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ON A FREE BOUNDARY PROBLEM FOR AN OPTIMAL INVESTMENT PROBLEM WITH DIFFERENT INTEREST RATES*

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Abstract. This paper discusses an investment problem for a single agent with higher borrowing interest rate than lending in the market. The objective is to maximize the expected discounted utility of terminal wealth by choosing portfolio of one risk asset and the bank account. The objective function is the solution of a free boundary problem with two nonlinear equations and one linear equation. The main contribution is that the existence of free boundary lines is proved in all situations and the design methods can be generally applied to other similar problems.

Keywords. fully nonlinear equation; free boundary lines; stochastic optimal control; optimal investment; different interest rates.

AMS subject classifications. 35R35; 91B70; 93E20.

1. Introduction

This paper treats an investment problem for a single agent in the market with inconsistent interest rates on deposits and loans. The investor holds wealth X_t at time t and distributes it to two assets. One is the bank account with lending rates r_1 and borrowing rate r_2. The other is a stock with price submitting to geometric Brownian motion with appreciation rate α and volatility σ. We assume that all the given market parameters r_1, r_2, α and σ are deterministic positive constants and satisfying $\alpha > r_2 > r_1$.

The objective is to maximize the total expected (discounted) utility at fixed terminal time T over a finite trading horizon $[t,T]$. Suppose the investor must keep the wealth nonnegative, i.e., bankruptcy never occurs, moreover, the agent is a "small investor" in that his or her decisions do not affect the asset prices and he or she does not pay transaction fees when trading.

For the specific HARA (hyperbolic absolute risk aversion) case, we find that the investor does not always keeping borrowing or always keeping lending in some cases and there exist free boundary lines which can be expressed as a functional form.

Specifically, if the index of relative risk aversion $\gamma \le a_2 := \frac{\alpha - r_2}{\sigma^2}$, the investor should keep borrowing no matter how much endowment he or she owns. If $a_2 < \gamma \le a_1 := \frac{\alpha - r_1}{\sigma^2}$, there exists borrowing free boundary line $B(t)$, when the investor's endowment $X_t < B(t)$, he or she should borrow from the bank, while when $X_t \ge B(t)$, he or she should neither borrow money from the bank nor deposit money in the bank. If $\gamma > a_1$, there exists borrowing free boundary line $B(t)$ and lending free boundary line $L(t)$, when $X_t < B(t)$, the investor should borrow money, when $B(t) \le X_t \le L(t)$, he or she should stop borrowing or lending, when $X_t > L(t)$, he or she should deposit money in the bank. (see Figures 6.1–6.3.)

The difficulties and contributions of this paper are listed below. First, it is a free boundary problem for two fully nonlinear equations and one linear equation. Second, the fully nonlinear equations do not satisfy the parabolic condition. Third, we prove that the solution belongs to $C^{3,2}$ space, and some useful conclusions such as its upper and

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lower bounds, monotonicity and concavity are obtained. Fourth, the main contribution is that we prove the existence of free boundary lines in all situations, and the design methods can be generally applied to other similar problems.

Our model is based on the famous consumption-investment model, let's briefly introduce the history of the model.

The famous article [19] by Harry M. Markowitz (1952,1959) and his book of the same title [20] inaugurated a new era in modern finance. While considering a single agent or individual investor, for whom, the impact of its consumption on lifetime should be taken into account. The paper [21] by Merton (1969), based on the works [2,25,26] etc., formulated the consumption-portfolio problem in continuous time with the risky asset yielding log-normally distributed returns. In a pioneering paper [22], Merton (1971) extended his 1969 paper to incorporate an important class of utility functions known as the HARA (Hyperbolic Absolute Risk Aversion) family. He used Itô's lemma as a tool for analyzing the dynamics of asset prices, wealth, and consumption. The problem concerning optimal consumption-investment decision involves the decisions endowed with some initial wealth and seeks to maximize the expected discounted utility of consumption over time. Using stochastic dynamic programming, he derived the Hamilton-Jacobi-Bellman (HJB) equations for the value functions of the problem. The problem according to Merton (1975) is the natural beginning point for the development of a theory of finance.

Samuelson and Merton's pioneering papers prompted researchers to contribute a considerable volume of new work on the subject in various directions. The literature has extensively covered the optimal consumption-investment problems in the financial markets that are subject to constraints and market imperfections. [11,30] considered the optimal consumption-investment problem with borrowing constraints. [9] considered the optimal consumption-investment problem with the constraint that the wealth process never falls below a fixed fraction of its running maximum. [1,6,7,27,29] considered proportional transaction costs in the study of optimal consumption-investment problems, etc.

For the problem of inconsistent interest rates on borrowing and lending, [3, 5, 16] consider some option pricing model with higher borrowing rate then lending. [8] uses backward differential equations to study the optimal investment-consumption problem with inconsistent interest rates on borrowing and lending. However, up to now, there is no literature studying the related free boundary problems.

On the other hand, on the free boundary problem for fully nonlinear equation and linear equation of parabolic type, [14, 15] consider dividend optimization/risk control problems, where, the free boundaries are reinsurance free boundary and dividend free boundary.

The remainder of the paper is organized as follows. In Section 2, the mathematical formulation of the model and the correspondent HJB equation with its terminal boundary conditions are presented. In Section 3, by dual transformation, we derive a fully nonlinear problem (6.2) which satisfies the parabolic condition. In Section 4, we prove the existence, uniqueness, regularity and a priori estimate of the solution to problem (6.2). In Section 5, we discuss the existence of the free boundaries to problem (6.2). In Section 6, we prove the existence of the solution to the original problem (3.1), and give the monotonicity, concavity and the estimation of upper and lower bounds. Moreover, we give the existence of the free boundaries of the original problem. In Appendix A, we prove Theorem 4.1. In Appendix B, we prove that (4.6) is the solution of problem (4.5).

2. Model formulation

Consider a financial market with a fixed filtered complete probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$. A standard $\mathcal{F}_t$-adapted one-dimensional Brownian motion $\{W_t, t\geq 0\}$ is defined on the space. Let $L^2_{\mathcal{F}}(0,T;\mathbb{R})$ denote the set of all $\mathbb{R}$-valued, $\mathcal{F}_t$-progressively measurable stochastic processes $f(t)$ with $\mathbb{E}\int_0^T |f(t)|^2 dt < +\infty$, where, the terminal time $T>0$ is given.

The market consists of two continuously traded securities. One is risk-free bond (or we can treat it as bank account) whose value process S^0_t is subject to the following ordinary differential equation (ODE):

$$\mathrm{d}S^0_t = \begin{cases} r_1 S^0_t \mathrm{d}t, \ S^0_t > 0; \\ r_2 S^0_t \mathrm{d}t, \ S^0_t < 0. \end{cases} \tag{2.1}$$

where r_1, r_2 represent the deposit and loan interest rates respectively. The other asset is a risky stock whose price process S^1_t satisfies the following stochastic differential equation (SDE):

$$\mathrm{d}S^1_t = S^1_t \big(\alpha \mathrm{d}t + \sigma \mathrm{d}W_t\big), \tag{2.2}$$

where α is the appreciation rate, and σ is the volatility or dispersion rate of the stock. We assume that all the given market parameters r_1, r_2, α and σ are deterministic positive constants and satisfy $\alpha > r_2 > r_1$.

Consider an agent with an initial endowment $X_t = x > 0$ and an investment horizon $[t,T]$. Assume that the trading of shares is self-financed and takes place continuously, where transaction cost and consumptions are not considered. Then X_s, $t\leq s\leq T$ satisfies

$$\begin{cases} \mathrm{d}X_s = \big[\big(r_1 \chi_{\{\pi_s < X_s\}} + r_2 \chi_{\{\pi_s > X_s\}}\big)(X_s - \pi_s) + \alpha \pi_s\big]\mathrm{d}s + \sigma \pi_s \mathrm{d}W_s, \ t\leq s\leq T, \\ X_t = x, \end{cases} \tag{2.3}$$

where π_s denotes the total market value of the agent's wealth in the stock at time s. We call the process π_s, $t\leq s\leq T$ a portfolio of the agent.

Define an admissible investment set as

$$\Pi_t := \big\{\pi_s \in L^2_{\mathcal{F}}([t,T];\mathbb{R}) \,\big|\, X_s \geq 0, t\leq s\leq T\big\}.$$

Under the so called HARA utility

$$\Phi(x) := \frac{(x+K)^{1-\gamma}}{1-\gamma}, \quad K>0, \quad \gamma>0, \gamma\neq 1,$$

the agent's objective is to find an admissible portfolio $\pi_s, t\leq s\leq T$ among all admissible portfolios such that

$$V(x,t) = \sup_{\pi\in\Pi_t} \mathbb{E}\Big[e^{-\beta(T-t)}\Phi(X_T)\Big| X_t = x\Big], \tag{2.4}$$

where $\beta>0$ is the discounted rate. We just need to study the case with $\beta=0$, since we can take the transformation of $\widehat{V} = e^{\beta(T-t)}V$.

If $K=0$, the explicit solution of (2.4) can be expressed by

$$\underline{V}(x,t) := e^{\rho(T-t)}\frac{x^{1-\gamma}}{1-\gamma},$$

where

$$\rho := \begin{cases} \left(\frac{\sigma^2 a_1^2}{2\gamma}+r_1\right)(1-\gamma), & \gamma > a_1, \\ \left(-\frac{\sigma^2}{2}\gamma+\alpha\right)(1-\gamma), & a_2 \le \gamma \le a_1, \\ \left(\frac{\sigma^2 a_2^2}{2\gamma}+r_2\right)(1-\gamma), & \gamma < a_2. \end{cases} \quad a_i := \frac{\alpha - r_i}{\sigma^2}, \; i=1,2. \tag{2.5}$$

In this case, the optimal investment is $\overline{\pi}_t := \kappa X_t$, where

$$\kappa := \begin{cases} \frac{a_1}{\gamma}, \; \gamma > a_1, \\ 1, \quad a_2 \le \gamma \le a_1, \\ \frac{a_2}{\gamma}, \; \gamma < a_2, \end{cases}$$

(see [28]).

3. Related equations

By using standard viscosity theory, one can prove that the value function is the unique viscosity solution of the following HJB equation with terminal condition.

$$\begin{cases} -V_t - \max_{\pi}\left(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \Big((r_1\chi_{\{\pi<x\}}+r_2\chi_{\{\pi>x\}})(x-\pi)+\alpha\pi\Big)V_x\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x>0, \, 0<t<T, \\ V(0,t) = \frac{K^{1-\gamma}}{1-\gamma}, \quad 0<t<T, \\ V(x,T) = \frac{(x+K)^{1-\gamma}}{1-\gamma}, \quad x>0, \end{cases} \tag{3.1}$$

(see [4] or [10]). Moreover, we will prove that the solution $V \in C^{3,2}\big((0,+\infty)\times(0,T)\big)$ (see Theorem 6.1).

In the following, we transform problem (3.1) into a fully nonlinear problem (3.8) satisfying the usual structural conditions. The process is heuristic because many a priori estimates of solution will be used, but in Section 6, we will rigorously prove the existence and uniqueness of the solution to problem (3.8) and the estimates of its solution, and then construct the solution of problem (3.1) and prove these a prior estimates that have been used in this section.

