta(u_\delta^N)\nabla u_\delta^N \to m'(u)\nabla u \quad \text{in } L^2(\Omega).$$

Hence

$$\int_{\Omega_T} \epsilon^2 \operatorname{div}(A(\nabla u_\delta^N)\nabla u_\delta^N) m'_\delta(u_\delta^N)\nabla u_\delta^N \cdot \Pi_N \eta \to \int_{\Omega_T} \epsilon^2 \operatorname{div}(A(\nabla u)\nabla u) m'(u)\nabla u \cdot \eta,$$

as $\delta \to 0$, where we used the fact that $\eta \in L^\infty(\Omega_T)$.

Finally the strong convergence of u_δ^N in $C([0,T];L^2(\Omega))$ proves $u_\delta^N(0) \to u_0$ in $L^2(\Omega)$, which shows that u solves the Cahn-Hilliard equation in the sense of Theorem 1.1. □

3. Discussion and outlook

We have proved the existence of weak solutions to the anisotropic Cahn-Hilliard Equation (1.5) with degenerate mobility under the assumption that the strength of the anisotropy is sufficiently small (see Lemma 2.3). The main difficulties arise in establishing the estimates of Lemma 2.5, in particular in view of the degenerate mobility and the non-linear anisotropy function. The limitation of sufficiently weak anisotropy enables to apply Lemma 2.3, given in the preliminary results of Section 2.1, at this point, which turns out to be of essential importance for the present existence proof. In addition to existence, we show that solutions $|u|$ are bounded by one without having a maximum principle.

There are still many open questions. The most important is whether the assumptions of Lemma 2.3 may be relaxed in order to obtain existence of solution in a more general case. In particular, the existence of solutions on different domains would be desirable.

Furthermore, it would be interesting to know if there exists a unique solution. We note that already in the isotropic case, studied by Elliott and Garcke [8] or Grün [10], this remains an open question. Since so far no uniqueness result for fourth order degenerate parabolic equations has been established, a corresponding existence result for the present problem is less obvious.

Besides studying the question of uniqueness we are also interested in the qualitative behavior of solutions, for example as $|u| \to 1$. Just as in the isotropic case we expect that for the present degenerate mobility the sets $\{u = -1\}$ and $\{u = 1\}$ develop an interior which implies a free boundary problem for $\partial\{u = -1\}$ and $\partial\{u = -1\}$, respectively. In addition, it would be interesting to study the asymptotic behavior of solutions in the case as $t \to \infty$.

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Appendix. Size of G. Recalling inequality (2.9) we know that G has to be chosen sufficiently small such that

$$0<\epsilon_G \leq \left(\gamma^2+\frac{c_1}{2}\sin(\alpha)\right). \tag{4.1}$$

Realizing that on the one hand we have

$$\epsilon_G = \frac{3}{4}\sqrt{C_1} \leq \frac{3}{4}\ Gn((2+n)+Gn)$$

and on the other hand

$$0<\frac{\gamma^2}{2} \leq \gamma^2+\frac{c_1}{2}\sin(\alpha),$$

which implies that if G is sufficiently small such that

$$\frac{3}{2}\ Gn((2+n)+Gn) \leq (1-G)^2 \leq \gamma^2,$$

then (4.1) clearly holds true as well.

Defining

$$H(G) := \frac{3}{2}\ Gn((2+n)+Gn)-(1-G)^2,$$

and calculating its zeros reveals that $H(G)$ has a positive zero at $G=G_z=G_z(n)$, where

$$G_z(n) = \frac{-4-6n-3n^2+\sqrt{3n\,(16+28n+12n^2+3n^3)}}{2(-2+3n^2)}.$$

This means, that if $G \leq G_z \leq G_0 = 1/(n^2-1)$, then (4.1) is satisfied as well. Realizing that $G_z(n)/G_0(n)$ is a monotonically increasing function in n, we obtain

$$G_z(n)/G_0(n) \geq G_z(2)/G_0(2) \geq \frac{1}{5},$$

which provides a greatest lower bound for $G_z(n)$, i.e. we may at least choose

$$G_z(n) = \frac{1}{5}\ G_0(n),$$

in order to satisfy condition (4.1).

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