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THE REGULARITY CRITERIA ON THE MAGNETIC FIELD TO THE $3D$ INCOMPRESSIBLE MHD EQUATIONS*

XIANGXIANG GUO†, YI DU‡, AND PENG LU§

Abstract. This note is devoted to studying the regularity conditions of the mild solution (u,B) to the $3D$ incompressible MHD equations. More precisely, for the $3D$ incompressible MHD equations, [He and Xin, J. Diff. Eqs., 213(2):235–254, 2005] (see also [Zhou, Discrete Contin. Dyn. Syst., 12:881–886, 2005]) proved that the velocity field is dominant in the MHD fluids; meanwhile, the effect of the magnetic field B is vague. In this note, we shall establish the regularity criteria for the MHD equations in terms of $\int_0^{T^*}\|u_3\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}(\mathbb{R}^3)}+\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|^p_{\dot{H}^{1+\frac{2}{p}+\delta}(\mathbb{R}^3)}\,\mathrm{d}s<\infty$, with $p\in(2,\infty)$, $\delta=3(\frac{1}{r}-\frac{1}{2})>0$, here r sufficiently close to 2. This result follows along the lines of [Chemin and Zhang, Ann. Sci. *Éc* Norm Supér, 49:131-167, 2016], [Chemin et al., Arch. Ration Mech. Anal., 224(3):871–905, 2017] and [Han et al., Arch. Ration. Mech. Anal., 231:939–970, 2019], which partially improved the works of [Yamazaki, Bull. Sci. Math., 140:575–614, 2016] and [Liu, J. Diff. Eqs., 260:6989–7019, 2016].

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1. Introduction

MHD system describes the motion of an electrically conducting fluid in the presence of the magnetic field. The $3D$ incompressible MHD system reads as

$$\begin{cases} u_t+(u\cdot\nabla)u-\Delta u+\nabla P=(B\cdot\nabla)B, & \\ B_t+(u\cdot\nabla)B-\Delta B=(B\cdot\nabla)u, & x\in\mathbb{R}^3, t>0,\\ \nabla\cdot u=\nabla\cdot B=0, & \\ (u,B)|_{t=0}=(u_0,B_0), & \end{cases} \tag{1.1}$$

where $u=(u_1,u_2,u_3)$ represents the fluid velocity field, $B=(B_1,B_2,B_3)$ denotes the magnetic field and P corresponds to the pressure.

If we take $B\equiv 0$, then the system (1.1) degenerates to the famous incompressible Navier-Stokes (N-S) equations. For the $3D$ incompressible N-S equations, whether the global solution exists or blows up in finite time is one of the most famous open problems.

To make our result clearer, we recall some relevant results about the incompressible N-S equations beforehand. In [16], Leray proved the existence (but not the uniqueness) of global weak solution to the N-S equations with initial data in $L^2(\mathbb{R}^3)$. After that, there are huge literatures to study the uniqueness of the weak solution. One of the most famous criteria is the so-called Prodi-Ladyzhenskaya-Serrin criteria (Ladyzhenskaya [17], Prodi [21] and Serrin [22]). That is, for $T^*>0$, if the weak solution of the incompressible N-S equations $u\in L^p(0,T^*;L^q(\mathbb{R}^n))$ exists with the pair (p,q) satisfying

$$\frac{2}{p}+\frac{n}{q}\le 1,\quad q\in(n,+\infty], \tag{1.2}$$

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†Department of Mathematics, JiNan University, Guangzhou, 510632, P.R. China (xiangxiangguo123@stu2016.jnu.edu.cn).

‡Department of Mathematics, JiNan University, Guangzhou, 510632, P.R. China (duyidy@jnu.edu.cn).

§School of Mathematics Science, Fudan University, Shanghai, P.R. China (17110180024@fudan.edu.cn).

then the weak solution should be unique. For the borderline case $\frac{2}{p}+\frac{n}{q}=1$, it is much subtle, see Escauriaza-Seregin-Sverak [6] for details.

Here, when $\frac{2}{p}+\frac{n}{q}=1$, the space $L^p(0,T^*;L^q(\mathbb{R}^n))$ is scaling invariant for the incompressible N-S equations. The scaling invariant space reveals the fundamental physical characteristics of the fluids, which means that if a pair of functions $(u(x,t),P(x,t))$ solve the incompressible N-S equations, then

$$(u_\lambda,P_\lambda)(x,t)=(\lambda u(\lambda x,\lambda^2 t),\lambda^2 P(\lambda x,\lambda^2 t)) \tag{1.3}$$

is also a pair of solutions to the N-S equations with initial data $(u_\lambda(x,0),P_\lambda(x,0))=(\lambda u_0(\lambda x),\lambda^2 P_0(\lambda x))$.

Recently, Chemin and Zhang [3] proved the regularity criteria of the velocity field u for the $3D$ incompressible N-S equations as $\int_0^{T^*}\|u_3\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}\mathrm{d}s<\infty$ with $p\in(4,6)$. This interesting result is scaling invariant. Subsequently, the work of [3] has been extended by [4] and [9] to $4\le p<\infty$ and $2\le p<\infty$ respectively.

Since the incompressible N-S equations and the incompressible MHD equations share a similar nonlinear structure, the MHD equations has an analogous difficulty as to the N-S equations. For instance, Duvaut and Lions [5] constructed a class of global weak solutions with finite energy. For the $2D$ case in [23], the smoothness and uniqueness of classical solutions have been proved. However, whether smooth solutions of the $3D$ MHD equations with smooth large data break down in finite time remains quite open. In [2], Caflisch et al. extended the well known result of Beale-Kato-Majda [1] for the incompressible Euler equations to the $3D$ ideal MHD Equations (1.1). More accurately, they proved that if the smooth solution (u,B) satisfies the condition

$$\int_0^{T^*}\|\nabla\times u\|_{L^\infty(\mathbb{R}^3)}+\|\nabla\times B\|_{L^\infty(\mathbb{R}^3)}\mathrm{d}s<\infty, \tag{1.4}$$

then the solution (u,B) can be extended beyond T^*.

Subsequently, for the $3D$ MHD Equations (1.1), He-Xin [8] proved that the weak solution is globally regular when $u\in L^p(0,T;L^q(\mathbb{R}^3))$ with $\frac{2}{p}+\frac{3}{q}\le 1$, $q>3$; simultaneously Zhou [28,29] obtained a similar result. These results imply that the velocity field is more significant compared with the magnetic field.

Recently, some researchers are interested in the problem of the so-called "regularity criteria via partial components", which aims at reducing the components in the previous works (see [10–13, 24, 30] and the references therein). Here let us state some results directly related to our work. Yamazaki [25] proved the following regularity criteria in a scaling invariant space as

$$\int_0^{T^*}\|u_3\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}(\mathbb{R}^3)}+\|B\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}(\mathbb{R}^3)}+\|B\|^{r_1}_{L^{p_1}(\mathbb{R}^3)}+\|B\|^{r_2}_{L^{p_2}(\mathbb{R}^3)}\mathrm{d}s<\infty,\ \text{with } p\in(4,6), \tag{1.5}$$

here $\frac{3}{p_1}+\frac{2}{r_1}=1$ and $\frac{3}{p_2}+\frac{2}{r_2}=2$. Then, Liu [18] sightly improved the work of Yamazaki [25] as

$$\int_0^{T^*}\|u_3\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}(\mathbb{R}^3)}+\|B\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}(\mathbb{R}^3)}\mathrm{d}s<\infty, \text{with } p\in[4,\infty). \tag{1.6}$$

Soon after, Jia-Zhou [14] established some regularity criteria for the weak solutions of the $3D$ incompressible MHD equations in terms of one component of the velocity field

u and horizontal components of the magnetic field B. The result of Jia-Zhou [14] is not a scaling-invariant case.

In this note, illustrated by the works of [3,4,9,14,28], we shall present a regularity criteria on one component of the velocity field u and horizontal components of the magnetic field B with $p\in(2,\infty)$ for the MHD Equations (1.1). Our work here is at the scaling-invariant level and the range of p is larger than the range of p discussed in the works of [18,25]. What's more, our result improved the works of [18,25] to some extent.

Before stating our result, we give the following notations beforehand.

$$\nabla=(\partial_x,\partial_y,\partial_z)=(\partial_1,\partial_2,\partial_3),\quad \nabla_h=(\partial_1,\partial_2).$$

For $u=(u_1,u_2,u_3)$ and $B=(B_1,B_2,B_3)$, we write

$$u_h=(u_1,u_2),\quad B_h=(B_1,B_2), \tag{1.7}$$

as well as

$$\omega_1=\partial_1u_2-\partial_2u_1,\quad \omega_2=\partial_1B_2-\partial_2B_1. \tag{1.8}$$

For $s\in\mathbb{R}$, $1\le q<\infty$, we denote the semi-norm: $\|f\|_{\dot{W}^{s,q}}\triangleq\||\nabla|^sf\|_{L^q}$. Particularly, we write $\dot{W}^{s,2}\triangleq\dot{H}^s$.

Our main result states as the following:

THEOREM 1.1. *Let* $(u_0,B_0)\in\dot{H}^{\frac{1}{2}}$ *and* (u,B) *be the local solution of* (1.1) *on* $[0,T^*)$ *satisfying*

$$(u,B)\in C([0,T^*);\dot{H}^{\frac{1}{2}})\cap L^2([0,T^*);\dot{H}^{\frac{3}{2}}).$$

Suppose that for a fixed constant $\delta=3(\frac{1}{r}-\frac{1}{2})>0$ *with* r *sufficiently close to* 2 *and satisfying the definition in* (3.1), *there holds*

$$\int_0^{T^*}\|u_3\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}(\mathbb{R}^3)}+\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|^p_{\dot{H}^{1+\frac{2}{p}+\delta}(\mathbb{R}^3)}\mathrm{d}s<\infty,\quad \text{with } p\in(2,\infty), \tag{1.9}$$

then (u,B) *should be regular up to time* T^*.

REMARK 1.1. The local well-posedness for the Equations (1.1) with initial data $(u_0,B_0)\in\dot{H}^{\frac{1}{2}}$ can follow along the lines of Fujita-Kato [7] or Miao-Yuan [20]. Here, for simplicity, we omit the proof.

REMARK 1.2. During the process of proving our main theorem, due to the strong coupling effect between the velocity filed u and the magnetic filed B, it requires careful computations to control the coupling terms and nonlinear terms. For example, to the term $D_5=\int_{\mathbb{R}^3}(B\cdot\nabla)\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x$ in (3.3), by decomposing B in the vertical direction and applying the divergence-free condition, we have $\int_{\mathbb{R}^3}|\partial_3|^{-1}\nabla_h\cdot B_h\partial_3\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x$. The embedding of $\||\partial_3|^{-1}|\nabla_h|B_h\|_{L^\infty_v(\mathbb{R}^1)}$ cannot be bounded by $\||\partial_3|^{-\frac{1}{2}}|\nabla_h|B_h\|_{L^2_v(\mathbb{R}^1)}$. For this reason we use $\||\partial_3|^{-\frac{1}{2}-\delta}|\nabla_h|B_h\|_{L^2_v(\mathbb{R}^1)}$ with $\delta=3(\frac{1}{r}-\frac{1}{2})>0$, where r is sufficiently close to 2 and satisfies the definition in (3.1).

The remaining part of this paper is organized as follows: In Section 2, we shall present some preliminary results. In Section 3 and Section 4, we shall prove the estimates on the horizontal term (ω_1,ω_2) and the vertical term (u_3,B_3) respectively. In Section 5, the main theorem will be proved.

Throughout this paper, we sometimes use the notation $A\lesssim B$ as an equivalent to $A\le CB$ with a uniform constant C. To simplify the notations, we will use $\|\cdot\|_{L^p}$ to denote $\|\cdot\|_{L^p(\mathbb{R}^3)}$.

2. Preliminaries

In our proof, we shall handle the velocity field and the magnetic field in the vertical and horizontal parts respectively. To do this, we give the following propositions beforehand.

PROPOSITION 2.1. *Denote* $\nabla_h^\perp=(-\partial_2,\partial_1)$ *and* $\Delta_h=\partial_1^2+\partial_2^2$. (ω_1,ω_2) *is defined in* (1.8), *then there holds*

$$\begin{cases} u_h=(u_1,u_2)=\nabla_h^\perp\Delta_h^{-1}\omega_1-\nabla_h\Delta_h^{-1}\partial_3u_3, \\ B_h=(B_1,B_2)=\nabla_h^\perp\Delta_h^{-1}\omega_2-\nabla_h\Delta_h^{-1}\partial_3B_3, \end{cases} \tag{2.1}$$

with

$$\begin{aligned} &(\omega_1)_t-\Delta\omega_1+(u\cdot\nabla)\omega_1-(B\cdot\nabla)\omega_2 \\ =&\nabla_h^\perp B_3\cdot\partial_3B_h-\nabla_h^\perp u_3\cdot\partial_3u_h+\partial_3u_3\omega_1-\partial_3B_3\omega_2, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &(\omega_2)_t-\Delta\omega_2+(u\cdot\nabla)\omega_2-(B\cdot\nabla)\omega_1 \\ =&\nabla_h^\perp B_3\cdot\partial_3u_h-\nabla_h^\perp u_3\cdot\partial_3B_h-\nabla_h^\perp\cdot(u_h\cdot\nabla_h)B_h+\nabla_h^\perp\cdot(B_h\cdot\nabla_h)u_h. \end{aligned} \tag{2.3}$$

Proof. See for example in [3] and [26]. □

We shall also need the following well-known interpolation inequality.

