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EXISTENCE OF MILD SOLUTIONS AND REGULARITY CRITERIA OF WEAK SOLUTIONS TO THE VISCOELASTIC NAVIER-STOKES EQUATION WITH DAMPING*

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Abstract. In this paper, we consider the viscoelastic Navier-Stokes equation (VNS) with damping in the whole space. We first show that there exist global mild solutions with small initial data in the scaling invariant space. The main technique we have used is implicit function theorem which yields necessarily continuous dependence of solutions on the initial data. Moreover, we derive the asymptotic stability of solutions as the time goes to infinity. As a byproduct of our construction of solutions in the weak L^p-spaces, the existence of self-similar solutions was established provided the initial data are small homogeneous functions. Next, we deduce the regularity criteria of weak solutions to VNS with damping. Sufficient conditions for the regularity of weak solutions are presented by imposing Serrin's-type growth conditions on the velocity field and deformation tensor in Lorentz spaces, multiplier spaces, bounded mean oscillation spaces and Besov spaces, respectively.

Keywords. Viscoelastic Navier-Stokes equation; mild solutions; asymptotic stability; self-similar solutions; regularity criteria.

AMS subject classifications. 76B03; 35Q30, 35Q35; 76A10; 76D03.

1. Introduction

In this paper, we mainly consider the following VNS with damping:

$$\begin{cases} \frac{\partial u}{\partial t} - \mu\Delta u + (u\cdot\nabla)u + \nabla p = \nabla\cdot ff^t & in \quad \mathbb{R}^N\times(0,\infty), \\ \frac{\partial f}{\partial t} - \nu\Delta f + (u\cdot\nabla)f = \nabla u f & in \quad \mathbb{R}^N\times(0,\infty), \\ \text{div } u = 0 & in \quad \mathbb{R}^N\times(0,\infty), \end{cases} \tag{1.1}$$

with initial data $u(x,0)=u_0(x), f(x,0)=f_0(x)$, where $u=u(x,t)=(u_1(x,t),u_2(x,t),\cdots,u_N(x,t))$, $f=f(x,t)=(f_{ij}(x,t))(i,j=1,2...,N)$, $p=p(x,t)$ denote the velocity, the local deformation tensor and the pressure, respectively. The i-th component of $\nabla\cdot ff^t$ equals to $\partial_j(f_{ik}f_{jk})$, $\nabla uf=\partial_i u_j f_{ki}=f^t\cdot\nabla u$, $\mu>0,\nu>0$ denote the viscosity constant and the damping constant (cf. [14]), respectively. Here, for simplicity, let $\mu=\nu=1$. Note that $(1.1)_2$ is simply the consequence of the chain law. It can also be regarded as the consistency condition of the flow trajectories obtained from the velocity field u and also of those obtained from the deformation tensor f, for more details one can refer [5,13–15,22,24] and the references therein.

The aim of the paper is to show the global existence of mild solutions to (1.1) in $\mathbb{R}^N(N\geq 2)$ and the regularity criteria of weak solutions to (1.1) under the assumption that $\text{div} f=0$ in $\mathbb{R}^3$. Here, we introduce some results about the existence and regularity of weak solutions to the viscoelastic equations with damping. For the incompressible system (1.1), Liu et al. in [14] constructed a local-in-time smooth solution in two or three dimensional bounded domains with smooth boundary as well as the whole space or periodic boxes. The authors proved global-in-time existence of solutions with small

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initial data in a two dimensional periodic box by introducing an auxiliary vector field to replace the transport variable f, or the whole plane which also indicates some stability of the trivial steady motion. Using the method in [7] for the damped wave equation, they also obtained the global existence of classical small solutions for the three dimensional case. For the strong solution, Zhang et al. [23] proved the global well posedness of the incompressible version of system (1.1) in the critical L^p framework which allows one to construct the unique global solution for highly oscillating initial velocity. Soon after, R. Hynd [4] obtained the partial regularity of suitable weak solutions under the case with space dimensions $N=3$. By the Stokes estimates and the energy estimates, J. Kim [8] gave a weak-L^p Serrin-type regularity criteria of weak solutions to the three-dimension viscoelastic Navier-Stokes equations with damping.

The second results of this paper are Theorem 1.2, and Corollary 1.1 about the global stability of mild solution to (1.1) under the small initial disturbance, and the existence of forward self-similar solution. Recently, Wang et al. [11] proved that there exists a global forward self-similar solution to (1.1) in R^3, that is smooth for $t>0$, and for any initial data that is homogeneous of degree -1 by applying the Leray-Schauder fixed point theorem. For the forward self-similar solutions, one can also refer to [6, 12, 19] and the references therein. Here, we show the existence of self-similar solutions in N-dimensions with $N\geq 3$ provided the initial data are small homogeneous functions. We shall note that the method we used in the proof of Theorem 1.1-1.2 was provided by Tan et al. [17], and H. Kozono et al. [9]. There are several new difficulties that arise when we try to establish the existence of mild solutions in Theorem 1.1 extending the main theorem of [17] and [9] to (1.1):

(I) The local deformation tensor f does not satisfy $\operatorname{div} f=0$ (cf. [17]), which makes it difficult to establish the continuous map $G(\cdot,\cdot,\cdot,\cdot)$ in Lemma 2.4;

(II) the existence and uniqueness of mild solutions of (1.2) are more delicate to establish than the case of MHD system in [17] or double-type Keller-Segel system in [9], since it is a complicated thing to analyze the relationship of parameters p,q,m,r in Theorem 1.1.

Now, we shall give the definition of mild solutions to (1.1):

Definition 1.1. *Let $N\geq 2$, and let the initial data $\{u_0,f_0\}$ satisfy $u_0\in L^N_w(\mathbb{R}^N), \nabla u_0\in L^{\frac{N}{2}}_w(\mathbb{R}^N), f_0\in L^N_w(\mathbb{R}^N), \nabla f_0\in L^{\frac{N}{2}}_w(\mathbb{R}^N)$. We call measurable functions $\{u,f\}$ on $\mathbb{R}^N\times(0,\infty)$ a mild solution of (1.1) on $(0,\infty)$ if $u,f\in L^q_{loc}(0,\infty;L^r(\mathbb{R}^N))$ for some $1\leq q,r\leq\infty$ and satisfy the following identities*

$$\begin{cases} u(t)=e^{t\Delta}u_0-\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[(u\cdot\nabla)u(\tau)-\nabla\cdot ff^t]d\tau,\\ f(t)=e^{t\Delta}f_0-\int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)f-\nabla uf]d\tau, \end{cases} \tag{1.2}$$

for $t\in(0,\infty)$, where $e^{t\Delta}$ denotes the heat semi-group defined by

$$(e^{t\Delta}g)(x)\equiv\int_{\mathbb{R}^N}L(x-y,t)g(y)dy \tag{1.3}$$

with $L(x,y)=\frac{1}{(4\pi t)^{N/2}}exp(-\frac{|x|^2}{4t})$ and $\mathbb{P}=\{\mathbb{P}_{jk}\}_{j,k=1,\cdots,N}$ denotes the projection operator onto the solenoidal vector fields with the expression (cf. [18])

$$\mathbb{P}_{jk}=\delta_{jk}+R_jR_k\quad (R_j\equiv\frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}:Riesz\quad operator), \tag{1.4}$$

where $j,k=1,2,\cdots,N$.

To begin with, we shall give the first result about unique global existence of mild solutions to (1.1). For simplicity, here and in what follows, we shall denote $\frac{p}{p-1}$, $\frac{q}{q-1}$, $\frac{Np}{p-N}$, $\frac{Nq}{q-N}$, $\frac{1}{\frac{1}{m}+\frac{1}{N}-\frac{1}{p}}$, $\frac{1}{\frac{1}{m}+\frac{1}{p}-\frac{1}{N}}$, $\frac{1}{\frac{2}{N}+\frac{1}{p}-\frac{1}{q}}$ by p', q', p_*, q_*, η_1, η_2, η_3, respectively.