We will prove that (see Theorem 6.1)

$$V_x > 0, \quad V_{xx} < 0, \quad \forall x>0, \, 0<t<T, \tag{3.2}$$

This leads to the optimal strategy to (3.1) satisfying (3.2) as follows

$$\pi^* = \operatorname*{argmax}_{\pi}\left(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \big((r_1\chi_{\{\pi<x\}}+r_2\chi_{\{\pi>x\}})(x-\pi)+\alpha\pi\big)V_x\right)$$
$$= \begin{cases} -a_1\frac{V_x}{V_{xx}}, & -a_1\frac{V_x}{V_{xx}} < x, \\ x, & -a_2\frac{V_x}{V_{xx}} \le x \le -a_1\frac{V_x}{V_{xx}}, \\ -a_2\frac{V_x}{V_{xx}}, & -a_2\frac{V_x}{V_{xx}} > x, \end{cases}$$

where, a_i, $i=1,2$ is defined in (2.5). Indeed, denote

$$f(\pi) = \frac{1}{2}\sigma^2\pi^2 V_{xx} + \big((r_1\chi_{\{\pi<x\}}+r_2\chi_{\{\pi>x\}})(x-\pi)+\alpha\pi\big)V_x,$$

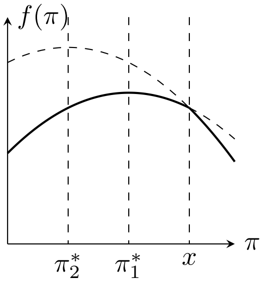

FIG. 3.1. $\pi_1^* < x$

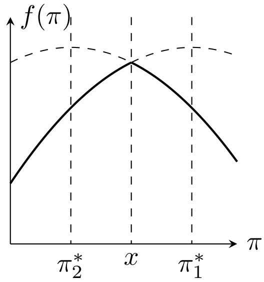

FIG. 3.2. $\pi_2^* \le x \le \pi_1^*$

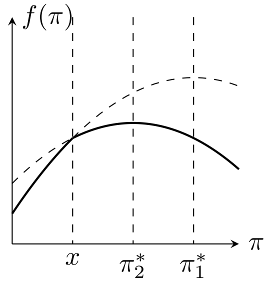

FIG. 3.3. $x < \pi_2^*$

and denote $\pi_i^* := -a_i \frac{V_x}{V_{xx}}, i=1,2$ as the symmetrical axes of corresponding quadratic function. The optimal $\pi^* = \operatorname*{argmax}_{\pi} f(\pi)$ can be obtained intuitively from Figures 3.1-3.3.

Then the equation in (3.1) can be rewritten as

$$\begin{cases} -V_t + \frac{\sigma^2 a_1^2}{2}\frac{V_x^2}{V_{xx}} + (\sigma^2 a_1 - \alpha)xV_x = 0, & -a_1\frac{V_x}{V_{xx}} < x, \\ -V_t - \frac{\sigma^2}{2}x^2 V_{xx} - \alpha x V_x = 0, & -a_2\frac{V_x}{V_{xx}} \le x \le -a_1\frac{V_x}{V_{xx}}, \\ -V_t + \frac{\sigma^2 a_2^2}{2}\frac{V_x^2}{V_{xx}} + (\sigma^2 a_2 - \alpha)xV_x = 0, & -a_2\frac{V_x}{V_{xx}} > x. \end{cases} \tag{3.3}$$

Therefore, (3.1) is a free boundary problem.

Since the fully nonlinear equations do not satisfy the parabolic condition, we make dual transformation of $V(x,t)$ (see [24]). Let

$$v(y,t) = \max_{x\ge 0}\big(V(x,t) - xy\big), \quad y>0,\, 0\le t\le T.$$

We will prove $V_x(\cdot,t)$ range in $(0,+\infty)$ for each $t\in(0,T)$ and $V_{xx}<0$ (see Theorem 6.1), then the optimal x corresponding to y is

$$x = x(y,t) := V_x^{-1}(\cdot,t)(y), \quad y>0,\, 0<t<T,$$

so we have the following correspondences between $v(y,t)$ and $V(x,t)$,

$$v(y,t) = V(x(y,t),t) - x(y,t)y,$$

$$v_y(y,t) = V_x(x(y,t),t)x_y(y,t) - yx_y(y,t) - x(y,t) = -x(y,t),$$

$$v_{yy}(y,t) = -x_y(y,t) = \frac{-1}{V_{xx}(x(y,t),t)},$$

$$v_t(y,t) = V_t(x(y,t),t) + V_x(x(y,t),t)x_t(y,t) - yx_t(y,t) = V_t(x(y,t),t).$$

Therefore, the equation on v can be derived from (3.3) as follows

$$\begin{cases} -v_t - \frac{1}{2}\sigma^2 a_1^2 y^2 v_{yy} - (\sigma^2 a_1 - \alpha)yv_y = 0, & -\frac{v_y}{v_{yy}} > a_1 y, \\ -v_t + \frac{\sigma^2}{2}\frac{v_y^2}{v_{yy}} + \alpha y v_y = 0, & a_2 y \le -\frac{v_y}{v_{yy}} \le a_1 y, \\ -v_t - \frac{1}{2}\sigma^2 a_2^2 y^2 v_{yy} - (\sigma^2 a_2 - \alpha)yv_y = 0, & -\frac{v_y}{v_{yy}} < a_2 y. \end{cases}$$

After differentiating w.r.t. y, we obtain $u:=-v_y$ satisfies

$$\begin{cases} -u_t-\frac{1}{2}\sigma^2a_1^2y^2u_{yy}+(-\sigma^2a_1^2-\sigma^2a_1+\alpha)yu_y+(\alpha-\sigma^2a_1)u=0, & -\frac{u}{u_y}>a_1y, \\ -u_t-\frac{\sigma^2}{2}\left(\frac{u}{u_y}\right)^2u_{yy}+\sigma^2u+\alpha yu_y+\alpha u=0, & a_2y\le-\frac{u}{u_y}\le a_1y, \\ -u_t-\frac{1}{2}\sigma^2a_2^2y^2u_{yy}+(-\sigma^2a_2^2-\sigma^2a_2+\alpha)yu_y+(\alpha-\sigma^2a_2)u=0, & -\frac{u}{u_y}<a_2y. \end{cases} \tag{3.4}$$

Furthermore, make a transformation that $s=T-t$, $z=\ln y$ and $u(y,t)=w(z,s)$, thus

$$u_t=-w_s,\quad u_y=w_z\frac{1}{y},\quad u_{yy}=\left(w_{zz}-w_z\right)\frac{1}{y^2},$$

we have

$$\begin{cases} w_s-\frac{1}{2}\sigma^2a_1^2w_{zz}+\left(-\frac{1}{2}\sigma^2a_1^2-\sigma^2a_1+\alpha\right)w_z+(\alpha-\sigma^2a_1)w=0, & -\frac{w}{w_z}>a_1, \\ w_s-\frac{\sigma^2}{2}\left(\frac{w}{w_z}\right)^2\left(w_{zz}-w_z\right)+\sigma^2w+\alpha w_z+\alpha w=0, & a_2\le-\frac{w}{w_z}\le a_1, \\ w_s-\frac{1}{2}\sigma^2a_2^2w_{zz}+\left(-\frac{1}{2}\sigma^2a_2^2-\sigma^2a_2+\alpha\right)w_z+(\alpha-\sigma^2a_2)w=0, & -\frac{w}{w_z}<a_2. \end{cases} \tag{3.5}$$

For convenience, we define the function

$$A(\xi)=\begin{cases} a_1, & -\xi>a_1, \\ -\xi, & a_2\le-\xi\le a_1, \\ a_2, & -\xi<a_2. \end{cases}$$

Then (3.5) can be merged into

$$w_s-\frac{1}{2}\sigma^2A^2\left(\frac{w}{w_z}\right)w_{zz}+\left(-\frac{1}{2}\sigma^2A^2\left(\frac{w}{w_z}\right)-\sigma^2A\left(\frac{w}{w_z}\right)+\alpha\right)w_z+\left(\alpha-\sigma^2A\left(\frac{w}{w_z}\right)\right)w=0. \tag{3.6}$$

Let's make dual transformation on terminal value $\Phi(x)=\frac{(x+K)^{1-\gamma}}{1-\gamma}$, define

$$\phi(y)=\max_{x\ge0}(\Phi(x)-xy),\quad y>0.$$

Then the optimal

$$x=x(y):=\begin{cases} y^{-\frac{1}{\gamma}}-K, & y\le K^{-\gamma}, \\ 0, & y>K^{-\gamma}, \end{cases}$$

Therefore,

$$v(y,T)=\phi(y)=\Phi(x(y))-x(y)y=\begin{cases} \frac{\gamma}{1-\gamma}y^{-\frac{1-\gamma}{\gamma}}+Ky, & y\le K^{-\gamma}, \\ \frac{1}{1-\gamma}K^{1-\gamma}, & y>K^{-\gamma}, \end{cases}$$

$$u(y,T)=-v_y(y,T)=-\phi'(y)=\begin{cases} y^{-\frac{1}{\gamma}}-K, & y\le K^{-\gamma}, \\ 0, & y>K^{-\gamma} \end{cases}=\left(y^{-\frac{1}{\gamma}}-K\right)^+, \tag{3.7}$$

$$w(z,0)=u(e^z,T)=\big(e^{-\frac{1}{\gamma}z}-K\big)^+,$$

Hence, we derive the initial value problem about w as follows,

$$\begin{cases} w_s-\mathcal{T}w=0, & \text{in} \quad Q_T:=\mathbb{R}\times(0,T) \\ w(z,0)=\big(e^{-\frac{1}{\gamma}z}-K\big)^+, & z\in\mathbb{R}, \end{cases} \tag{3.8}$$

where

$$\mathcal{T}w:=\frac{1}{2}\sigma^2A^2\big(\frac{w}{w_z}\big)w_{zz}-\Big(-\frac{1}{2}\sigma^2A^2\big(\frac{w}{w_z}\big)-\sigma^2A\big(\frac{w}{w_z}\big)+\alpha\Big)w_z-\Big(\alpha-\sigma^2A\big(\frac{w}{w_z}\big)\Big)w.$$

4. The properties of solution to (3.8)

Now, we begin with problem (3.8) and obtain that:

THEOREM 4.1. *There exists unique solution* $w\in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)\bigcap C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q_T}\setminus\{(-\gamma\ln K,0)\})\bigcap C(\overline{Q_T})$ *to problem* (3.8) *(for some* $\alpha\in(0,1)$*) satisfying*

$$0<w\le e^{\theta s}e^{-\frac{1}{\gamma}z}, \tag{4.1}$$

$$-\frac{1}{\gamma}e^{\theta s}e^{-\frac{1}{\gamma}z}\le w_z<0, \tag{4.2}$$

where, the constant

$$\theta=\frac{1}{2}\frac{\sigma^2a_1^2}{\gamma^2}+\frac{\alpha}{\gamma}+\frac{2\sigma^2a_1}{\gamma}+\frac{\sigma^2a_1^2}{\gamma}. \tag{4.3}$$

Moreover,

$$w_2\le w\le w_1, \tag{4.4}$$

where w_i, $i=1,2$ *are the solutions of the following linear problem*

$$\begin{cases} \partial_s w_i-\frac{1}{2}\sigma^2a_i^2\partial_{zz}w_i+\big(-\frac{1}{2}\sigma^2a_j^2-\sigma^2a_j+\alpha\big)\partial_z w_i+\big(\alpha-\sigma^2a_i\big)w_i=0 & \text{in} \quad Q_T, \\ w_i(z,0)=\big(e^{-\frac{1}{\gamma}z}-K\big)^+, \quad z\in\mathbb{R} \end{cases} \tag{4.5}$$

($j=1,2$, $j\neq i$*), which can be expressed by*

$$w_i(z,s)=\int_{-\infty}^{-\gamma\ln K}\Gamma_i(z-\xi,s)e^{p_i(z-\xi)+q_is}\big(e^{-\frac{1}{\gamma}\xi}-K\big)\mathrm{d}\xi, \tag{4.6}$$

where, $p_i=\frac{-\frac{1}{2}\sigma^2a_j^2-\sigma^2a_j+\alpha}{\sigma^2a_i^2}$, $q_i=\frac{1}{2}\sigma^2a_i^2p_i^2-\big(-\frac{1}{2}\sigma^2a_j^2-\sigma^2a_j+\alpha\big)-\alpha+\sigma^2a_i$ *are constants and*

$$\Gamma_i(x,s)=\frac{1}{\sigma a_i\sqrt{2\pi s}}e^{-\frac{x^2}{2\sigma^2a_i^2s}}$$

(see Appendix B).