PROPOSITION 2.2 (Gagliardo-Nirenberg-Sobolev Inequality). *Let* j,m *be any integers satisfying* $0\le j<m$. *Take* $q\le q_1,q_2\le\infty$, $\theta\ge1$ *and* $\frac{j}{m}\le a\le1$, *such that*

$$\frac{1}{\theta}-\frac{j}{n}=a\left(\frac{1}{q_2}-\frac{m}{n}\right)+(1-a)\frac{1}{q_1}. \tag{2.4}$$

Then for any $u\in W^{m,q_2}(\mathbb{R}^n)\cap L^{q_1}(\mathbb{R}^n)$, *there exists a positive constant* C, *such that the following inequality holds*

$$\|\nabla^j u\|_{L^\theta(\mathbb{R}^n)}\le C\|\nabla^m u\|_{L^{q_2}(\mathbb{R}^n)}^a\|u\|_{L^{q_1}(\mathbb{R}^n)}^{1-a}. \tag{2.5}$$

Proof. See for example in [27]. □

Let us recall the definition of homogenous Littlewood-Paley decomposition which we will use to prove our results.

Take ϕ_0 to be a radial function in $C_c^\infty(\mathbb{R}^n)$ and satisfy

$$0\le\phi_0\le1,\quad \phi_0(\xi)=1\quad\text{for}\quad |\xi|\le1,\quad \phi_0(\xi)=0\quad\text{for}\quad |\xi|\ge\frac{7}{6}.$$

Let $\phi(\xi)=\phi_0(\xi)-\phi_0(2\xi)$ which is supported in $\frac{1}{2}\le|\xi|\le\frac{7}{6}$. For any $f\in\mathcal{S}(\mathbb{R}^n)$, $j\in\mathbb{Z}$, define

$$\begin{aligned} &\widehat{P_{\le j}f}(\xi)=\phi_0(2^{-j}\xi)\hat f(\xi), \\ &\widehat{P_jf}(\xi)=\phi(2^{-j}\xi)\hat f(\xi),\quad \xi\in\mathbb{R}^n. \end{aligned}$$

We will also denote $P_{>j}=I-P_{\le j}$ (I is the identity operator) for any $-\infty<a<b<\infty$ and $P_{[a,b]}=\sum_{a\le j\le b}P_j$. For convenience, we apply the notations as $f_j=P_jf$, $f_{\le j}=P_{\le j}f$

and $f_{a\le\cdot\le b}=\sum_{j=a}^{b} f_j$. By using the support property of ϕ, it is obvious that $P_jP_{j'}=0$ when $|j-j'|>1$.

We have the following trilinear paraproduct decomposition as well.

LEMMA 2.1. *For any $f,g,h\in\mathcal{S}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^3} fgh\,\mathrm{d}x=\sum_j\int_{\mathbb{R}^3} f_jg_{[j-3,j+3]}h_{[j-10,j+5]}+f_jg_{[j-3,j+3]}h_{<j-10}\\ +f_jg_{<j-3}h_{[j-2,j+2]}+f_{<j-3}g_jh_{[j-2,j+2]}\,\mathrm{d}x.$$

Proof. See for example in [9]. □

To prove our main results, it is sufficient to give the H^1 bounds for (u,B).

PROPOSITION 2.3. *Let $(u,B)\in L^\infty(0,T;\dot H^{\frac12})\cap L^2(0,T;\dot H^{\frac32})$ be the local solution to the system (1.1), (ω_1,ω_2) is defined in (1.8). When $2<p\le 4$, write*

$$N(T)=\int_0^T\|u_3\|^p_{\dot H^{\frac12+\frac2p}}+\|B_h\|^p_{\dot H^{\frac12+\frac2p}}\,\mathrm{d}s+\int_0^T\|\omega_1\|^p_{\dot H^{-\frac12+\frac2p}}\,\mathrm{d}s,$$

when $4<p<\infty$, we replace $N(T)$ by

$$\widetilde N(T)=\int_0^T\|u_3\|^p_{\dot H^{\frac12+\frac2p}}+\|B_h\|^p_{\dot H^{\frac12+\frac2p}}\,\mathrm{d}s+T\cdot(1+\sup_{0\le t\le T}\|\omega_1\|_{L^r})^p,$$

where r satisfies $\frac12<\frac1r<\frac23(1-\frac1p)$. Then we have

$$\sup_{0\le t\le T}(\|u\|_{\dot H^1}+\|B\|_{\dot H^1})+\|\nabla u\|_{L^2([0,T],\dot H^1)}+\|\nabla B\|_{L^2([0,T],\dot H^1)}\lesssim e^{const\cdot N(T)}. \tag{2.6}$$

For $4<p<\infty$, we replace $N(T)$ with $\widetilde N(T)$.

Proof. By the standard energy estimates, we obtain

$$\begin{aligned}&\frac12\frac{d}{dt}(\|\nabla u\|^2_{L^2}+\|\nabla B\|^2_{L^2})+\|\Delta u\|^2_{L^2}+\|\Delta B\|^2_{L^2}\\ &=\int_{\mathbb{R}^3}(\partial_kB\cdot\nabla)B\cdot\partial_ku\,\mathrm{d}x-\int_{\mathbb{R}^3}(\partial_ku\cdot\nabla)u\cdot\partial_ku\,\mathrm{d}x+\int_{\mathbb{R}^3}(\partial_kB\cdot\nabla)u\cdot\partial_kB\,\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^3}(\partial_ku\cdot\nabla)B\cdot\partial_kB\,\mathrm{d}x\triangleq\sum_{i=1}^4 I_i.\end{aligned}$$

The structure of the above four terms are similar, therefore we just give details of I_1.

$$I_1=\int_{\mathbb{R}^3}\partial_kB_3\partial_3B\cdot\partial_ku\,\mathrm{d}x+\int_{\mathbb{R}^3}\partial_kB_h\partial_hB_3\partial_ku_3\,\mathrm{d}x+\int_{\mathbb{R}^3}\partial_kB_h\partial_hB_{\tilde h}\partial_ku_{\tilde h}\,\mathrm{d}x\triangleq\sum_{i=1}^3 K_i.$$

Here we used the Einstein's convention over repeated indices. The summation over k is from 1 to 3 and the summations over h and $\tilde h$ are from 1 to 2.

We first consider the terms K_2 and K_3. Applying the Sobolev embedding and the Gagliardo-Nirenberg inequalities, when $2<p\le 4$, we have

$$|K_2|+|K_3|\lesssim\|\nabla B_h\|_{L^{\frac{3p}{2p-2}}}\|\nabla B\|_{L^{\frac{6p}{p+4}}}\|\nabla u\|_{L^6}$$

$$\lesssim \|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla B\|_{L^2}^{\frac{2}{p}} \|\Delta B\|_{L^2}^{1-\frac{2}{p}} \|\Delta u\|_{L^2}. \tag{2.7}$$

When $4<p<\infty$, we use the fractional Leibniz rule (see [15]) and obtain

$$\begin{aligned} |K_2|+|K_3| &\lesssim \||\nabla|^{-\frac{1}{2}+\frac{2}{p}}\partial_k B_h\|_{L^2} \||\nabla|^{\frac{1}{2}-\frac{2}{p}}(\partial_k B\cdot\partial_k u)\|_{L^2} \\ &\lesssim \|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}\cdot(\||\nabla|^{\frac{1}{2}-\frac{2}{p}}\partial_k B\|_{L^3}\|\nabla u\|_{L^6}+\|\nabla B\|_{L^6}\||\nabla|^{\frac{1}{2}-\frac{2}{p}}\partial_k u\|_{L^3}) \\ &\lesssim \|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}\cdot(\|\nabla B\|_{L^2}^{\frac{2}{p}}\|\Delta B\|_{L^2}^{1-\frac{2}{p}}\|\Delta u\|_{L^2}+\|\nabla u\|_{L^2}^{\frac{2}{p}}\|\Delta u\|_{L^2}^{1-\frac{2}{p}}\|\Delta B\|_{L^2}). \end{aligned} \tag{2.8}$$

For the term K_1, since

$$\begin{aligned} K_1 &= \int_{\mathbb{R}^3} \partial_3 B_3 \partial_3 B\cdot\partial_3 u \mathrm{d}x + \int_{\mathbb{R}^3} \partial_h B_3 \partial_3 B \cdot \partial_h u \mathrm{d}x \\ &= \int_{\mathbb{R}^3} \partial_3 B_3 \partial_3 B\cdot\partial_3 u \mathrm{d}x + \int_{\mathbb{R}^3} \partial_h B_3 \partial_3 B_3 \partial_h u_3 \mathrm{d}x + \int_{\mathbb{R}^3} \partial_h B_3 \partial_3 B_h \partial_h u_h \mathrm{d}x \\ &= -\int_{\mathbb{R}^3} \nabla_h\cdot B_h \partial_3 B \partial_3 u \mathrm{d}x - \int_{\mathbb{R}^3} \partial_h B_3 \nabla_h\cdot B_h \partial_h u_3 \mathrm{d}x + \int_{\mathbb{R}^3} \partial_h B_3 \partial_3 B_h \partial_h u_h \,\mathrm{d}x, \end{aligned} \tag{2.9}$$

here we use the divergence-free condition $\partial_3 B_3 = -\nabla_h\cdot B_h$ and the notation $\partial_h=\partial_1$ or ∂_2. Thus we can estimate K_1 similarly as in (2.7) and (2.8). Then we get the bound of I_1 as (2.8). Analogously, we get the bounds of I_2, I_3 and I_4 as

$$I_2 \lesssim (\|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}+\|\omega_1\|_{\dot{H}^{-\frac{1}{2}+\frac{2}{p}}})\cdot\|\nabla u\|_{L^2}^{\frac{2}{p}}\|\Delta u\|_{L^2}^{2-\frac{2}{p}}, \tag{2.10}$$

and

$$\begin{aligned} I_3 \lesssim &(\|\omega_1\|_{\dot{H}^{-\frac{1}{2}+\frac{2}{p}}}+\|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}})\cdot\|\nabla B\|_{L^2}^{\frac{2}{p}}\|\Delta B\|_{L^2}^{2-\frac{2}{p}}+\|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \\ &\cdot(\|\nabla B\|_{L^2}^{\frac{2}{p}}\|\Delta B\|_{L^2}^{1-\frac{2}{p}}\|\Delta u\|_{L^2}+\|\nabla u\|_{L^2}^{\frac{2}{p}}\|\Delta u\|_{L^2}^{1-\frac{2}{p}}\|\Delta B\|_{L^2}), \end{aligned} \tag{2.11}$$

as well as

$$\begin{aligned} I_4 \lesssim &(\|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}+\|\omega_1\|_{\dot{H}^{-\frac{1}{2}+\frac{2}{p}}})\cdot\|\nabla B\|_{L^2}^{\frac{2}{p}}\|\Delta B\|_{L^2}^{2-\frac{2}{p}}+\|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \\ &\cdot(\|\nabla B\|_{L^2}^{\frac{2}{p}}\|\Delta B\|_{L^2}^{1-\frac{2}{p}}\|\Delta u\|_{L^2}+\|\nabla u\|_{L^2}^{\frac{2}{p}}\|\Delta u\|_{L^2}^{1-\frac{2}{p}}\|\Delta B\|_{L^2}). \end{aligned} \tag{2.12}$$

Summing up the above estimates from (2.7)- (2.12), then by using the Young's inequalities and the Grönwall's inequalities, we finish the proof. □

3. The estimate of the horizontal terms

Based on the Proposition 2.3, in order to prove our main result, it is sufficient to get the bound for $\int_0^T \|\omega_1\|_{\dot{H}^{-\frac{1}{2}+\frac{2}{p}}}^p \mathrm{d}s$.

When $2<p\le 4$ (that is $-\frac{1}{2}+\frac{2}{p}\ge 0$), we can replace $\|\omega_1\|_{\dot{H}^{-\frac{1}{2}+\frac{2}{p}}}$ by the weak norm $\||\nabla_h|^{-\frac{1}{2}+\frac{2}{p}}\omega_1\|_{L^2}$, then by using the Gagliardo-Nirenberg inequalities, we have $\frac{2}{3}(1-\frac{1}{p})<\frac{1}{r}<\frac{1}{3}(3-\frac{2}{p})$, there holds

$$\int_0^T \|\omega_1\|_{\dot{H}^{-\frac{1}{2}+\frac{2}{p}}}^p \mathrm{d}s \le C\int_0^T \|\omega_1\|_{L^r}^{(3-\frac{3}{r}-\frac{2}{p})p}\|\nabla\omega_1\|_{L^r}^{(\frac{2}{p}+\frac{3}{r}-2)p}\mathrm{d}s.$$

Recalling that $2<p<4$, the above estimates will hold for $\frac{1}{2}<\frac{1}{r}<\frac{2}{3}$.

When $4<p<\infty$ (that is $-\frac{1}{2}+\frac{2}{p}<0$), we obtain

$$\int_0^T \|\omega_1\|^p_{\dot{H}^{-\frac{1}{2}+\frac{2}{p}}}\mathrm{d}s\le T\cdot(1+\sup_{0\le t\le T}\|\omega_1\|_{L^r})^p,$$

where r satisfies $\frac{1}{2}<\frac{1}{r}<\frac{2}{3}(1-\frac{1}{p})$.

Therefore, it is sufficient to estimate $\sup_{0\le t\le T}\|\omega_1(t)\|_{L^r}$ with $\frac{1}{2}<\frac{1}{r}<\frac{2}{3}$ and $\frac{1}{2}<\frac{1}{r}<\frac{2}{3}(1-\frac{1}{p})$ for $2<p<4$ and $4<p<\infty$ respectively.