THEOREM 1.1. *For* $N\geq 2$, *suppose that the exponents* p,m,r *satisfy the following either (1), (2) or (3) for the case* $N<q<2N$,

(1) $\frac{2Nq}{q+N}<p<q$, $\max\{p',\frac{N}{2}\}\leq m<q_*$, $\max\{\frac{N}{2},\eta_1,p'\}<r<\min\{q_*,\eta_2\}$;
(2) $q\leq p\leq 2N$, $\max\{q',\eta_3\}<m<p_*$, $\max\{\frac{N}{2},\eta_1,q'\}<r<\min\{p_*,\eta_2\}$;
(3) $2N<p\leq 2q$, $\max\{q',\eta_3\}<m<p_*$, $\max\left\{\frac{N}{2},\frac{1}{\frac{1}{m}+\frac{1}{p}},q'\right\}<r<\min\left\{p_*,\frac{1}{\frac{1}{m}-\frac{1}{p}}\right\}$;

and for the other case $2N\leq q<\infty$, p,m,r *satisfy the following either (4), (5) or (6),*

(4) $\frac{2Nq}{q+N}\leq p\leq 2N$, $\max\{p',\frac{N}{2}\}\leq m<q_*$, $\max\{\frac{N}{2},\eta_1,p'\}<r<\min\{q_*,\eta_2\}$;
(5) $2N<p<q$, $\max\{p',\frac{N}{2}\}\leq m<q_*$, $\max\left\{\frac{N}{2},\frac{1}{\frac{1}{m}+\frac{1}{p}},p'\right\}<r<\min\left\{q_*,\frac{1}{\frac{1}{m}-\frac{1}{p}}\right\}$;
(6) $q\leq p\leq 2q$, $\max\{q',\eta_3\}<m<p_*$, $\max\left\{\frac{N}{2},\frac{1}{\frac{1}{m}+\frac{1}{p}},q'\right\}<r<\min\left\{p_*,\frac{1}{\frac{1}{m}-\frac{1}{p}}\right\}$;

and there exists a constant $\delta=\delta(N,p,q,m,r)$ *such that the initial data* $\{u_0,f_0\}$ *satisfy the following conditions*

$$\|u_0\|_{L^N_w(\mathbb{R}^N)}+\|\nabla u_0\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)}+\|f_0\|_{L^N_w(\mathbb{R}^N)}+\|\nabla f_0\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)}<\delta \quad \textit{for } N\geq 3, \tag{1.5}$$

$$\|u_0\|_{L^2_w(\mathbb{R}^2)}+\|\nabla u_0\|_{L^1(\mathbb{R}^2)}+\|f_0\|_{L^2_w(\mathbb{R}^2)}+\|\nabla f_0\|_{L^1(\mathbb{R}^2)}<\delta \quad \textit{for } N=2, \tag{1.6}$$

then there exists a mild solution u,f *of (1.1) such that*

$$t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}u\in BC_w([0,\infty);L^q(\mathbb{R}^N)), \tag{1.7}$$

$$t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\nabla u\in BC_w([0,\infty);L^r(\mathbb{R}^N)), \tag{1.8}$$

$$t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}f\in BC_w([0,\infty);L^p(\mathbb{R}^N)), \tag{1.9}$$

$$t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\nabla f\in BC_w([0,\infty);L^m(\mathbb{R}^N)), \tag{1.10}$$

where $BC_w([0,\infty);Y)$ *denotes the set of bounded weakly-star continuous functions. Meanwhile, if the norms corresponding to the spaces (1.7)-(1.10) are sufficiently small, the mild solution* $\{u,f\}$ *to (1.1) is unique. Furthermore, as* $t\to\infty$, *the mild solution* $\{u,f\}$ *has the following asymptotic behavior*

$$\|u(t)-e^{t\Delta}u_0\|_{L^q(\mathbb{R}^N)}=O\left(t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\right), \tag{1.11}$$

$$\|\nabla u(t)-\nabla e^{t\Delta}u_0\|_{L^r(\mathbb{R}^N)}=O\left(t^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\right), \tag{1.12}$$

$$\|f(t)-e^{t\Delta}f_0\|_{L^p(\mathbb{R}^N)}=O\left(t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\right), \tag{1.13}$$

$$\|\nabla f(t)-\nabla e^{t\Delta}f_0\|_{L^m(\mathbb{R}^N)}=O\left(t^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\right). \tag{1.14}$$

Next, we proceed to study the global stability of our mild solution under the initial disturbance in the scaling invariant class and establish the existence of forward self-similar solutions to (1.1).

Theorem 1.2. *Let the exponents p,q,m,r be as in Theorem 1.1. Suppose that $\delta=\delta(N,p,q,m,r)$ is the same constant as in (1.5). For any $\varepsilon>0$, there is a constant $\theta=\theta(N,p,q,m,r,\varepsilon)>0$ with the following property: The initial data $\{u_0,f_0\}$ and $\{\tilde{u}_0,\tilde{f}_0\}$ satisfy that*

$$\|u_0\|_{L^N_w(\mathbb{R}^N)}+\|\nabla u_0\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)}+\|f_0\|_{L^N_w(\mathbb{R}^N)}+\|\nabla f_0\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)}<\delta, \tag{1.15}$$

$$\|\tilde{u}_0\|_{L^N_w(\mathbb{R}^N)}+\|\nabla \tilde{u}_0\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)}+\|\tilde{f}_0\|_{L^N_w(\mathbb{R}^N)}+\|\nabla \tilde{f}_0\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)}<\delta, \tag{1.16}$$

for $N\geq 3$, and that (1.15)-(1.16) with $L^{\frac{N}{2}}_w$ replaced by $L^1(\mathbb{R}^2)$ for $N=2$. Assume that $\{u,f\}$ and $\{\tilde{u},\tilde{f}\}$ are mild solutions to (1.1) on $[0,\infty)$ given by Theorem 1.1 with the initial data $\{u_0,f_0\}$ and $\{\tilde{u}_0,\tilde{f}_0\}$ in the class (1.7) and (1.10), respectively. If it holds that

$$\|u_0-\tilde{u}_0\|_{L^N_w(\mathbb{R}^N)}+\|\nabla u_0-\nabla\tilde{u}_0\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)}$$
$$+\|f_0-\tilde{f}_0\|_{L^N_w(\mathbb{R}^N)}+\|\nabla f_0-\nabla\tilde{f}_0\|_{L^{\frac{N}{2}}_w(\mathbb{R}^N)}<\theta, \tag{1.17}$$

for $N\geq 3$, and that (1.17) with $L^{\frac{N}{2}}_w(\mathbb{R}^N)$ replaced by $L^1(\mathbb{R}^2)$ for $N=2$. Then we have

$$\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(t)-\tilde{u}(t)\|_{L^q(\mathbb{R}^N)}+\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u(t)-\nabla\tilde{u}(t)\|_{L^r(\mathbb{R}^N)}$$
$$+\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(t)-\tilde{f}(t)\|_{L^p(\mathbb{R}^N)}+\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla f(t)-\nabla\tilde{f}(t)\|_{L^m(\mathbb{R}^N)}<\varepsilon \tag{1.18}$$

From the construction of solutions in the weak L^p spaces, we have the following corollary of the forward self-similar solution.

Corollary 1.1. *Let $N\geq 3$ and assume the initial data $\{u_0,f_0\}$ satisfies $u_0\in L^N_w(\mathbb{R}^N), f_0\in L^N_w(\mathbb{R}^N)$ and $\{u_0,f_0\}$ are homogeneous functions with degree -1, i.e. $u_0(rx)=r^{-1}u_0(x)$, $f_0(rx)=r^{-1}f_0(x)$, for all $x\in\mathbb{R}^N$ and all $r>0$. Moreover, the initial data $\{u_0,f_0\}$ satisfy the condition (1.5), then the solution $\{u,f\}$ given by Theorem 1.1 is a forward self-similar one, i.e. it holds that*

$$u(rx,r^2t)=r^{-1}u(x,t) \quad f(rx,r^2t)=r^{-1}f(x,t), \tag{1.19}$$

for all $r>0$ and for all $x\in\mathbb{R}^N,t>0$. When we assume further that $\operatorname{div} f=0$, and $\mu=\nu=1$, then (1.1) can be rewritten as

$$\begin{cases} \frac{\partial u}{\partial t}-\Delta u+(u\cdot\nabla)u+\nabla p=\nabla\cdot ff^t \\ \frac{\partial f}{\partial t}-\Delta f+(u\cdot\nabla)f=\nabla uf \\ \operatorname{div}\ u=0,\operatorname{div}\ f=0,\ t>0,\ x\in\mathbb{R}^3, \end{cases} \tag{1.20}$$

with initial data $u(x,0)=u_0(x),f(x,0)=f_0(x)$. Before we start to introduce the regularity criteria, we would like to recall the definition of weak solutions to (1.20) (cf. [11]).

Definition 1.2. *Let $T>0$, $(u_0,f_0)\in L^2(\mathbb{R}^3)$ and $\operatorname{div}\ u_0=0,\operatorname{div}\ f_0=0$. Then we call (u,f) a weak solution to (1.20) on $(0,T)$ if and only if (u,f) satisfies the following properties:*

- $(u,f)\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$;

- *(u,f) satisfies (1.20) in the sense of distributions.*

The last results of this paper are Theorems 1.3–1.5 about the regularity criteria of weak solution to (1.20) in $\mathbb{R}^3$. Such results were inspired by [8] and [3]. The motivation for the study we undertake in the fourth part of this paper is two fold:

(a) We aim to extend the previous theorems by J.M. Kim [8] to more general function spaces, e.g., Lorentz spaces, multiplier spaces, bounded mean oscillation spaces and Besov spaces, respectively;

(b) We propose to remove the precondition about $\operatorname{div} f=0$ in the proof of Theorems 1.3–1.4 under the Assumption (1.21), (1.23)-(1.24), and we state it in Remark 4.1.

