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A REMARK ON THE CONTACT WAVE FOR THE 1-D COMPRESSIBLE NAVIER-STOKES EQUATIONS*

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Abstract. We revisit the classical work of Huang-Matsumura-Xin [F.M. Huang, A. Matsumura, and Z.P. Xin, Arch. Ration. Mech. Anal., 179:55–77, 2006] and Huang-Xin-Yang [F.M. Huang, Z.P. Xin, and T. Yang, Adv. Math., 219:1246–1297, 2008] for contact wave of the one-dimensional compressible Navier-Stokes equations. By using Huang-Matsumura-Xin-Yang's approach and a detailed energy analysis, we prove the large-time asymptotic stability of a contact wave pattern with a better convergence rate for compressible Navier-Stokes equations under non-zero mass condition on the perturbation. This improves previous results of [F.M. Huang, A. Matsumura, and Z.P. Xin, Arch. Ration. Mech. Anal., 179:55–77, 2006] and [F.M. Huang, Z.P. Xin, and T. Yang, Adv. Math., 219:1246–1297, 2008].

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1. Introduction

We study the one-dimensional compressible Navier-Stokes equations in Lagrangian coordinates which read

$$\begin{cases} v_t - u_x = 0, \qquad x \in \mathbb{R} = (-\infty, +\infty),\ t > 0, \\ u_t + p_x = \mu(\frac{u_x}{v})_x, \\ (e + \frac{|u|^2}{2})_t + (pu)_x = \kappa(\frac{\theta_x}{v})_x + \mu(\frac{uu_x}{v})_x, \end{cases} \tag{1.1}$$

where $v = v(t,x) > 0$, $u = u(t,x)$, $\theta = \theta(t,x) > 0$ and $e = e(t,x) > 0$ denote the specific volume, the fluid velocity, the absolute temperature and the internal energy, respectively. The constants $\mu > 0$ and $\kappa > 0$ denote the viscosity and heat conduction coefficients, respectively. Here we study the perfect fluids so that the pressure p and e are given by $p = \frac{R\theta}{v}$ and $e = \frac{R}{\gamma-1}\theta + const.$, where $R > 0$ is the gas constant and $\gamma > 1$ is the adiabatic exponent.

The purpose of this paper is to establish a better time decay rate of the solutions to the Cauchy problem for Equations (1.1) supplemented with the following initial values and far field conditions

$$\begin{cases} (v,u,\theta)(0,x) = (v_0,u_0,\theta_0)(x), \quad x \in \mathbb{R}, \\ (v,u,\theta)(t,x) \to (v_\pm,u_\pm,\theta_\pm), \quad \text{when } x \to \pm\infty, \quad t > 0, \end{cases} \tag{1.2}$$

in which $v_\pm > 0$, $u_\pm = 0$, $\theta_\pm > 0$ are given constants. Here, the two constant states $(v_\pm,u_\pm,\theta_\pm)$ are connected by the contact discontinuity wave solution to the Riemann problem of the corresponding 1D compressible Euler system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (e + \frac{|u|^2}{2})_t + (pu)_x = 0, \end{cases} \tag{1.3}$$

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with the Riemann initial data

$$(v_0,u_0,\theta_0)(x)=\begin{cases}(v_+,u_+,\theta_+), & x>0,\\ (v_-,u_-,\theta_-), & x<0.\end{cases} \tag{1.4}$$

There have been some results on the contact wave for system (1.1) and (1.3). Xin in [15] first proved the metastability of a weak contact discontinuity for the compressible Euler system with uniform viscosity. This was later generalized by Liu-Xin in [13] to show the local stability of the contact discontinuities for a class of general system of viscous conservation laws with artificial viscosity. But these methods do not apply to the compressible Navier-Stokes system because the viscosity matrix in (1.1) is only semi-positive definite. Recently, Huang-Matsumura-Xin in [7] used the anti-derivative method, motivated by the theory of viscous shock waves in [12], to obtain not only the stability of the viscous contact wave, but also the convergence rate for compressible Navier-Stokes Equations (1.1) under zero mass condition on the perturbation. Then Huang-Xin-Yang in [8] removed this zero mass condition on the perturbation. The elementary energy method different from the anti-derivative method used in [7, 8] was proposed by Huang-Li-Matsumura in [4] where they succeeded in obtaining a new estimate on heat equations and applied it to prove the stability of the viscous contact waves for system (1.1), but the results do not obtain the convergence rate. By the way, there are also some studies on the composite waves of compressible Navier-Stokes equations, cf. [4, 6, 9] and the references therein.

In [7, 8] with or without zero mass condition on the perturbation, the authors obtained that the same convergence rate toward the contact wave is $(1+t)^{-\frac{1}{4}}$, where they left a problem: can we show a faster convergence rate? Later, Huang-Wang-Wang in [10] improved the convergence rate to $(1+t)^{-\frac{1}{2}}$ under zero mass condition on the perturbation, but the results do not contain non-zero mass case as [8]. In this paper, we basically follow Huang-Matsumura-Xin-Yang's approach and use a detailed energy analysis, improve the convergence rate to $(1+t)^{-\frac{3}{8}}$ under non-zero mass condition on the perturbation. This removes the zero mass condition on the perturbation, which is a crucially restrictive condition in [7, 10]. Thus, our results improves the results of [7, 8].

The paper is arranged as follows: The next section contains the statement of the main theorem and some notations. Section 3 is devoted to get a priori estimates for the compressible Navier-Stokes equations. The proofs of the main theorem will be given in last section.

2. The main theorem

To state our main results, we first recall the contact wave $(\bar v,\bar u,\bar\theta)(t,x)$ for the compressible Navier-Stokes Equations (1.1) defined in [7]. According to [14], one sees that the Riemann problem (1.3) and (1.4) admits a contact discontinuity solution

$$(\overline{V},\overline{U},\overline{\Theta})(t,x)=\begin{cases}(v_+,u_+,\theta_+), & x>0,\\ (v_-,u_-,\theta_-), & x<0,\end{cases} \tag{2.1}$$

on the condition that

$$p_-:=R\frac{\theta_-}{v_-}=R\frac{\theta_+}{v_+}=:p_+. \tag{2.2}$$

In the setting of the compressible Navier-Stokes Equations (1.1), the corresponding wave $(\bar v,\bar u,\bar\theta)(t,x)$ to the contact discontinuity $(\overline{V},\overline{U},\overline{\Theta})(t,x)$ becomes smooth and behaves as a diffusion wave due to the dissipation effect. From [7], the pressure of the profile $(\bar{v},\bar{u},\bar{\theta})(t,x)$ is almost constant, that is

$$\bar{p}=R\frac{\bar{\theta}}{\bar{v}}\approx p_+, \tag{2.3}$$

which indicates the leading part of the energy equation $(1.1)_3$ is

$$\frac{R}{\gamma-1}\theta_t+p_+u_x=\kappa(\frac{\theta_x}{v})_x. \tag{2.4}$$

In view of (2.3), (2.4) and $(1.1)_1$, we obtain a nonlinear diffusion equation as follows

$$\theta_t=a(\frac{\theta_x}{\theta})_x,\quad a=\frac{\gamma-1}{\gamma R^2}\kappa p_+>0. \tag{2.5}$$

Applying the same arguments as in [1, 2], one sees that (2.5) admits a unique self-similarity solution $\Theta(\xi)$, $\xi=\frac{x}{\sqrt{1+t}}$ with the boundary conditions $\Theta(t,\pm\infty)=\theta_\pm$. Additionally, it turns out that $\Theta(\xi)$ is a monotone function, increasing if $\theta_+>\theta_-$ and decreasing if $\theta_+<\theta_-$, and more importantly, there exists some positive constant δ, such that for $\delta=|\theta_+-\theta_-|$, satisfies (see [3])

$$\begin{cases}|\Theta_x|=O(\delta)(1+t)^{-\frac{1}{2}}e^{-\frac{\theta_\pm x^2}{4a(1+t)}}, & \text{as} \quad |x|\to\infty,\\ (1+t)|\Theta_{xx}|+(1+t)^{1/2}|\Theta_x|+|\Theta-\theta_\pm|\le C\delta e^{-\frac{c_1x^2}{1+t}}, & \text{as} \quad |x|\to\infty,\end{cases} \tag{2.6}$$

where $c_1>0$ is given constant. After Θ is determined, we can define the contact wave profile $(\bar{v},\bar{u},\bar{\theta})(t,x)$ as follows

$$\bar{v}=\frac{R}{p_+}\Theta,\quad \bar{u}=\frac{Ra}{p_+\Theta}\Theta_x,\quad \bar{\theta}=\Theta-\frac{\gamma-1}{2R}\bar{u}^2. \tag{2.7}$$

Then $(\bar{v},\bar{u},\bar{\theta})$ satisfies

$$\|(\bar{v}-\overline{V},\bar{u}-\overline{U},\bar{\theta}-\overline{\Theta})\|_{L^p}=O(\kappa^{\frac{1}{2p}})(1+t)^{\frac{1}{2p}},\quad p\ge 1, \tag{2.8}$$

and

$$\begin{cases}\bar{v}_t-\bar{u}_x=0,\\ \bar{u}_t+\bar{p}_x=\mu(\frac{\bar{u}_x}{\bar{v}})_x+\mathcal{R}_{1x},\\ (\bar{e}+\frac{|\bar{u}|^2}{2})_t+(\bar{p}\bar{u})_x=\kappa(\frac{\bar{\theta}_x}{\bar{v}})_x+\mu(\frac{\bar{u}}{\bar{v}}\bar{u}_x)_x+\mathcal{R}_{2x},\end{cases} \tag{2.9}$$

where

$$\begin{cases}\mathcal{R}_1=(\frac{\gamma-1}{\gamma R}\kappa-\mu)\frac{\bar{u}_x}{\bar{v}}+\bar{p}-p_+=O(\delta)(1+t)^{-1}e^{-\frac{\theta_\pm x^2}{4a(1+t)}}, & \text{as} \quad |x|\to\infty,\\ \mathcal{R}_2=(\frac{\gamma-1}{\gamma R}\kappa-\mu)\frac{\bar{u}\bar{u}_x}{\bar{v}}+(\bar{p}-p_+)\bar{u}=O(\delta)(1+t)^{-\frac{3}{2}}e^{-\frac{\theta_\pm x^2}{4a(1+t)}}, & \text{as} \quad |x|\to\infty.\end{cases} \tag{2.10}$$