Proposition 3.1. *Let (ω_1,ω_2) be as defined in (1.8) and $2<p<\infty$, take $\delta=3(\frac{1}{r}-\frac{1}{2})$ with r sufficiently close to 2 and satisfying*

$$\begin{cases}\frac{1}{2}<\frac{1}{r}<min\{\frac{2}{p},\frac{5}{8},\frac{5}{6}-\frac{2}{3p}\}, & with \quad 2<p<4;\\ \frac{1}{2}<\frac{1}{r}<\frac{5}{8}, & with \quad p=4;\\ \frac{1}{2}<\frac{1}{r}<min\{\frac{2}{3}(1-\frac{1}{p}),\frac{5}{8},\frac{1}{2}+\frac{1}{p}\}, & with \quad 4<p<\infty.\end{cases} \tag{3.1}$$

Then there holds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\||\omega_1|^{\frac{r}{2}}\|^{\frac{4}{r}}_{L^2}+\||\omega_2|^{\frac{r}{2}}\|^{\frac{4}{r}}_{L^2})+\frac{4(r-1)}{r^2}(\|\nabla\omega_1\|^2_{L^r}+\|\nabla\omega_2\|^2_{L^r})\\ \le& C(\|u_3\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}+\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|^p_{\dot{H}^{1+\frac{2}{p}+\delta}})\cdot(\||\omega_1|^{\frac{r}{2}}\|^{\frac{4}{r}}_{L^2}+\||\omega_2|^{\frac{r}{2}}\|^{\frac{4}{r}}_{L^2})\\ &+\frac{1}{10}(\||\nabla_h|^{-\delta}\partial_k^2u_3\|^2_{L^2}+\||\nabla_h|^{-\delta}\partial_k^2B_3\|^2_{L^2}).\end{aligned} \tag{3.2}$$

Proof. Here and hereafter, we denote $f(x_1,x_2,x_3)\triangleq f(x_h,x_3)$ and the mixed norm for $1\le p,q<\infty$ as

$$\Big\|\,\|f(x_h,x_3)\|_{L^p_{x_h}(\mathbb{R}^2)}\Big\|_{L^q_{x_3}(\mathbb{R}^1)}=\Big\|\,\|f\|_{L^p_h}\Big\|_{L^q_v};$$

and

$$\Big\|\,\|f(x_h,x_3)\|_{L^q_{x_3}(\mathbb{R}^1)}\Big\|_{L^p_{x_h}(\mathbb{R}^2)}=\Big\|\,\|f\|_{L^q_v}\Big\|_{L^p_h}.$$

By the standard L^r estimate to (2.2) and (2.3) respectively, we yield

$$\begin{aligned}&\frac{1}{r}\frac{d}{dt}\|\omega_1\|^r_{L^r}+\frac{4(r-1)}{r^2}\|\nabla|\omega_1|^{\frac{r}{2}}\|^2_{L^2}\\ =&\int_{\mathbb{R}^3}\partial_3u_3|\omega_1|^r\mathrm{d}x-\int_{\mathbb{R}^3}\nabla_h^\perp u_3\cdot\partial_3u_h\omega_1|\omega_1|^{r-2}\mathrm{d}x\\ &-\int_{\mathbb{R}^3}\partial_3B_3\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x+\int_{\mathbb{R}^3}\nabla_h^\perp B_3\cdot\partial_3B_h\omega_1|\omega_1|^{r-2}\mathrm{d}x\\ &+\int_{\mathbb{R}^3}(B\cdot\nabla)\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x\triangleq\sum_{i=1}^5D_i,\end{aligned} \tag{3.3}$$

and

$$\frac{1}{r}\frac{d}{dt}\|\omega_2\|^r_{L^r}+\frac{4(r-1)}{r^2}\|\nabla|\omega_2|^{\frac{r}{2}}\|^2_{L^2}$$

$$
\begin{aligned}
&=\int_{\mathbb{R}^3}\nabla_h^\perp\cdot(B_h\cdot\nabla_h)u_h\omega_2|\omega_2|^{r-2}\mathrm{d}x-\int_{\mathbb{R}^3}\nabla_h^\perp\cdot(u_h\cdot\nabla_h)B_h\omega_2|\omega_2|^{r-2}\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}\nabla_h^\perp u_3\cdot\partial_3B_h\omega_2|\omega_2|^{r-2}\mathrm{d}x+\int_{\mathbb{R}^3}\nabla_h^\perp B_3\cdot\partial_3u_h\omega_2|\omega_2|^{r-2}\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}(B\cdot\nabla)\omega_1\omega_2|\omega_2|^{r-2}\mathrm{d}x\triangleq\sum_{i=6}^{10}D_i.
\end{aligned}\tag{3.4}
$$

First of all, we estimate the term D_1. By denoting $g=|\omega_1|^{\frac{r}{2}}$ and applying integration by parts, when $2<p<\infty$, we get the bound by

$$
|D_1|\lesssim\|u_3\|_{L^{\frac{3p}{p-2}}}\|\partial_3g\|_{L^2}\|g\|_{L^{\frac{6p}{p+4}}}\lesssim\|u_3\|_{\dot{H}^{\frac12+\frac2p}}\||\omega_1|^{\frac r2}\|_{L^2}^{\frac2p}\|\nabla|\omega_1|^{\frac r2}\|_{L^2}^{2-\frac2p}.\tag{3.5}
$$

Then, we handle the term D_2. By applying $(2.1)_1$, D_2 can be rewritten as

$$
\begin{aligned}
D_2=&\int_{\mathbb{R}^3}\nabla_h^\perp u_3\cdot\nabla_h\Delta_h^{-1}\partial_3^2u_3\omega_1|\omega_1|^{r-2}\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\nabla_h^\perp u_3\cdot\nabla_h^\perp\Delta_h^{-1}\partial_3\omega_1\omega_1|\omega_1|^{r-2}\mathrm{d}x\triangleq D_{21}+D_{22}.
\end{aligned}\tag{3.6}
$$

In the following estimates, we use the homogeneous horizontal Besov norm $\|\cdot\|_{\dot{B}^{h,s}_{p,q}}$ for a function $f(x_h,x_3)=f(x_1,x_2,x_3)$ as

$$
\|f(\cdot,x_3)\|_{\dot{B}^{h,s}_{p,q}}:=\Big\|\,(2^{js}\|P_j^hf(\cdot,x_3)\|_{L_h^p})\,\Big\|_{l_j^q},\tag{3.7}
$$

where $s\in\mathbb{R}$, $1\le p,q\le\infty$, and P_j^h is the Littlewood-Paley projection operator in the horizontal variable.

When $2<p<4$, applying the Hölder's inequalities and the Sobolev embedding, one can deduce that

$$
\begin{aligned}
|D_{21}|\lesssim&\Big\|\,\||\nabla_h|u_3\|_{L_h^{\frac{6}{3-\delta}}}\Big\|_{L_v^{\frac{2pr}{pr+2p-4r}}}\Big\|\,\||\nabla_h|^{-1}\partial_3^2u_3\|_{L_h^{\frac2\delta}}\Big\|_{L_v^2}\\
&\cdot\Big\|\,\||\omega_1|^{r-1}\|_{L_h^{\frac{r}{r-1}}}\Big\|_{L_v^{\frac{pr}{2r-p}}}\lesssim\|u_3\|_{\dot{H}^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_3^2u_3\|_{L^2}\Big\|\,\||\omega_1|^{\frac r2}\|_{L_v^{\frac{2p(r-1)}{2r-p}}}\Big\|_{L_h^2}^{\frac{2(r-1)}{r}}\\
\lesssim&\|u_3\|_{\dot{H}^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_3^2u_3\|_{L^2}\||\omega_1|^{\frac r2}\|_{L^2}^{1-\frac2r+\frac2p}\|\nabla|\omega_1|^{\frac r2}\|_{L^2}^{1-\frac2p}.
\end{aligned}\tag{3.8}
$$

In the above, we take r sufficiently close to 2 and satisfying

$$
\frac12<\frac1r<min\{\frac2p,\frac23\}.
$$

When $p=4$, we have

$$
\begin{aligned}
|D_{21}|\lesssim&\||\nabla_h|u_3\|_{L^2}\Big\|\,\||\nabla_h|^{-1}\partial_3^2u_3\|_{L_h^{\frac2\delta}}\Big\|_{L_v^2}\Big\|\,\||\omega_1|^{r-1}\|_{L_h^{\frac{2}{1-\delta}}}\Big\|_{L_v^\infty}\\
\lesssim&\|u_3\|_{\dot{H}^1}\||\nabla_h|^{-\delta}\partial_3^2u_3\|_{L^2}\Big\|\,\||\omega_1|^{\frac r2}\|_{L_h^{\frac{8(r-1)}{5r-6}}}\Big\|_{L_v^\infty}^{\frac{2(r-1)}{r}}\\
\lesssim&\|u_3\|_{\dot{H}^1}\||\nabla_h|^{-\delta}\partial_3^2u_3\|_{L^2}\||\omega_1|^{\frac r2}\|_{L^2}^{\frac32-\frac2r}\|\nabla|\omega_1|^{\frac r2}\|_{L^2}^{\frac12},
\end{aligned}\tag{3.9}
$$

where for $\Big\| \||\omega_1|^{\frac{r}{2}}\|_{L_h^{\frac{8(r-1)}{5r-6}}} \Big\|_{L_v^\infty}$, we used the interpolation inequalities to get

$$\begin{aligned}
&\Big\| \||\omega_1|^{\frac{r}{2}}\|_{L_h^{\frac{8(r-1)}{5r-6}}} \Big\|_{L_v^\infty} \lesssim \Big\| \||\nabla_h|^{\frac{2-r}{4(r-1)}}|\omega_1|^{\frac{r}{2}}\|_{L_h^2} \Big\|_{L_v^\infty} \\
&\lesssim \Big\| \||\nabla_h|^{\frac{2-r}{4(r-1)}}|\omega_1|^{\frac{r}{2}}\|_{L_v^2}^{\frac{3r-4}{5r-6}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L_v^2}^{\frac{2r-2}{5r-6}} \Big\|_{L_h^2} \\
&\lesssim \||\nabla_h|^{\frac{2-r}{4(r-1)}}|\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{3r-4}{5r-6}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{2r-2}{5r-6}} \\
&\lesssim \||\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{3r-4}{4(r-1)}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{r}{4(r-1)}}. \qquad (3.10)
\end{aligned}$$

In the estimates of (3.9) and (3.10), we take r sufficiently close to 2 and satisfying $\frac{1}{2} < \frac{1}{r} < \frac{3}{4}$.

When $4 < p < \infty$, applying the fractional Leibniz rule [15] and the Hölder's inequalities, we get

$$\begin{aligned}
|D_{21}| &\lesssim \Big\| \||\nabla_h|^{\frac{1}{2}+\frac{2}{p}-\frac{\delta}{3}} u_3\|_{L_h^2} \Big\|_{L_v^{\frac{6}{3-2\delta}}} \\
&\cdot \Big(\Big\| \||\nabla_h|^{-1}\partial_3^2 u_3\|_{L_h^{\frac{2}{\delta}}} \Big\|_{L_v^2} \Big\| \||\nabla_h|^{\frac{1}{2}-\frac{2}{p}+\frac{\delta}{3}}(\omega_1|\omega_1|^{r-2})\|_{L_h^{\frac{2}{1-\delta}}} \Big\|_{L_v^{\frac{3}{\delta}}} \\
&+ \Big\| \||\nabla_h|^{-\frac{1}{2}-\frac{2}{p}+\frac{\delta}{3}}\partial_3^2 u_3\|_{L_h^{\frac{12p}{3p-12+8p\delta}}} \Big\|_{L_v^2} \Big\| \||\omega_1|^{r-1}\|_{L_h^{\frac{12p}{3p+12-8p\delta}}} \Big\|_{L_v^{\frac{3}{\delta}}} \Big) \\
&\lesssim \|u_3\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}} \||\nabla_h|^{-\delta}\partial_3^2 u_3\|_{L^2} \Big(\Big\| \||\nabla_h|^{\frac{\frac{1}{2}-\frac{2}{p}+\frac{2\delta}{3}}{r-1}}\omega_1\|_{L_h^r}^{r-1} + \|\omega_1\|^{r-1}_{\dot B^{h,\frac{\frac{1}{2}-\frac{2}{p}+\frac{2\delta}{3}}{r-1}}_{r,2(r-1)}} \Big\|_{L_v^{\frac{3(r-1)}{\delta}}} \Big) \\
&\lesssim \|u_3\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}} \||\nabla_h|^{-\delta}\partial_3^2 u_3\|_{L^2} \||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}. \qquad (3.11)
\end{aligned}$$

Again, by using the interpolation inequalities, we need r sufficiently close to 2 and satisfying

$$\frac{1}{2} < \frac{1}{r} < min\{\frac{5}{8}+\frac{1}{2p}, \frac{2}{3}(1-\frac{1}{p})\}, \quad (4 < p < \infty).$$

As to the term D_{22}, when $2 < p < 4$, by the Hölder's inequalities and the Sobolev embedding, we have

$$\begin{aligned}
|D_{22}| &\lesssim \Big\| \||\nabla_h| u_3\|_{L_h^2} \Big\|_{L_v^{\frac{p}{p-2}}} \Big\| \||\nabla_h|^{-1}\partial_3\omega_1\|_{L_h^{\frac{3}{\delta}}} \Big\|_{L_v^r} \Big\| \||\omega_1|^{r-1}\|_{L_h^{\frac{r}{r-1}}} \Big\|_{L_v^{\frac{pr}{2r-p}}} \\
&\lesssim \|u_3\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}} \|\partial_3\omega_1\|_{L^r} \Big\| \||\omega_1|^{\frac{r}{2}}\|_{L_h^2}^{\frac{2(r-1)}{r}} \Big\|_{L_v^{\frac{2p(r-1)}{2r-p}}} \\
&\lesssim \|u_3\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}} \||\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{2}{p}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{2-\frac{2}{p}}, \qquad (3.12)
\end{aligned}$$

here, we need r sufficiently close to 2 and satisfying $\frac{1}{2} < \frac{1}{r} < min\{\frac{2}{p}, \frac{2}{3}\}$.