More precisely, we have

THEOREM 1.3. *Let $T>0$, (u,f) be a weak solution to (1.20) with the initial data $(u_0,f_0)\in H^1(\mathbb{R}^3)$. Then the weak solution (u,f) is regular on $[0,T]$ if any one of the following conditions*

$$u,f\in L^p(0,T;L^{q,\infty}(\mathbb{R}^3)),\quad \frac{2}{p}+\frac{3}{q}=1,\quad 3<q\le\infty, \tag{1.21}$$

$$\nabla u\in L^p(0,T;L^{q,\infty}(\mathbb{R}^3)),\quad \frac{2}{p}+\frac{3}{q}=2,\quad \frac{3}{2}<q\le\infty, \tag{1.22}$$

$$u,f\in L^{2/(1-r)}(0,T;\dot{H}^{-r}(\mathbb{R}^3)),\quad r\in(0,1], \tag{1.23}$$

is satisfied.

For the limit case $q=\infty$, we also have the following regularity criteria in BMO and Besov spaces.

THEOREM 1.4. *Let $T>0$, (u,f) be a weak solution to (1.20) with the initial data $(u_0,f_0)\in H^1(\mathbb{R}^3)$. Then the weak solution (u,f) is regular on $[0,T]$ if the following condition*

$$u,f\in L^4(0,T;BMO(\mathbb{R}^3)) \tag{1.24}$$

holds true.

THEOREM 1.5. *Let $T>0$, (u,f) be a weak solution to (1.20) with the initial data $(u_0,f_0)\in H^1(\mathbb{R}^3)$. Then the weak solution (u,f) is regular on $[0,T]$ if the following condition*

$$\nabla u\in L^2(0,T;\dot{B}^0_{3,1}(\mathbb{R}^3)) \tag{1.25}$$

holds true.

The rest of this paper is organized as follows. In Section 2, we present some auxiliary lemmas and the definition of some function spaces. In Section 3, we prove the existence and uniqueness of mild solutions to (1.1), and we also show the existence of forward self-similar solutions to (1.1). Lastly, in Section 4, we prove the regularity criteria of weak solutions to (1.20) in various spaces.

2. Preliminaries and auxiliary lemmas

In this section, we shall present some notations and some auxiliary results which will be used in the following parts of the paper. At first, we may like to introduce several usual function spaces.

2.1. Notation. Throughout this paper, we use $L^p(\mathbb{R}^N)$, $1\le p\le\infty$, to denote the usual $L^p(\mathbb{R}^N)$ spaces with norm $\|\cdot\|_{L^p}$. $L^{p,q}(\mathbb{R}^N)$ denotes the Lorentz space associated with norm

$$\|f\|_{L^{p,q}}=\begin{cases}\left(q\int_0^\infty t^q(\mu\{x\in\mathbb{R}^N:|f(x)|>t\})^{\frac{q}{p}}\frac{dt}{t}\right)^{\frac{1}{q}} & 1\le p,q<\infty,\\ \operatorname{ess\,sup}_{t\ge 0} t(\mu\{x\in\mathbb{R}^N:|f(x)|>t\})^{\frac{1}{p}} & p<q=\infty,\\ \|f\|_{L^\infty} & p=q=\infty,\end{cases}$$

where $\mu(\cdot)$ denotes the Lebesgue measure of $\mathbb{R}^N$. $BMO(\mathbb{R}^N)$ denotes the homogeneous space of bounded mean oscillations associated with the norm

$$\|f\|_{BMO(\mathbb{R}^N)}=\sup_{x\in\mathbb{R}^N,r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)}\left|f(y)-\frac{1}{B_r(y)}\int_{B_r(y)}f(z)dz\right|dy.$$

$\dot{H}^{-r}$ denotes the homogeneous Banach space of bounded linear multipliers $f:\dot{H}^r(\mathbb{R}^N)\mapsto L^2(\mathbb{R}^N)$ associated with the norm

$$\|f\|_{\dot{H}^{-r}(\mathbb{R}^N)}=\sup_{\|g\|_{\dot{H}^r}\le 1}\left(\int_{\mathbb{R}^N}|fg|^2dx\right)^{\frac{1}{2}}.$$

We then introduce the homogeneous Besov space, let $\varphi\in C_0^\infty(\mathbb{R}^d_\xi)$ be a cut-off function such that $\varphi(\xi)=1$ with $|\xi|\le 1$, and $\varphi(\xi)=0$ when $|\xi|\ge 2$. Let $\psi(\xi)=\varphi(\xi)-\varphi(2\xi)$ and $\psi_j(\xi)=\psi(2^{-j}\xi)$ for $j\in\mathbb{Z}$. Then, by the construction $\sum_{j\in\mathbb{Z}}\psi_j(\xi)=1$ if $\xi\ne 0$, we define Littlewood-Paley projection operator $\dot{\Delta}_j f:=\mathscr{F}^{-1}(\psi_j)*f$, then, for $s\in\mathbb{R}$, we define the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^N)$ with norm $\|\cdot\|_{\dot{B}^s_{p,q}}$ by

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^N)}=\begin{cases}\left(\sum_{-\infty<j<\infty}2^{jsq}\|\dot{\Delta}_j f\|^q_{L^p(\mathbb{R}^N)}\right)^{\frac{1}{q}} & 1\le p\le\infty, 1\le q<\infty,\\ \operatorname{ess\,sup}_{j\in\mathbb{Z}}2^{js}\|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^N)} & 1\le p\le\infty, q=\infty,\end{cases}$$

Actually, $\dot{\Delta}_j$ can be regarded as a frequency projection to the annulus $|\xi|\sim 2^j$, which implies the following homogeneous Littlewood-Paley decomposition

$$f=\sum_{j=-\infty}^{\infty}\dot{\Delta}_j f$$

in the sense of distributions.

2.2. Auxiliary Lemmas. In this subsection, we collect some helpful results, some of which have been proven elsewhere. Firstly, we shall need the Hölder inequality in Lorentz spaces.

LEMMA 2.1. *Let* $f\in L^{p_2,q_2}(\mathbb{R}^3)$, $g\in L^{p_3,q_3}(\mathbb{R}^3)$ *with*

$$1\le p_2,p_3\le\infty,\ 1\le q_2,q_3\le\infty.$$

Then, $fg\in L^{p_1,q_1}(\mathbb{R}^3)$ *with*

$$\frac{1}{p_1}=\frac{1}{p_2}+\frac{1}{p_3},\ \frac{1}{q_1}=\frac{1}{q_2}+\frac{1}{q_3}$$

and the Hölder inequality of Lorentz spaces

$$\|fg\|_{L^{p_1,q_1}} \le c\|f\|_{L^{p_2,q_2}}\|g\|_{L^{p_3,q_3}}$$

is valid for a constant c.

A proof can be retrieved e.g. from [16]. The following Lemma is a well known result, ensuring $\|\cdot\|_{L^r}$ bound in terms of $\|\cdot\|_{BMO}$ (cf. [10]).

LEMMA 2.2. *Let $1<r<\infty$. Then, we have*

$$\|f\cdot g\|_{L^r} \le c(\|f\|_{L^r}\|g\|_{BMO} + \|g\|_{L^r}\|f\|_{BMO}) \tag{2.1}$$

for all $f,g \in L^r \cap BMO$ with $c=c(r)$.

In addition, we shall use the following Bernstein inequality (cf. [1]).

LEMMA 2.3. *Let $\alpha \in \mathbb{N}$. Then, for all $1 \le p \le q \le \infty$,*

$$\sup_{|\alpha|=k} \|\partial^\alpha \Delta_j f\|_{L^q} \le c2^{jk+3j(1/p-1/q)}\|\Delta_j f\|_{L^p} \tag{2.2}$$

with c as a positive constant independent of f,j.

Next, we shall introduce two function spaces W and Z defined by

$$W \equiv \left\{ \{u_0,f_0\}; u_0 \in L^N_w, \nabla u_0 \in L^{\frac{N}{2}}_w, f_0 \in L^N_w, \nabla f_0 \in L^{\frac{N}{2}}_w \right\}$$

with the norm

$$\|\{u_0,f_0\}\|_W \equiv \|u_0\|_{L^N_w} + \|\nabla u_0\|_{L^{\frac{N}{2}}_w} + \|f_0\|_{L^N_w} + \|\nabla f_0\|_{L^{\frac{N}{2}}_w}$$

and

$$\begin{aligned} Z \equiv \Big\{ \ &\{u,f\}; t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}u(\cdot) \in BC_w([0,\infty);L^q(\mathbb{R}^N)), t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\nabla u(\cdot) \\ &\in BC_w([0,\infty);L^r(\mathbb{R}^N)), \\ &t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}f(\cdot) \in BC_w([0,\infty);L^p(\mathbb{R}^N)), t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\nabla f(\cdot) \in BC_w([0,\infty);L^m(\mathbb{R}^N)) \Big\} \end{aligned}$$

with the norm

$$\begin{aligned} \|\{u,f\}\|_Z \equiv &\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(t)\|_{L^q} + \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u(t)\|_{L^r} \\ &+ \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(t)\|_{L^p} + \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla f(t)\|_{L^m}, \end{aligned}$$

respectively. Here, L^p_w denotes the weak-L^p space with the norm $\|\cdot\|_{L^p_w}$ defined by

$$\|f\|_{L^p_w} = \sup_{\alpha>0} \alpha\mu\left\{x \in \mathbb{R}^N; |f(x)| > \alpha\right\}^{\frac{1}{p}}.$$