Motivated by [8], we denote the conserved quantities as

$$m(t,x)=(v,u,\theta+\frac{\gamma-1}{2R}u^2)^t,\quad \bar{m}(t,x)=(\bar{v},\bar{u},\bar{\theta}+\frac{\gamma-1}{2R}\bar{u}^2)^t,$$

where $(\cdot,\cdot,\cdot)^t$ is the transpose of the vector $(\cdot,\cdot,\cdot)$ and let

$$A(v,u,\theta)=\begin{pmatrix} 0 & -1 & 0 \\ -\frac{p}{v} & 0 & \frac{R}{v} \\ -\frac{(\gamma-1)pu}{Rv} & \frac{(\gamma-1)p}{R} & \frac{(\gamma-1)u}{v} \end{pmatrix}$$

be the Jacobi matrix of the flux $(-u,p,\frac{\gamma-1}{R}pu)^t$. The first eigenvalue of $A(v_-,0,\theta_-)$ is $\lambda_1^-=-\sqrt{\frac{\gamma p_-}{v_-}}$ with right eigenvector $r_1^-=(-1,\lambda_1^-,\frac{\gamma-1}{R}p_-)^t$. Similarly, the third eigenvalue and right eigenvector of $A(v_+,0,\theta_+)$ are $\lambda_3^+=\sqrt{\frac{\gamma p_+}{v_+}}$ and $r_3^+=(-1,\lambda_3^+,\frac{\gamma-1}{R}p_+)^t$, respectively. By strict hyperbolicity, the vectors r_1^-, $m_+-m_-=(v_+-v_-,0,\theta_+-\theta_-)^t$ and r_3^+ are linearly independent in $\mathbb{R}^3$. Hence, there exist unique constants $\bar{\theta}_i$ $(i=1,2,3)$ such that

$$\int_{-\infty}^{+\infty}\big(m(0,x)-\bar{m}(0,x)\big)dx=\bar{\theta}_1 r_1^-+\bar{\theta}_2(m_+-m_-)+\bar{\theta}_3 r_3^+. \tag{2.11}$$

As in [8], we define

$$\widetilde{m}(t,x)=\bar{m}(t,x+\bar{\theta}_2)+\bar{\theta}_1\theta_1 r_1^-+\bar{\theta}_3\theta_3 r_3^+, \tag{2.12}$$

where

$$\theta_1(t,x)=\frac{1}{\sqrt{4\pi(1+t)}}e^{-\frac{(x-\lambda_1^-(1+t))^2}{4(1+t)}},\quad \theta_3(t,x)=\frac{1}{\sqrt{4\pi(1+t)}}e^{-\frac{(x-\lambda_3^+(1+t))^2}{4(1+t)}}, \tag{2.13}$$

satisfying

$$\theta_{1t}+\lambda_1^-\theta_{1x}=\theta_{1xx},\quad \theta_{3t}+\lambda_3^+\theta_{3x}=\theta_{3xx},$$

and $\int_{\mathbb{R}}\theta_i(t,x)dx=1$ for $i=1,3$ and any $t\geq 0$. Precisely, the $\widetilde{m}(t,x)$ can be written as

$$\widetilde{m}(t,x)=(\hat{v},\hat{u},\hat{\theta}+\frac{\gamma-1}{2R}\hat{u}^2)^t(t,x), \tag{2.14}$$

with

$$\begin{cases} \hat{v}(t,x)=\bar{v}(t,x+\bar{\theta}_2)-\bar{\theta}_1\theta_1-\bar{\theta}_3\theta_3, \\ \hat{u}(t,x)=\bar{u}(t,x+\bar{\theta}_2)+\lambda_1^-\bar{\theta}_1\theta_1+\lambda_3^+\bar{\theta}_3\theta_3, \\ \hat{\theta}(t,x)=\bar{\theta}(t,x+\bar{\theta}_2)+\frac{\gamma-1}{2R}\bar{u}^2(t,x+\bar{\theta}_2)+\frac{\gamma-1}{R}p_+(\bar{\theta}_1\theta_1+\bar{\theta}_3\theta_3)-\frac{\gamma-1}{2R}\hat{u}^2. \end{cases} \tag{2.15}$$

Furthermore, one has

$$\begin{aligned}\int_{-\infty}^{+\infty}\big(m(0,x)-\widetilde{m}(0,x)\big)dx&=\int_{-\infty}^{+\infty}\big(m(0,x)-\bar{m}(0,x)\big)dx+\int_{-\infty}^{+\infty}\big(\bar{m}(0,x)-\widetilde{m}(0,x)\big)dx \\ &=\bar{\theta}_2(m_+-m_-)+\int_{-\infty}^{+\infty}\big(\bar{m}(0,x)-\bar{m}(0,x+\bar{\theta}_2)\big)dx=0.\end{aligned} \tag{2.16}$$

Without loss of generality, we can assume that $\bar{\theta}_2=0$ from now on. A direct computation shows that

$$\begin{cases} \hat{v}_t-\hat{u}_x=\hat{R}_{1x}, \\ \hat{u}_t+\hat{p}_x=\mu(\frac{\hat{u}_x}{\hat{v}})_x+\hat{R}_{2x}, \\ (\hat{e}+\frac{|\hat{u}|^2}{2})_t+(\hat{p}\hat{u})_x=\kappa(\frac{\hat{\theta}_x}{\hat{v}})_x+\mu(\frac{\hat{u}}{\hat{v}}\hat{u}_x)_x+\hat{R}_{3x}. \end{cases} \tag{2.17}$$

Here $\hat{R}_1$, $\hat{R}_2$, $\hat{R}_3$ are given by

$$\begin{cases} \hat{R}_1 = -\bar{\theta}_1\theta_{1x} - \bar{\theta}_3\theta_{3x}, \\ \hat{R}_2 = \mathcal{R}_1 + \mu(\frac{\bar{u}_x}{\bar{v}} - \frac{\hat{u}_x}{\hat{v}}) + (\lambda_1^-\bar{\theta}_1\theta_{1x} + \lambda_3^+\bar{\theta}_3\theta_{3x}) + (\hat{p} - \bar{p} - (\lambda_1^-)^2\bar{\theta}_1\theta_1 - (\lambda_3^+)^2\bar{\theta}_3\theta_3), \\ \hat{R}_3 = \mathcal{R}_2 + \kappa(\frac{\bar{\theta}_x}{\bar{v}} - \frac{\hat{\theta}_x}{\hat{v}}) + \mu(\frac{\bar{u}\bar{u}_x}{\bar{v}} - \frac{\hat{u}\hat{u}_x}{\hat{v}}) + p_+(\bar{\theta}_1\theta_{1x} + \bar{\theta}_3\theta_{3x}) \\ \qquad + (\hat{p}\hat{u} - \bar{p}\bar{u} - p_+\lambda_1^-\bar{\theta}_1\theta_1 - p_+\lambda_3^+\bar{\theta}_3\theta_3). \end{cases}$$

Then using the same arguments as (2.a31) in [8], for some constant $c_2 > 0$, $\hat{R}_i$ $(i = 1,2,3)$ satisfy

$$\hat{R}_i = O(\delta + |\bar{\theta}_1| + |\bar{\theta}_3|)\frac{1}{1+t}\{e^{-\frac{c_2x^2}{1+t}} + e^{-\frac{c_2(x-\lambda_1^-(1+t))^2}{1+t}} + e^{-\frac{c_2(x-\lambda_3^+(1+t))^2}{1+t}}\}. \tag{2.18}$$

To present the results in this paper, the following notations are needed. Several positive generic constants are denoted by C (generally large) and c (generally small) without confusion. c_1, c_2 etc. denote fixed positive constants. For functional space, H^k $(k \geq 0)$ denotes the usual Sobolev space $W^{k,2}$ with the norm $\|\cdot\|_{H^k}$ and $\|\cdot\|_{L^p}$ denoting the usual L^p-norm. Denote the perturbation around the $(\hat{v},\hat{u},\hat{\theta})$ by

$$(\widetilde{v},\widetilde{u},\widetilde{\theta})(t,x) = (v - \hat{v}, u - \hat{u}, \theta - \hat{\theta})(t,x), \tag{2.19}$$

and then set the anti-derivative variables as

$$(\widetilde{V},\widetilde{U},\widetilde{W})(t,x) = \int_{-\infty}^{x}(\widetilde{v},\widetilde{u}, e + \frac{|u|^2}{2} - \hat{e} - \frac{|\hat{u}|^2}{2})(t,y)dy, \tag{2.20}$$

which satisfy $(\widetilde{V},\widetilde{U},\widetilde{W})(0,\pm\infty) = 0$. For $0 \leq T \leq +\infty$, we define a function space $X(0,+\infty)$ as follows

$$X(0,T) = \Big\{(\widetilde{V},\widetilde{U},\widetilde{W},\widetilde{v},\widetilde{u},\widetilde{\theta})\,|\,(\widetilde{V},\widetilde{U},\widetilde{W}) \in C(0,T;H^2), \\ \widetilde{v} \in L^2(0,T;H^1), (\widetilde{u},\widetilde{\theta}) \in L^2(0,T;H^2)\Big\}.$$

With the above preparation, the main results can be stated as follows.