When $p = 4$, we take r sufficiently close to 2 and satisfying $\frac{1}{2} < \frac{1}{r} < 1$, we get

$$\begin{aligned}
|D_{22}| &\lesssim \||\nabla_h| u_3\|_{L^2} \Big\| \||\nabla_h|^{-1}\partial_3\omega_1\|_{L_h^{\frac{3}{\delta}}} \||\omega_1|^{r-1}\|_{L_h^{\frac{r}{r-1}}} \Big\|_{L_v^2} \\
&\lesssim \|\partial_2 u_3\|_{L^2} \Big\| \|\partial_3|\omega_1|^{\frac{r}{2}}\|_{L_h^2} \||\omega_1|^{\frac{r}{2}}\|_{L_h^2} \Big\|_{L_v^2}
\end{aligned}$$

$$\lesssim \|u_3\|_{\dot{H}^1}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}\Big\|\,\||\omega_1|^{\frac{r}{2}}\|_{L_h^2}\Big\|_{L_v^\infty}$$
$$\lesssim \|u_3\|_{\dot{H}^1}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{3}{2}}\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{1}{2}}. \tag{3.13}$$

when $4<p<\infty$, we have

$$\begin{aligned}|D_{22}|\lesssim&\Big\|\,\||\nabla_h|^{\frac{2}{p}+\frac{\delta}{3}}u_3\|_{L_h^2}\Big\|_{L_v^{\frac{3}{\delta}}}\\&\cdot\Big(\Big\|\,\||\nabla_h|^{-\frac{2}{p}-\frac{\delta}{3}}\partial_3\omega_1\|_{L_h^{\frac{6pr}{6p-6r-pr\delta}}}\Big\|_{L_v^r}\Big\|\,\||\omega_1|^{r-1}\|_{L_h^{\frac{6p}{6-p\delta}}}\Big\|_{L_v^{\frac{6}{3-4\delta}}}\\&+\Big\|\,\||\nabla_h|^{-1}\partial_3\omega_1\|_{L_h^{\frac{3}{\delta}}}\Big\|_{L_v^r}\Big\|\,\||\nabla_h|^{1-\frac{2}{p}-\frac{\delta}{3}}(\omega_1|\omega_1|^{r-2})\|_{L_h^{\frac{r}{r-1}}}\Big\|_{L_v^{\frac{6}{3-4\delta}}}\Big)\\\lesssim&\|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}\|\nabla\omega_1\|_{L^r}\Big(\Big\|\,\||\nabla_h|^{\frac{1-\frac{2}{p}-\frac{\delta}{3}}{r-1}}\omega_1\|_{L_h^r}^{r-1}+\|\omega_1\|^{r-1}_{\dot{B}^{h,\frac{1-\frac{2}{p}-\frac{\delta}{3}}{r-1}}_{r,2(r-1)}}\Big\|_{L_v^{\frac{6(r-1)}{3-4\delta}}}\Big)\\\lesssim&\|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{2}{p}}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{2-\frac{2}{p}}.\end{aligned} \tag{3.14}$$

Similar to (3.11), here we take r sufficiently close to 2 and satisfying

$$\frac{1}{2}<\frac{1}{r}<min\{\frac{1}{2}+\frac{2}{p},\frac{2}{3}(1-\frac{1}{p})\},\quad(4<p<\infty).$$

Combining the estimates from (3.8)-(3.14), we get the bound of D_2.

Afterwards, using the divergence-free condition $\partial_3B_3=-\nabla_h\cdot B_h$, the terms D_4, D_8 and D_9 can be bounded by a similar process of D_2. We show the results as following:

$$\begin{aligned}|D_4|\lesssim&\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|_{\dot{H}^{1+\frac{2}{p}+\delta}}\||\nabla_h|^{-\delta}\partial_3^2B_3\|_{L^2}\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}\\&+\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|_{\dot{H}^{1+\frac{2}{p}+\delta}}\|\nabla\omega_2\|_{L^r}\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}},\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}|D_8|\lesssim&\|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{\frac{2}{p}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{2-\frac{2}{p}}\\&+\|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}\||\nabla_h|^{-\delta}\partial_3^2B_3\|_{L^2}\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}},\end{aligned} \tag{3.16}$$

as well as

$$\begin{aligned}|D_9|\lesssim&\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|_{\dot{H}^{1+\frac{2}{p}+\delta}}\||\nabla_h|^{-\delta}\partial_3^2u_3\|_{L^2}\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}\\&+\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|_{\dot{H}^{1+\frac{2}{p}+\delta}}\|\nabla\omega_1\|_{L^r}\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}.\end{aligned} \tag{3.17}$$

Subsequently, we consider the term D_3. When $2<p<4$, using the divergence-free condition $\partial_3B_3=-\nabla_h\cdot B_h$ and the Gagliardo-Nirenberg inequalities gives

$$\begin{aligned}|D_3|\lesssim&\Big\|\,\||\nabla_h|B_h\|_{L_h^2}\Big\|_{L_v^{\frac{p}{p-2}}}\Big\|\,\|\omega_2\|_{L_h^{\frac{2r}{2-r}}}\Big\|_{L_v^r}\Big\|\,\||\omega_1|^{r-1}\|_{L_h^{\frac{r}{r-1}}}\Big\|_{L_v^{\frac{pr}{2r-p}}}\\\lesssim&\|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}\|\nabla\omega_2\|_{L^r}\Big\|\,\||\omega_1|^{\frac{r}{2}}\|_{L_h^2}\Big\|_{L_v^{\frac{2p(r-1)}{2r-p}}}^{\frac{2(r-1)}{r}}\\\lesssim&\|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}\|\nabla\omega_2\|_{L^r}\||\nabla|^{\frac{(p-2)r}{2p(r-1)}}|\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{2(r-1)}{r}}\end{aligned}$$

$$\lesssim \|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla\omega_2\|_{L^r} \||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}, \tag{3.18}$$

here, we take r sufficiently close to 2 and satisfying

$$\frac{1}{2} < \frac{1}{r} < min\{\frac{2}{p}, \frac{2}{3}\}, \quad (2<p<4).$$

When $p=4$, we have

$$\begin{aligned} |D_3| &\lesssim \||\nabla_h|B_h\|_{L^2} \|\omega_2\|_{L^{\frac{3r}{3-r}}} \||\omega_1|^{r-1}\|_{L^{\frac{6r}{5r-6}}} \\ &\lesssim \|B_h\|_{\dot{H}^1} \|\nabla\omega_2\|_{L^r} \||\omega_1|^{\frac{r}{2}}\|_{L^{\frac{12(r-1)}{5r-6}}}^{\frac{2(r-1)}{r}} \\ &\lesssim \|B_h\|_{\dot{H}^1} \|\nabla\omega_2\|_{L^r} \||\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{3}{2}-\frac{2}{r}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{1}{2}}, \end{aligned} \tag{3.19}$$

here r is sufficiently close to 2 and satisfies $\frac{1}{2} < \frac{1}{r} < \frac{3}{4}$.

When $4<p<\infty$, there holds

$$\begin{aligned} |D_3| \lesssim & \||\nabla_h|^{\frac{1}{2}+\frac{2}{p}} B_h\|_{L^2} \cdot \Big(\Big\| \||\nabla_h|^{\frac{1}{2}-\frac{2}{p}}\omega_2\|_{L_h^{\frac{6p}{3p-6+2p\delta}}} \Big\|_{L_v^{\frac{3}{\delta}}} \Big\| \||\omega_1|^{r-1}\|_{L_h^{\frac{3p}{3-p\delta}}} \Big\|_{L_v^{\frac{6}{3-2\delta}}} \\ &+ \Big\| \|\omega_2\|_{L_h^{\frac{12}{3+4\delta}}} \Big\|_{L_v^{\frac{3}{\delta}}} \Big\| \||\nabla_h|^{\frac{1}{2}-\frac{2}{p}}(\omega_1|\omega_1|^{r-2})\|_{L_h^{\frac{12}{3-4\delta}}} \Big\|_{L_v^{\frac{6}{3-2\delta}}} \Big) \\ \lesssim & \|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla\omega_2\|_{L^r} \Big\| \|\omega_1\|^{r-1}_{\dot{B}^{h,\frac{1-\frac{2}{p}}{r-1}}_{r,2(r-1)}} \Big\|_{L_v^{\frac{6(r-1)}{3-2\delta}}} \\ \lesssim & \|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla\omega_2\|_{L^r} \||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}, \end{aligned} \tag{3.20}$$

here, by using the Gagliardo-Nirenberg inequalities, we take r sufficiently close to 2 and satisfying

$$\frac{1}{2} < \frac{1}{r} < min\{\frac{1}{2}+\frac{1}{p}, \frac{2}{3}(1-\frac{1}{p})\}, \quad (4<p<\infty).$$

Combining the estimates from (3.18)-(3.20), we get the bound of D_3.

Furthermore, to the term D_5, we need to estimate it in two cases. On the one hand, to the term $|\int_{\mathbb{R}^3} B_3\partial_3\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x|$, when $2<p<4$, applying the divergence-free condition $\partial_3 B_3 = -\nabla_h \cdot B_h$ and the Sobolev embedding, we have

$$\begin{aligned} &|\int_{\mathbb{R}^3} B_3\partial_3\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x| \\ \lesssim & \Big\| \||\partial_3|^{-1}|\nabla_h|B_h\|_{L_h^{\frac{4pr}{5pr-4r-6p}}} \Big\|_{L_v^{\frac{2r}{6-3r}}} \|\nabla\omega_2\|_{L^r} \Big\| \||\omega_1|^{r-1}\|_{L_h^{\frac{4pr}{2p-pr+4r}}} \Big\|_{L_v^{\frac{2r}{5r-8}}} \\ \lesssim & \||\partial_3|^{-\frac{1}{2}-\delta} B_h\|_{\dot{H}^{1+\frac{2}{p}+\delta}} \|\nabla\omega_2\|_{L^r} \Big\| \||\nabla_h|^{\frac{5pr-6p-4r}{4p(r-1)}} |\omega_1|^{\frac{r}{2}}\|_{L_h^2} \Big\|_{L_v^{\frac{4(r-1)}{5r-8}}}^{\frac{2(r-1)}{r}} \\ \lesssim & \||\partial_3|^{-\frac{1}{2}-\delta} B_h\|_{\dot{H}^{1+\frac{2}{p}+\delta}} \|\nabla\omega_2\|_{L^r} \||\nabla|^{\frac{pr-2r}{2p(r-1)}} |\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{2(r-1)}{r}} \\ \lesssim & \||\partial_3|^{-\frac{1}{2}-\delta} B_h\|_{\dot{H}^{1+\frac{2}{p}+\delta}} \|\nabla\omega_2\|_{L^r} \||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}. \end{aligned} \tag{3.21}$$

In the above, there needs

$$\frac{1}{2} < \frac{1}{r} < min\{\frac{5}{8}, \frac{5}{6}-\frac{2}{3p}\}, \quad (2<p<4).$$

When $p=4$, we get

$$
\begin{aligned}
&|\int_{\mathbb{R}^3} B_3\partial_3\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x| \\
\lesssim &\Big\| \||\partial_3|^{-1}|\nabla_h|B_h\|_{L_h^{\frac{4}{1-2\delta}}} \Big\|_{L_v^{\frac{1}{\delta}}} \|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2} \Big\| \||\omega_2|^{1-\frac{r}{2}}|\omega_1|^{r-1}\|_{L_h^{\frac{4}{1+2\delta}}} \Big\|_{L_v^{\frac{2}{1-2\delta}}} \\
\lesssim &\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|_{\dot{H}^{\frac{3}{2}+\delta}}\|\nabla\omega_2\|_{L^r}\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{3}{2}-\frac{2}{r}}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{1}{2}}, \qquad (3.22)
\end{aligned}
$$

here r is sufficiently close to 2 and satisfies $\frac{1}{2}<\frac{1}{r}<\frac{5}{8}$.

When $4<p<\infty$, we have

$$
\begin{aligned}
&|\int_{\mathbb{R}^3} B_3\partial_3\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x| \\
\lesssim &\Big\| \||\partial_3|^{-1}|\nabla_h|B_h\|_{L_h^{\frac{2p}{p-2-p\delta}}} \Big\|_{L_v^{\frac{1}{\delta}}} \|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2} \Big\| \||\omega_2|^{1-\frac{r}{2}}|\omega_1|^{r-1}\|_{L_h^{\frac{2p}{2+p\delta}}} \Big\|_{L_v^{\frac{2}{1-2\delta}}} \\
\lesssim &\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|_{\dot{H}^{1+\frac{2}{p}+\delta}}\|\nabla\omega_2\|_{L^r}\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}, \qquad (3.23)
\end{aligned}
$$

here for $\Big\| \||\omega_2|^{1-\frac{r}{2}}|\omega_1|^{r-1}\|_{L_h^{\frac{2p}{2+p\delta}}} \Big\|_{L_v^{\frac{2}{1-2\delta}}}$, we use the Gagliardo-Nirenberg inequalities to get

$$
\begin{aligned}
&\Big\| \||\omega_2|^{1-\frac{r}{2}}|\omega_1|^{r-1}\|_{L_h^{\frac{2p}{2+p\delta}}} \Big\|_{L_v^{\frac{2}{1-2\delta}}} \lesssim \Big\| \||\omega_2|^{\frac{r}{2}}\|_{L_h^2}^{\frac{2-r}{r}} \||\omega_1|^{\frac{r}{2}}\|_{L_h^{\frac{4p(r-1)}{2r+pr\delta-2p+pr}}}^{\frac{2(r-1)}{r}} \Big\|_{L_v^{\frac{2}{1-2\delta}}} \\
\lesssim &\Big\| \||\omega_2|^{\frac{r}{2}}\|_{L_h^2}^{\frac{2-r}{r}} \Big\|_{L_v^{\frac{2r}{2-r}}} \Big\| \||\omega_1|^{\frac{r}{2}}\|_{L_h^{\frac{4p(r-1)}{2r+pr\delta-2p+pr}}}^{\frac{2(r-1)}{r}} \Big\|_{L_v^{\frac{r}{r-r\delta-1}}} \\
\lesssim &\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{\frac{2-r}{r}} \||\nabla|^{\frac{pr-2r}{2p(r-1)}}|\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{2(r-1)}{r}} \lesssim \||\omega_2|^{\frac{r}{2}}\|_{L^2}^{\frac{2-r}{r}} \||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}. \quad (3.24)
\end{aligned}
$$

In the estimates of(3.23)-(3.24), we need r sufficiently close to 2 and satisfying

$$
\frac{1}{2}<\frac{1}{r}<min\{\frac{5}{8},\frac{2}{3}(1-\frac{1}{p})\}, \quad (4<p<\infty).
$$

On the other hand, to $|\int_{\mathbb{R}^3} B_h\cdot\nabla_h\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x|$, using the Hölder's inequalities and the Gagliardo-Nirenberg inequalities, when $2<p<4$, there holds

$$
\begin{aligned}
&|\int_{\mathbb{R}^3} B_h\cdot\nabla_h\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x| \\
\lesssim &\|B_h\|_{L^{\frac{3p}{p-2}}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}\||\omega_2|^{1-\frac{r}{2}}|\omega_1|^{r-1}\|_{L^{\frac{6p}{p+4}}} \\
\lesssim &\|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{\frac{2}{r}-1}\||\omega_1|^{\frac{r}{2}}\|_{L^{\frac{6p(r-1)}{2pr+2r-3p}}}^{\frac{2(r-1)}{r}} \\
\lesssim &\|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}\|\nabla\omega_2\|_{L^r}\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}, \qquad (3.25)
\end{aligned}
$$

here, we take r sufficiently close to 2 and satisfying $\frac{1}{2}<\frac{1}{r}<\frac{2}{3}$.