Clearly, W and Z equipped with the norm $\|\cdot\|_W$ and $\|\cdot\|_Z$ are Banach spaces. For $\{u_0,f_0\} \in W$ and $\{u,f\} \in Z$, we define

$$G(u_0,f_0,u,f) \equiv \{U,F\}, \tag{2.3}$$

with

$$\begin{cases} U(t)=u(t)-e^{t\Delta}u_0+\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[(u\cdot\nabla)u(\tau)-\nabla\cdot ff^t]d\tau, & 0<t<\infty;\\ F(t)=f(t)-e^{t\Delta}f_0+\int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)f-\nabla uf]d\tau, & 0<t<\infty. \end{cases} \tag{2.4}$$

Moreover, the Fréchet derivative of G in the direction of $\{u,f\}$ is usually denoted by $G^{'}_{(u,f)}$, and the Fréchet derivative of $G(\cdot,\cdot,\cdot,\cdot)$ at point $\{u_0,b_0,u,f\}\in W\times Z$ in the direction of $\{u,f\}$ whose value at point $(\tilde{u},\tilde{f})$ is defined by $\langle G^{'}_{\{u,f\}}(u_0,b_0,u,f),(\tilde{u},\tilde{f})\rangle$.

From above definition, it is clear that, in the following lemma, we mainly study the property of the map $G(\cdot,\cdot,\cdot,\cdot)$. The basic techniques are the estimate of heat semi-group and the property of Beta function. Furthermore, we derive the Fréchet derivative of $G(\cdot,\cdot,\cdot,\cdot)$ at point $\{u_0,b_0,u,f\}\in W\times Z$ in the direction of $\{u,f\}$.

LEMMA 2.4. *Suppose that the exponents p,m,r satisfy the following either (1), (2) or (3) for the case $N<q<2N$,*

(1) $\frac{2Nq}{q+N}<p<q$, $\max\{p',\frac{N}{2}\}\le m<q_*$, $\max\{\frac{N}{2},\eta_1,p'\}<r<\min\{q_*,\eta_2\}$;
(2) $q\le p\le 2N$, $\max\{q',\eta_3\}<m<p_*$, $\max\{\frac{N}{2},\eta_1,q'\}<r<\min\{p_*,\eta_2\}$;
(3) $2N<p\le 2q$, $\max\{q',\eta_3\}<m<p_*$, $\max\left\{\frac{N}{2},\frac{1}{\frac{1}{m}+\frac{1}{p}},q'\right\}<r<\min\left\{p_*,\frac{1}{\frac{1}{m}-\frac{1}{p}}\right\}$;

and p,m,r satisfy the following either (4), (5) or (6) for the other case $2N\le q<\infty$,

(4) $\frac{2Nq}{q+N}\le p\le 2N$, $\max\{p',\frac{N}{2}\}\le m<q_*$, $\max\{\frac{N}{2},\eta_1,p'\}<r<\min\{q_*,\eta_2\}$;
(5) $2N<p<q$, $\max\{p',\frac{N}{2}\}\le m<q_*$, $\max\left\{\frac{N}{2},\frac{1}{\frac{1}{m}+\frac{1}{p}},p'\right\}<r<\min\left\{q_*,\frac{1}{\frac{1}{m}-\frac{1}{p}}\right\}$;
(6) $q\le p\le 2q$, $\max\{q',\eta_3\}<m<p_*$, $\max\left\{\frac{N}{2},\frac{1}{\frac{1}{m}+\frac{1}{p}},q'\right\}<r<\min\left\{p_*,\frac{1}{\frac{1}{m}-\frac{1}{p}}\right\}$;

for $N\ge 2$. Then we obtain

(i) *The map G defined by (2.3) is a continuous map from $W\times Z$ into Z.*

(ii) *For each initial data $\{u_0,f_0\}\in W$, the map $G(u_0,f_0,\cdot,\cdot)$ is of class C^1 from Z into itself.*

Proof. (i). Firstly, we shall prove that

$$t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}U(t)\in BC_w([0,\infty);L^q),$$

and

$$t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\nabla U(t)\in BC_w([0,\infty);L^r).$$

By using the $L^q-L^N_w$ estimate of the heat semi-group, it holds that

$$\|e^{t\Delta}u_0\|_{L^q}\le ct^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u_0\|_{L^N_w}, \tag{2.5}$$

where $c=c(N,q)$. Similarly, we obtain

$$\begin{cases} \|\nabla e^{t\Delta}u_0\|_{L^r}\le ct^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u_0\|_{L_w^{\frac{N}{2}}} & N\ge 3,\\ \|\nabla e^{t\Delta}u_0\|_{L^r}\le ct^{-(1-\frac{1}{r})}\|\nabla u_0\|_{L^1} & N=2, \end{cases} \tag{2.6}$$

where $c=c(N,r)$. From above, we can deduce that

$$t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}e^{t\Delta}u_0\in BC_w([0,\infty);L^q),$$

and

$$t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\nabla e^{t\Delta}u_0\in BC_w([0,\infty);L^r).$$

Taking into account the conditions (1)-(6) on p,q,m,r, we obtain $\frac{1}{2}-\frac{N}{2q}>0$, and since the projection operator $\tilde{P}$ is bounded in L^q, we have the following estimate

$$\begin{aligned}
&\left\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(u\cdot\nabla)u(\tau)d\tau\right\|_{L^q}\\
&=\left\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(\nabla\cdot(u\otimes u))(\tau)d\tau\right\|_{L^q}\\
&\le c\int_0^t \|\nabla\cdot e^{(t-\tau)\Delta}(u\otimes u)(\tau)\|_{L^q}d\tau\\
&\le c\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{2}{q}-\frac{1}{q})-\frac{1}{2}}\|u\otimes u(\tau)\|_{L^{\frac{q}{2}}}d\tau\\
&\le c\left(\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(\tau)\|_{L^q}\right)^2\cdot\int_0^t (t-\tau)^{\frac{1}{2}-\frac{N}{2q}-1}\tau^{\frac{N}{q}-1}d\tau\\
&=cB\left(\frac{1}{2}-\frac{N}{2q},\frac{N}{q}\right)\left(\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(\tau)\|_{L^q}\right)^2\cdot t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{q})} \qquad (2.7)
\end{aligned}$$

for all $t>0$ with $c=c(N,q)$, and $B(\cdot,\cdot)$ denotes the Beta function.

Similarly, since $\frac{p}{2}\le q, \frac{1}{2}-\frac{N}{2}(\frac{2}{p}-\frac{1}{q})>0$, we have

$$\begin{aligned}
&\left\|\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(\nabla\cdot ff^t)(\tau)d\tau\right\|_{L^q}\\
&=\left\|\int_0^t \mathbb{P}\nabla\cdot e^{(t-\tau)\Delta}(ff^t)(\tau)d\tau\right\|_{L^q}\le c\int_0^t\|\nabla\cdot e^{(t-\tau)\Delta}(ff^t)(\tau)\|_{L^q}d\tau\\
&\le c\left(\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(\tau)\|_{L^p}\right)^2\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N}{2}(\frac{2}{p}-\frac{1}{q})}\tau^{\frac{N}{p}-1}d\tau\\
&=cB\left(\frac{1}{2}-\frac{N}{2}\left(\frac{2}{p}-\frac{1}{q}\right),\frac{N}{p}\right)\left(\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(\tau)\|_{L^p}\right)^2\cdot t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}, \qquad (2.8)
\end{aligned}$$

for all $t>0$ with $c=c(N,p,q)$.