THEOREM 2.1. *Let $(\hat{v},\hat{u},\hat{\theta})(t,x)$ be the contact wave defined by (2.15) with $\delta = |\theta_+ - \theta_-|$. Then there exist small constants $\delta_0 > 0$ and $\epsilon_0 > 0$ such that if $\delta + |\bar{\theta}_1| + |\bar{\theta}_3| \leq \delta_0$ and the initial data (v_0,u_0,θ_0) satisfies*

$$\|(\widetilde{V},\widetilde{U},\widetilde{W})(0,x)\| + \|(\widetilde{v},\widetilde{u},\widetilde{\theta})(0,x)\|_{H^1} \leq \epsilon_0, \tag{2.21}$$

then the initial value problem (1.1) and (1.2) admits a unique global solution $(v,u,\theta)(t,x)$ satisfying

$$(\widetilde{V},\widetilde{U},\widetilde{W},\widetilde{v},\widetilde{u},\widetilde{\theta})(t,x) \in X(0,+\infty). \tag{2.22}$$

Moreover, we have the following time decay rates

$$\|(\widetilde{v},\widetilde{u},\widetilde{\theta})(t)\| \leq C\big(\epsilon_0 + \sqrt{\delta_0}\big)(1+t)^{-\frac{1}{4}}, \tag{2.23}$$

$$\|(\widetilde{v},\widetilde{u},\widetilde{\theta})_x(t)\| \leq C\big(\epsilon_0 + \sqrt{\delta_0}\big)(1+t)^{-\frac{1}{2}}, \tag{2.24}$$

and

$$\|(\widetilde{v},\widetilde{u},\widetilde{\theta})(t)\|_{L^\infty}\leq C\big(\epsilon_0+\sqrt{\delta_0}\big)(1+t)^{-\frac{3}{8}}. \tag{2.25}$$

We give a few remarks on the Theorem 2.1: First, compared to the results of [8], the convergence rate in (2.24) and (2.25) are faster than the one established in [8]. This improves the results of [8]. Moreover, we expect that our arguments can be applied to general systems such as Boltzmann equation [8,11], radiative hydrodynamic system and other related systems to get a better convergence rate. Second, it should be noted that $\|(\widetilde{v},\widetilde{u},\widetilde{\theta})_x\|$ decay faster than $\|(\widetilde{v},\widetilde{u},\widetilde{\theta})\|$. Therefore, a faster decay rate for the higher-order derivatives of $(\widetilde{v},\widetilde{u},\widetilde{\theta})$ can be obtained provided that the initial data have the same order regularity by our method.

3. Energy estimates

This section is devoted to the energy analysis for compressible Navier-Stokes Equations (1.1) and (1.2). We first reformulate Navier-Stokes equations in terms of (2.19) and (2.20). Then we derive the lower order energy estimates for $(\widetilde{V},\widetilde{U},W)$ in Section 3.2. And Section 3.3 is devoted to the derivative estimates of $(\widetilde{V},\widetilde{U},W)$. The main energy estimates will be given in Section 3.4.

3.1. Reformulated system. In view of (2.19) and (2.20), one can deduce that

$$(\widetilde{v},\widetilde{u})=(\widetilde{V}_x,\widetilde{U}_x),\quad\text{and}\quad\frac{R}{\gamma-1}\widetilde{\theta}+\frac{1}{2}|\widetilde{U}_x|^2+\hat{u}\widetilde{U}_x=\widetilde{W}_x. \tag{3.1}$$

Subtracting (2.17) from system (1.1) and integrating the resulting system, we get

$$\begin{cases}\widetilde{V}_t-\widetilde{U}_x=-\hat{R}_1,\\ \widetilde{U}_t+p-\hat{p}=\frac{\mu}{v}u_x-\frac{\mu}{\hat{v}}\hat{u}_x-\hat{R}_2,\\ \widetilde{W}_t+pu-\hat{p}\hat{u}=\frac{\kappa}{v}\theta_x-\frac{\kappa}{\hat{v}}\hat{\theta}_x+\mu\frac{u}{v}u_x-\mu\frac{\hat{u}}{\hat{v}}\hat{u}_x-\hat{R}_3.\end{cases} \tag{3.2}$$

From [8], we introduce another variable related to the temperature

$$W=\frac{\gamma-1}{R}(\widetilde{W}-\bar{u}\widetilde{U}). \tag{3.3}$$

It follows from (3.1) and (3.3) that

$$\widetilde{\theta}=W_x-Y,\quad\text{and}\quad Y=\frac{\gamma-1}{R}\big(\frac{1}{2}|\widetilde{U}_x|^2-\hat{u}_x\widetilde{U}\big). \tag{3.4}$$

Using the new variable W, we can rewrite the Equations (3.2) as

$$\begin{cases}\widetilde{V}_t-\widetilde{U}_x=-\hat{R}_1,\\ \widetilde{U}_t-\frac{p_+}{\hat{v}}\widetilde{V}_x+\frac{R}{\hat{v}}W_x=\frac{\mu}{\hat{v}}\widetilde{U}_{xx}+Q_1,\\ \frac{R}{\gamma-1}W_t+p_+\widetilde{U}_x=\frac{\kappa}{\hat{v}}W_{xx}+Q_2,\end{cases} \tag{3.5}$$

where

$$\begin{cases}Q_1=\big\{\frac{\hat{p}-p_+}{\hat{v}}\widetilde{V}_x-(p-\hat{p}+\frac{\hat{p}}{\hat{v}}\widetilde{V}_x-\frac{R}{\hat{v}}\widetilde{\theta})\big\}+(\frac{\mu}{v}-\frac{\mu}{\hat{v}})u_x+\frac{R}{\hat{v}}Y-\hat{R}_2,\\ Q_2=(p_+-p)\widetilde{U}_x+(\frac{\kappa}{v}-\frac{\kappa}{\hat{v}})\theta_x+\frac{\mu u_x}{v}\widetilde{U}_x-\hat{u}_t\widetilde{U}-\frac{\kappa}{\hat{v}}Y_x+\hat{u}\hat{R}_2-\hat{R}_3.\end{cases} \tag{3.6}$$

Since the local existence of the solution to system (3.5) is well known, which is similar to that in [5], the details are omitted. To prove the global existence of the solution in Theorem 2.1, it only suffices to close the following a priori estimates:

$$\mathcal{E}_{ns}(T)=\sup_{0\le t\le T}\{\|(\widetilde{V},\widetilde{U},W)(t)\|_{L^\infty}^2+\|(\widetilde{v},\widetilde{u},\widetilde{\theta})(t)\|_{H^1}^2\}\le\varepsilon_0^2, \tag{3.7}$$

where $\varepsilon_0>0$ is a small constant depending on the initial data and the strength of the contact discontinuity. By (2.11), it is obvious that $|\bar{\theta}_1|+|\bar{\theta}_3|\le C\varepsilon_0$ for some positive constant C. For brevity, from now on until the end of this paper, we always assume $\bar{\delta}=\delta+|\bar{\theta}_1|+|\bar{\theta}_3|$ and ε_0 small enough such that $\bar{\delta}+\varepsilon_0\ll 1$.

3.2. Lower estimates. Next, we derive the lower order energy estimates.

LEMMA 3.1. *Suppose that* $(\widetilde{V},\widetilde{U},\widetilde{W},\widetilde{v},\widetilde{u},\widetilde{\theta})\in X(0,T)$ *and* $\bar{\delta}>0$, $\mathcal{E}_{ns}(t)>0$ *small enough. Then for* $t\in[0,T]$, *we have*

$$\mathcal{E}_{1t}(t)+c\mathcal{D}_1(t)\le C(\bar{\delta}+\varepsilon_0)\|(\widetilde{v},\widetilde{u},\widetilde{\theta})_x\|^2+C\bar{\delta}(1+t)^{-1}\mathcal{E}_1(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}. \tag{3.8}$$

Here $\mathcal{E}_1(t)$ *and* $\mathcal{D}_1(t)$ *are given by*

$$\mathcal{E}_1(t)=C_1\mathcal{E}_0(t)+\int(\frac{\mu}{2\hat{v}}\widetilde{V}_x^2-\widetilde{V}_x\widetilde{U})dx,\quad \mathcal{D}_1(t)=\|(\widetilde{V},\widetilde{U},W)_x\|^2, \tag{3.9}$$

for some large constant $C_1>1$ *so that* $\mathcal{E}_1(t)\ge\mathcal{E}_0(t)$ *with*

$$\mathcal{E}_0(t)=\int(\frac{p_+}{2}\widetilde{V}^2+\frac{\hat{v}}{2}\widetilde{U}^2+\frac{R^2}{2(\gamma-1)p_+}W^2)dx. \tag{3.10}$$

Proof. Multiplying $(3.5)_1$, $(3.5)_2$ and $(3.5)_3$ by $p_+\widetilde{V}$, $\hat{v}\widetilde{U}$, and $\frac{R}{p_+}W$, respectively, adding the resulting equations, we obtain

$$\begin{aligned}&\Big(\frac{p_+}{2}\widetilde{V}^2+\frac{\hat{v}}{2}\widetilde{U}^2+\frac{R^2}{2(\gamma-1)p_+}W^2\Big)_t+\mu\widetilde{U}_x^2+\frac{R\kappa}{\hat{v}p_+}W_x^2\\&=\frac{1}{2}\hat{v}_t\widetilde{U}^2+\hat{v}\widetilde{U}Q_1+\frac{R}{p_+}WQ_2-(\frac{R\kappa}{\hat{v}p_+})_xWW_x-\hat{R}_1p_+\widetilde{V}+(\cdots)_x.\end{aligned} \tag{3.11}$$