When $p=4$, we have

$$
|\int_{\mathbb{R}^3} B_h\cdot\nabla_h\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x|
$$

$$
\begin{aligned}
&\lesssim \Big\| \|B_h\|_{L_h^8} \Big\|_{L_v^4} \|\nabla\omega_2\|_{L^r} \Big\| \||\omega_1|^{r-1}\|_{L_h^{\frac{8r}{7r-8}}} \Big\|_{L_v^{\frac{4r}{3r-4}}} \\
&\lesssim \|B_h\|_{\dot H^1} \|\nabla\omega_2\|_{L^r} \||\nabla|^{\frac{r}{4(r-1)}} |\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{2(r-1)}{r}} \\
&\lesssim \|B_h\|_{\dot H^1} \|\nabla\omega_2\|_{L^r} \||\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{3}{2}-\frac{2}{r}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{1}{2}},
\end{aligned} \tag{3.26}
$$

where r sufficiently close to 2 and satisfying $\frac{1}{2}<\frac{1}{r}<\frac{3}{4}$.

When $4<p<\infty$, one has

$$
\begin{aligned}
&|\int_{\mathbb{R}^3} B_h\cdot\nabla_h\omega_2\omega_1|\omega_1|^{r-2}\mathrm{d}x| \\
&\lesssim \|B_h\|_{L^{\frac{3p}{p-2}}} \|\nabla\omega_2\|_{L^r} \||\omega_1|^{r-1}\|_{L^{\frac{3pr}{2pr+2r-3p}}} \\
&\lesssim \|B_h\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}} \|\nabla\omega_2\|_{L^r} \||\omega_1|^{\frac{r}{2}}\|_{L^{\frac{6p(r-1)}{2pr+2r-3p}}}^{\frac{2(r-1)}{r}} \\
&\lesssim \|B_h\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}} \|\nabla\omega_2\|_{L^r} \||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}} \|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}},
\end{aligned} \tag{3.27}
$$

here, we take r sufficiently close to 2 and satisfying $\frac{1}{2}<\frac{1}{r}<min\{\frac{1}{2}+\frac{1}{p},\frac{2}{3}(1-\frac{1}{p})\}$.

Combining the estimates from (3.21)-(3.27), we get the bound of D_5.

Likewise, we give the bound of D_{10} as

$$
|D_{10}| \lesssim \||\partial_3|^{-\frac{1}{2}-\delta}B_h\|_{\dot H^{1+\frac{2}{p}+\delta}} \|\nabla\omega_1\|_{L^r} \||\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}} \|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}. \tag{3.28}
$$

Finally, we handle the terms D_6 and D_7. Owing to

$$
\partial_h u_h = \mathcal{R}_2(\omega_1)+\mathcal{R}_2(\partial_3 u_3), \tag{3.29}
$$

where $\mathcal{R}_2$ is a Riesz transform, which is a bounded operator with $1<p<\infty$. Thus we get

$$
D_6+D_7 \lesssim \int_{\mathbb{R}^3} \partial_h B_h \mathcal{R}_2(\omega_1)\omega_2|\omega_2|^{r-2}\mathrm{d}x + \int_{\mathbb{R}^3} \partial_h B_h \mathcal{R}_2(\partial_3 u_3)\omega_2|\omega_2|^{r-2}\mathrm{d}x \triangleq \sum_{i=1}^{2} F_i.
$$

To the term F_1, it is similar to the term D_3. For $2<p<\infty$, we give the result as follow

$$
|F_1| \lesssim \|B_h\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}} \|\nabla\omega_1\|_{L^r} \||\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}} \|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}. \tag{3.30}
$$

As to the term F_2, when $2<p<4$, using the Gagliardo-Nirenberg inequalities, we get

$$
\begin{aligned}
|F_2| &\lesssim \Big\| \||\nabla_h|B_h\|_{L_h^{\frac{4p}{3p-4}}} \Big\|_{L_v^2} \Big\| \|\mathcal{R}_2(\partial_3 u_3)\|_{L_h^4} \Big\|_{L_v^{\frac{1}{\delta}}} \Big\| \||\omega_2|^{r-1}\|_{L_h^p} \Big\|_{L_v^{\frac{2}{1-2\delta}}} \\
&\lesssim \|B_h\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}} \||\nabla_h|^{-\delta}\partial_3^2 u_3\|_{L^2} \Big\| \||\nabla_h|^{\frac{rp-p-r}{(r-1)p}} |\omega_2|^{\frac{r}{2}}\|_{L_h^2} \Big\|_{L_v^{\frac{4(r-1)}{(1-2\delta)r}}}^{\frac{2(r-1)}{r}} \\
&\lesssim \|B_h\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}} \||\nabla_h|^{-\delta}\partial_3^2 u_3\|_{L^2} \||\nabla|^{\frac{rp-2r}{2p(r-1)}} |\omega_2|^{\frac{r}{2}}\|_{L^2}^{\frac{2(r-1)}{r}} \\
&\lesssim \|B_h\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}} \||\nabla_h|^{-\delta}\partial_3^2 u_3\|_{L^2} \||\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}} \|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}},
\end{aligned} \tag{3.31}
$$

Here, we need r sufficiently close to 2 and satisfying $\frac{1}{2}<\frac{1}{r}<\frac{2}{3}$.

When $p=4$, we have

$$
\begin{aligned}
|F_2| &\lesssim \||\nabla_h|B_h\|_{L^2}\Big\|\|\partial_3 u_3\|_{L^4_h}\Big\|_{L_v^{\frac{2r}{6-3r}}}\Big\|\||\omega_2|^{r-1}\|_{L^4_h}\Big\|_{L_v^{\frac{r}{2r-3}}}\\
&\lesssim \|B_h\|_{\dot H^1}\||\nabla_h|^{-\delta}\partial_3^2 u_3\|_{L^2}\Big\|\||\nabla_h|^{\frac{3r-4}{4(r-1)}}|\omega_2|^{\frac{r}{2}}\|_{L^2_h}\Big\|^{\frac{2(r-1)}{r}}_{L_v^{\frac{2(r-1)}{2r-3}}}\\
&\lesssim \|B_h\|_{\dot H^1}\||\nabla_h|^{-\delta}\partial_3^2 u_3\|_{L^2}\||\nabla|^{\frac{r}{4(r-1)}}|\omega_2|^{\frac{r}{2}}\|_{L^2}^{\frac{2(r-1)}{r}}\\
&\lesssim \|u_3\|_{\dot H^1}\||\nabla_h|^{-\delta}\partial_3^2 u_3\|_{L^2}\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{\frac{3}{2}-\frac{2}{r}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{\frac{1}{2}},
\end{aligned}
\tag{3.32}
$$

here, we have r sufficiently close to 2 and satisfying $\frac{1}{2}<\frac{1}{r}<\frac{2}{3}$.

When $4<p<\infty$, we get

$$
\begin{aligned}
|F_2| \lesssim &\||\nabla_h|^{-\frac{1}{2}+\frac{2}{p}}\partial_h B_h\|_{L^2}\\
&\cdot\Big(\Big\|\||\nabla_h|^{\frac{1}{2}-\frac{2}{p}}\mathcal{R}_2(\partial_3 u_3)\|_{L_h^{\frac{2p}{p-2}}}\Big\|_{L_v^{\frac{1}{\delta}}}\Big\|\||\omega_2|^{r-1}\|_{L^p_h}\Big\|_{L_v^{\frac{2}{1-2\delta}}}\\
&+\Big\|\|\mathcal{R}_2(\partial_3 u_3)\|_{L^4_h}\Big\|_{L_v^{\frac{1}{\delta}}}\Big\|\||\nabla_h|^{\frac{1}{2}-\frac{2}{p}}(\omega_2|\omega_2|^{r-2})\|_{L^4_h}\Big\|_{L_v^{\frac{2}{1-2\delta}}}\Big)\\
\lesssim &\|B_h\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}}\||\nabla_h|^{-\delta}\partial_3^2 u_3\|_{L^2}\Big(\Big\|\||\nabla_h|^{\frac{1-\frac{2}{p}-\frac{2\delta}{3}}{r-1}}\omega_1\|^{r-1}_{L^r_h}+\|\omega_2\|^{r-1}_{\dot B^{h,\frac{1-\frac{2}{p}-\frac{2\delta}{3}}{r-1}}_{r,2(r-1)}}\Big\|_{L_v^{\frac{2(r-1)}{1-2\delta}}}\Big)\\
\lesssim &\|B_h\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}}\||\nabla_h|^{-\delta}\partial_3^2 u_3\|_{L^2}\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}},
\end{aligned}
\tag{3.33}
$$

here, we need r sufficiently close to 2 and satisfying $\frac{1}{2}<\frac{1}{r}<min\{\frac{1}{2}+\frac{1}{p},\frac{2}{3}(1-\frac{1}{p})\}$.

Combining the estimates from (3.30)-(3.33), we get the bound of D_6 and D_7.

Summing up the bounds of D_1-D_{10}, we get

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\|\omega_1\|^r_{L^r}+\frac{4(r-1)}{r^2}\|\nabla|\omega_1|^{\frac{r}{2}}\|^2_{L^2}\\
\lesssim &\|u_3\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}}\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{2}{p}}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{2-\frac{2}{p}}\\
&+\|u_3\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}\\
&+\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|_{\dot H^{1+\frac{2}{p}+\delta}}\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}\\
&+\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|_{\dot H^{1+\frac{2}{p}+\delta}}\|\nabla\omega_2\|_{L^r}\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_1|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}},
\end{aligned}
\tag{3.34}
$$

and

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\|\omega_2\|^r_{L^r}+\frac{4(r-1)}{r^2}\|\nabla|\omega_2|^{\frac{r}{2}}\|^2_{L^2}\\
\lesssim &\|u_3\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}}\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{\frac{2}{p}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{2-\frac{2}{p}}\\
&+\|u_3\|_{\dot H^{\frac{1}{2}+\frac{2}{p}}}\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}\\
&+\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|_{\dot H^{1+\frac{2}{p}+\delta}}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}\\
&+\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|_{\dot H^{1+\frac{2}{p}+\delta}}\|\nabla\omega_1\|_{L^r}\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{r}+\frac{2}{p}}\|\nabla|\omega_2|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}}.
\end{aligned}
\tag{3.35}
$$

Then, multiplying the inequalities (3.34) and (3.35) by $\||\omega_1|^{\frac{r}{2}}\|_{L^2}^{\frac{4}{r}-2}$ and $\||\omega_2|^{\frac{r}{2}}\|_{L^2}^{\frac{4}{r}-2}$ respectively; what's more, summing up the two inequalities together and using the Young's inequalities, we complete the proof of the proposition. □

4. The estimate of the vertical terms

Recalling the Proposition 2.3 and Proposition 3.1, we still need to get the bounds for $\||\nabla_h|^{-\delta}\nabla u_3\|_{L^2}$ and $\||\nabla_h|^{-\delta}\nabla B_3\|_{L^2}$.