Proof. We arrange it in Appendix A. ∎

LEMMA 4.1. *For any* $s\in(0,T)$*, the solution to problem* (3.8) *satisfies*

$$\lim_{z\to-\infty}w(z,s)=+\infty \tag{4.7}$$

and for any $\beta > 0$,

$$\lim_{z\to+\infty} e^{\beta z} w(z,s) = 0. \tag{4.8}$$

Proof. Since $w \geq w_2$, where w_2 is defined in Theorem 4.1, we have

$$\begin{aligned}
w(z,s) \geq & w_2(z,s) \\
= & \int_{-\infty}^{-\gamma\ln K} \Gamma(z-\xi,s) e^{p_2(z-\xi)+q_2 s}\left(e^{-\frac{1}{\gamma}\xi} - K\right)\mathrm{d}\xi \\
= & e^{q_2 s + p_2 z}\int_{-\infty}^{-\gamma\ln K} \frac{1}{\sigma a_2\sqrt{2\pi s}} e^{-\frac{(z-\xi)^2}{2\sigma^2 a_2^2 s}}\left(e^{-(p_2+\frac{1}{\gamma})\xi} - K e^{-p_2\xi}\right)\mathrm{d}\xi \\
= & e^{q_2 s + p_2 z}\Big[e^{\frac{1}{2}\sigma^2 a_2^2 (p_2+\frac{1}{\gamma})^2 s - (p_2+\frac{1}{\gamma})z}\int_{-\infty}^{-\gamma\ln K} \frac{1}{\sigma a_2\sqrt{2\pi s}} e^{-\frac{\left(\xi - z + \sigma^2 a_2^2 (p_2+\frac{1}{\gamma})s\right)^2}{2\sigma^2 a_2^2 s}}\mathrm{d}\xi \\
& \qquad - K e^{\frac{1}{2}\sigma^2 a_2^2 p_2^2 s - p_2 z}\int_{-\infty}^{-\gamma\ln K} \frac{1}{\sigma a_2\sqrt{2\pi s}} e^{-\frac{\left(\xi - z + \sigma^2 a_2^2 p_2 s\right)^2}{2\sigma^2 a_2^2 s}}\mathrm{d}\xi\Big] \\
= & e^{q_2 s}\Big[e^{\frac{1}{2}\sigma^2 a_2^2 (p_2+\frac{1}{\gamma})^2 s - \frac{1}{\gamma}z}\int_{-\infty}^{-\gamma\ln K - z + \sigma^2 a_2^2 (p_2+\frac{1}{\gamma})s} \frac{1}{\sigma a_2\sqrt{2\pi s}} e^{-\frac{\eta^2}{2\sigma^2 a_2^2 s}}\mathrm{d}\eta \\
& \qquad - K e^{\frac{1}{2}\sigma^2 a_2^2 p_2^2 s}\int_{-\infty}^{-\gamma\ln K - z + \sigma^2 a_2^2 p_2 s} \frac{1}{\sigma a_2\sqrt{2\pi s}} e^{-\frac{\eta^2}{2\sigma^2 a_2^2 s}}\mathrm{d}\eta\Big] \\
& \longrightarrow +\infty, \quad z \to -\infty,
\end{aligned}$$

so (4.7) holds.

Similarly, due to $w \leq w_1$, we have

$$\begin{aligned}
e^{\beta z} w(z,s) \leq & e^{\beta z} w_1(z,s) \\
= & e^{q_1 s + \beta z}\Big[e^{\frac{1}{2}\sigma^2 a_1^2 (p_1+\frac{1}{\gamma})^2 s - \frac{1}{\gamma}z}\int_{-\infty}^{-\gamma\ln K - z + \sigma^2 a_1^2 (p_1+\frac{1}{\gamma})s} \frac{1}{\sigma a_1\sqrt{2\pi s}} e^{-\frac{\eta^2}{2\sigma^2 a_1^2 s}}\mathrm{d}\eta \\
& \qquad - K e^{\frac{1}{2}\sigma^2 a_1^2 p_1^2 s}\int_{-\infty}^{-\gamma\ln K - z + \sigma^2 a_1^2 p_1 s} \frac{1}{\sigma a_1\sqrt{2\pi s}} e^{-\frac{\eta^2}{2\sigma^2 a_1^2 s}}\mathrm{d}\eta\Big],
\end{aligned}$$

and then (4.8) can be proved by using L'Hospital's rule.

Indeed, let's see a reduced form $e^z\int_{-\infty}^{-z} e^{-\eta^2}\mathrm{d}\eta$. Since $e^z \to +\infty$ and $\int_{-\infty}^{-z} e^{-\eta^2}\mathrm{d}\eta \to 0$ when $z \to +\infty$, so we can use L'Hospital's rule such that

$$\lim_{z\to+\infty}\frac{\int_{-\infty}^{-z} e^{-\eta^2}\mathrm{d}\eta}{e^{-z}} = \lim_{z\to+\infty}\frac{-e^{-z^2}}{-e^{-z}} = \lim_{z\to+\infty} e^{-z^2+z} = 0.$$

By this way, we can prove $\lim\limits_{z\to+\infty} e^{\beta z} w_1(z,s) = 0$, thus $\lim\limits_{z\to+\infty} e^{\beta z} w(z,s) = 0$. ∎

5. The free boundary of (3.8)

Suppose w is the solution to problem (3.8) defined in Theorem 4.1, define

$$\mathcal{B} := \left\{(z,s)\in Q_T \,\middle|\, -\frac{w}{w_z} < a_2\right\}, \quad \textbf{borrowing set},$$
$$\mathcal{L} := \left\{(z,s)\in Q_T \,\middle|\, -\frac{w}{w_z} > a_1\right\}, \quad \textbf{lending set},$$
$$\mathcal{S} := \left\{(z,s)\in Q_T \,\middle|\, a_2 \le -\frac{w}{w_z} \le a_1\right\}, \quad \textbf{stop borrowing or lending set}.$$

LEMMA 5.1. *For any* $s\in(0,T)$,

$$\sup\{z\in\mathbb{R} \,|\, (z,s)\in\mathcal{B}\} = +\infty. \tag{5.1}$$

And if $\gamma > a_2$, *for any* $s\in(0,T)$,

$$\inf\{z\in\mathbb{R} \,|\, (z,s)\notin\mathcal{B}\} = -\infty. \tag{5.2}$$

If $\gamma > a_1$,

$$\inf\{z\in\mathbb{R} \,|\, (z,s)\in\mathcal{L}\} = -\infty. \tag{5.3}$$

Proof. We first prove (5.1). By contradiction, if (5.1) is not true, there exists $s\in(0,T)$ and $z_0\in\mathbb{R}$ such that $\{z\ge z_0 \,|\, (z,s)\in\mathcal{B}\}=\emptyset$, i.e. $-\frac{w(z,s)}{w_z(z,s)} \ge a_2$, $\forall z\ge z_0$, then we have

$$\ln w(z,s) - \ln w(z_0,s) = \int_{z_0}^{z} \frac{w_z(\xi,s)}{w(\xi,s)}\mathrm{d}\xi \ge -\int_{z_0}^{z} \frac{1}{a_2}\mathrm{d}\xi = \frac{z_0 - z}{a_2}, \quad \forall z\ge z_0.$$

By Lemma 4.1, we can choose $\beta = \frac{2}{a_2}$ to let $w(z,s)\le e^{-\frac{2}{a_2}z}$ when z is sufficient large, so we have

$$-\frac{2z}{a_2} - \ln w(z_0,s) \ge \ln w(z,s) - \ln w(z_0,s) \ge \frac{z_0 - z}{a_2},$$

taking the limit through $z\to+\infty$ on both sides we get a contradiction $-\frac{2}{a_2} \ge -\frac{1}{a_2}$. So (5.1) holds.

Now, we prove (5.2). By contradiction, if $\gamma > a_2$ but (5.2) is not true, there exists $s\in(0,T)$ and $z_0\in\mathbb{R}$ such that $(z,s)\in\mathcal{B}$, $\forall z\le z_0$, i.e. $-\frac{w(z,s)}{w_z(z,s)} < a_2$, $\forall z\le z_0$, then we have

$$\ln w(z_0,s) - \ln w(z,s) = \int_{z}^{z_0} \frac{w_z(\xi,s)}{w(\xi,s)}\mathrm{d}\xi < -\int_{z}^{z_0} \frac{1}{a_2}\mathrm{d}\xi = \frac{z - z_0}{a_2}, \quad \forall z\le z_0.$$

By (4.1), $w(z,s)\le e^{\theta s}e^{-\frac{1}{\gamma}z}$, so we have

$$\ln w(z_0,s) - \theta s + \frac{z}{\gamma} \le \ln w(z_0,s) - \ln w(z,s) \le \frac{z - z_0}{a_2},$$

taking the limit through $z\to-\infty$ on both sides we get a contradiction $-\frac{1}{\gamma} \le -\frac{1}{a_2}$. So (5.2) holds.

The proof of (5.3) is similar to (5.2), so we omit it. ∎

Define

$$b(s) := \sup\left\{z\in\mathbb{R} \,\middle|\, w(z,s) + a_2 w_z(z,s) \ge 0\right\}, \quad s\in(0,T),$$

$$l(s):=\inf\Big\{z\in\mathbb{R}\,\Big|\,w(z,s)+a_1w_z(z,s)\le 0\Big\},\quad s\in(0,T).$$

LEMMA 5.2. *If $b_*(s_0-)<b^*(s_0+)$ for some $s_0\in(0,T)$, then*

$$w(z,s_0)+a_2w_z(z,s_0)=0,\quad \forall z\in(b_*(s_0-),b^*(s_0+)),\tag{5.4}$$

where, $b_*(s_0-):=\liminf\limits_{s\to s_0-}b(s)$, $b^*(s_0+):=\limsup\limits_{s\to s_0+}b(s)$.