Moreover, by the conditions (1)-(6) on p,q,m,r, we may verify that $-\frac{N}{2}(\frac{1}{N}-\frac{1}{q}-\frac{1}{r})>0$, $\frac{1}{2}-\frac{N}{2q}>0, \frac{1}{q}+\frac{1}{r}\le 1$, $\frac{1}{m}+\frac{1}{p}\le 1, \frac{N}{2m}+\frac{N}{2p}-\frac{1}{2}>0, \frac{1}{m}+\frac{1}{p}-\frac{1}{r}\ge 0$ and $\frac{1}{2}-\frac{N}{2}(\frac{1}{m}+\frac{1}{p}-\frac{1}{r})>0$, which follows

$$\begin{aligned}
&\left\|\nabla\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[(u\cdot\nabla)u(\tau)-\nabla\cdot ff^t]d\tau\right\|_{L^r}\\
&\le c\int_0^t\|\nabla e^{(t-\tau)\Delta}(u\cdot\nabla u)(\tau)\|_{L^r}d\tau+c\int_0^t\|\nabla e^{(t-\tau)\Delta}(\nabla\cdot ff^t)(\tau)\|_{L^r}d\tau\\
&\le c\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}+\frac{1}{r}-\frac{1}{r})}\|u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^r}d\tau\\
&\quad +c\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{m}+\frac{1}{p}-\frac{1}{r})}\|f(\tau)\|_{L^p}\|\nabla f(\tau)\|_{L^m}d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq c \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(\tau)\|_{L^q} \cdot \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u(\tau)\|_{L^r} \\
&\quad \times \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N}{2q}} \tau^{-\frac{N}{2}(\frac{3}{N}-\frac{1}{q}-\frac{1}{r})} d\tau \\
&\quad + c \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(\tau)\|_{L^p} \cdot \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla f(\tau)\|_{L^m} \\
&\quad \times \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{m}+\frac{1}{p}-\frac{1}{r})} \tau^{-\frac{N}{2}(\frac{3}{N}-\frac{1}{m}-\frac{1}{p})} d\tau \\
&= cB\left(-\frac{N}{2}\left(\frac{1}{N}-\frac{1}{q}-\frac{1}{r}\right), \frac{1}{2}-\frac{N}{2q}\right) \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(\tau)\|_{L^q} \\
&\quad \times \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u(\tau)\|_{L^r} t^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{r})} \\
&\quad + cB\left(\frac{N}{2m}+\frac{N}{2p}-\frac{1}{2}, \frac{1}{2}-\frac{N}{2}\left(\frac{1}{m}+\frac{1}{p}-\frac{1}{r}\right)\right) \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(\tau)\|_{L^p} \\
&\quad \times \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla f(\tau)\|_{L^m} t^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}.
\end{aligned}
\tag{2.9}
$$

Now, combine $(2.4)_1$ with (2.5)-(2.9), to obtain

$$
t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}U(t) \in BC_w([0,\infty);L^q) \quad \text{and} \quad t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{q})}\nabla U(t) \in BC_w([0,\infty);L^r),
$$

and with the estimate

$$
\begin{aligned}
&\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|U(t)\|_{L^q} \leq c\|u_0\|_{L^N_w} + c \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(t)\|_{L^q} \\
&+ c\left(\sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(\tau)\|_{L^q}\right)^2 + c\left(\sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(\tau)\|_{L^p}\right)^2,
\end{aligned}
\tag{2.10}
$$

where $c = c(N,p,q)$, and

$$
\begin{aligned}
\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla U(t)\|_{L^r} &\leq c\|\nabla u_0\|_{L^{\frac{N}{2}}_w} + c \sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u(t)\|_{L^r} \\
&\quad + c \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(\tau)\|_{L^q} \times \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u(\tau)\|_{L^r} \\
&\quad + c \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla f(\tau)\|_{L^m} \cdot \sup_{0<\tau<\infty} \tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(\tau)\|_{L^p},
\end{aligned}
\tag{2.11}
$$

where $c = c(N,p,q,m,r)$ and for $N = 2$, we replace $\|\nabla u_0\|_{L^{\frac{N}{2}}_w}$ by $\|\nabla u_0\|_{L^1}$.

Next, we propose to show

$$
t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}F(t) \in BC_w([0,\infty);L^p(\mathbb{R}^N)),
$$

and

$$
t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\nabla F(t) \in BC_w([0,\infty);L^m(\mathbb{R}^N)).
$$

Indeed, it holds that

$$
\|e^{t\Delta} f_0\|_{L^p} \leq ct^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f_0\|_{L^N_w}, \tag{2.12}
$$

and

$$
\begin{cases}
\|\nabla e^{t\Delta} f_0\|_{L^m} \leq ct^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla f_0\|_{L^{\frac{N}{2}}_w} & N \geq 3, \\
\|\nabla e^{t\Delta} f_0\|_{L^m} \leq ct^{-(1-\frac{1}{m})}\|f_0\|_{L^1} & N = 2.
\end{cases}
\tag{2.13}
$$

Since $\frac{1}{m}+\frac{1}{q}\leq 1, \frac{N}{2q}+\frac{N}{2m}-\frac{1}{2}>0, 1-\frac{N}{2}(\frac{1}{q}+\frac{1}{m}-\frac{1}{p})>0, \frac{1}{p}+\frac{1}{r}\leq 1, \frac{N}{2p}+\frac{N}{2r}-\frac{1}{2}>0$ and $1-\frac{N}{2r}>0$, appealing to (1)-(6) and Hölder's inequality, we arrive at

$$\begin{aligned}
&\left\|\int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)f-\nabla uf](\tau)d\tau\right\|_{L^p}\\
\leq& c\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}+\frac{1}{m}-\frac{1}{p})}\|u(\tau)\|_{L^q}\|\nabla f\|_{L^m}d\tau\\
&+c\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{p}+\frac{1}{r}-\frac{1}{p})}\|f(\tau)\|_{L^p}\cdot\|\nabla u(\tau)\|_{L^r}d\tau\\
\leq& c\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(\tau)\|_{L^q}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla f(\tau)\|_{L^m}\\
&\times B\left(\frac{N}{2q}+\frac{N}{2m}-\frac{1}{2},1-\frac{N}{2}\left(\frac{1}{q}+\frac{1}{m}-\frac{1}{p}\right)\right)t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\\
&+c\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(\tau)\|_{L^p}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u(\tau)\|_{L^r}\\
&\times B\left(\frac{N}{2p}+\frac{N}{2r}-\frac{1}{2},1-\frac{N}{2r}\right)t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{p})},
\end{aligned}\tag{2.14}$$

for all $t>0$ with $c=c(N,p,q,m,r)$.

Similarly, since $\frac{1}{m}+\frac{1}{q}\leq 1, \frac{1}{p}+\frac{1}{r}\leq 1, \frac{N}{2q}+\frac{N}{2m}-\frac{1}{2}>0, \frac{N}{2p}+\frac{N}{2r}-\frac{1}{2}>0, \frac{1}{2}-\frac{N}{2}(\frac{1}{p}+\frac{1}{r}-\frac{1}{m})>0, \frac{1}{2}-\frac{N}{2q}>0$, we can obtain the following estimate

$$\begin{aligned}
&\left\|\nabla\int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)f+\nabla uf](\tau)d\tau\right\|_{L^m}\\
\leq& c\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}+\frac{1}{m}-\frac{1}{m})}\|u(\tau)\|_{L^q}\\
&\cdot\|\nabla f\|_{L^m}d\tau+c\int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p}+\frac{1}{r}-\frac{1}{m})}\|\nabla u(\tau)\|_{L^r}\|f\|_{L^p}d\tau\\
\leq& c\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(\tau)\|_{L^q}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla f(\tau)\|_{L^m}\\
&\times B\left(\frac{N}{2q}+\frac{N}{2m}-\frac{1}{2},\frac{1}{2}-\frac{N}{2q}\right)t^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\\
&+c\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u(\tau)\|_{L^r}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(\tau)\|_{L^p}\\
&\times B\left(\frac{N}{2p}+\frac{N}{2r}-\frac{1}{2},\frac{1}{2}-\frac{N}{2}(\frac{1}{p}+\frac{1}{r}-\frac{1}{m})\right)t^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{m})},
\end{aligned}\tag{2.15}$$

for all $t>0$ with $c=c(N,p,q,m,r)$.

Taking together $(2.4)_2$ with (2.12)-(2.15), we are in a position to find that

$$t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}F(t)\in BC_w([0,\infty);L^p)\quad\text{and}\quad t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\nabla F(t)\in BC_w([0,\infty);L^m),$$

with the estimates

$$\begin{aligned}
&\sup_{0<t<\infty}t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|F(t)\|_{L^p}\\
\leq& c\|f_0\|_{L^N_w}+c\sup_{0<t<\infty}t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(t)\|_{L^p}
\end{aligned}$$

$$
\begin{aligned}
&+c\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(t)\|_{L^q}\cdot\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla f(t)\|_{L^m}\\
&+c\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(t)\|_{L^p}\cdot\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u(t)\|_{L^r} \qquad (2.16)
\end{aligned}
$$

for all $t>0$, where $c=c(N,p,q,m,r)$, and

$$
\begin{aligned}
&\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla F(t)\|_{L^m}\\
\le & c\|\nabla f_0\|_{L_w^{\frac{N}{2}}}+c\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla f(t)\|_{L^m}\\
&+c\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(t)\|_{L^q}\cdot\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla f(t)\|_{L^m}\\
&+c\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(t)\|_{L^p}\cdot\sup_{0<t<\infty} t^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u(t)\|_{L^r} \qquad (2.17)
\end{aligned}
$$

for all $t>0$, where $c=c(N,p,q,m,r)$. Here, for $N=2$, we substitute $L_w^{\frac{N}{2}}$ by L^1.

Taking into account (2.10)-(2.11) and (2.16)-(2.17), we obtain the conclusion that $G(u_0,f_0,u,f)\equiv\{U,F\}\in Z$, with the estimate

$$
\|G(u_0,f_0,u,f)\|_Z\le c\|\{u_0,f_0\}\|_W+c\|\{u,f\}\|_Z(1+\|\{u,f\}\|_Z)
$$

where $c=c(N,p,q,m,r)$.