PROPOSITION 4.1. *Let $\delta=3(\frac{1}{r}-\frac{1}{2})$, with r sufficiently close to 2 and satisfying the definition in (3.1), take (ω_1,ω_2) as defined in (1.8). For $2<p<\infty$, then there exists a uniform constant C, such that*

$$\begin{aligned}&\frac{d}{dt}\sum_{k=1}^{3}(\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^2+\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^2)+\sum_{k=1}^{3}(\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^2+\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^2)\\&\leq\frac{1}{10}(\|\nabla\omega_1\|_{L^r}^2+\|\nabla\omega_2\|_{L^r}^2)+C(\|u_3\|_{\dot H^{\frac12+\frac2p}}^p+\||\partial_3|^{-\frac12-\delta}B_h\|_{\dot H^{1+\frac2p+\delta}}^p)\\&\quad\cdot(\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^2+\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^2+\||\omega_1|^{\frac r2}\|_{L^2}^{\frac4r}+\||\omega_2|^{\frac r2}\|_{L^2}^{\frac4r}).\end{aligned}\tag{4.1}$$

Proof. Applying ∂_k $(k=1,2,3)$ to the third component of (1.1) yields

$$\begin{aligned}&\partial_t\partial_k u_3+(u\cdot\nabla)\partial_k u_3-(B\cdot\nabla)\partial_k B_3-\Delta\partial_k u_3\\&=(\partial_k B\cdot\nabla)B_3-(\partial_k u\cdot\nabla)u_3-\partial_3\partial_k\Delta^{-1}\sum_{i,j=1}^{3}(\partial_j B_i\partial_i B_j-\partial_j u_i\partial_i u_j),\end{aligned}\tag{4.2}$$

and

$$\partial_t\partial_k B_3+(u\cdot\nabla)\partial_k B_3-(B\cdot\nabla)\partial_k u_3-\Delta\partial_k B_3=-(\partial_k u\cdot\nabla)B_3+(\partial_k B\cdot\nabla)u_3.\tag{4.3}$$

Taking the L^2 inner product of Equations (4.2) and (4.3) with $|\nabla_h|^{-2\delta}\partial_k u_3$ and $|\nabla_h|^{-2\delta}\partial_k B_3$ respectively, then adding them together, we have

$$\begin{aligned}&\frac{d}{dt}\sum_{k=1}^{3}(\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^2+\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^2)+\sum_{k=1}^{3}\||\nabla_h|^{-\delta}\partial_k\nabla u_3\|_{L^2}^2+\sum_{k=1}^{3}\||\nabla_h|^{-\delta}\partial_k\nabla B_3\|_{L^2}^2\\&=\sum_{k=1}^{3}\Big(-\int_{\mathbb{R}^3}(\partial_k u\cdot\nabla)u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x\\&\quad-\int_{\mathbb{R}^3}(u\cdot\nabla)\partial_k u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x+\int_{\mathbb{R}^3}(\partial_k B\cdot\nabla)B_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x\\&\quad-\int_{\mathbb{R}^3}\partial_3\partial_k P\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x+\int_{\mathbb{R}^3}(B\cdot\nabla)\partial_k B_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x\\&\quad+\int_{\mathbb{R}^3}(B\cdot\nabla)\partial_k u_3\cdot|\nabla_h|^{-2\delta}\partial_k B_3\mathrm{d}x-\int_{\mathbb{R}^3}(\partial_k u\cdot\nabla)B_3\cdot|\nabla_h|^{-2\delta}\partial_k B_3\mathrm{d}x\\&\quad-\int_{\mathbb{R}^3}(u\cdot\nabla)\partial_k B_3\cdot|\nabla_h|^{-2\delta}\partial_k B_3\mathrm{d}x+\int_{\mathbb{R}^3}(\partial_k B\cdot\nabla)u_3\cdot|\nabla_h|^{-2\delta}\partial_k B_3\mathrm{d}x\Big)\triangleq\sum_{i=1}^{9}E_i.\end{aligned}$$

It is obvious that the structure of the terms E_1, E_7, E_9, the terms E_2, E_8 and the terms E_5, E_6 are similar respectively. Therefore, we combine them to estimate together.

We start with the term E_1. Decomposing u in horizontal and vertical direction respectively, we have $\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x$ and $\int_{\mathbb{R}^3}(\partial_k u_h\cdot\nabla_h)u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x$.

To $\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x$, for $2<p<4$, by using the fractional Leibniz rule [15] and the Gagliardo-Nirenberg inequalities, we have

$$\begin{aligned}&|\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x|\\ \lesssim&\Big\|\,\|\partial_k u_3\|_{L_h^{\frac{4p}{3p-4}}}\Big\|_{L_v^2}\Big\|\,\|\partial_3 u_3\|_{L_h^4}\Big\|_{L_v^{\frac{1}{\delta}}}\Big\|\,\||\nabla_h|^{-2\delta}\partial_k u_3\|_{L_h^p}\Big\|_{L_v^{\frac{2}{1-2\delta}}}\\ \lesssim&\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^{2-\frac2p}.\end{aligned}\tag{4.4}$$

When $4\le p<\infty$, we obtain

$$\begin{aligned}&|\int_{\mathbb{R}^3}\partial_k u_3\partial_3 u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x|\lesssim\||\nabla|^{-\frac12+\frac2p}\partial_k u_3\|_{L^2}\\ &\cdot\Big(\Big\|\,\||\nabla|^{\frac12-\frac2p}\partial_3 u_3\|_{L_h^{\frac{8}{3+2\delta}}}\Big\|_{L_v^{\frac{4}{1+2\delta}}}\Big\|\,\||\nabla_h|^{-2\delta}\partial_k u_3\|_{L_h^{\frac{8}{1-2\delta}}}\Big\|_{L_v^{\frac{4}{1-2\delta}}}\\ &+\Big\|\,\|\partial_3 u_3\|_{L_h^4}\Big\|_{L_v^{\frac1\delta}}\Big\|\,\||\nabla_h|^{-2\delta}|\nabla|^{\frac12-\frac2p}\partial_k u_3\|_{L_h^4}\Big\|_{L_v^{\frac{2}{1-2\delta}}}\Big)\\ \lesssim&\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^{2-\frac2p}.\end{aligned}\tag{4.5}$$

As to the term $\int_{\mathbb{R}^3}(\partial_k u_h\cdot\nabla_h)u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3 dx$, which has the following two cases: $\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}\omega_1\cdot\nabla_h)u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x$ and $\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}\partial_3 u_3\cdot\nabla_h)u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3 dx$. Applying the Littlewood-Paley decomposition on the horizontal direction and P_j^h denotes projection in the horizontal variable, one has

$$\begin{aligned}&\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}\omega_1\cdot\nabla_h)u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3\mathrm{d}x\\ =&\sum_{j=1}\Big[\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}P_j^h\omega_1\cdot\nabla_h)P_{\le j}^h u_3\cdot|\nabla_h|^{-2\delta}\partial_k P_j^h u_3\mathrm{d}x &(4.6a)\\ &+\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}P_j^h\omega_1\cdot\nabla_h)P_j^h u_3\cdot|\nabla_h|^{-2\delta}\partial_k P_{\le j}^h u_3\mathrm{d}x &(4.6b)\\ &+\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}P_{\le j}^h\omega_1\cdot\nabla_h)P_j^h u_3\cdot|\nabla_h|^{-2\delta}\partial_k P_j^h u_3\mathrm{d}x &(4.6c)\\ &+\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}P_j^h\omega_1\cdot\nabla_h)P_j^h u_3\cdot|\nabla_h|^{-2\delta}\partial_k P_j^h u_3\mathrm{d}x\Big]. &(4.6d)\end{aligned}$$

By taking r sufficiently close to 2, for $2<p<\infty$, there holds

$$\begin{aligned}(4.6a)\lesssim&\|2^j\partial_k|\nabla_h|^{-1}P_j^h\omega_1\|_{L^r l_j^2}\Big\|\,\|2^{(\frac{24-15p+8p\delta}{12p})j}|\nabla_h|P_{\le j}^h u_3\|_{L_v^{\frac{3}{2\delta}}}\Big\|_{L_h^{\frac83}l_j^\infty}\\ &\cdot\Big\|\,\|2^{(\frac{3p-24-8p\delta}{12p})j}|\nabla_h|^{-2\delta}\partial_k P_j^h u_3\|_{L_v^{\frac{2}{1-2\delta}}}\Big\|_{L_h^{\frac{8r}{5r-8}}l_j^2}\\ \lesssim&\|\nabla\omega_1\|_{L^r}\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^{1-\frac2p}\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^{\frac2p},\end{aligned}\tag{4.7}$$

and

$$(4.6c)\lesssim\Big\|\,\|\partial_k|\nabla_h|^{-1}P_{\le j}^h\omega_1\|_{L_v^r}\Big\|_{L_h^{\frac{3}{\delta}}l_j^\infty}\Big\|\,\|2^{j(\frac{-3p+6+p\delta}{3p})}|\nabla_h|P_j^h u_3\|_{L_v^{\frac{3}{2\delta}}}\Big\|_{L_h^{\frac{6}{3-\delta}}l_j^2}$$

$$
\begin{aligned}
&\cdot\Big\| \|2^{(\frac{3p-6-p\delta}{3p})j}|\nabla_h|^{-2\delta}\partial_k P_j^h u_3\|_{L_v^{\frac{2}{1-2\delta}}}\Big\|_{L_h^{\frac{6}{3-\delta}} l_j^2}\\
&\lesssim \|\nabla\omega_1\|_{L^r}\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^{1-\frac2p}\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^{\frac2p}. \qquad (4.8)
\end{aligned}
$$

The remaining terms (4.6b), (4.6d) can be handled similar to (4.7)-(4.8).

With regard to the term $\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}\partial_3 u_3\cdot\nabla_h)u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3 \mathrm{d}x$, similarly, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}\partial_3 u_3\cdot\nabla_h)u_3\cdot|\nabla_h|^{-2\delta}\partial_k u_3 \mathrm{d}x\\
=&\sum_{j=1}\Big[\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}P_j^h\partial_3 u_3\cdot\nabla_h)P_{\le j}^h u_3\cdot|\nabla_h|^{-2\delta}\partial_k P_j^h u_3 \mathrm{d}x \qquad (4.9a)\\
&+\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}P_j^h\partial_3 u_3\cdot\nabla_h)P_j^h u_3\cdot|\nabla_h|^{-2\delta}\partial_k P_{\le j}^h u_3 \mathrm{d}x \qquad (4.9b)\\
&+\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}P_{\le j}^h\partial_3 u_3\cdot\nabla_h)P_j^h u_3\cdot|\nabla_h|^{-2\delta}\partial_k P_j^h u_3 \mathrm{d}x \qquad (4.9c)\\
&+\int_{\mathbb{R}^3}(\partial_k|\nabla_h|^{-1}P_j^h\partial_3 u_3\cdot\nabla_h)P_j^h u_3\cdot|\nabla_h|^{-2\delta}\partial_k P_j^h u_3 \mathrm{d}x\Big]. \qquad (4.9d)
\end{aligned}
$$

For $2<p<\infty$, there holds

$$
\begin{aligned}
(4.9a)\lesssim&\|2^{j(1-\delta)}\partial_k|\nabla_h|^{-1}P_j^h\partial_3 u_3\|_{L^2 l_j^2}\Big\|\|2^{j(\frac{2-p}{p})}|\nabla_h|P_{\le j}^h u_3\|_{L_v^{\frac{3}{2\delta}}}\Big\|_{L_h^{\frac{6}{3-2\delta}} l_j^\infty}\\
&\cdot\Big\|\|2^{j\frac{-2+p\delta}{p}}|\nabla_h|^{-2\delta}\partial_k P_j^h u_3\|_{L_v^{\frac{6}{3-4\delta}}}\Big\|_{L_h^{\frac{3}{\delta}} l_j^2}\\
\lesssim&\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^{2-\frac2p}, \qquad (4.10)
\end{aligned}
$$

and

$$
\begin{aligned}
(4.9b)\lesssim&\|2^{j(1-\delta)}\partial_k|\nabla_h|^{-1}P_j^h\partial_3 u_3\|_{L^2 l_j^2}\Big\|\|2^{j\frac{24-15p+8p\delta}{12p}}|\nabla_h|P_j^h u_3\|_{L_v^{\frac{3}{2\delta}}}\Big\|_{L_h^{\frac83} l_j^2}\\
&\cdot\Big\|\|2^{(\frac{3p-24+4p\delta}{12p})j}|\nabla_h|^{-2\delta}\partial_k P_{\le j}^h u_3\|_{L_v^{\frac{6}{3-4\delta}}}\Big\|_{L_h^8 l_j^\infty}\\
\lesssim&\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^{2-\frac2p}, \qquad (4.11)
\end{aligned}
$$

as well as

$$
\begin{aligned}
(4.9c)\lesssim&\Big\|\|\partial_k|\nabla_h|^{-1}P_{\le j}^h\partial_3 u_3\|_{L_v^2}\Big\|_{L_h^{\frac2\delta} l_j^\infty}\Big\|\|2^{j(\frac{12-6p+p\delta}{6p})}|\nabla_h|P_j^h u_3\|_{L_v^{\frac{3}{2\delta}}}\Big\|_{L_h^{\frac{4}{2-\delta}} l_j^2}\\
&\cdot\Big\|\|2^{j(\frac{-12+6p-p\delta}{6p})}|\nabla_h|^{-2\delta}\partial_k P_j^h u_3\|_{L_v^{\frac{6}{3-4\delta}}}\Big\|_{L_h^{\frac{4}{2-\delta}} l_j^2}\\
\lesssim&\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^{2-\frac2p}. \qquad (4.12)
\end{aligned}
$$

Analogously, we get the bound of (4.9d) from (4.10) − (4.12).

Combining the estimates from (4.4)-(4.12), we get the bound for E_1.

Similar to E_2, we get the bounds of E_7 and E_9 as

$$
|E_7|\lesssim\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{2-\frac2p}\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac2p}+\||\partial_3|^{-\frac12-\delta}B_h\|_{\dot H^{1+\frac2p+\delta}}
$$

$$\cdot(\|\nabla\omega_1\|_{L^r}+\||\nabla_h|^{-\delta}\partial_k^2u_3\|_{L^2})\||\nabla_h|^{-\delta}\partial_kB_3\|_{L^2}^{\frac{2}{p}}\||\nabla_h|^{-\delta}\partial_k^2B_3\|_{L^2}^{1-\frac{2}{p}}, \tag{4.13}$$

and

$$\begin{aligned}|E_9|\lesssim&\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_kB_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2B_3\|_{L^2}^{2-\frac2p}\\&+\|u_3\|_{\dot H^{\frac12+\frac2p}}(\|\nabla\omega_2\|_{L^r}+\||\nabla_h|^{-\delta}\partial_k^2B_3\|_{L^2})\||\nabla_h|^{-\delta}\partial_kB_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2B_3\|_{L^2}^{1-\frac2p}.\end{aligned} \tag{4.14}$$

Next, we estimate the term E_2. Noting that $-\int_{\mathbb{R}^3}(u\cdot\nabla)\partial_ku_3\cdot|\nabla_h|^{-2\delta}\partial_ku_3\mathrm{d}x=\int_{\mathbb{R}^3}u\partial_ku_3\cdot\nabla|\nabla_h|^{-2\delta}\partial_ku_3\mathrm{d}x$, then it is sufficient to estimate $\int_{\mathbb{R}^3}u_3\partial_ku_3\partial_3|\nabla_h|^{-2\delta}\partial_ku_3\mathrm{d}x$ and $\int_{\mathbb{R}^3}u_h\partial_ku_3\cdot\nabla_h|\nabla_h|^{-2\delta}\partial_ku_3\mathrm{d}x$.