Proof. By the continuity of $w+a_2w_z$ and the definition of $b(s)$, we have

$$w(z,s_0)+a_2w_z(z,s_0)\le 0,\quad z\ge b_*(s_0-).\tag{5.5}$$

Now, if (5.4) is not true, then there exits $z_0\in(b_*(s_0-),b^*(s_0+))$, such that $w(z_0,s_0)+a_2w_z(z_0,s_0)<0$. Since $w+a_2w_z$ is continuous, we have

$$w(z_0,s)+a_2w_z(z_0,s)<0,\, s\in[s_0,s_0+\varepsilon]\tag{5.6}$$

for a small $\varepsilon>0$. Denote the region $\mathcal{D}:=[z_0,+\infty)\times(s_0,s_0+\varepsilon)$, In the following, we prove $w+a_2w_z<0$ in $\mathcal{D}$.

Indeed, suppose ψ is the solution to

$$\begin{cases}\psi_s-\frac12\sigma^2a_2^2\psi_{zz}+\big(-\frac12\sigma^2a_2^2-\sigma^2a_2+\alpha\big)\psi_z+(\alpha-\sigma^2a_2)\psi=0 \quad \text{in}\quad \mathcal{D},\\ \big(\psi+a_2\psi_z\big)(z_0,s)=\big(w+a_2w_z\big)(z_0,s),\quad s\in[s_0,s_0+\varepsilon],\\ \psi(z,s_0)=w(z,s_0),\quad z\ge z_0.\end{cases}\tag{5.7}$$

satisfying the growth condition (4.1) and (4.2).

Differentiating the equation in (5.7) w.r.t. z, we have

$$\psi_{zs}-\frac12\sigma^2a_2^2\psi_{zzz}+\big(-\frac12\sigma^2a_2^2-\sigma^2a_2+\alpha\big)\psi_{zz}+(\alpha-\sigma^2a_2)\psi_z=0\quad\text{in}\quad\mathcal{D},$$

So $\Psi=\psi+a_2\psi_z$ satisfies

$$\begin{cases}\Psi_s-\frac12\sigma^2a_2^2\Psi_{zz}+\big(-\frac12\sigma^2a_2^2-\sigma^2a_2+\alpha\big)\Psi_z+(\alpha-\sigma^2a_2)\Psi=0 \quad \text{in}\quad \mathcal{D},\\ \Psi(z_0,s)=\big(w+a_2w_z\big)(z_0,s),\quad s\in[s_0,s_0+\varepsilon],\\ \Psi(z,s_0)=\big(w+a_2w_z\big)(z,s_0),\quad z\ge z_0.\end{cases}$$

By strong maximum principle, $\Psi<0$ in $\mathcal{D}$.

On the other hand, denote $\varphi=e^{\frac{z}{a_2}}\psi$, then φ satisfies

$$\begin{cases}\varphi_s-\frac12\sigma^2a_2^2\varphi_{zz}+\big(\sigma^2a_2^3-\frac12\sigma^2a_2^2-\sigma^2a_2+\alpha\big)\varphi_z\\ \qquad\qquad+\Big(-\frac12\sigma^2a_2^4-\big(-\frac12\sigma^2a_2^2-\sigma^2a_2+\alpha\big)a_2+\alpha-\sigma^2a_2\Big)\varphi=0\quad\text{in}\quad\mathcal{D},\\ \varphi_z(z_0,s)=e^{\frac{z}{a_2}}\frac{1}{a_2}\big(w+a_2w_z\big)(z_0,s)<0,\quad s\in[s_0,s_0+\varepsilon],\\ \varphi(z,s_0)=e^{\frac{z}{a_2}}w(z,s_0)>0,\quad z\ge z_0.\end{cases}\tag{5.8}$$

By strong maximum principle with second boundary condition, we have $\varphi>0$ in $\mathcal{D}$. So $\psi>0$ in $\mathcal{D}$. Together with $\Psi<0$, it then follows that $\psi_z<0$.

So we have $A(\frac{\psi}{\psi_z})=a_2$ in $\mathcal{D}$, thus (5.7) can be rewritten as

$$\begin{cases} \psi_s-\frac{1}{2}\sigma^2A(\frac{\psi}{\psi_z})^2\psi_{zz}+\big(-\frac{1}{2}\sigma^2A(\frac{\psi}{\psi_z})^2-\sigma^2A(\frac{\psi}{\psi_z})+\alpha\big)\psi_z+\big(\alpha-\sigma^2A(\frac{\psi}{\psi_z})\big)\psi=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in}\quad \mathcal{D}, \\ \big(\psi+a_2\psi_z\big)(z_0,s)=\big(w+a_2w_z\big)(z_0,s),\quad s\in[s_0,s_0+\varepsilon] \\ \psi(z,s_0)=w(z,s_0),\quad z\geq z_0. \end{cases} \tag{5.9}$$

By the uniqueness of solution to problem (5.9), we have $\psi=w$ in $\mathcal{D}$. So $w+a_2w_z<0$ in $\mathcal{D}$.

Therefore, by the definition of $b(s_0)$, we have $b(s)\leq z_0$, $s\in(s_0,s_0+\varepsilon)$, i.e., $b^*(s_0+)\leq z_0$, which is a contradiction to $z_0\in(b_*(s_0-),b^*(s_0+))$. □

Similarly, we can prove the following lemma.

LEMMA 5.3. *If $l_*(s_0+)<l^*(s_0-)$ for some $s_0\in(0,T)$, then*

$$w(z,s)+a_1w_z(z,s)=0,\quad \forall z\in(l_*(s_0+),l^*(s_0-)).$$

where, $l_*(s_0+):=\liminf\limits_{s\to s_0+}l(s)$, $l^*(s-):=\limsup\limits_{s\to s_0-}l(s)$.

THEOREM 5.1. *$b(s)$ is the borrowing free boundary line, i.e.,*

$$\mathcal{B}=\Big\{(z,s)\;\Big|\;z>b(s),\,s\in(0,T)\Big\}; \tag{5.10}$$

Moreover, if $\gamma>a_2$, then $b(s)$ is bounded in $(0,T)$; Otherwise, if $\gamma\leq a_2$, then $b(s)=-\infty,\forall s\in(0,T)$, i.e. $\mathcal{B}=Q_T$. (See Figures 5.1–5.3.)

Proof. Denote $\mathcal{C}:=\{(z,s)\,|\,z>b(s),\,s\in(0,T)\}$, by the definition of $b(s)$ we know $\mathcal{C}\subseteq\mathcal{B}$. In the following, we prove $\mathcal{C}\supseteq\mathcal{B}$. By contradiction, if $\mathcal{C}\supseteq\mathcal{B}$ is not true, suppose $\mathcal{D}$ is a nonempty subset of $\mathcal{B}$ contained in $\{(z,s)\,|\,z\leq b(s),\,s\in[0,T)\}$, where

$$b(0):=\inf\big\{z\in\mathbb{R}\,\big|\,w(z,0)+a_2w_z(z,0)\leq 0\big\}=\begin{cases}-\gamma\ln\big(\frac{K}{1-\frac{a_2}{\gamma}}\big), & \gamma>a_2;\\ -\infty, & \gamma\leq a_2.\end{cases}$$

Since $w+a_2w_z$ is continuous in $\{(z,s)\,|\,z\leq b(s),\,s\in[0,T)\}$, we could suppose $\mathcal{D}$ is an open connected component of $\mathcal{B}\cap\{(z,s)\,|\,z\leq b(s),\,s\in[0,T)\}$, then we have $w+a_2w_z<0$ in $\mathcal{D}$ and $w+a_2w_z=0$ on $\partial_p\mathcal{D}$, where $\partial_p\mathcal{D}$ is the parabolic boundary of $\mathcal{D}$ (which involves Lemma 5.2). Therefore,

$$w_s-\tfrac{1}{2}\sigma^2a_2^2w_{zz}+\big(-\tfrac{1}{2}\sigma^2a_2^2-\sigma^2a_2+\alpha\big)w_z+(\alpha-\sigma^2a_2)w=0\quad\text{in}\quad\mathcal{D}.$$

Differentiating w.r.t. z we get

$$w_{zs}-\tfrac{1}{2}\sigma^2a_2^2w_{zzz}+\big(-\tfrac{1}{2}\sigma^2a_2^2-\sigma^2a_2+\alpha\big)w_{zz}+(\alpha-\sigma^2a_2)w_z=0\quad\text{in}\quad\mathcal{D}.$$

So $\Psi:=w+a_2w_z$ satisfies

$$\begin{cases}\Psi_s-\frac{1}{2}\sigma^2a_2^2\Psi_{zz}+\big(-\frac{1}{2}\sigma^2a_2^2-\sigma^2a_2+\alpha\big)\Psi_z+(\alpha-\sigma^2a_2)\Psi=0\quad\text{in}\quad\mathcal{D},\\ \Psi=0\quad\text{on}\quad\partial_p\mathcal{D}.\end{cases}$$

By maximum principle, we have $\Psi=0$ in $\mathcal{D}$, which leads to a contradiction. So $\mathcal{C}\supseteq\mathcal{B}$.

Furthermore, if $\gamma>a_2$, by Lemma 5.1 we know $-\infty<b(s)<\infty$. Now, we come to prove that $b(s)$ is bounded in $(0,T)$. If not, there exist $s_n\to s_0\in[0,T]$ such that $b(s_n)\to+\infty$(or $-\infty$). By the definition of $b(s)$ and the continuity of $w+a_2w_z$ we have

$$\big(w+a_2w_z\big)(z,s_0)\geq 0(\text{or}\leq 0),\quad z\in\mathbb{R},$$

but from Lemma 5.1 and

$$\big(w+a_2w_z\big)(z,0)=\Phi'(z)+a_2\Phi'(z)=\begin{cases}(1-\frac{a_2}{\gamma})e^{-\frac{1}{\gamma}z}-K, & z<-\gamma\ln K,\\ 0, & z>-\gamma\ln K,\end{cases}$$

we know it is impossible. So $b(s)$ is bounded in $(0,T)$.