(ii). In order to prove the conclusion of (**ii**), we define a linear map $\langle H_{\{u,f\}},(\tilde u,\tilde f)\rangle=\{\tilde U,\tilde F\}$ on Z by

$$
\begin{cases}
\tilde U(t)=\tilde u(t)+\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[(u\cdot\nabla)\tilde u+(\tilde u\cdot\nabla)u-\nabla\cdot\tilde f f^t-\nabla\cdot f\tilde f^t]d\tau, & 0<t<\infty\\
\tilde F(t)=\tilde f(t)+\int_0^t e^{(t-\tau)\Delta}[(u\cdot\nabla)\tilde f+(\tilde u\cdot\nabla)f-\nabla\tilde u f-\nabla u\tilde f]d\tau, & 0<t<\infty
\end{cases}
\qquad (2.18)
$$

for $\{u,f\}\in Z$. Moreover, we define $\{\mathfrak{U},\mathfrak{F}\}$ by

$$
\{\mathfrak{U},\mathfrak{F}\}\equiv G(u_0,f_0,u+\tilde u,f+\tilde f)-G(u_0,f_0,u,f)-\langle H_{\{u,f\}},(\tilde u,\tilde f)\rangle.
$$

Hence, by the definitions of G and $H_{\{u,f\}}$, we arrive at

$$
\begin{aligned}
\mathfrak{U}(t)=&u(t)+\tilde u(t)-e^{t\Delta}u_0+\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[((u+\tilde u)\cdot\nabla)(u+\tilde u)-\nabla\cdot(f+\tilde f)(f+\tilde f)^t]d\tau\\
&-\left(u(t)-e^{t\Delta}u_0+\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[(u\cdot\nabla)u-\nabla\cdot ff^t]d\tau\right)\\
&-\left(\tilde u(t)+\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[(u\cdot\nabla)\tilde u+(\tilde u\cdot\nabla)u-\nabla\cdot\tilde f f^t-\nabla\cdot f\tilde f^t]d\tau\right)\\
=&\int_0^t e^{(t-\tau)\Delta}\mathbb{P}[(\tilde u\cdot\nabla)\tilde u-\nabla\cdot\tilde f\tilde f^t](\tau)d\tau. \qquad (2.19)
\end{aligned}
$$

Therefore, combining the estimates (2.7) with (2.8), we obtain

$$
\begin{aligned}
\|\mathfrak{U}(t)\|_{L^q}\le & cB(\frac{1}{2}-\frac{N}{2q},\frac{N}{q})\left(\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|\tilde u(\tau)\|_{L^q}\right)^2 t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\\
&+cB\left(\frac{1}{2}-\frac{N}{2}(\frac{2}{p}-\frac{1}{q}),\frac{N}{p}\right)\left(\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|\tilde f(\tau)\|_{L^p}\right)^2 t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}, \qquad (2.20)
\end{aligned}
$$

for any $t>0$, and by the estimate (2.9), we further obtain

$$\begin{aligned}\|\nabla\mathfrak{U}(t)\|_{L^r}\leq & cB\left(-\frac{N}{2}(\frac{1}{N}-\frac{1}{q}-\frac{1}{r}),\frac{1}{2}-\frac{N}{2q}\right)\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|\tilde{u}(\tau)\|_{L^q}\\ &\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla u(\tau)\|_{L^r}t^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\\ &+cB\left(\frac{N}{2m}+\frac{N}{2p}-\frac{1}{2},\frac{1}{2}-\frac{N}{2}(\frac{1}{m}+\frac{1}{p}-\frac{1}{r})\right)\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|f(\tau)\|_{L^p}\\ &\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla\tilde{f}(\tau)\|_{L^m}t^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\end{aligned}\tag{2.21}$$

for any $t>0$. By the same way, from the definition of $\mathfrak{F}$, (2.14) and (2.15), we have that

$$\mathfrak{F}(t)=\int_0^t e^{(t-\tau)\Delta}[(\tilde{u}\cdot\nabla)\tilde{f}-\nabla\tilde{u}\tilde{f}]d\tau\tag{2.22}$$

with the estimate

$$\begin{aligned}\|\mathfrak{F}(t)\|_{L^p}\leq & c\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|u(\tau)\|_{L^q}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla\tilde{f}(\tau)\|_{L^m}\\ &\times B\left(\frac{N}{2q}+\frac{N}{2m}-\frac{1}{2},1-\frac{N}{2}\left(\frac{1}{q}+\frac{1}{m}-\frac{1}{p}\right)\right)t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\\ &+c\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|\tilde{f}(\tau)\|_{L^p}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla\tilde{u}(\tau)\|_{L^r}\\ &\times B\left(\frac{N}{2p}+\frac{N}{2r}-\frac{1}{2},1-\frac{N}{2r}\right)t^{-\frac{N}{2}(\frac{1}{N}-\frac{1}{p})},\end{aligned}\tag{2.23}$$

for any $t>0$, and

$$\begin{aligned}\|\nabla\mathfrak{F}(t)\|_{L^m}\leq & c\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|\tilde{u}(\tau)\|_{L^q}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla\tilde{f}(\tau)\|_{L^m}\\ &\times B\left(\frac{N}{2q}+\frac{N}{2m}-\frac{1}{2},\frac{1}{2}-\frac{N}{2q}\right)t^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\\ &+c\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla\tilde{u}(\tau)\|_{L^r}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|\tilde{f}(\tau)\|_{L^p}\\ &\times B\left(\frac{N}{2p}+\frac{N}{2r}-\frac{1}{2},\frac{1}{2}-\frac{N}{2}(\frac{1}{p}+\frac{1}{r}-\frac{1}{m})\right)t^{-\frac{N}{2}(\frac{2}{N}-\frac{1}{m})},\end{aligned}\tag{2.24}$$

for any $t>0$.

Appealing to the estimates (2.20), (2.21), (2.23) and (2.24), we are in a position to obtain that

$$\begin{aligned}&\lim_{\|\{\tilde{u},\tilde{f}\}\|_Z\to 0}\frac{\|\{\mathfrak{U},\mathfrak{F}\}\|_Z}{\|\{\tilde{u},\tilde{f}\}\|_Z}\\ =&\lim_{\|\{\tilde{u},\tilde{f}\}\|_Z\to 0}\Big(\|G(u_0,f_0,u+\tilde{u},f+\tilde{f})-G(u_0,f_0,u,f)-\langle H_{\{u,f\}},(\tilde{u},\tilde{f})\rangle\|_Z\Big)/\|\{\tilde{u},\tilde{f}\}\|_Z\\ \leq & c\lim_{\|\{\tilde{u},\tilde{f}\}\|_Z\to 0}\Big[(\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|\tilde{u}(\tau)\|_{L^q})^2+(\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|\tilde{f}(\tau)\|_{L^p})^2\\ &+\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla\tilde{u}(\tau)\|_{L^r}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|\tilde{f}(\tau)\|_{L^p}\end{aligned}$$

$$+\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla\tilde{f}(\tau)\|_{L^m}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|\tilde{u}(\tau)\|_{L^q}$$
$$+\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{r})}\|\nabla\tilde{u}(\tau)\|_{L^r}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{q})}\|\tilde{u}(\tau)\|_{L^q}$$
$$+\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{2}{N}-\frac{1}{m})}\|\nabla\tilde{f}(\tau)\|_{L^m}\cdot\sup_{0<\tau<\infty}\tau^{\frac{N}{2}(\frac{1}{N}-\frac{1}{p})}\|\tilde{f}(\tau)\|_{L^p}\Bigg]/\|\{\tilde{u},\tilde{f}\}\|_Z=0, \tag{2.25}$$

for each $\{u_0,f_0\}\in W$ and each $\{u,f\}\in Z$. The argument of (2.25) implies that $\langle G'_{\{u,f\}}(u_0,b_0,u,f),(\tilde{u},\tilde{f})\rangle=\langle H_{\{u,f\}},(\tilde{u},\tilde{f})\rangle$. This completes the proof of Lemma 2.4. ∎

3. Existence of mild solutions and forward self-similar solutions

Based on the conclusions of Lemma 2.4, we now proceed to prove Theorem 1.1-1.2 and Corollary 1.1. Firstly, we propose to prove the global existence of mild solution to (1.1), and the key point is to prove the bijective $H_{\{0,0\}}$.