To $\int_{\mathbb{R}^3}u_3\partial_ku_3\partial_3|\nabla_h|^{-2\delta}\partial_ku_3\mathrm{d}x$, for $2<p<\infty$, using the Gagliardo-Nirenberg inequalities and the Hölder's inequalities gives

$$\begin{aligned}&|\int_{\mathbb{R}^3}u_3\partial_ku_3\partial_3|\nabla_h|^{-2\delta}\partial_ku_3\mathrm{d}x|\lesssim\Big\|\||\nabla_h|^{-\delta}\partial_ku_3\|_{L_h^{\frac{4p}{p+2}}}\Big\|_{L_v^p}\\&\quad\cdot\Big(\Big\|\||\nabla_h|^{\delta}u_3\|_{L_h^{\frac{4p}{p-2+2p\delta}}}\Big\|_{L_v^{\frac{2p}{p-2}}}\Big\|\||\nabla_h|^{-2\delta}\partial_3\partial_ku_3\|_{L_h^{\frac{2}{1-\delta}}}\Big\|_{L_v^2}\\&\quad+\Big\|\|u_3\|_{L_h^{\frac{4p}{p-2}}}\Big\|_{L_v^{\frac{2p}{p-2}}}\||\nabla_h|^{-\delta}\partial_3\partial_ku_3\|_{L^2}\Big)\\&\lesssim\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_ku_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2u_3\|_{L^2}^{2-\frac2p}.\end{aligned} \tag{4.15}$$

To the term $\int_{\mathbb{R}^3}u_h\partial_ku_3\cdot\nabla_h|\nabla_h|^{-2\delta}\partial_ku_3\mathrm{d}x$, using a commutator estimate (see [19]), we get

$$\begin{aligned}&2\int_{\mathbb{R}^3}u_h\partial_ku_3\cdot\nabla_h|\nabla_h|^{-2\delta}\partial_ku_3\mathrm{d}x\\&=-\int_{\mathbb{R}^3}|\nabla_h|^{-2\delta}\nabla_h\cdot(u_h\partial_ku_3)\partial_ku_3\mathrm{d}x+\int_{\mathbb{R}^3}u_h\cdot\nabla_h|\nabla_h|^{-2\delta}\partial_ku_3\partial_ku_3\mathrm{d}x\\&\lesssim|\int_{\mathbb{R}^3}|\nabla_h|^{1-2\delta}u_h\partial_ku_3\partial_ku_3\mathrm{d}x|.\end{aligned}$$

For $2<p<\infty$, noting $(2.1)_1$ and applying the Gagliardo-Nirenberg inequalities, we have the following two situations:

$$\begin{aligned}&|\int_{\mathbb{R}^3}|\nabla_h|^{-2\delta}\omega_1\partial_ku_3\partial_ku_3\mathrm{d}x|\\&\lesssim\Big\|\||\nabla_h|^{-2\delta}\omega_1\|_{L_h^{\frac{4p}{p+2-2p\delta}}}\Big\|_{L_v^p}\Big\|\|\partial_ku_3\|_{L_h^{\frac{8p}{3p-2+2p\delta}}}\Big\|_{L_v^{\frac{2p}{p-1}}}^2\\&\lesssim\|\omega_1\|_{L^r}^{\frac2p}\|\nabla\omega_1\|_{L^r}^{1-\frac2p}\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k^2u_3\|_{L^2},\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}&|\int_{\mathbb{R}^3}|\nabla_h|^{-2\delta}\partial_3u_3\partial_ku_3\partial_ku_3\mathrm{d}x|\\&\lesssim\Big\|\||\nabla_h|^{-2\delta}|\partial_3|^{\frac12}u_3\|_{L_h^{\frac{2p}{p-2-p\delta}}}\Big\|_{L_v^{\frac{2}{1-2\delta}}}\Big\|\||\partial_3|^{\frac12}\partial_ku_3\|_{L_h^2}\Big\|_{L_v^{\frac1\delta}}\Big\|\|\partial_ku_3\|_{L_h^{\frac{2p}{2+p\delta}}}\Big\|_{L_v^2}\end{aligned}$$

$$\lesssim \|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^{\frac{2}{p}} \||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^{2-\frac{2}{p}}. \tag{4.17}$$

Combining the estimates from (4.15)-(4.17), we get the bound for E_2.

Likewise, to the term E_8, it is similar to E_2. Here we just give the details for the part of E_8: $\int_{\mathbb{R}^3} |\nabla_h|^{-2\delta}\omega_1\partial_k B_3\partial_k B_3 \mathrm{d}x$, which is different from the estimate of E_2. For $2<p<\infty$, we have

$$\begin{aligned}
|\int_{\mathbb{R}^3} |\nabla_h|^{-2\delta}\omega_1\partial_k B_3\partial_k B_3 \mathrm{d}x| \lesssim & \Big\| \||\nabla_h|^{\delta}|\partial_3|^{-1}\partial_k B_h\|_{L_h^{\frac{2p}{p-2}}} \Big\|_{L_v^{\frac{1}{\delta}}} \\
&\cdot \Big(\Big\| \||\nabla_h|^{1-3\delta}\omega_1\|_{L_h^{\frac{2}{1-\delta}}} \Big\|_{L_v^{\frac{2}{1-2\delta}}} \Big\| \|\partial_k B_3\|_{L_h^{\frac{2p}{2+p\delta}}} \Big\|_{L_v^2} \\
&+ \Big\| \||\nabla_h|^{-2\delta}\omega_1\|_{L_h^p} \Big\|_{L_v^{\frac{2}{1-2\delta}}} \||\nabla_h|^{1-\delta}\partial_k B_3\|_{L^2} \Big) \\
\lesssim & \||\partial_3|^{-\frac{1}{2}-\delta} B_h\|_{\dot{H}^{1+\frac{2}{p}+\delta}} (\|\nabla\omega_1\|_{L^r} \||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac{2}{p}} \||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{1-\frac{2}{p}} \\
&+ \|\omega_1\|_{L^r}^{\frac{2}{p}} \|\nabla\omega_1\|_{L^r}^{1-\frac{2}{p}} \||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}).
\end{aligned} \tag{4.18}$$

Thus we can further bound by

$$\begin{aligned}
|E_8| \lesssim & \|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{2-\frac{2}{p}} \||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac{2}{p}} \\
&+ \||\partial_3|^{-\frac{1}{2}-\delta} B_h\|_{\dot{H}^{1+\frac{2}{p}+\delta}} \|\omega_1\|_{L^r}^{\frac{2}{p}} \|\nabla\omega_1\|_{L^r}^{1-\frac{2}{p}} \||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2} \\
&+ \||\partial_3|^{-\frac{1}{2}-\delta} B_h\|_{\dot{H}^{1+\frac{2}{p}+\delta}} \|\nabla\omega_1\|_{L^r} \||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac{2}{p}} \||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{1-\frac{2}{p}}.
\end{aligned} \tag{4.19}$$

Furthermore, we consider E_3. It can be divided into $\int_{\mathbb{R}^3} \partial_k B_3 \partial_3 B_3 \cdot |\nabla_h|^{-2\delta}\partial_k u_3 \mathrm{d}x$ and $\int_{\mathbb{R}^3} (\partial_k B_h\cdot\nabla_h) B_3 \cdot |\nabla_h|^{-2\delta}\partial_k u_3 \mathrm{d}x$. By using the fractional Leibniz rule [15] and the Gagliardo-Nirenberg inequalities, for $2<p<\infty$, we obtain

$$\begin{aligned}
|\int_{\mathbb{R}^3} \partial_k B_3 \partial_3 B_3 \cdot |\nabla_h|^{-2\delta}\partial_k u_3 \mathrm{d}x| \lesssim & \Big\| \||\nabla|^{\frac{1}{2}}\partial_k B_3\|_{L_h^2} \Big\|_{L_v^{\frac{1}{\delta}}} \\
&\cdot \Big\| \|\partial_k B_3\|_{L_h^{\frac{2p}{2+p\delta}}} \Big\|_{L_v^2} \Big\| \||\nabla_h|^{-2\delta}|\nabla|^{\frac{1}{2}} u_3\|_{L_h^{\frac{2p}{p-2-p\delta}}} \Big\|_{L_v^{\frac{2}{1-2\delta}}} \\
\lesssim & \||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{2-\frac{2}{p}} \||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac{2}{p}} \|u_3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}},
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
|\int_{\mathbb{R}^3} (\partial_k B_h\cdot\nabla_h) B_3 \cdot |\nabla_h|^{-2\delta}\partial_k u_3 \mathrm{d}x| \lesssim & \Big\| \||\nabla|^{\frac{1}{2}} B_h\|_{L_h^{\frac{2p}{p-1}}} \Big\|_{L_v^{\frac{2p}{p-2}}} \\
&\cdot \Big(\Big\| \||\nabla|^{\frac{1}{2}}|\nabla_h| B_3\|_{L_h^{\frac{4}{1+2\delta}}} \Big\|_{L_v^2} \Big\| \||\nabla_h|^{-2\delta}\partial_k u_3\|_{L_h^{\frac{4p}{p+2-2p\delta}}} \Big\|_{L_v^p} \\
&+ \Big\| \||\nabla_h| B_3\|_{L_h^{\frac{4p}{p+2+p\delta}}} \Big\|_{L_v^{\frac{2p}{2+p\delta}}} \Big\| \||\nabla_h|^{-2\delta}|\nabla|^{\frac{1}{2}}\partial_k u_3\|_{L_h^{\frac{4}{1-\delta}}} \Big\|_{L_v^{\frac{2}{1-\delta}}} \Big) \\
\lesssim & \|B_h\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot (\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{1-\frac{2}{p}} \||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac{2}{p}} \||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2} \\
&+ \||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^{1-\frac{2}{p}} \||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^{\frac{2}{p}} \||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}).
\end{aligned} \tag{4.21}$$

Combining the estimates of (4.20)-(4.21), we get the bound for E_3.

With regard to the term E_4, we infer that
$\int_{\mathbb{R}^3}\partial_3\partial_k P|\nabla_h|^{-2\delta}\partial_k u_3 \mathrm{d}x=\int_{\mathbb{R}^3}(\sum_{l,m=1}^3\partial_l B_m\partial_m B_l)\partial_3\partial_k\Delta^{-1}(|\nabla_h|^{-2\delta}\partial_k u_3)\mathrm{d}x$
$-\int_{\mathbb{R}^3}(\sum_{l,m=1}^3\partial_l u_m\partial_m u_l)\partial_3\partial_k\Delta^{-1}(|\nabla_h|^{-2\delta}\partial_k u_3)\mathrm{d}x\triangleq E_{41}+E_{42}$.

Due to the similar structure of E_{41} and E_{42}, it is only necessary to estimate E_{41}. Noting that $l,m\in 1,2$, for $2<p\le 4$, we have

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}(\partial_l B_m\partial_m B_l)\mathcal{R}_3(|\nabla_h|^{-2\delta}\partial_k u_3)\mathrm{d}x|\\
\lesssim&\Big(\Big\|\|\omega_2\|_{L_h^{\frac{4}{1+2\delta}}}\Big\|_{L_v^p}^2+\Big\|\|\partial_3 B_3\|_{L_h^{\frac{4}{1+2\delta}}}\Big\|_{L_v^p}^2\Big)\cdot\Big\|\|\mathcal{R}_3(|\nabla_h|^{-2\delta}\partial_k u_3)\|_{L_h^{\frac{2}{1-2\delta}}}\Big\|_{L_v^{\frac{p}{p-2}}}\\
\lesssim&\|u_3\|_{\dot H^{\frac12+\frac2p}}\|\omega_2\|_{L^r}^{\frac2p}\|\nabla\omega_2\|_{L^r}^{2-\frac2p}+\|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{2-\frac2p}. \qquad (4.22)
\end{aligned}
$$