If $\gamma\leq a_2$, we come to prove that $\mathcal{B}=Q_T$. suppose ψ is the solution to

$$\begin{cases}\psi_s-\frac{1}{2}\sigma^2a_2^2\psi_{zz}+\big(-\frac{1}{2}\sigma^2a_2^2-\sigma^2a_2+\alpha\big)\psi_z+(\alpha-\sigma^2a_2)\psi=0 \quad \text{in} \quad Q_T,\\ \psi(z,0)=w(z,0),\quad z\in\mathbb{R}.\end{cases}\tag{5.11}$$

So $\Psi=\psi+a_2\psi_z$ satisfies

$$\begin{cases}\Psi_s-\frac{1}{2}\sigma^2a_2^2\Psi_{zz}+\big(-\frac{1}{2}\sigma^2a_2^2-\sigma^2a_2+\alpha\big)\Psi_z+(\alpha-\sigma^2a_2)\Psi=0 \quad \text{in} \quad Q_T,\\ \Psi(z,0)=\big(w+a_2w_z\big)(z,0),\quad z\in\mathbb{R}.\end{cases}$$

Since

$$\big(w+a_2w_z\big)(z,0)=\begin{cases}(e^{-\frac{1}{\gamma}z}-K)-\frac{a_2}{\gamma}e^{-\frac{1}{\gamma}z}, & z<-\gamma\ln K,\\ 0, & z>-\gamma\ln K\end{cases}\leq 0.$$

By strong maximum principle, $\Psi<0$ in Q_T, so we have $A\big(\frac{\psi}{\psi_z}\big)=a_2$ in $\mathcal{D}$, thus (5.11) can be rewritten as

$$\begin{cases}\psi_s-\frac{1}{2}\sigma^2A\big(\frac{\psi}{\psi_z}\big)^2\psi_{zz}+\big(-\frac{1}{2}\sigma^2A\big(\frac{\psi}{\psi_z}\big)^2-\sigma^2A\big(\frac{\psi}{\psi_z}\big)+\alpha\big)\psi_z+\big(\alpha-\sigma^2A\big(\frac{\psi}{\psi_z}\big)\big)\psi=0\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad Q_T,\\ \psi(z,0)=w(z,0),\quad z\in\mathbb{R}.\end{cases}\tag{5.12}$$

By the uniqueness of solution to (5.12), we have $\psi=w$ in Q_T. So $w+a_2w_z<0$ in Q_T, i.e. $\mathcal{B}=Q_T$. □

Theorem 5.2. *$l(s)$ is the lending free boundary line, i.e.,*

$$\mathcal{L}=\Big\{(z,s)\,\Big|\,z<l(s),\,s\in(0,T)\Big\}.\tag{5.13}$$

Moreover, if $\gamma>a_1$, then $l(s)$ is bounded in $(0,T)$; Otherwise, if $\gamma\leq a_1$, then $l(s)=-\infty,\forall s\in(0,T)$ and $\mathcal{L}=\emptyset$. (See Figures 5.1–5.3.)

Proof. The proof of (5.13) and boundedness of $l(s)$ if $\gamma>a_1$ is similar to Lemma 5.1, so we omit it. Now, we prove $\mathcal{L}=\emptyset$ if $\gamma\leq a_1$. If not, $\mathcal{L}\neq\emptyset$, we could suppose $\mathcal{D}$ is an open connected component of $\mathcal{L}$, then we have $w+a_1w_z>0$ in $\mathcal{D}$. Note that

$$w(z,0)+a_1w_z(z,0)=\begin{cases}(e^{-\frac{1}{\gamma}z}-K)-\frac{a_1}{\gamma}e^{-\frac{1}{\gamma}z}, & z<-\gamma\ln K,\\ 0, & z>-\gamma\ln K\end{cases}\leq 0,$$

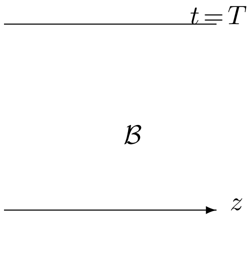

FIG. 5.1. $\gamma \le a_2$

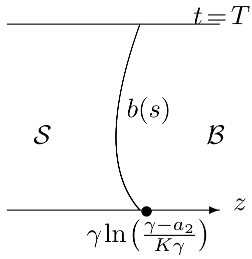

FIG. 5.2. $a_2 < \gamma \le a_1$

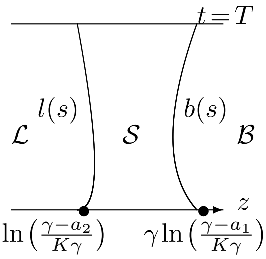

FIG. 5.3. $\gamma > a_1$

so $w + a_1 w_z = 0$ on $\partial_p \mathcal{D}$. Therefore,

$$w_s - \tfrac{1}{2}\sigma^2 a_1^2 w_{zz} + \big(-\tfrac{1}{2}\sigma^2 a_1^2 - \sigma^2 a_1 + \alpha\big) w_z + (\alpha - \sigma^2 a_1) w = 0 \quad \text{in} \quad \mathcal{D}.$$

So $\Psi := w + a_1 w_z$ satisfies

$$\begin{cases} \Psi_s - \tfrac{1}{2}\sigma^2 a_1^2 \Psi_{zz} + \big(-\tfrac{1}{2}\sigma^2 a_1^2 - \sigma^2 a_1 + \alpha\big)\Psi_z + (\alpha - \sigma^2 a_1)\Psi = 0 & \text{in} \quad \mathcal{D}, \\ \Psi = 0 \quad \text{on} \quad \partial_p \mathcal{D}. \end{cases}$$

By maximum principle, we have $\Psi = 0$, which leads to a contradiction. So $\mathcal{L} = \emptyset$. ∎

6. The solution and the free boundaries of original problem (3.1)

First, we rewrite the problem of u and v as the following

$$\begin{cases} -u_t - \mathcal{J}u = 0 \quad \text{in} \quad (0,+\infty)\times(0,T), \\ u(y,T) = \big(y^{-\frac{1}{\gamma}} - K\big)^+, \quad y > 0, \end{cases} \tag{6.1}$$

where

$$\mathcal{J}u := \frac{1}{2}\sigma^2 A\Big(\frac{u}{yu_y}\Big)^2 y^2 u_{yy} - \Big(-\sigma^2 A^2\Big(\frac{u}{yu_y}\Big) - \sigma^2 A\Big(\frac{u}{yu_y}\Big) + \alpha\Big) y u_y - \Big(\alpha - \sigma^2 A\Big(\frac{u}{yu_y}\Big)\Big) u.$$

$$\begin{cases} -v_t - \mathcal{H}v = 0 \quad \text{in} \quad (0,+\infty)\times(0,T), \\ v(y,T) = \phi(y) = \begin{cases} \frac{\gamma}{1-\gamma} y^{-\frac{1-\gamma}{\gamma}} + Ky, \; 0 < y \le K^{-\gamma}, \\ \frac{1}{1-\gamma} K^{1-\gamma}, \qquad y > K^{-\gamma}, \end{cases} \end{cases} \tag{6.2}$$

where

$$\mathcal{H}v := \frac{1}{2}\sigma^2 A^2\Big(\frac{v_y}{yv_{yy}}\Big) y^2 v_{yy} - \Big(\alpha - \sigma^2 A\Big(\frac{v_y}{yv_{yy}}\Big)\Big) y v_y.$$

According to Theorem 4.1 and Lemma 4.1, we have:

LEMMA 6.1. *There exists a unique solution* $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}\big((0,+\infty)\times(0,T)\big) \bigcap C\big((0,+\infty)\times[0,T]\big)$ *to problem* (6.1) *satisfying*

$$0 < u \le e^{\theta(T-t)} y^{-\frac{1}{\gamma}}, \tag{6.3}$$

$$-\frac{1}{\gamma}e^{\theta(T-t)}y^{\frac{\gamma-1}{\gamma}} \le u_y < 0 \tag{6.4}$$

in $(0,+\infty)\times(0,T)$. *Moreover, for any* $t\in(0,T)$,

$$\lim_{y\to 0+} u = +\infty, \qquad \lim_{y\to+\infty} u = 0, \qquad \lim_{y\to+\infty} yu = 0. \tag{6.5}$$

LEMMA 6.2. *There exists a solution* $v\in C^{3,2}\big((0,+\infty)\times(0,T)\big)\bigcap C\big((0,+\infty)\times[0,T]\big)$ *to problem* (6.2) *satisfying*

$$\phi \le v \le \frac{K^{1-\gamma}}{1-\gamma} + e^{\kappa(T-t)}y^{-\lambda}, \tag{6.6}$$

$$-e^{\theta(T-t)}y^{-\frac{1}{\gamma}} \le v_y < 0, \tag{6.7}$$

$$0 < v_{yy} \le \frac{1}{\gamma}e^{\theta(T-t)}y^{\frac{1-\gamma}{\gamma}} \tag{6.8}$$

in $(0,+\infty)\times(0,T)$, *where* κ, λ *are large constants to let* $\frac{K^{1-\gamma}}{1-\gamma}+y^{-\lambda}\ge\phi(y)$, $\forall y>0$ *and* $\kappa\ge\frac{1}{2}\sigma^2a_1^2\lambda(\lambda+1)+\alpha\lambda$.

Moreover, for any $t\in(0,T)$,

$$\lim_{y\to 0+} v_y = -\infty, \qquad \lim_{y\to+\infty} v_y = 0, \qquad \lim_{y\to+\infty} yv_y = 0, \tag{6.9}$$

$$\lim_{y\to+\infty} v = \frac{K^{1-\gamma}}{1-\gamma}. \tag{6.10}$$

Proof. Suppose u is the solution to problem (6.1), define

$$v(y,t) := -\int_{K^{-\gamma}}^{y} u(\xi,t)\mathrm{d}\xi + \int_t^T h(\tau)\mathrm{d}\tau + \frac{K^{1-\gamma}}{1-\gamma},$$

where

$$h(\tau) := \Big(-\frac{1}{2}\sigma^2A^2\Big(\frac{u}{yu_y}\Big)y^2u_y - \Big(\sigma^2A\Big(\frac{u}{yu_y}\Big)-\alpha\Big)yu\Big)(K^{-\gamma},\tau),$$

then $v_y=-u$, so we have

$$\begin{aligned}
&\partial_y(-v_t-\mathcal{H}v)\\
&= -v_{ty} - \partial_y\Big[\frac{1}{2}\sigma^2A^2\Big(\frac{v_y}{yv_{yy}}\Big)y^2v_{yy} - \Big(\alpha-\sigma^2A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)yv_y\Big]\\
&= -v_{ty} - \Big[\frac{1}{2}\sigma^2A^2\Big(\frac{v_y}{yv_{yy}}\Big)(2yv_{yy}+y^2v_{yyy}) - \Big(\alpha-\sigma^2A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)(v_y+yv_{yy})\Big]\\
&\quad +\sigma^2\partial_y\Big(A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)\Big(A\Big(\frac{v_y}{yv_{yy}}\Big)y^2v_{yy}+yv_y\Big)\\
&= -v_{ty} - \Big[\frac{1}{2}\sigma^2A^2\Big(\frac{v_y}{yv_{yy}}\Big)(2yv_{yy}+y^2v_{yyy}) - \Big(\alpha-\sigma^2A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)(v_y+yv_{yy})\Big]\\
&= -v_{ty} - \Big[\frac{1}{2}\sigma^2A^2\Big(\frac{v_y}{yv_{yy}}\Big)y^2v_{yyy} + \Big(\sigma^2A^2\Big(\frac{v_y}{yv_{yy}}\Big) - (\alpha-\sigma^2A\Big(\frac{v_y}{yv_{yy}}\Big))\Big)yv_{yy}\\
&\quad -\Big(\alpha-\sigma^2A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)v_y\Big]
\end{aligned}$$

$$
\begin{aligned}
&= u_t + \Big[\frac{1}{2}\sigma^2 A^2\Big(\frac{u}{yu_y}\Big)y^2 u_{yy} + \Big(\sigma^2 A^2\Big(\frac{u}{yu_y}\Big) - \big(\alpha - \sigma^2 A\Big(\frac{u}{yu_y}\Big)\big)\Big) yu_y \\
&\quad - \Big(\alpha - \sigma^2 A\Big(\frac{u}{yu_y}\Big)\Big)u\Big] - \Big(\alpha - \sigma^2 A\Big(\frac{u}{yu_y}\Big)\Big)u\Big] \\
&= u_t + \mathcal{T}u,
\end{aligned}
$$

where, in the third equation, $\partial_y\Big(A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)\Big(A\Big(\frac{v_y}{yv_{yy}}\Big)y^2 v_{yy} + yv_y\Big) = 0$ because

$$
\begin{cases}
A\Big(\frac{v_y}{yv_{yy}}\Big)y^2 v_{yy} + yv_y = 0 & \text{if} \quad a_2 < -\frac{v_y}{yv_{yy}} < a_1, \\
\partial_y\Big(A\Big(\frac{v_y}{yv_{yy}}\Big)\Big) = 0 & \text{if} \quad -\frac{v_y}{yv_{yy}} \le a_2 \text{ or } -\frac{v_y}{yv_{yy}} \ge a_1.
\end{cases}
$$

Note that

$$(-v_t - \mathcal{H}v)(K^{-\gamma}, t) = 0,$$

so

$$(-v_t - \mathcal{H}v)(y,t) = (-v_t - \mathcal{H}v)(K^{-\gamma}, t) + \int_{K^{-\gamma}}^{y} \partial_y(-v_t - \mathcal{H}v)(\xi, t)\mathrm{d}\xi = 0.$$

Therefore, v is a solution to problem (6.2).