Proof. **(Proof of Theorem 1.1.)** In order to proof this theorem, we shall first show that the Fréchet derivative $H_{\{u,f\}}$ at $\{u,f\}=\{0,0\}$ is a bijection from Z onto Z. On the one hand, we notice from Lemma 2.4, for each $\{\tilde{u},\tilde{f}\}\in Z$, we infer that $\langle H_{\{0,0\}},(\tilde{u},\tilde{f})\rangle=\{\tilde{U}_0,\tilde{F}_0\}$ as

$$\tilde{U}_0(t)=\tilde{u}(t),\quad \tilde{F}_0(t)=\tilde{f}(t).$$

Thus, $\tilde{U}_0(t)=\tilde{F}_0(t)=0$ implies that $\tilde{u}(t)=\tilde{f}(t)=0$, which means $H_{\{0,0\}}$ is a injection from Z into Z. On the other hand, for each $\{\tilde{U}_0,\tilde{F}_0\}\in Z$, we may take $\{\tilde{u},\tilde{f}\}\in Z$ as

$$\tilde{u}(t)=\tilde{U}_0(t),\quad \tilde{f}(t)=\tilde{F}_0(t),$$

then, it satisfies $\langle H_{\{0,0\}},(\tilde{u},\tilde{f})\rangle=\{\tilde{U}_0,\tilde{F}_0\}$. Therefore, it follows that $H_{\{0,0\}}$ is a surjection from Z onto Z.

Now, using the Banach implicit function theorem, we can see that there exists a C^1-map h

$$h:W_\delta:=\{\{u_0,f_0\}\in W;\|\{u_0,f_0\}\|_W<\delta\}\to Z_\delta:=\{\{u,f\}\in Z;\|\{u,f\}\|_Z<\delta\},$$

for some $\delta(N,p,q,m,r)>0$, which satisfies $h(0,0)=\{0,0\}$ and $G(u_0,f_0,h(u_0,f_0))=\{0,0\}$, for all $\{u_0,f_0\}\in W_\delta$. Thus, under the condition (1.5)-(1.6), one can see that the function $h(u_0,f_0)$ gives the unique solution of (1.2) with the properties (1.7)-(1.10).

From above, we obtain the uniqueness of solutions $\{u,f\}$ of (1.2) with the small norms corresponding to the class of (1.7)-(1.10), since the existence of the C^1- map h from W_δ to Z_δ. We also obtain the asymptotic behavior (1.11)-(1.14), which follows from the estimates (2.7)-(2.9) and (2.14)-(2.15), respectively. ∎

Proof. **(Proof of Theorem 1.2.)** Appealing to the continuity of the map h, we are in a position to claim that the estimate of stability (1.18) holds, under (1.17). So we have completed the proof of Theorem 1.2. ∎

Proof. **(Proof of Corollary 1.1.)** Let $\{u,f\}$ be the solution of (1.2) which is given by Theorem 1.1. Then, we get

$$u(x,t)=u_1(x,t)-J(u,f)(x,t),\quad f(x,t)=f_1(x,t)-K(u,f)(x,t),$$

$$u_1(x,t)=\int_{\mathbb{R}^N}L(x-y,t)u_0(y)dy,\quad f_1(x,t)=\int_{\mathbb{R}^N}L(x-y,t)f_0(y)dy,$$

and we have the expressions that

$$J_i(u,f)(x,t)=\int_0^t\int_{\mathbb{R}^N}L(x-y,t-\tau)\sum_{j=1}^N S_{ij}(x-y,t-\tau)$$
$$\cdot\left\{\sum_{k=1}^N u_k\left(\frac{\partial u_j}{\partial y_k}(y,\tau)-\sum_{l=1}^n\frac{\partial f_{jl}f_{kl}}{\partial y_k}(y,\tau)\right)\right\}dyd\tau,$$
$$K(u,f)(x,t)=\int_0^t\int_{\mathbb{R}^N}L(x-y,t-\tau)((u\cdot\nabla)f(y,\tau)-\nabla uf(y,\tau))dyd\tau,$$

for $i=1,2,\cdots,N$, with

$$S_{ij}(y,\tau)=L(y,\tau)\delta_{ij}+\frac{\partial^2}{\partial y_j\partial y_i}(L(\cdot,\tau)*\Gamma)(y)\qquad(i,j=1,2,\cdots,N),$$

and

$$\Gamma(y)=\frac{1}{(N-2)w_N}|y|^{2-N}\qquad(N\geq 3).$$

By using the condition of homogeneity of u_0,f_0, we obtain

$$\begin{aligned}u_1(rx,r^2t)&=\int_{\mathbb{R}^N}L(rx-y,r^2t)u_0(y)dy\\&=\int_{\mathbb{R}^N}\frac{1}{(4\pi r^2t)^{N/2}}e^{\frac{-|rx-y|^2}{4r^2t}}u_0(y)dy\\&=\int_{\mathbb{R}^N}\frac{r^{-N}}{(4\pi t)^{N/2}}e^{\frac{-|x-w|^2}{4t}}u_0(rw)r^Ndw=r^{-1}u_1(x,t),\end{aligned}\tag{3.1}$$

where in the third equality, we have used the fact $w:=\frac{y}{r}$. Similarly, we have $f_1(rx,r^2t)=r^{-1}f_1(x,t)$ for all $x\in\mathbb{R}^N$, $t>0$. Since the solution $\{u,f\}$ of (1.2) in Theorem 1.1 is given by the mapping $h:W_\delta\to Z_\delta$, in order to prove Corollary 1.1, we will use the following proposition.

PROPOSITION 3.1. *Let $N\geq 3$ and $\{u,f\}$ satisfy (1.19). Then we have*

$$rJ(u,f)(rx,r^2t)=J(u,f)(x,t),\quad rK(u,f)(rx,r^2t)=K(u,f)(x,t)$$

for all $x\in\mathbb{R}^N$, $t>0$ and all $r>0$.

We notice that the proof of the proposition is rather standard, we may just omit it. Thus, we have proved Corollary 1.1. □

4. Regularity criteria of weak solutions

In this section, we shall show the regularity of (u,f) to (1.20). To begin with, we deduce such results under the Assumptions (1.21)-(1.23).

Proof. **(Proof of Theorem 1.3.)** Taking the inner product of $(1.20)_1$ with Δu, employing the divergence-free property and integration by parts, we have that

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2+2\|\Delta u\|_{L^2}^2=-2\int_{\mathbb{R}^3}(f\cdot\nabla f)\cdot\Delta udx+2\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\Delta udx.\tag{4.1}$$

Similarly, taking the inner product of $(1.20)_2$ with Δf, we have that

$$\frac{d}{dt}\|\nabla f\|_{L^2}^2+2\|\Delta f\|_{L^2}^2=-2\int_{\mathbb{R}^3}(f\cdot\nabla u)\cdot\Delta f dx+2\int_{\mathbb{R}^3}(u\cdot\nabla f)\cdot\Delta f dx. \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2)+2(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)\\ =&-2\int_{\mathbb{R}^3}(f\cdot\nabla f)\cdot\Delta u dx+2\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\Delta u dx\\ &-2\int_{\mathbb{R}^3}(f\cdot\nabla u)\cdot\Delta f dx+2\int_{\mathbb{R}^3}(u\cdot\nabla f)\cdot\Delta f dx.\end{aligned} \tag{4.3}$$

Now, we shall estimate (4.3) by Assumptions (1.21)-(1.23), respectively. We start to estimate (4.3) based on (1.21). By Young's inequality, (4.3) implies

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2)+2(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)\\ \le& c(\|f\cdot\nabla f\|_{L^2}^2+\|f\cdot\nabla u\|_{L^2}^2+\|u\cdot\nabla u\|_{L^2}^2+\|u\cdot\nabla f\|_{L^2}^2)+\frac{1}{4}(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2).\end{aligned} \tag{4.4}$$

In order to estimate the right-hand side of (4.4), we may use the following Gagliardo-Nirenberg inequality in the Lorentz space,

$$\|\nabla g\|_{L^{2q/(q-2),2}}\le c\|\nabla g\|_{L^2}^{(q-3)/q}\|\Delta g\|_{L^2}^{3/q}, \tag{4.5}$$

which, for $q=\infty$, is the classical Gagliardo-Nirenberg inequality. For $3<q<\infty$, the inequality (4.5) can be derived from the interpolation

$$L^{\frac{2q}{q-2},2}(\mathbb{R}^3)=(L^{\frac{2q_1}{q_1-2}}(\mathbb{R}^3),L^{\frac{2q_2}{q_2-2}}(\mathbb{R}^3))_{\theta,2}$$

with

$$\frac{q-2}{2q}=\frac{(1-\theta)(q_1-2)}{2q_1}+\frac{\theta(q_2-2)}{2q_2},\quad 1\le q_1<q<q_2\le\infty,$$

and Sobolev imbedding inequality

$$\|\nabla g\|_{L^{\frac{2q_i}{q_i-2}}(\mathbb{R}^3)}\le c\|\nabla g\|_{L^2(\mathbb{R}^3)}^{\frac{q_i-3}{q_i}}\|\Delta g\|_{L^2(\mathbb{R}^3)}^{\frac{3}{q_i}}$$