Here and in the sequel the notation $(\cdots)_x$ represents the term in the conservative form so that it vanishes after integration. By (2.18), (3.10) and the elementary inequalities, we get

$$\begin{aligned}&\int|\frac{1}{2}\hat{v}_t\widetilde{U}^2|dx+\int|(\frac{R\kappa}{\hat{v}p_+})_xWW_x|dx+\int|\hat{R}_1p_+\widetilde{V}|dx\\&\le C\|\hat{v}_t\|_{L^\infty}\|\widetilde{U}\|^2+C\|\hat{v}_x\|_{L^\infty}\|W\|\|W_x\|+C\|\hat{R}_1\|\|\widetilde{V}\|\\&\le C\bar{\delta}(1+t)^{-1}\mathcal{E}_0(t)+C\bar{\delta}\|W_x\|^2+C\bar{\delta}(1+t)^{-\frac{1}{2}},\end{aligned}$$

where we have used the fact that

$$\|\hat{v}_t\|_{L^\infty}\le C\bar{\delta}(1+t)^{-1},\quad \|\hat{v}_x\|_{L^\infty}\le C\bar{\delta}(1+t)^{-\frac{1}{2}},\quad \|\hat{R}_1\|\le C\bar{\delta}(1+t)^{-\frac{3}{4}}.$$

We will deal with the term involving Q_1, since

$$\frac{\hat{p}-p_+}{\hat{v}}\widetilde{V}_x-(p-\hat{p}+\frac{\hat{p}}{\hat{v}}\widetilde{V}_x-\frac{R}{\hat{v}}\widetilde{\theta})=O(1)(\widetilde{V}_x^2+W_x^2+Y^2+|\hat{u}|^4).$$

It follows from this and (3.7) that

$$\begin{aligned}&\int|\hat{v}\widetilde{U}||\frac{\hat{p}-p_+}{\hat{v}}\widetilde{V}_x-(p-\hat{p}+\frac{\hat{p}}{\hat{v}}\widetilde{V}_x-\frac{R}{\hat{v}}\widetilde{\theta})|dx\\ \le& C\|\widetilde{U}\|_{L^\infty}\int(\widetilde{V}_x^2+W_x^2+Y^2+|\hat{u}|^4)dx\\ \le& C\varepsilon_0\|(\widetilde{V},\widetilde{U},W)_x\|^2+C\bar{\delta}(1+t)^{-1}\mathcal{E}_0(t)+C\bar{\delta}(1+t)^{-\frac{5}{2}}.\end{aligned}$$

In view of (2.18), (3.4), (3.10) and the elementary inequalities, we obtain

$$\begin{aligned}&\int|\hat{v}\widetilde{U}||(\frac{\mu}{v}-\frac{\mu}{\hat{v}})u_x+\frac{R}{\hat{v}}Y-\hat{R}_2|dx\\ \le& C(\bar{\delta}+\varepsilon_0)\|(\widetilde{V},\widetilde{U},W)_x\|^2+C\bar{\delta}(1+t)^{-1}\mathcal{E}_0(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}+C\varepsilon_0\|\widetilde{u}_x\|^2.\end{aligned}$$

By the expression of Q_1 in (3.6), we have from the above two estimates that

$$\begin{aligned}\int|\hat{v}\widetilde{U}Q_1|dx\le& C(\bar{\delta}+\varepsilon_0)\|(\widetilde{V},\widetilde{U},W)_x\|^2+C\varepsilon_0\|\widetilde{u}_x\|^2\\ &+C\bar{\delta}(1+t)^{-1}\mathcal{E}_0(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}.\end{aligned}$$

Similar arguments as the above imply

$$\begin{aligned}\int|\frac{R}{p_+}WQ_2|dx\le& C(\bar{\delta}+\varepsilon_0)\{\|(\widetilde{V},\widetilde{U},W)_x\|^2+\|(\widetilde{v},\widetilde{u},\widetilde{\theta})_x\|^2\}\\ &+C\bar{\delta}(1+t)^{-1}\mathcal{E}_0(t)+C\bar{\delta}(1+t)^{-\frac{3}{2}}.\end{aligned}$$

Hence, collecting the above estimates and then using the smallness of $\bar{\delta}$ and ε_0, one has

$$\begin{aligned}\mathcal{E}_{0t}(t)+c\|(\widetilde{U},W)_x\|^2\le& C(\bar{\delta}+\varepsilon_0)\{\|\widetilde{V}_x\|^2+\|(\widetilde{v},\widetilde{u},\widetilde{\theta})_x\|^2\}\\ &+C\bar{\delta}(1+t)^{-1}\mathcal{E}_0(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}.\end{aligned}\tag{3.12}$$

Since the dissipation $\|\widetilde{V}_x\|^2$ is not included in (3.12), to complete the lower order estimates, we rewrite $(3.5)_2$ by using $\widetilde{V}_t=\widetilde{U}_x-\hat{R}_1$ such that

$$\frac{\mu}{\hat{v}}\widetilde{V}_{xt}-\widetilde{U}_t+\frac{p_+}{\hat{v}}\widetilde{V}_x=\frac{R}{\hat{v}}W_x-Q_1-\frac{\mu}{\hat{v}}\hat{R}_{1x}.\tag{3.13}$$

Multiplying (3.13) by $\widetilde{V}_x$ and using $\widetilde{V}_x\widetilde{U}_t=(\widetilde{V}_x\widetilde{U})_t+\widetilde{U}_x^2-(\widetilde{V}_t\widetilde{U})_x-\hat{R}_1\widetilde{U}_x$, we can arrive at

$$\Big(\int\frac{\mu}{2\hat{v}}\widetilde{V}_x^2-\widetilde{V}_x\widetilde{U}dx\Big)_t+c\|\widetilde{V}_x\|^2\le C\|(\widetilde{U},W)_x\|^2+C\varepsilon_0\|\widetilde{u}_x\|^2+C\bar{\delta}(1+t)^{-\frac{3}{2}}.\tag{3.14}$$

In summary, choosing some large constant $C_1>0$ and using the smallness of $\bar{\delta}$ and ε_0, then the summation of $(3.12)\times C_1$ and (3.14) give (3.8). This completes the proof of Lemma 3.1. □

3.3. Derivative estimates. Now, we turn to consider the estimates of $(\widetilde{v},\widetilde{u},\widetilde{\theta})$.

LEMMA 3.2. *Under the assumptions of Lemma 3.1. Then for $t\in[0,T]$, we have*

$$\mathcal{E}_{2t}(t)+c\mathcal{D}_2(t)\le C\bar{\delta}(1+t)^{-1}\mathcal{D}_1(t)+C\bar{\delta}(1+t)^{-\frac{3}{2}}+C\varepsilon_0\|\widetilde{u}_{xx}\|^2.\tag{3.15}$$

Here $\mathcal{E}_2(t)$ and $\mathcal{D}_2(t)$ are given by

$$\mathcal{E}_2(t)=C_2\widetilde{\mathcal{E}}_2(t)+\int(\frac{\mu}{2\hat{v}}\widetilde{v}_x^2-\widetilde{v}_x\widetilde{u})dx,\quad \mathcal{D}_2(t)=\|(\widetilde{v},\widetilde{u},\widetilde{\theta})_x\|^2, \tag{3.16}$$

for some large constant $C_2>1$ so that $\mathcal{E}_2(t)\geq\widetilde{\mathcal{E}}_2(t)$ with

$$\widetilde{\mathcal{E}}_2(t)=\int\big(R\hat{\theta}\Phi(\frac{v}{\hat{v}})+\frac{1}{2}\widetilde{u}^2+\frac{R}{\gamma-1}\hat{\theta}\Phi(\frac{\theta}{\hat{\theta}})\big)dx, \tag{3.17}$$

Proof. Since $p-\hat{p}=\frac{R\widetilde{\theta}}{v}-\frac{\hat{p}}{v}\widetilde{v}$, we have from (1.1) and (2.17) that

$$\begin{cases}\widetilde{v}_t-\widetilde{u}_x=-\hat{R}_{1x},\\ \widetilde{u}_t+\big(\frac{R\widetilde{\theta}}{v}-\frac{\hat{p}}{v}\widetilde{v}\big)_x=(\frac{\mu}{v}u_x-\frac{\mu}{\hat{v}}\hat{u}_x)_x-\hat{R}_{2x},\\ \frac{R}{\gamma-1}\widetilde{\theta}_t+pu_x-\hat{p}\hat{u}_x=(\frac{\kappa}{v}\theta_x-\frac{\kappa}{\hat{v}}\hat{\theta}_x)_x+Q_3,\end{cases} \tag{3.18}$$

where

$$Q_3=\frac{\mu}{v}u_x^2+\hat{u}\hat{u}_t+\hat{p}_x\hat{u}-(\frac{\mu\hat{u}\hat{u}_x}{\hat{v}})_x-\hat{R}_{3x}. \tag{3.19}$$