Since $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}\big((0,+\infty)\times(0,T)\big) \bigcap C\big((0,+\infty)\times[0,T]\big)$, we have $v, v_y \in C^{2+\alpha,1+\frac{\alpha}{2}}\big((0,+\infty)\times[0,T]\big) \bigcap C\big((0,+\infty)\times[0,T]\big)$. Differentiating the left-hand side of the equation in (6.2) w.r.t. t,

$$
\begin{aligned}
\partial_t(-v_t - \mathcal{H}v) &= -v_{ty} - \partial_t\Big[\frac{1}{2}\sigma^2 A^2\Big(\frac{v_y}{yv_{yy}}\Big)y^2 v_{yy} - \Big(\alpha - \sigma^2 A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)yv_y\Big] \\
&= -v_{ty} - \Big[\frac{1}{2}\sigma^2 A^2\Big(\frac{v_y}{yv_{yy}}\Big)y^2 v_{tyy} - \Big(\alpha - \sigma^2 A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)yv_{ty}\Big] \\
&\quad + \sigma^2 \partial_t\Big(A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)\Big(A\Big(\frac{v_y}{yv_{yy}}\Big)y^2 v_{yy} + yv_y\Big) \\
&= -v_{ty} - \frac{1}{2}\sigma^2 A^2\Big(\frac{v_y}{yv_{yy}}\Big)y^2 v_{tyy} + \Big(\alpha - \sigma^2 A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)yv_{ty},
\end{aligned}
$$

where, in the third equation, $\partial_t\Big(A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)\Big(A\Big(\frac{v_y}{yv_{yy}}\Big)y^2 v_{yy} + yv_y\Big) = 0$ is due to the fact that either $A\Big(\frac{v_y}{yv_{yy}}\Big)y^2 v_{yy} + yv_y = 0$ or $\partial_t\Big(A\Big(\frac{v_y}{yv_{yy}}\Big)\Big) = 0$. So we have

$$-v_{ty} - \frac{1}{2}\sigma^2 A^2\Big(\frac{v_y}{yv_{yy}}\Big)y^2 v_{tyy} + \Big(\alpha - \sigma^2 A\Big(\frac{v_y}{yv_{yy}}\Big)\Big)yv_{ty} = 0.$$

By using Schauder interior estimation (see [17]), we get $v_t \in C^{2+\alpha,1+\frac{\alpha}{2}}\big((0,+\infty)\times(0,T)\big)$, therefore, we have $v \in C^{3,2}\big((0,+\infty)\times(0,T)\big) \bigcap C\big((0,+\infty)\times[0,T]\big)$.

Now, we prove (6.6). Note that

$$\phi_t = 0, \quad \phi_y \le 0, \quad \phi_{yy} \ge 0,$$

so

$$-\phi_t - \frac{1}{2}\sigma^2 A^2\big(\frac{v_y}{yv_{yy}}\big)y^2 \phi_{yy} + \Big(\alpha - \sigma^2 A\big(\frac{v_y}{yv_{yy}}\big)\Big)y\phi_y \le 0.$$

By using comparison principle we could prove $v\geq\phi$. Denote $\Psi:=\frac{K^{1-\gamma}}{1-\gamma}+e^{\kappa(T-t)}y^{-\lambda}$, note that

$$\begin{aligned}&-\Psi_t-\frac{1}{2}\sigma^2A^2(\frac{v_y}{yv_{yy}})y^2\Psi_{yy}+\Big(\alpha-\sigma^2A(\frac{v_y}{yv_{yy}})\Big)y\Psi_y\\&=e^{\kappa(T-t)}y^{-\lambda}\Big(\kappa-\frac{1}{2}\sigma^2A^2(\frac{v_y}{yv_{yy}})\lambda(\lambda+1)-\Big(\alpha-\sigma^2A(\frac{v_y}{yv_{yy}})\Big)\lambda\Big)\\&\geq e^{\kappa(T-t)}y^{-\lambda}\Big(\kappa-\frac{1}{2}\sigma^2a_1^2\lambda(\lambda+1)-\alpha\lambda\Big)\\&=0.\end{aligned}$$

By using comparison principle for linear equations we have $v\leq\Psi$.

Moreover, (6.7), (6.8) and (6.9) are the direct conclusions of (6.3), (6.4) and (6.5) respectively and (6.6) implies (6.10). □

THEOREM 6.1. *There exists unique solution* $V\in C^{3,2}\big((0,+\infty)\times(0,T)\big)\bigcap C\big([0,+\infty)\times[0,T]\big)$ *to problem* (3.1) *satisfying*

$$\Phi\leq V\leq\frac{K^{1-\gamma}}{1-\gamma}+C_Tx^{\frac{\lambda}{\lambda+1}}, \tag{6.11}$$

$$V_x>0, \tag{6.12}$$

$$V_{xx}<0 \tag{6.13}$$

in $(0,+\infty)\times(0,T)$*, where, the constant* $\lambda>0$ *is defined in Lemma 6.2,* $C_T>0$ *only depends on* T*. Moreover,*

$$\lim_{x\to0+}V_x=+\infty,\qquad\lim_{x\to+\infty}V_x=0,\quad\forall t\in(0,T). \tag{6.14}$$

Proof. Suppose v is the solution to problem (6.2), define

$$V(x,t):=\inf_{y>0}(v(y,t)+xy),\quad x>0,t\in[0,T]. \tag{6.15}$$

We come to prove that V defined in (6.15) is the solution to problem (3.1).

Equations (6.8) and (6.9) imply that $v_y(\cdot,t)$ is strictly increasing and ranges in $(-\infty,0)$ for each $t\in(0,T)$, so the optimal

$$y^*=\arg\min_{y>0}(v(y,t)+xy)=J(x,t):=(v_y(\cdot,t))^{-1}(-x),\quad\forall x>0,0<t<T,$$

and

$$V(x,t)=v(J(x,t),t)+xJ(x,t), \tag{6.16}$$

where, $J(x,t)\in C((0,+\infty)\times(0,T))$ and is decreasing w.r.t. x. Note that

$$\begin{aligned}&V_x(x,t)=v_y(J(x,t),t)J_x(x,t)+xJ_x(x,t)+J(x,t)=J(x,t)>0,\\&V_{xx}(x,t)=J_x(x,t)=\partial_x[(v_y(\cdot,t))^{-1}(x)]=\frac{-1}{v_{yy}(J(x,t),t)}<0,\\&V_t(x,t)=v_y(J(x,t),t)J_t(x,t)+v_t(J(x,t),t)+xJ_t(x,t)=v_t(J(x,t),t),\end{aligned}$$

Since $v\in C^{3,2}\big((0,+\infty)\times(0,T)\big)$, so $V\in C^{3,2}\big((0,+\infty)\times(0,T)\big)$.

From (6.9) we know for any $t\in(0,T)$,

$$\lim_{x\to 0+} J(x,t)=+\infty,\quad \lim_{x\to 0+} xJ(x,t)=0,\quad \lim_{x\to +\infty} J(x,t)=0. \tag{6.17}$$

So (6.14) holds.

Moreover, (6.17) and (6.16) imply $V(0+,t)=v(+\infty,t)=\frac{K^{1-\gamma}}{1-\gamma}$, so the boundary condition in (3.1) holds.

Now, we verify the terminal condition. Thanks to (6.6), which implies $v(y,t)\geq \phi(y)$, thus

$$V(x,t)=\inf_{y>0}(v(y,t)+xy)\geq \inf_{y>0}(\phi(y)+xy)=\Phi(x). \tag{6.18}$$

On the other hand, note that

$$V(x,t)=\inf_{y>0}(v(y,t)+xy)\leq v(\Phi'(x),t)+x\Phi'(x).$$

Let $t\to T-$ we get

$$\limsup_{t\to T-} V(x,t)\leq \lim_{t\to T-} v(\Phi'(x),t)+x\Phi'(x)=\phi(\Phi'(x))+x\Phi'(x)=\Phi(x). \tag{6.19}$$

Equations (6.18) and (6.19) imply that V satisfies the terminal condition in (3.1).

Moreover, (6.6) implies (6.11). □

Now, we give the properties of the free boundary to original problem (3.1). Suppose V is the solution to problem (3.8), $b(s)$ and $l(s)$ are the free boundaries of problem (3.8), define

$$\mathfrak{B}:=\Big\{(x,t)\in(0,+\infty)\times(0,T)\;\Big|\;-a_2\frac{V_x}{V_{xx}}>x\Big\},\quad \textbf{borrowing set},$$

$$\mathfrak{L}:=\Big\{(x,t)\in(0,+\infty)\times(0,T)\;\Big|\;-a_1\frac{V_x}{V_{xx}}<x\Big\},\quad \textbf{lending set},$$

$$\mathfrak{S}:=\Big\{(x,t)\in(0,+\infty)\times(0,T)\;\Big|\;-a_2\frac{V_x}{V_{xx}}\leq x\leq -a_1\frac{V_x}{V_{xx}}\Big\},$$

stop borrowing or lending set.