For $4<p<\infty$, there holds

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}(\partial_l B_m\partial_m B_l)\mathcal{R}_3(|\nabla_h|^{-2\delta}\partial_k u_3)\mathrm{d}x|\\
\lesssim&|\int_{\mathbb{R}^3}R_2(\partial_3 B_3)\mathcal{R}_2(\partial_3 B_3)\mathcal{R}_3(|\nabla_h|^{-2\delta}\partial_k u_3)\mathrm{d}x| \qquad (4.23a)\\
&+|\int_{\mathbb{R}^3}\mathcal{R}_2(\omega_2)\mathcal{R}_2(\partial_3 B_3)\mathcal{R}_3(|\nabla_h|^{-2\delta}\partial_k u_3)\mathrm{d}x| \qquad (4.23b)\\
&+|\int_{\mathbb{R}^3}\mathcal{R}_2(\omega_2)\mathcal{R}_2(\omega_2)\mathcal{R}_3(|\nabla_h|^{-2\delta}\partial_k u_3)\mathrm{d}x|, \qquad (4.23c)
\end{aligned}
$$

where $\mathcal{R}_2$ and $\mathcal{R}_3$ are Riesz transforms in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. For $4<p<\infty$, we have

$$
\begin{aligned}
(4.23a)\lesssim&\Big\|\|\partial_3 B_3\|_{L_h^{\frac{8}{1+6\delta}}}\Big\|_{L_v^{\frac{4}{1-2\delta}}}\Big\|\||\nabla|^{\frac12-\frac2p}\partial_3 B_3\|_{L_h^{\frac{8}{3+2\delta}}}\Big\|_{L_v^{\frac{4}{1+2\delta}}}\\
&\cdot\Big\|\||\nabla_h|^{-2\delta}|\nabla|^{-\frac12+\frac2p}\partial_k u_3\|_{L_h^{\frac{2}{1-2\delta}}}\Big\|_{L_v^2}\\
\lesssim&\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{2-\frac2p}\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac2p}\|u_3\|_{\dot H^{\frac12+\frac2p}}, \qquad (4.24)
\end{aligned}
$$

and

$$
\begin{aligned}
(4.23b)\lesssim&\Big\|\||\nabla_h|^{-2\delta}|\nabla|^{-\frac12+\frac2p}\partial_k u_3\|_{L_h^{\frac{2}{1-2\delta}}}\Big\|_{L_v^2}\cdot\Big(\Big\|\||\nabla|^{\frac12-\frac2p}\partial_3 B_3\|_{L_h^{\frac{8}{3+2\delta}}}\Big\|_{L_v^{\frac{4}{1+2\delta}}}\\
&\cdot\Big\|\|\omega_2\|_{L_h^{\frac{8}{1+6\delta}}}\Big\|_{L_v^{\frac{4}{1-2\delta}}}+\Big\|\|\partial_3 B_3\|_{L_h^{\frac{3}{2\delta}}}\Big\|_{L_v^{\frac{6}{3-2\delta}}}\Big\|\||\nabla|^{\frac12-\frac2p}\omega_2\|_{L_h^{\frac{6}{3+2\delta}}}\Big\|_{L_v^{\frac{3}{\delta}}}\Big)\\
\lesssim&\|u_3\|_{\dot H^{\frac12+\frac2p}}(\|\nabla\omega_2\|_{L^r}\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{1-\frac2p}\\
&+\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}\|\nabla\omega_2\|_{L^r}^{1-\frac2p}\|\omega_2\|_{L^r}^{\frac2p}), \qquad (4.25)
\end{aligned}
$$

as well as

$$
\begin{aligned}
(4.23c)\lesssim&\|\partial_k\omega_2\|_{L^{\frac{6}{3+2\delta}}}\Big\|\|\omega_2\|_{L_h^{\frac{12p}{3p+12+2p\delta}}}\Big\|_{L_v^{\frac{3}{2\delta}}}\cdot\Big\|\||\nabla_h|^{-2\delta}u_3\|_{L_h^{\frac{4p}{p-4-2p\delta}}}\Big\|_{L_v^{\frac{2}{1-2\delta}}}\\
\lesssim&\|\nabla\omega_2\|_{L^r}^{2-\frac2p}\|\omega_2\|_{L^r}^{\frac2p}\|u_3\|_{\dot H^{\frac12+\frac2p}}. \qquad (4.26)
\end{aligned}
$$

Then, noting E_{41} with $l=3$ or $m=3$, which is analogous to E_3, there hold similar bounds from (4.20)-(4.21). Then, we give the bound for the term E_{42} as

$$\begin{aligned} E_{42}\lesssim & \|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^{2-\frac2p} \\ & +\|u_3\|_{\dot H^{\frac12+\frac2p}}\cdot(\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}+\|\nabla\omega_1\|_{L^r})\|\omega_1\|_{L^r}^{\frac2p}\|\nabla\omega_1\|_{L^r}^{1-\frac2p}. \end{aligned} \tag{4.27}$$

Combining the estimates of (4.22)-(4.27), we get the bound for E_4.
Finally, using a commutator estimate (see [19]) for E_5, E_6 gives

$$\begin{aligned} E_5+E_6 =& -\int_{\mathbb{R}^3} B\partial_k B_3\cdot\nabla|\nabla_h|^{-2\delta}\partial_k u_3 \mathrm{d}x-\int_{\mathbb{R}^3} B\partial_k u_3\cdot\nabla|\nabla_h|^{-2\delta}\partial_k B_3\mathrm{d}x \\ =& -\int_{\mathbb{R}^3} B_3\partial_k B_3|\nabla_h|^{-2\delta}\partial_3\partial_k u_3\mathrm{d}x-\int_{\mathbb{R}^3} B_3\partial_k u_3|\nabla_h|^{-2\delta}\partial_3\partial_k B_3\mathrm{d}x \\ &+\int_{\mathbb{R}^3}(|\nabla_h|^{-2\delta}\nabla_h\cdot(B_h\partial_k B_3)-B_h\cdot\nabla_h|\nabla_h|^{-2\delta}\partial_k B_3)\partial_k u_3\mathrm{d}x \\ \lesssim & \left|\int_{\mathbb{R}^3} B_3\partial_k B_3|\nabla_h|^{-2\delta}\partial_3\partial_k u_3\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3} B_3\partial_k u_3|\nabla_h|^{-2\delta}\partial_3\partial_k B_3\mathrm{d}x\right| \\ &+\left|\int_{\mathbb{R}^3}|\nabla_h|^{1-2\delta}B_h\partial_k u_3\partial_k B_3\mathrm{d}x\right|. \end{aligned} \tag{4.28}$$

To the term $|\int_{\mathbb{R}^3} B_3\partial_k B_3|\nabla_h|^{-2\delta}\partial_3\partial_k u_3\mathrm{d}x|$, using the Gagliardo-Nirenberg inequalities, for $2<p<\infty$, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} B_3\partial_k B_3|\nabla_h|^{-2\delta}\partial_3\partial_k u_3\mathrm{d}x\right| \\ \lesssim & \Big\|\||\partial_3|^{-1}|\nabla_h|B_h\|_{L_h^{\frac{2p}{p-2-p\delta}}}\Big\|_{L_v^{\frac1\delta}}\Big\|\|\partial_k B_3\|_{L_h^{\frac{p}{1+p\delta}}}\Big\|_{L_v^{\frac{2}{1-2\delta}}}\Big\|\||\nabla_h|^{-2\delta}\partial_3\partial_k u_3\|_{L_h^{\frac{2}{1-\delta}}}\Big\|_{L_v^2} \\ \lesssim & \||\partial_3|^{-\frac12-\delta}B_h\|_{\dot H^{1+\frac2p+\delta}}\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{1-\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}, \end{aligned} \tag{4.29}$$

As to the term $|\int_{\mathbb{R}^3} B_3\partial_k u_3|\nabla_h|^{-2\delta}\partial_3\partial_k B_3\mathrm{d}x|$. For $2<p<4$, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} B_3\partial_k u_3|\nabla_h|^{-2\delta}\partial_3\partial_k B_3\mathrm{d}x\right| \\ \lesssim & \Big\|\|B_3\|_{L_h^{\frac2\delta}}\Big\|_{L_v^{\frac{2p}{-p+4}}}\Big\|\|\partial_k u_3\|_{L_h^2}\Big\|_{L_v^{\frac{p}{p-2}}}\Big\|\||\nabla_h|^{-2\delta}\partial_3\partial_k B_3\|_{L_h^{\frac{2}{1-\delta}}}\Big\|_{L_v^2} \\ \lesssim & \|u_3\|_{\dot H^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{2-\frac2p}, \end{aligned} \tag{4.30}$$

for $4\le p<\infty$, using the divergence-free condition $\partial_3 B_3=-\nabla_h\cdot B_h$, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} B_3\partial_k u_3|\nabla_h|^{-2\delta}\partial_3\partial_k B_3\mathrm{d}x\right|\lesssim\Big\|\||\partial_3|^{-1}|\nabla_h|B_h\|_{L_h^{\frac{2p}{p-2-p\delta}}}\Big\|_{L_v^{\frac1\delta}} \\ &\cdot\Big\|\|\partial_k u_3\|_{L_h^{\frac{p}{1+p\delta}}}\Big\|_{L_v^{\frac{2}{1-2\delta}}}\Big\|\||\nabla_h|^{-2\delta}\partial_3\partial_k B_3\|_{L_h^{\frac{2}{1-\delta}}}\Big\|_{L_v^2} \\ \lesssim & \||\partial_3|^{-\frac12-\delta}B_h\|_{\dot H^{1+\frac2p+\delta}}\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^{\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 u_3\|_{L^2}^{1-\frac2p}\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}. \end{aligned} \tag{4.31}$$

The term $|\int_{\mathbb{R}^3}|\nabla_h|^{1-2\delta}B_h\partial_k u_3\partial_k B_3\mathrm{d}x|$, can be separated into $|\int_{\mathbb{R}^3}|\nabla_h|^{-2\delta}\omega_2\partial_k u_3\partial_k B_3\mathrm{d}x|$ and $|\int_{\mathbb{R}^3}|\nabla_h|^{-2\delta}\partial_3 B_3\partial_k u_3\partial_k B_3\mathrm{d}x|$ through using $(2.1)_2$.

To the term $|\int_{\mathbb{R}^3}|\nabla_h|^{-2\delta}\omega_2\partial_k u_3\partial_k B_3 \mathrm{d}x|$, applying the Gagliardo-Nirenberg inequalities, for $2<p<4$, we have

$$\begin{aligned}&|\int_{\mathbb{R}^3}|\nabla_h|^{-2\delta}\omega_2\partial_k u_3\partial_k B_3 \mathrm{d}x|\\ \lesssim&\Big\|\,\||\nabla_h|^{-2\delta}\omega_2\|_{L_h^{\frac{6p}{6-p\delta}}}\Big\|_{L_v^{\frac{6}{3-4\delta}}}\Big\|\,\|\partial_k u_3\|_{L_h^{\frac{4p}{3p-4}}}\Big\|_{L_v^2}\Big\|\,\|\partial_k B_3\|_{L_h^{\frac{12}{3+2\delta}}}\Big\|_{L_v^{\frac{3}{2\delta}}}\\ \lesssim&\|\omega_2\|_{L^r}^{\frac{2}{p}}\|\nabla\omega_2\|_{L^r}^{1-\frac{2}{p}}\|u_3\|_{\dot{H}^{\frac12+\frac2p}}\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2},\end{aligned}\tag{4.32}$$

for $4\le p<\infty$, one has

$$\begin{aligned}&|\int_{\mathbb{R}^3}|\nabla_h|^{-2\delta}\omega_2\partial_k u_3\partial_k B_3 \mathrm{d}x|\lesssim\||\nabla|^{\frac12+\frac2p}u_3\|_{L^2}\\ &\cdot\Big(\Big\|\,\||\nabla_h|^{-2\delta}|\nabla|^{\frac12-\frac2p}\omega_2\|_{L_h^{\frac{4p}{p-2-p\delta}}}\Big\|_{L_v^{\frac{2p}{p-2-p\delta}}}\Big\|\,\|\partial_k B_3\|_{L_h^{\frac{4p}{p+2+p\delta}}}\Big\|_{L_v^{\frac{2p}{2+p\delta}}}\\ &+\Big\|\,\||\nabla_h|^{-2\delta}\omega_2\|_{L_h^{\frac{8}{1-2\delta}}}\Big\|_{L_v^{\frac{4}{1-2\delta}}}\Big\|\,\||\nabla|^{\frac12-\frac2p}\partial_k B_3\|_{L_h^{\frac{8}{3+2\delta}}}\Big\|_{L_v^{\frac{4}{1+2\delta}}}\Big)\\ \lesssim&\|u_3\|_{\dot{H}^{\frac12+\frac2p}}\|\nabla\omega_2\|_{L^r}\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{1-\frac2p}\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac2p}.\end{aligned}\tag{4.33}$$

As to the term $|\int_{\mathbb{R}^3}|\nabla_h|^{-2\delta}\partial_3 B_3\partial_k u_3\partial_k B_3 \mathrm{d}x|$, which is similar to (4.4)-(4.5), here we just give the result as

$$\begin{aligned}&|\int_{\mathbb{R}^3}|\nabla_h|^{-2\delta}\partial_3 B_3\partial_k u_3\partial_k B_3 \mathrm{d}x|\\ \lesssim&\||\nabla_h|^{-\delta}\partial_k^2 B_3\|_{L^2}^{2-\frac2p}\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^{\frac2p}\|u_3\|_{\dot{H}^{\frac12+\frac2p}}.\end{aligned}\tag{4.34}$$

Combining the estimates of (4.29)-(4.34), we get the bounds for E_5 and E_6.

Adding the bounds from E_1 to E_9 and using the Young's inequalities, we proved this proposition. □

5. Proof of the main theorem

Proof. **(Proof of Theorem 1.1.)** By summing up (3.2) in Proposition 3.1 and (4.1) in Proposition 4.1, and using the Young's inequalities, one yields

$$\begin{aligned}&\frac{d}{dt}\sum_{k=1}^{3}(\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^2+\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^2)+\||\omega_1|^{\frac r2}\|_{L^2}^{\frac4r}+\||\omega_2|^{\frac r2}\|_{L^2}^{\frac4r}\\ \le&C(\|u_3\|_{\dot{H}^{\frac12+\frac2p}}^p+\||\partial_3|^{-\frac12-\delta}B_h\|_{\dot{H}^{1+\frac2p+\delta}}^p)\\ &\cdot\Big(\sum_{k=1}^{3}(\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^2+\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^2)+\||\omega_1|^{\frac r2}\|_{L^2}^{\frac4r}+\||\omega_2|^{\frac r2}\|_{L^2}^{\frac4r}\Big).\end{aligned}\tag{5.1}$$

Then, using the standard Grönwall's inequalities shows that

$$\begin{aligned}&\sum_{k=1}^{3}(\||\nabla_h|^{-\delta}\partial_k u_3\|_{L^2}^2+\||\nabla_h|^{-\delta}\partial_k B_3\|_{L^2}^2)+\||\omega_1|^{\frac r2}\|_{L^2}^{\frac4r}+\||\omega_2|^{\frac r2}\|_{L^2}^{\frac4r}\\ \le&\Big(\sum_{k=1}^{3}(\||\nabla_h|^{-\delta}\partial_k u_3(0)\|_{L^2}^2+\||\nabla_h|^{-\delta}\partial_k B_3(0)\|_{L^2}^2)+\||\omega_1(0)|^{\frac r2}\|_{L^2}^{\frac4r}\end{aligned}$$

$$+\||\omega_2(0)|^{\frac{r}{2}}\|_{L^2}^{\frac{4}{r}}\Big)\cdot exp\{C\int_0^t \|u_3\|^p_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}+\||\partial_3|^{-\frac{1}{2}-\delta}B_h\|^p_{\dot{H}^{1+\frac{2}{p}+\delta}}\mathrm{d}s\}. \tag{5.2}$$

We finished the proof. □

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