Similar to [8], multiplying $(3.18)_1$, $(3.18)_2$ and $(3.18)_3$ by $\frac{\hat{p}}{v}\widetilde{v}$, $\widetilde{u}$, and $\frac{\widetilde{\theta}}{\theta}$, respectively, then adding the resulting equations together, we can arrive at

$$\begin{aligned}&\big(R\hat{\theta}\Phi(\frac{v}{\hat{v}})+\frac{1}{2}\widetilde{u}^2+\frac{R}{\gamma-1}\hat{\theta}\Phi(\frac{\theta}{\hat{\theta}})\big)_t+\frac{\mu}{v}\widetilde{u}_x^2+\frac{\kappa}{v\theta}\widetilde{\theta}_x^2\\ =&\hat{v}\hat{p}_t\Phi(\frac{v}{\hat{v}})-\hat{p}\hat{v}_t\Phi(\frac{\hat{v}}{v})-\frac{R}{\gamma-1}\hat{\theta}_t\Phi(\frac{\hat{\theta}}{\theta})-(\frac{\mu}{v}-\frac{\mu}{\hat{v}})\hat{u}_x\widetilde{u}_x+R\hat{\theta}(\frac{1}{v}-\frac{1}{\hat{v}})\hat{R}_{1x}-\hat{R}_{2x}\widetilde{u}\\ &+\frac{\widetilde{\theta}}{\theta}(\hat{p}-p)\hat{u}_x+\frac{\kappa\widetilde{v}\widetilde{\theta}_x}{\hat{v}v\theta}\hat{\theta}_x+\frac{\widetilde{\theta}\theta_x}{\theta^2}(\frac{\kappa\theta_x}{v}-\frac{\kappa\hat{\theta}_x}{\hat{v}})+\frac{\widetilde{\theta}}{\theta}Q_3+(\cdots)_x,\end{aligned} \tag{3.20}$$

where

$$\Phi(s)=s-\ln s-1. \tag{3.21}$$

It is easy to check that $\Phi'(1)=0$ and $\Phi(s)$ is strictly convex around $s=1$. Thus, there exists $c_3>1$ such that

$$c_3^{-1}\widetilde{v}^2\leq\Phi(\frac{v}{\hat{v}})\leq c_3\widetilde{v}^2,\quad c_3^{-1}\widetilde{\theta}^2\leq\Phi(\frac{\theta}{\hat{\theta}})\leq c_3\widetilde{\theta}^2. \tag{3.22}$$

For any function $g(x)\in H^1(\mathbb{R})\subset L^\infty(\mathbb{R})$, the following Sobolev imbedding inequality holds

$$\|g(x)\|_{L^\infty}\leq C\|g(x)\|^{\frac{1}{2}}\|g_x(x)\|^{\frac{1}{2}}. \tag{3.23}$$

This together with

$$\|\widetilde{\theta}\|^2\leq\|W_x\|^2+C\|\widetilde{U}_x\|^3\|\widetilde{U}_{xx}\|+C\bar{\delta}(1+t)^{-\frac{3}{2}}\|\widetilde{U}\|_{L^\infty}^2 \tag{3.24}$$

due to (3.4), one can deduce from (3.22) that

$$\int\big|\hat{v}\hat{p}_t\Phi(\frac{v}{\hat{v}})-\hat{p}\hat{v}_t\Phi(\frac{\hat{v}}{v})-\frac{R}{\gamma-1}\hat{\theta}_t\Phi(\frac{\hat{\theta}}{\theta})\big|dx\leq C\bar{\delta}(1+t)^{-1}\mathcal{D}_1(t)+C\bar{\delta}(1+t)^{-\frac{5}{2}}.$$

From (2.18), (3.23) and (3.24), we find that

$$\begin{aligned}&\int|(\frac{\mu}{v}-\frac{\mu}{\hat v})\hat u_x\widetilde u_x+\hat R_{2x}\widetilde u-R\hat\theta(\frac{1}{v}-\frac{1}{\hat v})\hat R_{1x}|dx\\ \le& C\bar\delta\|\widetilde u_x\|^2+C\bar\delta(1+t)^{-1}\mathcal D_1(t)+C\bar\delta(1+t)^{-\frac32}.\end{aligned}$$

By (3.1), (3.4), (3.24) and the Sobolev imbedding inequality, we get

$$\begin{aligned}&\int|\frac{\widetilde\theta}{\theta}(\hat p-p)\hat u_x+\frac{\kappa\widetilde v\widetilde\theta_x}{\hat v v\theta}\hat\theta_x+\frac{\widetilde\theta\theta_x}{\theta^2}(\frac{\kappa\theta_x}{v}-\frac{\kappa\hat\theta_x}{\hat v})|dx\\ \le& C\bar\delta(1+t)^{-1}\mathcal D_1(t)+C(\bar\delta+\varepsilon_0)\|\widetilde\theta_x\|^2+C\bar\delta(1+t)^{-\frac52}.\end{aligned}$$

On the other hand, one finds by (3.19) and the elementary inequalities that

$$\int|\frac{\widetilde\theta}{\theta}Q_3|dx\le C\bar\delta(1+t)^{-1}\mathcal D_1(t)+C(\bar\delta+\varepsilon_0)\|\widetilde u_x\|^2+C\bar\delta(1+t)^{-\frac32}.$$

Hence, it holds by those above estimates that

$$\widetilde{\mathcal E}_{2t}(t)+c\|(\widetilde u,\widetilde\theta)_x\|^2\le C\bar\delta(1+t)^{-1}\mathcal D_1(t)+C\bar\delta(1+t)^{-\frac32}. \tag{3.25}$$

As before, we need to estimate $\|\widetilde v_x\|^2$. For this, multiplying $(3.18)_2$ by $\widetilde v_x$ yields

$$\begin{aligned}&(\frac{\mu}{2\hat v}\widetilde v_x^2)_t-(\frac{\mu}{2\hat v})_t\widetilde v_x^2-\widetilde u_t\widetilde v_x-(\frac{R\widetilde\theta}{v}-\frac{\hat p}{v}\widetilde v)_x\widetilde v_x\\ =&(\frac{\mu\widetilde v}{\hat v v}u_x)_x\widetilde v_x-(\frac{\mu}{\hat v})_x\widetilde u_x\widetilde v_x+\hat R_{2x}\widetilde v_x-\frac{\mu}{\hat v}\hat R_{1xx}\widetilde v_x,\end{aligned} \tag{3.26}$$

where we have used the fact that

$$\frac{\mu}{\hat v}\widetilde u_{xx}\widetilde v_x=\frac{\mu}{\hat v}\widetilde v_{tx}\widetilde v_x+\frac{\mu}{\hat v}\hat R_{1xx}\widetilde v_x=(\frac{\mu}{2\hat v}\widetilde v_x^2)_t-(\frac{\mu}{2\hat v})_t\widetilde v_x^2+\frac{\mu}{\hat v}\hat R_{1xx}\widetilde v_x.$$

Integrating (3.26) with respect to x, then we estimate the resulting equation term by term. First of all, one has from the Sobolev imbedding inequality that

$$\int|(\frac{\mu}{2\hat v})_t\widetilde v_x^2|dx\le C\|\hat v_t\|_{L^\infty}\|\widetilde v_x\|^2\le C\bar\delta(1+t)^{-1}\|\widetilde v_x\|^2.$$

Using $(3.18)_1$ and the integration by parts, we can claim that

$$\begin{aligned}\int\widetilde u_t\widetilde v_xdx=&\frac{d}{dt}\int\widetilde u\widetilde v_xdx+\int\widetilde u_x^2dx-\int\hat R_{1x}\widetilde u_xdx\\ \le&\frac{d}{dt}\int\widetilde u\widetilde v_xdx+\int\widetilde u_x^2dx+C\bar\delta\|\widetilde u_x\|^2+C\bar\delta(1+t)^{-\frac52}.\end{aligned}$$

By using the elementary inequalities, (3.24) and $\frac{\hat p}{v}\ge c$, we have

$$\begin{aligned}-\int(\frac{R\widetilde\theta}{v}-\frac{\hat p}{v}\widetilde v)_x\widetilde v_xdx=&\int\frac{\hat p}{v}\widetilde v_x^2dx-\int\frac{R}{v}\widetilde\theta_x\widetilde v_xdx+\int(\frac{\hat p}{v})_x\widetilde v\widetilde v_xdx-\int(\frac{R}{v})_x\widetilde\theta\widetilde v_xdx\\ \ge& c\|\widetilde v_x\|^2-C\|\widetilde\theta_x\|^2-C\bar\delta(1+t)^{-1}\mathcal D_1(t)-C\bar\delta(1+t)^{-\frac52}.\end{aligned}$$

On the other hand, we get

$$\int |(\frac{\mu\widetilde{v}}{\hat{v}v}u_x)_x\widetilde{v}_x|dx \le C(\bar{\delta}+\varepsilon_0)\|(\widetilde{v},\widetilde{u})_x\|^2+C\bar{\delta}(1+t)^{-1}\mathcal{D}_1(t)+C\varepsilon_0\|\widetilde{u}_{xx}\|^2,$$

where we have used the fact that

$$\int |\widetilde{v}_x^2\widetilde{u}_x|dx \le C\|\widetilde{u}_x\|_{L^\infty}\|\widetilde{v}_x\|^2 \le C\|\widetilde{u}_x\|^{\frac{1}{2}}\|\widetilde{u}_{xx}\|^{\frac{1}{2}}\|\widetilde{v}_x\|^2 \le C\varepsilon_0(\|\widetilde{v}_x\|^2+\|\widetilde{u}_{xx}\|^2).$$

The last three terms in (3.26) are bounded by

$$\int |(\frac{\mu}{\hat{v}})_x\widetilde{u}_x\widetilde{v}_x|dx+\int |\hat{R}_{2x}\widetilde{v}_x|dx+\int |\frac{\mu}{\hat{v}}\hat{R}_{1xx}\widetilde{v}_x|dx \le C\bar{\delta}\|(\widetilde{v},\widetilde{u})_x\|^2+C\bar{\delta}(1+t)^{-\frac{5}{2}}.$$

Substituting the above estimates into (3.26), we can conclude that

$$\begin{aligned}&\frac{d}{dt}\int (\frac{\mu}{2\hat{v}}\widetilde{v}_x^2-\widetilde{u}\widetilde{v}_x)dx+c\|\widetilde{v}_x\|^2\\ \le& C\|(\widetilde{u},\widetilde{\theta})_x\|^2+C\bar{\delta}(1+t)^{-1}\mathcal{D}_1(t)+C\bar{\delta}(1+t)^{-\frac{5}{2}}+C\varepsilon_0\|\widetilde{u}_{xx}\|^2.\end{aligned}\tag{3.27}$$

In summary, choosing some large constant $C_2>0$, then the summation of $(3.25)\times C_2$ and (3.27) gives (3.15). This completes the proof of Lemma 3.2. □

Next, we will move to the estimates of $(\widetilde{v}_x,\widetilde{u}_x,\widetilde{\theta}_x)$. We design a suitable linear combination of the derivative estimates of $(3.18)_1$, $(3.18)_2$ and $(3.18)_3$ and consider the most delicate calculations for each term.