According to the correspondence

$$x=-v_y(y,t)=u(y,t)=w(z,s),$$

we can define the free boundary lines to problem (3.1) as

$$B(t):=w(b(T-t),T-t),\quad L(t):=w(l(T-t),T-t).$$

According to Lemma 5.1 and Lemma 5.2, we have

THEOREM 6.2. *$B(s)$ and $L(s)$ are the borrowing and lending free boundary lines to problem (3.1), i.e.,*

$$\mathfrak{B}=\Big\{(x,s)\;\Big|\;x<B(t),\,t\in(0,T)\Big\},\quad \mathfrak{L}=\Big\{(x,s)\;\Big|\;x>L(s),\,t\in(0,T)\Big\}.$$

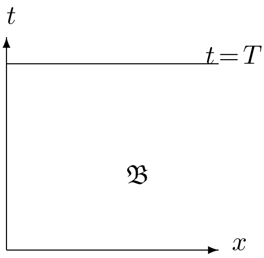

FIG. 6.1. $\gamma \leq a_2$

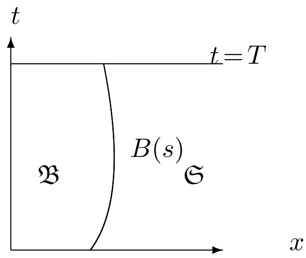

FIG. 6.2. $a_2 < \gamma \leq a_1$

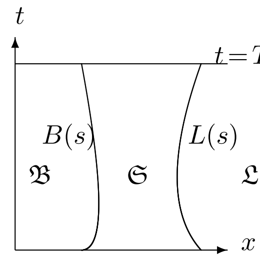

FIG. 6.3. $\gamma > a_1$

Moreover, if $\gamma \leq a_2$, then $B(t) = L(t) = +\infty, \forall t \in (0,T)$. If $a_2 < \gamma \leq a_1$, then $B(t)$ is bounded in $(0,T)$ and $L(t) = +\infty, \forall t \in (0,T)$. If $\gamma > a_1$, then both $B(t)$ and $L(t)$ are bounded in $(0,T)$ satisfying $B(t) < L(t), \forall t \in (0,T)$.

Appendix A. In this section, we prove Theorem 4.1.

Firstly, let's see the function

$$G(\xi,\eta) := A\Big(\frac{\xi}{\eta}\Big),$$

We claim that it is Lipschitz continuous in $[\varepsilon,\infty] \times [-\infty,\infty]$ for any fixed $\varepsilon > 0$. Indeed,

$$G_\xi(\xi,\eta) = A'\Big(\frac{\xi}{\eta}\Big)\frac{1}{\eta} = \begin{cases} -\frac{1}{\eta} = -\frac{\xi}{\eta}\frac{1}{\xi} \in (0, \frac{a_1}{\varepsilon}), \; a_2 < -\frac{\xi}{\eta} < a_1 \\ 0, \qquad -\frac{\xi}{\eta} > a_1 \quad \text{or} \quad -\frac{\xi}{\eta} < a_2. \end{cases}$$

$$G_\eta(\xi,\eta) = -A'\Big(\frac{\xi}{\eta}\Big)\frac{1}{\eta^2} = \begin{cases} \frac{1}{\eta^2} = \frac{\xi^2}{\eta^2}\frac{1}{\xi^2} \in (0, \frac{a_1^2}{\varepsilon^2}), \; a_2 < -\frac{\xi}{\eta} < a_1 \\ 0, \qquad -\frac{\xi}{\eta} > a_1 \quad \text{or} \quad -\frac{\xi}{\eta} < a_2. \end{cases}$$

Now, consider the approximation problem in bounded domain

$$\begin{cases} w_s^{\varepsilon,N} - \frac{1}{2}\sigma^2 A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big) w_{zz}^{\varepsilon,N} + \Big(-\frac{1}{2}\sigma^2 A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big) - \sigma^2 A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big) + \alpha\Big) w_z^{\varepsilon,N} \\ \qquad + \Big(\alpha - \sigma^2 A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Big) w^{\varepsilon,N} = 0 \quad \text{in} \quad Q_T^N := (-N,N)\times(0,T), \\ w_z^{\varepsilon,N}(-N,s) = -\frac{1}{\gamma}e^{\frac{1}{\gamma}N}, \quad w_z^{\varepsilon,N}(N,s) = 0, \quad s \in (0,T), \\ w^{\varepsilon,N}(z,0) = \pi_\varepsilon(e^{-\frac{1}{\gamma}z} - K), \quad z \in \mathbb{R}, \end{cases} \tag{A.1}$$

where $\pi_\varepsilon(x) \in C^2(\mathbb{R})$ is defined as

$$\pi_\varepsilon(x) = \begin{cases} 0, \quad x \leq -\varepsilon; \\ \nearrow, \; -\varepsilon < x < \varepsilon; \\ x, \quad x \geq \varepsilon. \end{cases} \tag{A.2}$$

(See Figures A.1-A.3).

The Leray-Schauder fixed point theorem (see [12, 13]) and embedding theorem (see [12]) implies the existence of $C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q_T^N})$ solution to problem (A.1). Moreover, Schauder estimation (see [17]) implies $w^{\varepsilon,N} \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T^N})$.

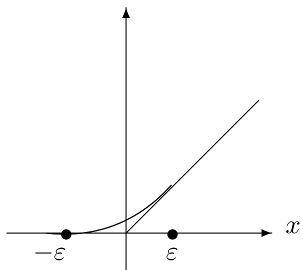

FIG. A.1. $\pi_\varepsilon(x)$

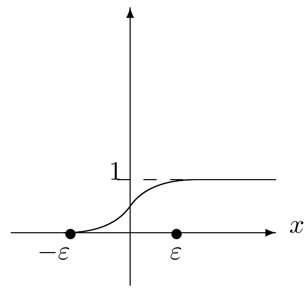

FIG. A.2. $\pi'_\varepsilon(x)$

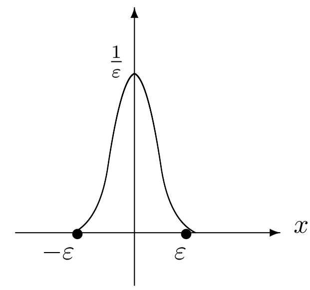

FIG. A.3. $\pi''_\varepsilon(x)$

Now, we prove

$$0\leq w^{\varepsilon,N}\leq e^{\theta s}e^{-\frac{1}{\gamma}z}. \tag{A.3}$$

The first inequality can immediately be proved by maximum principle. Denoting $\varphi = e^{\theta s}e^{-\frac{1}{\gamma}z}$, note that

$$\begin{aligned}
&\varphi_s-\frac{1}{2}\sigma^2A^2(\cdot)\varphi_{zz}+\Big(-\frac{1}{2}\sigma^2A^2(\cdot)-\sigma^2A(\cdot)+\alpha\Big)\varphi_z+\Big(\alpha-\sigma^2A(\cdot)\Big)\varphi\\
=&e^{\theta s}e^{-\frac{1}{\gamma}z}\Big[\theta-\frac{1}{2}\sigma^2A^2(\cdot)\frac{1}{\gamma^2}-\Big(-\frac{1}{2}\sigma^2A^2(\cdot)-\sigma^2A(\cdot)+\alpha\Big)\frac{1}{\gamma}+\Big(\alpha-\sigma^2A(\cdot)\Big)\Big]\\
\geq&e^{\theta s}e^{-\frac{1}{\gamma}z}\Big[\theta-\frac{1}{2}\sigma^2a_1^2\frac{1}{\gamma^2}-\alpha\frac{1}{\gamma}+\big(\alpha-\sigma^2a_1\big)\Big]\\
\geq&0.
\end{aligned}$$

Moreover,

$$\begin{cases}
\varphi(z,0)=e^{-\frac{1}{\gamma}z}\geq\pi_\varepsilon(e^{-\frac{1}{\gamma}z}-K)=w^{\varepsilon,N}(z,0), & z\in\mathbb{R},\\
\varphi_z(-N,s)=-\frac{1}{\gamma}e^{\theta s}e^{\frac{1}{\gamma}N}\leq-\frac{1}{\gamma}e^{\frac{1}{\gamma}N}=w_z^{\varepsilon,N}(-N,s), & s\in(0,T),\\
\varphi_z(N,s)=0=w_z^{\varepsilon,N}(N,s), & s\in(0,T),
\end{cases}$$

by comparison principle, the second inequality in (A.3) holds.

In the following, we prove

$$-\frac{1}{\gamma}e^{\theta s}e^{-\frac{1}{\gamma}z}\leq w_z^{\varepsilon,N}\leq 0. \tag{A.4}$$

Differentiating the equation in (A.1) we obtain the equation on $w_z^{\varepsilon,N}$ in divergence form as follows

$$\begin{aligned}
&\partial_s w_z^{\varepsilon,N}-\frac{\sigma^2}{2}\partial_z\Big(A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\partial_z w_z^{\varepsilon,N}\Big)\\
&+\Big(-\frac{1}{2}\sigma^2A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)-\sigma^2A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)+\alpha\Big)\partial_z w_z^{\varepsilon,N}+\Big(\alpha-\sigma^2A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Big)w_z^{\varepsilon,N}\\
&\qquad-\sigma^2A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Big(1-\frac{w^{\varepsilon,N}+\varepsilon}{(w_z^{\varepsilon,N})^2}w_{zz}^{\varepsilon,N}\Big)\Big(A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)+1\Big)w_z^{\varepsilon,N}
\end{aligned}$$

$$-\sigma^2 A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Big(1-\frac{w^{\varepsilon,N}+\varepsilon}{(w_z^{\varepsilon,N})^2}w_{zz}^{\varepsilon,N}\Big)w^{\varepsilon,N}=0,$$

after reorganizing, we get

$$\begin{aligned}\partial_s w_z^{\varepsilon,N}-\frac{\sigma^2}{2}\partial_z\Big(A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\partial_z w_z^{\varepsilon,N}\Big)&\\+\Big(-\frac{1}{2}\sigma^2A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)-\sigma^2A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)+\alpha\Big)\partial_z w_z^{\varepsilon,N}&\\+\Big(\alpha-\sigma^2A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Big)w_z^{\varepsilon,N}-\sigma^2A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Big(A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)+1\Big)w_z^{\varepsilon,N}&\\+\sigma^2A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\Big(A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)+1\Big)\partial_z w_z^{\varepsilon,N}&\\+\sigma^2A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\frac{(w^{\varepsilon,N}+\varepsilon)w^{\varepsilon,N}}{(w_z^{\varepsilon,N})^2}\partial_z w_z^{\varepsilon,N}=\sigma^2A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)w^{\varepsilon,N}\le 0.\quad&\text{(A.5)}\end{aligned}$$

Note that $|A(\xi)|\le a_1$, $|A'(\xi)|\le 1$, $|A'(\xi)\xi|\le a_1$ and $|A'(\xi)\xi^2|\le a_1^2$. Using maximum principle for divergence form (see [12] or [23]), we obtain $w_z^{\varepsilon,N}\le 0$.