Lemma 3.3. *Under the assumptions of Lemma 3.1, for $t\in[0,T]$, one has*

$$\begin{aligned}\mathcal{E}_{3t}(t)+c\|(\widetilde{u},\widetilde{\theta})_{xx}\|^2 \le& C\bar{\delta}(1+t)^{-\frac{3}{2}}\mathcal{D}_1(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}\mathcal{D}_2(t)\\ &+C\{\mathcal{E}_2(t)+\mathcal{E}_3(t)\}\mathcal{D}_2(t)+C\bar{\delta}(1+t)^{-\frac{5}{2}}.\end{aligned}\tag{3.28}$$

Here $\mathcal{E}_3(t)$ is defined by

$$\mathcal{E}_3(t)=\frac{1}{2}\int \frac{\hat{p}}{\hat{v}}|\widetilde{v}_x|^2dx+\frac{1}{2}\int |\widetilde{u}_x|^2dx+\frac{1}{2}\int \frac{R}{\gamma-1}\frac{1}{\hat{\theta}}|\widetilde{\theta}_x|^2dx.\tag{3.29}$$

Proof. Differentiating $(3.18)_1$ with respect to x, then multiplying the resulting equation by $\frac{\hat{p}}{\hat{v}}\widetilde{v}_x$ yields

$$(\frac{\hat{p}}{2\hat{v}}\widetilde{v}_x^2)_t-(\frac{\hat{p}}{2\hat{v}})_t\widetilde{v}_x^2-\frac{\hat{p}}{\hat{v}}\widetilde{v}_x\widetilde{u}_{xx}=-\frac{\hat{p}}{\hat{v}}\widetilde{v}_x\hat{R}_{1xx}.\tag{3.30}$$

Integrating (3.30) with respect to x, then direct computations show that

$$\frac{1}{2}\frac{d}{dt}\int \frac{\hat{p}}{\hat{v}}|\widetilde{v}_x|^2dx-\int \frac{\hat{p}}{\hat{v}}\widetilde{v}_x\widetilde{u}_{xx}dx \le C\bar{\delta}(1+t)^{-1}\|\widetilde{v}_x\|^2+C\bar{\delta}(1+t)^{-\frac{5}{2}}.\tag{3.31}$$

In what follows we concentrate on the equation $(3.18)_2$. Multiplying $(3.18)_2$ by $-\widetilde{u}_{xx}$ and integrating with respect to x over $\mathbb{R}$ gives

$$\frac{1}{2}\frac{d}{dt}\int |\widetilde{u}_x|^2dx+\int \frac{\hat{p}}{\hat{v}}\widetilde{v}_x\widetilde{u}_{xx}dx-\int \frac{R}{\hat{v}}\widetilde{\theta}_x\widetilde{u}_{xx}dx+\int (\frac{\hat{p}}{v}-\frac{\hat{p}}{\hat{v}})\widetilde{v}_x\widetilde{u}_{xx}dx$$

$$
\begin{aligned}
&-\int(\frac{R}{v}-\frac{R}{\hat{v}})\widetilde{\theta}_x\widetilde{u}_{xx}dx-\int(\frac{R}{v})_x\widetilde{\theta}\widetilde{u}_{xx}dx+\int(\frac{\hat{p}}{v})_x\widetilde{v}\widetilde{u}_{xx}dx+\int\frac{\mu}{v}\widetilde{u}_{xx}^2dx\\
=&-\int\{(\frac{\mu}{v})_x\widetilde{u}_x+(\frac{\mu}{v}-\frac{\mu}{\hat{v}})_x\hat{u}_x+(\frac{\mu}{v}-\frac{\mu}{\hat{v}})\hat{u}_{xx}-\hat{R}_{2x}\}\widetilde{u}_{xx}dx.
\end{aligned}\tag{3.32}
$$

Equation (3.23) and the Cauchy inequality imply

$$
\begin{aligned}
&|\int(\frac{\hat{p}}{v}-\frac{\hat{p}}{\hat{v}})\widetilde{v}_x\widetilde{u}_{xx}dx|+|\int(\frac{R}{v}-\frac{R}{\hat{v}})\widetilde{\theta}_x\widetilde{u}_{xx}dx|\\
\leq&C\|\widetilde{v}\|_{L^\infty}\|\widetilde{v}_x\|\|\widetilde{u}_{xx}\|+C\|\widetilde{v}\|_{L^\infty}\|\widetilde{\theta}_x\|\|\widetilde{u}_{xx}\|\\
\leq&C\varepsilon\|\widetilde{u}_{xx}\|^2+C_\varepsilon\big\{\mathcal{E}_2(t)+\mathcal{E}_3(t)\big\}\mathcal{D}_2(t).
\end{aligned}\tag{3.33}
$$

We use an integration by parts about x to get

$$
\int(\frac{R}{v})_x\widetilde{\theta}\widetilde{u}_{xx}dx=-\int\frac{R}{v^2}\widetilde{v}_x\widetilde{\theta}\widetilde{u}_{xx}dx+\int(\frac{R}{v^2}\hat{v}_x\widetilde{\theta})_x\widetilde{u}_xdx.
$$

Similar arguments as (3.33) imply

$$
|\int\frac{R}{v^2}\widetilde{v}_x\widetilde{\theta}\widetilde{u}_{xx}dx|\leq\varepsilon\|\widetilde{u}_{xx}\|^2+C_\varepsilon\big\{\mathcal{E}_2(t)+\mathcal{E}_3(t)\big\}\mathcal{D}_2(t).
$$

By the Sobolev imbedding inequality and (3.24), we obtain

$$
\begin{aligned}
&|\int(\frac{R}{v^2}\hat{v}_x\widetilde{\theta})_x\widetilde{u}_xdx|\leq C\int\big\{|\hat{v}_x\widetilde{\theta}_x\widetilde{u}_x|+|\hat{v}_{xx}\widetilde{\theta}\widetilde{u}_x|+|v_x\hat{v}_x\widetilde{\theta}\widetilde{u}_x|\big\}dx\\
\leq&C\|\hat{v}_x\|_{L^\infty}\|\widetilde{\theta}_x\|\|\widetilde{u}_x\|+C\|\hat{v}_{xx}\|_{L^\infty}\|\widetilde{\theta}\|\|\widetilde{u}_x\|\\
&+C\|\hat{v}_x^2\|_{L^\infty}\|\widetilde{\theta}\|\|\widetilde{u}_x\|+C\|\hat{v}_x\widetilde{\theta}\|_{L^\infty}\|\widetilde{v}_x\|\|\widetilde{u}_x\|\\
\leq&C\bar{\delta}(1+t)^{-\frac{3}{2}}\mathcal{D}_1(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}\mathcal{D}_2(t)+C\bar{\delta}(1+t)^{-\frac{5}{2}}.
\end{aligned}
$$

It follows from the above three estimates that

$$
\begin{aligned}
|\int(\frac{R}{v})_x\widetilde{\theta}\widetilde{u}_{xx}dx|\leq&C\varepsilon\|\widetilde{u}_{xx}\|^2+C_\varepsilon\big\{\mathcal{E}_2(t)+\mathcal{E}_3(t)\big\}\mathcal{D}_2(t)\\
&+C\bar{\delta}(1+t)^{-\frac{3}{2}}\mathcal{D}_1(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}\mathcal{D}_2(t)+C\bar{\delta}(1+t)^{-\frac{5}{2}}.
\end{aligned}\tag{3.34}
$$

Similar arguments as (3.34) imply

$$
\begin{aligned}
|\int(\frac{\hat{p}}{v})_x\widetilde{v}\widetilde{u}_{xx}dx|\leq&C\varepsilon\|\widetilde{u}_{xx}\|^2+C_\varepsilon\big\{\mathcal{E}_2(t)+\mathcal{E}_3(t)\big\}\mathcal{D}_2(t)\\
&+C\bar{\delta}(1+t)^{-\frac{3}{2}}\mathcal{D}_1(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}\mathcal{D}_2(t)+C\bar{\delta}(1+t)^{-\frac{5}{2}}.
\end{aligned}\tag{3.35}
$$

On the other hand, one can deduce that

$$
\begin{aligned}
&|\int(\frac{\mu}{v})_x\widetilde{u}_x\widetilde{u}_{xx}dx|\leq\varepsilon\|\widetilde{u}_{xx}\|^2+C_\varepsilon\|\hat{v}_x\|_{L^\infty}^2\|\widetilde{u}_x\|^2+C_\varepsilon\|\widetilde{v}_x\|^2\|\widetilde{u}_x\|_{L^\infty}^2\\
\leq&\varepsilon\|\widetilde{u}_{xx}\|^2+C_\varepsilon\bar{\delta}(1+t)^{-1}\mathcal{D}_2(t)+C_\varepsilon\mathcal{E}_3(t)\mathcal{D}_2(t)+C_\varepsilon\varepsilon_0\|\widetilde{u}_{xx}\|^2,
\end{aligned}\tag{3.36}
$$

and

$$
|\int(\frac{\mu}{v}-\frac{\mu}{\hat{v}})_x\hat{u}_x\widetilde{u}_{xx}dx|+|\int(\frac{\mu}{v}-\frac{\mu}{\hat{v}})\hat{u}_{xx}\widetilde{u}_{xx}dx|+|\int\hat{R}_{2x}\widetilde{u}_{xx}dx|
$$

$$\leq C\bar{\delta}(1+t)^{-\frac{3}{2}}\mathcal{D}_1(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}\mathcal{D}_2(t)+C\bar{\delta}(1+t)^{-\frac{5}{2}}+C\bar{\delta}\|\widetilde{u}_{xx}\|^2. \tag{3.37}$$

Hence, substituting (3.33)-(3.37) into (3.32) and taking $\varepsilon>0$ small enough, we get the desired estimate

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\widetilde{u}_x|^2dx+\int\frac{\hat{p}}{\hat{v}}\widetilde{v}_x\widetilde{u}_{xx}dx-\int\frac{R}{\hat{v}}\widetilde{\theta}_x\widetilde{u}_{xx}dx+\int\frac{\mu}{2v}\widetilde{u}_{xx}^2dx\\ \leq& C\bar{\delta}(1+t)^{-\frac{3}{2}}\mathcal{D}_1(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}\mathcal{D}_2(t)+C\{\mathcal{E}_2(t)+\mathcal{E}_3(t)\}\mathcal{D}_2(t)+C\bar{\delta}(1+t)^{-\frac{5}{2}}.\end{aligned} \tag{3.38}$$

We still deal with Equation $(3.18)_3$. Differentiating $(3.18)_3$ with respect to x yields that

$$\frac{R}{\gamma-1}\widetilde{\theta}_{xt}+\hat{p}\widetilde{u}_{xx}+\hat{p}_x\widetilde{u}_x+[(p-\hat{p})u_x]_x=(\frac{\kappa}{v}\widetilde{\theta}_x)_{xx}+[(\frac{\kappa}{v}-\frac{\kappa}{\hat{v}})\hat{\theta}_x]_{xx}+Q_{3x}. \tag{3.39}$$

Multiplying (3.39) by $\frac{1}{\hat{\theta}}\widetilde{\theta}_x$ and integrating with respect to x over $\mathbb{R}$ yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\frac{R}{\gamma-1}\frac{1}{\hat{\theta}}|\widetilde{\theta}_x|^2dx-\int\frac{R}{\gamma-1}(\frac{1}{2\hat{\theta}})_t|\widetilde{\theta}_x|^2dx+\int\frac{R}{\hat{v}}\widetilde{\theta}_x\widetilde{u}_{xx}dx+\int\hat{p}_x\widetilde{u}_x\frac{1}{\hat{\theta}}\widetilde{\theta}_xdx\\ &+\int[(p-\hat{p})u_x]_x\frac{1}{\hat{\theta}}\widetilde{\theta}_xdx=\int\{(\frac{\kappa}{v}\widetilde{\theta}_x)_{xx}+[(\frac{\kappa}{v}-\frac{\kappa}{\hat{v}})\hat{\theta}_x]_{xx}+Q_{3x}\}\frac{1}{\hat{\theta}}\widetilde{\theta}_xdx.\end{aligned} \tag{3.40}$$

We will estimate each term for (3.40). First of all, we use the Sobolev imbedding inequality and Hölder inequality to obtain

$$\begin{aligned}|\int(\frac{1}{2\hat{\theta}})_t|\widetilde{\theta}_x|^2dx|+|\int\hat{p}_x\widetilde{u}_x\frac{1}{\hat{\theta}}\widetilde{\theta}_xdx|&\leq C\|\hat{\theta}_t\|_{L^\infty}\|\widetilde{\theta}_x\|^2+C\|\hat{p}_x\|_{L^\infty}\|\widetilde{u}_x\|\|\widetilde{\theta}_x\|\\ &\leq C\bar{\delta}(1+t)^{-\frac{1}{2}}\mathcal{D}_2(t).\end{aligned} \tag{3.41}$$

Using the integration by parts, the elementary inequalities and (3.24), one has

$$\begin{aligned}&|\int[(p-\hat{p})u_x]_x\frac{1}{\hat{\theta}}\widetilde{\theta}_xdx|\leq\varepsilon\|(\frac{1}{\hat{\theta}}\widetilde{\theta}_x)_x\|^2+C_\varepsilon\|(p-\hat{p})u_x\|^2\\ \leq& C\varepsilon\|\widetilde{\theta}_{xx}\|^2+C_\varepsilon\bar{\delta}(1+t)^{-\frac{3}{2}}\mathcal{D}_1(t)+C_\varepsilon\bar{\delta}(1+t)^{-\frac{1}{2}}\mathcal{D}_2(t)\\ &+C_\varepsilon\{\mathcal{E}_2(t)+\mathcal{E}_3(t)\}\mathcal{D}_2(t)+C_\varepsilon\bar{\delta}(1+t)^{-\frac{5}{2}},\end{aligned} \tag{3.42}$$

where we have used the fact that

$$\|(\frac{1}{\hat{\theta}}\widetilde{\theta}_x)_x\|^2\leq C\|\widetilde{\theta}_{xx}\|^2+C\|\hat{\theta}_x\|_{L^\infty}^2\|\widetilde{\theta}_x\|^2\leq C\|\widetilde{\theta}_{xx}\|^2+C\bar{\delta}(1+t)^{-1}\|\widetilde{\theta}_x\|^2. \tag{3.43}$$

We use an integration by parts about x to get

$$\int(\frac{\kappa}{v}\widetilde{\theta}_x)_{xx}\frac{1}{\hat{\theta}}\widetilde{\theta}_xdx=-\int\frac{\kappa}{v}\frac{1}{\hat{\theta}}\widetilde{\theta}_{xx}^2dx-\int\frac{\kappa}{v}\widetilde{\theta}_{xx}(\frac{1}{\hat{\theta}})_x\widetilde{\theta}_xdx-\int(\frac{\kappa}{v})_x\widetilde{\theta}_x(\frac{1}{\hat{\theta}}\widetilde{\theta}_x)_xdx. \tag{3.44}$$

In view of the Sobolev imbedding inequality and (3.43), one obtains

$$|\int\frac{\kappa}{v}\widetilde{\theta}_{xx}(\frac{1}{\hat{\theta}})_x\widetilde{\theta}_xdx|\leq C\bar{\delta}(1+t)^{-1}\|\widetilde{\theta}_x\|^2+C\bar{\delta}\|\widetilde{\theta}_{xx}\|^2, \tag{3.45}$$

and

$$|\int(\frac{\kappa}{v})_x\widetilde{\theta}_x(\frac{1}{\hat{\theta}}\widetilde{\theta}_x)_xdx|\leq C\varepsilon\|(\frac{1}{\hat{\theta}}\widetilde{\theta}_x)_x\|^2+C_\varepsilon\|(\frac{\kappa}{v})_x\widetilde{\theta}_x\|^2$$

$$\leq C\varepsilon\|\widetilde{\theta}_{xx}\|^2+C_\varepsilon\bar{\delta}(1+t)^{-1}\mathcal{D}_2(t)+C_\varepsilon\mathcal{E}_3(t)\mathcal{D}_2(t)+C_\varepsilon\varepsilon_0\|\widetilde{\theta}_{xx}\|^2. \tag{3.46}$$

By the estimates from (3.44) to (3.46), we can arrive at

$$\int(\frac{\kappa}{v}\widetilde{\theta}_x)_{xx}\frac{1}{\hat{\theta}}\widetilde{\theta}_x dx\leq -c\|\widetilde{\theta}_{xx}\|^2+C\bar{\delta}(1+t)^{-1}\mathcal{D}_2(t)+C\mathcal{E}_3(t)\mathcal{D}_2(t). \tag{3.47}$$

Similarly, it holds that

$$\begin{aligned}&|\int[(\frac{\kappa}{v}-\frac{\kappa}{\hat{v}})\hat{\theta}_x]_{xx}\frac{1}{\hat{\theta}}\widetilde{\theta}_x dx|=|\int[(\frac{\kappa}{v}-\frac{\kappa}{\hat{v}})\hat{\theta}_x]_x(\frac{1}{\hat{\theta}}\widetilde{\theta}_x)_x dx|\\ &\leq C\varepsilon\|\widetilde{\theta}_{xx}\|^2+C_\varepsilon\bar{\delta}(1+t)^{-\frac{3}{2}}\mathcal{D}_1(t)+C_\varepsilon\bar{\delta}(1+t)^{-\frac{1}{2}}\mathcal{D}_2(t).\end{aligned} \tag{3.48}$$

Applying the integration by parts, (3.19) and (3.43) yields

$$\begin{aligned}|\int Q_{3x}\frac{1}{\hat{\theta}}\widetilde{\theta}_x dx|=|\int Q_3(\frac{1}{\hat{\theta}}\widetilde{\theta}_x)_x dx|&\leq C\varepsilon\|\widetilde{\theta}_{xx}\|^2+C_\varepsilon\bar{\delta}(1+t)^{-1}\mathcal{D}_2(t)\\ &+C_\varepsilon\mathcal{E}_3(t)\mathcal{D}_2(t)+C_\varepsilon\bar{\delta}(1+t)^{-\frac{5}{2}}+C_\varepsilon\varepsilon_0\|\widetilde{u}_{xx}\|^2.\end{aligned} \tag{3.49}$$