Denoting $\Psi=e^{\frac{1}{\gamma}z}w_z^{\varepsilon,N}$, according to (A.5), we have

$$\begin{aligned}\partial_s\Psi-\frac{\sigma^2}{2}\partial_z\Big(A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Psi_z\Big)+\frac{1}{\gamma}\sigma^2A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Psi_z+\frac{1}{\gamma^2}\frac{\sigma^2}{2}A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Psi&\\+\frac{1}{\gamma}\sigma^2A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Big(A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Psi-A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big(\Psi_z-\frac{1}{\gamma}\Psi\big)\Big)&\\+\Big(-\frac{1}{2}\sigma^2A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)-\sigma^2A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)+\alpha\Big)\big(\Psi_z-\frac{1}{\gamma}\Psi\big)&\\+\Big(\alpha-\sigma^2A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Big)\Psi-\sigma^2A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Big(A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)+1\Big)\Psi&\\+\sigma^2A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\Big(A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)+1\Big)\big(\Psi_z-\frac{1}{\gamma}\Psi\big)&\\+\sigma^2A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\frac{(w^{\varepsilon,N}+\varepsilon)w^{\varepsilon,N}}{(w_z^{\varepsilon,N})^2}\big(\Psi_z-\frac{1}{\gamma}\Psi\big)&\\=e^{\frac{1}{\gamma}z}\sigma^2A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)w^{\varepsilon,N}\ge-\sigma^2e^{\theta s}.&\end{aligned}$$

where, the last inequality is due to (A.3). Let $\psi:=-\frac{1}{\gamma}e^{\theta s}$, for sufficiently large θ ($\theta\ge\frac{2\sigma^2a_1}{\gamma}+\frac{\alpha}{\gamma}+\frac{\sigma^2a_1^2}{\gamma}$), we have

$$\begin{aligned}\partial_s\psi-\frac{\sigma^2}{2}\partial_z\Big(A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\psi_z\Big)+\frac{1}{\gamma}\sigma^2A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\psi_z+\frac{1}{\gamma^2}\frac{\sigma^2}{2}A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\psi&\\+\frac{1}{\gamma}\sigma^2A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\Big(A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\psi-A'\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big(\psi_z-\frac{1}{\gamma}\psi\big)\Big)&\\+\Big(-\frac{1}{2}\sigma^2A^2\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)-\sigma^2A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)+\alpha\Big)\big(\psi_z-\frac{1}{\gamma}\psi\big)&\end{aligned}$$

$$+\Big(\alpha-\sigma^2A(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}})\Big)\psi-\sigma^2A'(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}})\Big(A(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}})+1\Big)\psi$$
$$+\sigma^2A'(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}})\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\Big(A(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}})+1\Big)(\psi_z-\frac{1}{\gamma}\psi)$$
$$+\sigma^2A'(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}})\frac{(w^{\varepsilon,N}+\varepsilon)w^{\varepsilon,N}}{(w_z^{\varepsilon,N})^2}(\psi_z-\frac{1}{\gamma}\psi)\leq-\sigma^2e^{\theta s}.$$

Moreover,

$$\begin{cases}\Psi(z,0)=-\pi_\varepsilon'(e^{-\frac{1}{\gamma}z}-K)\frac{1}{\gamma}\geq-\frac{1}{\gamma}=\psi(z,0), & z\in\mathbb{R},\\ \Psi(-N,s)=-\frac{1}{\gamma}e^{\theta s}=\psi(-N,s), & s\in(0,T),\\ \Psi(N,s)=0\geq-\frac{1}{\gamma}e^{\theta s}=\psi(N,s), & s\in(0,T).\end{cases}$$

Using maximum principle of divergence form (see [12] or [23]), we obtain $\Psi\geq\psi$, i.e., $e^{\frac{1}{\gamma}z}w_z^{\varepsilon,N}\geq-\frac{1}{\gamma}e^{\theta s}$, so the first inequality in (A.4) holds.

For each $a<b$, $r>0$, taking $C^{\alpha,\frac{\alpha}{2}}$ interior estimate (see [17] or [18]) to the equation in (A.1) and (A.5) respectively we obtain

$$|w^{\varepsilon,N}|_{C^{\alpha,\frac{\alpha}{2}}([a,b]\times[0,T])},\quad |w_z^{\varepsilon,N}|_{C^{\alpha,\frac{\alpha}{2}}([a,b]\times[0,T]\setminus B_r)}\leq C,$$

where $B_r:=\{(z,s)\,|\,(z+\gamma\ln K)^2+s^2<r^2\}$, C does not depend on ε, N.

And then taking Schauder interior estimate to the equation in (A.1) we get

$$|w^{\varepsilon,N}|_{C^{2+\alpha,1+\frac{\alpha}{2}}([a,b]\times[0,T])}\leq C_\varepsilon$$

where C_ε does not depend on N.

Therefore, there exists $w^\varepsilon\in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q_T})$ such that, for any region $Q=(a,b)\times(0,T)\subset Q_T$, there exists a subsequence of $w^{\varepsilon,N}$, which we still denote by $w^{\varepsilon,N}$, such that $w^{\varepsilon,N}\to w^\varepsilon$ in $C^{2,1}(\overline{Q})$, $N\to\infty$. So w^ε satisfies

$$\begin{cases}w_s^\varepsilon-\frac{1}{2}\sigma^2A^2(\frac{w^\varepsilon+\varepsilon}{w_z^\varepsilon})w_{zz}^\varepsilon+\Big(-\frac{1}{2}\sigma^2A^2(\frac{w^\varepsilon+\varepsilon}{w_z^\varepsilon})-\sigma^2A(\frac{w^\varepsilon+\varepsilon}{w_z^\varepsilon})+\alpha\Big)w_z^\varepsilon\\ \qquad\qquad+\Big(\alpha-\sigma^2A(\frac{w^\varepsilon+\varepsilon}{w_z^\varepsilon})\Big)w^\varepsilon=0 \quad \text{in} \quad Q_T,\\ w^\varepsilon(z,0)=\pi_\varepsilon(e^{-\frac{1}{\gamma}z}-K), \quad z\in\mathbb{R},\end{cases} \tag{A.6}$$

In the following, we prove

$$w_2-\varepsilon\leq w^\varepsilon\leq w_1, \tag{A.7}$$

By direct calculation or maximum principle,

$$w_1\geq0,\quad \partial_zw_1\leq0,\quad \partial_{zz}w_1\geq0,$$

so

$$\partial_sw_1-\frac{1}{2}\sigma^2A^2(\cdot)\partial_{zz}w_1+\Big(-\frac{1}{2}\sigma^2A^2(\cdot)-\sigma^2A(\cdot)+\alpha\Big)\partial_zw_1+\Big(\alpha-\sigma^2A(\cdot)\Big)w_1$$
$$\geq\partial_sw_1-\frac{1}{2}\sigma^2a_1^2\partial_{zz}w_1+\Big(-\frac{1}{2}\sigma^2a_2^2-\sigma^2a_2+\alpha\Big)\partial_zw_1+\Big(\alpha-\sigma^2a_1\Big)w_1$$
$$=0.$$

So by comparison principle, we have $w^\varepsilon \le w_1$. Similar method can prove $w^\varepsilon \ge w_2 - \varepsilon$.

Since $w_2 > 0$ in any region $(a,b)\times(\tau,T)$ (for any $\tau>0$), From (A.7) we know w^ε has a positive lower bound which is independent of ε in $(a,b)\times(\tau,T)$, so $A\big(\frac{w^{\varepsilon,N}+\varepsilon}{w_z^{\varepsilon,N}}\big)$ is uniform Hölder continuous in this region (see the discussion in the beginning of this section), by Schauder interior estimate we have

$$|w^\varepsilon|_{C^{2+\alpha,1+\frac{\alpha}{2}}([a,b]\times[\tau,T])} \le C$$

where, C is independent of ε.

Therefore, there exists $w \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)\bigcap C^{1+\alpha,\frac{1+\alpha}{2}}\big(\overline{Q_T}\setminus\{(-\gamma\ln K,0)\}\big)\bigcap C\big(\overline{Q_T}\big)$ such that, for any region $Q=(a,b)\times(0,T)$, $Q_r=\big((a,b)\times(0,T)\big)\setminus B_r$, $Q^\tau=(a,b)\times(\tau,T)\subset Q_T$, there exists a subsequence of w^ε, which we still denote by w^ε, such that $w^\varepsilon \to w$ in $C^{2,1}(\overline{Q^\tau})\bigcap C^{1,0}(\overline{Q_r})\bigcap C(\overline{Q})$, $\varepsilon\to 0$. So w satisfies the equation and the initial condition of problem (3.8).

Finally, (4.1), (4.2) and (4.4) are the direct consequences of (A.3), (A.4) and (A.7) together with strong maximum principle.

Appendix B. In this section, we prove that (4.6) is the solution of problem (4.5). Take the transformation

$$w_i = e^{p_i z + q_i s} u_i, \quad i=1,2,$$

then

$$\begin{aligned}
\partial_s w_i &= e^{p_i z+q_i s}\big(\partial_s u_i + q_i u_i\big),\\
\partial_z w_i &= e^{p_i z+q_i s}\big(\partial_z u_i + p_i u_i\big),\\
\partial_{zz} w_i &= e^{p_i z+q_i s}\big(\partial_{zz} u_i + 2p_i\partial_z u_i + p_i^2 u_i\big).
\end{aligned}$$

According to (4.5), u_i satisfies

$$\begin{cases}
\partial_s u_i - \frac{1}{2}\sigma^2 a_i^2\partial_{zz}u_i + \Big(-\sigma^2 a_i^2 p_i - \frac{1}{2}\sigma^2 a_j^2 - \sigma^2 a_j + \alpha\Big)\partial_z u_i \\
\qquad + \Big(q_i - \frac{1}{2}\sigma^2 a_i^2 p_i^2 + \Big(-\frac{1}{2}\sigma^2 a_j^2 - \sigma^2 a_j + \alpha\Big)p_i + \alpha - \sigma^2 a_i\Big)u_i = 0 \quad \text{in} \quad Q_T,\\
u_i(z,0) = e^{-p_i z}\big(e^{-\frac{1}{\gamma}z} - K\big)^+, \quad z\in\mathbb{R}.
\end{cases} \tag{B.1}$$

Let

$$\begin{cases}
-\sigma^2 a_i^2 p_i - \frac{1}{2}\sigma^2 a_j^2 - \sigma^2 a_j + \alpha = 0,\\
q_i - \frac{1}{2}\sigma^2 a_i^2 p_i^2 + \Big(-\frac{1}{2}\sigma^2 a_j^2 - \sigma^2 a_j + \alpha\Big)p_i + \alpha - \sigma^2 a_i = 0,
\end{cases}$$

i.e.,

$$\begin{cases}
p_i - \frac{-\frac{1}{2}\sigma^2 a_j^2 - \sigma^2 a_j + \alpha}{\sigma^2 a_i^2},\\
q_i = \frac{1}{2}\sigma^2 a_i^2 p_i^2 - \Big(-\frac{1}{2}\sigma^2 a_j^2 - \sigma^2 a_j + \alpha\Big)p_i - \alpha + \sigma^2 a_i,
\end{cases}$$

then we have

$$\begin{cases}
\partial_s u_i - \frac{1}{2}\sigma^2 a_i^2 \partial_{zz} u_i = 0 \quad \text{in} \quad Q_T,\\
u_i(z,0) = e^{-p_i z}\big(e^{-\frac{1}{\gamma}z} - K\big)^+, \quad z\in\mathbb{R}.
\end{cases}$$

By the Poisson formula of heat equation, we have

$$\begin{aligned} u_i(z,s) &= \int_{-\infty}^{+\infty} \Gamma_i(z-\xi,s)e^{-p_i\xi}\big(e^{-\frac{1}{\gamma}\xi}-K\big)^+ \mathrm{d}\xi \\ &= \int_{-\infty}^{-\gamma\ln K} \Gamma_i(z-\xi,s)e^{-p_i\xi}\big(e^{-\frac{1}{\gamma}\xi}-K\big)\mathrm{d}\xi. \end{aligned}$$

Therefore,

$$w_i(z,s) = e^{p_i z + q_i s} u_i(z,s) = \int_{-\infty}^{-\gamma\ln K} \Gamma_i(z-\xi,s)e^{p_i(z-\xi)+q_i s}\big(e^{-\frac{1}{\gamma}\xi}-K\big)\mathrm{d}\xi.$$

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