Substituting (3.41)-(3.42) and (3.47)-(3.49) into (3.40) and taking $\varepsilon>0$ small enough, one can show that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\frac{R}{\gamma-1}\frac{1}{\hat{\theta}}|\widetilde{\theta}_x|^2dx+\int\frac{R}{\hat{v}}\widetilde{\theta}_x\widetilde{u}_{xx}dx+c\|\widetilde{\theta}_{xx}\|^2\leq C\bar{\delta}(1+t)^{-\frac{3}{2}}\mathcal{D}_1(t)\\ &\quad+C\bar{\delta}(1+t)^{-\frac{1}{2}}\mathcal{D}_2(t)+C\{\mathcal{E}_2(t)+\mathcal{E}_3(t)\}\mathcal{D}_2(t)+C\bar{\delta}(1+t)^{-\frac{5}{2}}+C\varepsilon_0\|\widetilde{u}_{xx}\|^2.\end{aligned} \tag{3.50}$$

In summary, adding (3.31), (3.38) and (3.50) together, then using the smallness of $\bar{\delta}$ and ε_0, we get (3.28). Now the proof is completed. □

3.4. Main estimates. Finally, we give the main energy estimates as follows.

LEMMA 3.4. *Under the assumptions of Lemma 3.1, there exists some large constant $C_3>1$ and for any $t\in[0,T]$ such that*

$$\overline{\mathcal{E}}_t(t)+c\overline{\mathcal{D}}(t)\leq C_3\bar{\delta}(1+t)^{-1}\overline{\mathcal{E}}(t)+C_3\bar{\delta}(1+t)^{-\frac{1}{2}}. \tag{3.51}$$

Here $\overline{\mathcal{E}}(t)$ and $\overline{\mathcal{D}}(t)$ are given by

$$\overline{\mathcal{E}}(t)=\mathcal{E}_1(t)+\mathcal{E}_2(t)+\mathcal{E}_3(t),\quad \overline{\mathcal{D}}(t)=\mathcal{D}_1(t)+\mathcal{D}_2(t)+\|(\widetilde{u},\widetilde{\theta})_{xx}\|^2. \tag{3.52}$$

Proof. The proof of Lemma 3.4 is a direct consequence of Lemmas 3.1-3.3. □

4. Stability and convergence rate

In this section, we prove our main result Theorem 2.1. Multiplying (3.51) by $(1+t)^{-C_3\bar{\delta}}$ with $C_3\bar{\delta}<1/2$ and integrating the resulting equation with respect to t yields

$$\overline{\mathcal{E}}(t)\leq C\big(\overline{\mathcal{E}}(0)+\bar{\delta}\big)(1+t)^{\frac{1}{2}},\quad \int_0^t\overline{\mathcal{D}}(s)ds\leq C\big(\overline{\mathcal{E}}(0)+\bar{\delta}\big)(1+t)^{\frac{1}{2}}. \tag{4.1}$$

In view of (3.9), (3.10) and (3.52), we can see that there exists $c_4>0$ such that

$$\overline{\mathcal{E}}(t)\geq c_4\|(\widetilde{V},\widetilde{U},W)\|^2. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\|(\widetilde{V},\widetilde{U},W)\|^2 \le C\big(\overline{\mathcal{E}}(0)+\bar{\delta}\big)(1+t)^{\frac{1}{2}}. \tag{4.3}$$

Combining (3.15) and (3.28) yields

$$\mathcal{E}_{4t}(t)+c\mathcal{D}_3(t)\le C\bar{\delta}(1+t)^{-1}\mathcal{D}_1(t)+C\bar{\delta}(1+t)^{-\frac{3}{2}}, \tag{4.4}$$

where we have used the fact that $\mathcal{E}_2(t)+\mathcal{E}_3(t)\le C\varepsilon_0$ and

$$\mathcal{E}_4(t)=\mathcal{E}_2(t)+\mathcal{E}_3(t),\quad \mathcal{D}_3(t)=\mathcal{D}_2(t)+\|(\widetilde{u},\widetilde{\theta})_{xx}\|^2. \tag{4.5}$$

Multiplying (4.4) by $(1+t)$ and using

$$\begin{aligned}\mathcal{E}_4(t)\le C\|(\widetilde{v},\widetilde{u},\widetilde{\theta})\|_{H^1}^2 &\le C\big\{\|(\widetilde{V},\widetilde{U},W)_x\|^2+\|(\widetilde{v},\widetilde{u},\widetilde{\theta})_x\|^2\big\}+C\bar{\delta}(1+t)^{-\frac{3}{2}}\\ &\le C\overline{\mathcal{D}}(t)+C\bar{\delta}(1+t)^{-\frac{3}{2}},\end{aligned} \tag{4.6}$$

then we have from (4.4) and (4.6) that

$$\begin{aligned}[(1+t)\mathcal{E}_4(t)]_t+c(1+t)\mathcal{D}_3(t) &\le C\mathcal{D}_1(t)+C\mathcal{E}_4(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}\\ &\le C\overline{\mathcal{D}}(t)+C\bar{\delta}(1+t)^{-\frac{1}{2}}.\end{aligned} \tag{4.7}$$

Integrating (4.7) with respect to t and using (4.1) yields

$$(1+t)\mathcal{E}_4(t)+c\int_0^t(1+s)\mathcal{D}_3(s)ds\le C\big(\overline{\mathcal{E}}(0)+\bar{\delta}\big)(1+t)^{\frac{1}{2}}. \tag{4.8}$$

It follows from (4.8) that

$$\mathcal{E}_4(t)\le C(\overline{\mathcal{E}}(0)+\bar{\delta})(1+t)^{-\frac{1}{2}}. \tag{4.9}$$

Furthermore, $\mathcal{E}_4(t)$ has the following lower bound

$$\mathcal{E}_4(t)\ge c_5\|(\widetilde{v},\widetilde{u},\widetilde{\theta})\|_{H^1}^2\ge c\|(\widetilde{V},\widetilde{U},W)_x\|^2+c_5\|(\widetilde{v},\widetilde{u},\widetilde{\theta})_x\|^2-c_5\bar{\delta}(1+t)^{-\frac{3}{2}}, \tag{4.10}$$

for constant $c_5>0$. We have from (4.9), (4.10) and (4.3) that

$$\|(\widetilde{V},\widetilde{U},W)\|_{L^\infty}\le C\|(\widetilde{V},\widetilde{U},W)\|^{\frac{1}{2}}\|(\widetilde{V},\widetilde{U},W)_x\|^{\frac{1}{2}}\le C\big(\overline{\mathcal{E}}(0)+\bar{\delta}\big)^{\frac{1}{2}}. \tag{4.11}$$

Similarly, from (3.3) and (4.11), one also can deduce that

$$\|(\widetilde{V},\widetilde{U},\widetilde{W})\|_{L^\infty}\le C\big(\overline{\mathcal{E}}(0)+\bar{\delta}\big)^{\frac{1}{2}}. \tag{4.12}$$

We now concentrate on the decay rate for $\|(\widetilde{v},\widetilde{u},\widetilde{\theta})\|_{L^\infty}$. From (3.28), (4.5) and (4.9), one finds that

$$\begin{aligned}\mathcal{E}_{3t}(t)+c\|(\widetilde{u},\widetilde{\theta})_{xx}\|^2 \le& C\bar{\delta}(1+t)^{-\frac{3}{2}}\mathcal{D}_1(t)\\ &+C(\overline{\mathcal{E}}(0)+\bar{\delta})(1+t)^{-\frac{1}{2}}\mathcal{D}_2(t)+C\bar{\delta}(1+t)^{-\frac{5}{2}}.\end{aligned} \tag{4.13}$$

Multiplying (4.13) by $(1+t)^{\frac{3}{2}}$, we obtain

$$[(1+t)^{\frac{3}{2}}\mathcal{E}_3(t)]_t+c(1+t)^{\frac{3}{2}}\|(\widetilde{u},\widetilde{\theta})_{xx}\|^2\le C\mathcal{D}_1(t)+C(1+t)\mathcal{D}_2(t)+C\bar{\delta}(1+t)^{-1}, \tag{4.14}$$

due to the fact that $\overline{\mathcal{E}}(0)+\bar{\delta}\leq C$ and $\mathcal{E}_3(t)\leq C\mathcal{D}_2(t)$.

Integrating (4.14) with respect to t and using (4.1), (4.8), we have

$$(1+t)^{\frac{3}{2}}\mathcal{E}_3(t)+c\int_0^t(1+s)^{\frac{3}{2}}\|(\widetilde{u},\widetilde{\theta})_{xx}\|^2ds\leq C\big(\overline{\mathcal{E}}(0)+\bar{\delta}\big)(1+t)^{\frac{1}{2}}. \tag{4.15}$$

Notice that $\overline{\mathcal{E}}(0)\leq C(\epsilon_0^2+\bar{\delta})$ due to (2.21) and (3.52), we thus have from (4.9) and (4.15) that

$$\|(\widetilde{v},\widetilde{u},\widetilde{\theta})\|\leq C\mathcal{E}_4^{\frac{1}{2}}(t)\leq C\big(\epsilon_0^2+\bar{\delta}\big)^{\frac{1}{2}}(1+t)^{-\frac{1}{4}}, \tag{4.16}$$

$$\|(\widetilde{v},\widetilde{u},\widetilde{\theta})_x\|\leq C\mathcal{E}_3^{\frac{1}{2}}(t)\leq C\big(\epsilon_0^2+\bar{\delta}\big)^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}. \tag{4.17}$$

As a consequence, one has from (3.23), (4.16) and (4.17) that

$$\|(\widetilde{v},\widetilde{u},\widetilde{\theta})\|_{L^\infty}\leq C\|(\widetilde{v},\widetilde{u},\widetilde{\theta})\|^{\frac{1}{2}}\|(\widetilde{v},\widetilde{u},\widetilde{\theta})_x\|^{\frac{1}{2}}\leq C\big(\epsilon_0^2+\bar{\delta}\big)^{\frac{1}{2}}(1+t)^{-\frac{3}{8}}. \tag{4.18}$$

Hence, we obtain the convergence rate of (2.23)-(2.25) and close the a priori estimates (3.7). This completes the proof of Theorem 2.1.

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