with $i=1,2$. Thus, applying Lemma 2.1, Young's inequality and (4.5), we have

$$\begin{aligned}&\|f\cdot\nabla f\|_{L^2}^2+\|f\cdot\nabla u\|_{L^2}^2+\|u\cdot\nabla u\|_{L^2}^2+\|u\cdot\nabla f\|_{L^2}^2\\ \le& c\|f\|_{L^{q,\infty}}^2\|\nabla f\|_{L^{2q/(q-2),2}}^2+c\|f\|_{L^{q,\infty}}^2\|\nabla u\|_{L^{2q/(q-2),2}}^2\\ &+c\|u\|_{L^{q,\infty}}^2\|\nabla u\|_{L^{2q/(q-2),2}}^2+c\|u\|_{L^{q,\infty}}^2\|\nabla f\|_{L^{2q/(q-2),2}}^2\\ \le& c\|f\|_{L^{q,\infty}}^2\|\nabla f\|_{L^2}^{[2(q-3)]/q}\|\Delta f\|_{L^2}^{6/q}+c\|f\|_{L^{q,\infty}}^2\|\nabla u\|_{L^2}^{[2(q-3)]/q}\|\Delta u\|_{L^2}^{6/q}\\ &+c\|u\|_{L^{q,\infty}}^2\|\nabla u\|_{L^2}^{[2(q-3)]/q}\|\Delta u\|_{L^2}^{6/q}+c\|u\|_{L^{q,\infty}}^2\|\nabla f\|_{L^2}^{[2(q-3)]/q}\|\Delta f\|_{L^2}^{6/q}\\ \le& c\|f\|_{L^{q,\infty}}^{2q/(q-3)}(\|\nabla f\|_{L^2}^2+\|\nabla u\|_{L^2}^2)+\frac{1}{8}(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)\end{aligned}$$

$$+c\|u\|_{L^{q,\infty}}^{2q/(q-3)}(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2)+\frac{1}{8}(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2). \tag{4.6}$$

Inserting (4.6) into (4.4), we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2)+\frac{3}{2}(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)\\ \le& c(\|u\|_{L^{q,\infty}}^{2q/(q-3)}+\|f\|_{L^{q,\infty}}^{2q/(q-3)})(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2).\end{aligned} \tag{4.7}$$

From Gronwall's inequality, it follows that

$$\begin{aligned}&\sup_{0\le t<T}(\|\nabla u(t)\|_{L^2}^2+\|\nabla f(t)\|_{L^2}^2)+\frac{3}{2}\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)dt\\ \le& c(\|\nabla u_0\|_{L^2}^2+\|\nabla f_0\|_{L^2}^2)exp\left\{\int_0^T(\|u\|_{L^{q,\infty}}^p+\|f\|_{L^{q,\infty}}^p)dt\right\},\end{aligned} \tag{4.8}$$

where $p=\frac{2q}{q-3}$.

Next, we carry out the estimation of (4.3) based on the assumption described by (1.22). Taking into account the divergence-free property of u, f, Lemma 2.1, (4.3), and integration by parts, we have

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2)+2(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)\\ =&-2\int_{\mathbb{R}^3}(f\cdot\nabla f)\cdot\Delta u dx+2\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\Delta u dx\\ &-2\int_{\mathbb{R}^3}(f\cdot\nabla u)\cdot\Delta f dx+2\int_{\mathbb{R}^3}(u\cdot\nabla f)\cdot\Delta f dx\\ =&2\sum_{i=1}^3\int_{\mathbb{R}^3}(\partial_i f\cdot\nabla f)\cdot\partial_i u dx-2\sum_{i=1}^3\int_{\mathbb{R}^3}(\partial_i u\cdot\nabla u)\cdot\partial_i u dx\\ &+2\sum_{i=1}^3\int_{\mathbb{R}^3}(\partial_i f\cdot\nabla u)\cdot\partial_i f dx-2\sum_{i=1}^3\int_{\mathbb{R}^3}(\partial_i u\cdot\nabla f)\cdot\partial_i f dx\\ \le&\|\nabla u\|_{L^{q,\infty}}(2\|\nabla u\|_{L^{2q/(q-1),2}}^2+6\|\nabla f\|_{L^{2q/(q-1),2}}^2).\end{aligned} \tag{4.9}$$

Likewise, the argument in (4.5) implies the following Gagliardo-Nirenberg inequality in Lorentz space:

$$\|\nabla g\|_{L^{2q/(q-1),2}}\le c\|\nabla g\|_{L^2}^{(2q-3)/2q}\|\Delta g\|_{L^2}^{3/2q}.$$

This, together with (4.9) and Young's inequality, implies

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2)+2(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)\\ \le& c\|\nabla u\|_{L^{q,\infty}}(\|\nabla u\|_{L^{2q/(q-1),2}}^2+\|\nabla f\|_{L^{2q/(q-1),2}}^2)\\ \le& c\|\nabla u\|_{L^{q,\infty}}(\|\nabla u\|_{L^2}^{(2q-3)/q}\|\Delta u\|_{L^2}^{3/q}+\|\nabla f\|_{L^2}^{(2q-3)/q}\|\Delta f\|_{L^2}^{3/q})\\ \le&\frac{1}{2}(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)+c\|\nabla u\|_{L^{q,\infty}}^{2q/(2q-3)}(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2).\end{aligned} \tag{4.10}$$

Hence, from Gronwall's inequality, we conclude that

$$\sup_{0\le t<T}(\|\nabla u(t)\|_{L^2}^2+\|\nabla f(t)\|_{L^2}^2)+\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)dt$$

$$\leq c(\|\nabla u_0\|_{L^2}^2+\|\nabla f_0\|_{L^2}^2)exp\left\{\int_0^T \|\nabla u\|_{L^{q,\infty}}^p dt\right\} \tag{4.11}$$

with $p=2q/(2q-3)$.

Finally, we proceed to estimate (4.3) based on the assumption described by (1.23). Appealing to Hölder's inequality, then (4.3) yields

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2)+2(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)\\
&\leq c\|f\cdot\nabla f\|_{L^2}\|\Delta u\|_{L^2}+c\|u\cdot\nabla u\|_{L^2}\|\Delta u\|_{L^2}\\
&\quad +c\|f\cdot\nabla u\|_{L^2}\|\Delta f\|_{L^2}+c\|u\cdot\nabla f\|_{L^2}\|\Delta f\|_{L^2}\\
&\leq c\|f\|_{\dot H^{-r}}\|\nabla f\|_{\dot H^r}\|\Delta u\|_{L^2}+c\|u\|_{\dot H^{-r}}\|\nabla u\|_{\dot H^r}\|\Delta u\|_{L^2}\\
&\quad +c\|f\|_{\dot H^{-r}}\|\nabla u\|_{\dot H^r}\|\Delta f\|_{L^2}+c\|u\|_{\dot H^{-r}}\|\nabla f\|_{\dot H^r}\|\Delta f\|_{L^2}\\
&\leq c\|f\|_{\dot H^{-r}}\|\nabla f\|_{L^2}^{1-r}\|\Delta u\|_{L^2}^{1+r}+c\|u\|_{\dot H^{-r}}\|\nabla u\|_{L^2}^{1-r}\|\Delta u\|_{L^2}^{1+r}\\
&\quad +c\|f\|_{\dot H^{-r}}\|\nabla u\|_{L^2}^{1-r}\|\Delta f\|_{L^2}^{1+r}+c\|u\|_{\dot H^{-r}}\|\nabla f\|_{L^2}^{1-r}\|\Delta f\|_{L^2}^{1+r}\\
&\leq c(\|f\|_{\dot H^{-r}}^{2/(1-r)}+\|u\|_{\dot H^{-r}}^{2/(1-r)})(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2)+\frac{1}{2}(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2),
\end{aligned} \tag{4.12}$$

where in the third inequality, we have taken into account the fact that

$$\|f\|_{\dot H^\alpha}\leq c\|f\|_{L^2}^{1-\alpha}\|\nabla f\|_{L^2}^\alpha,$$

Making use of Gronwall's inequality, from (4.12), we obtain

$$\begin{aligned}
&\sup_{0\leq t<T}(\|\nabla u(t)\|_{L^2}^2+\|\nabla f(t)\|_{L^2}^2)+\frac{3}{2}\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)dt\\
&\leq c(\|\nabla u_0\|_{L^2}^2+\|\nabla f_0\|_{L^2}^2)exp\left\{\int_0^T(\|f\|_{\dot H^{-r}}^{2/(1-r)}+\|u\|_{\dot H^{-r}}^{2/(1-r)})dt\right\}.
\end{aligned} \tag{4.13}$$

Taking into account (4.8), (4.11), (4.13), thus, we have completed the proof of Theorem 1.3. □

Next, we propose to prove the regularity criteria of weak solutions to (1.20) under the Assumption (1.24).

Proof. **(Proof of Theorem 1.4.)** Taking the inner product of $(1.20)_1$ with $u|u|^2$ and $(1.20)_2$ with $f|f|^2$, respectively, we can see that

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}(\|u\|_{L^4}^4+\|f\|_{L^4}^4)+3(\||u|\nabla u\|_{L^2}^2+\||f|\nabla f\|_{L^2}^2)\\
&=\int_{\mathbb{R}^3}(f\cdot\nabla f)\cdot u|u|^2dx-\int_{\mathbb{R}^3}(u\cdot\nabla p)|u|^2dx+\int_{\mathbb{R}^3}(f\cdot\nabla u)\cdot f|f|^2dx\\
&=:I_1+I_2+I_3,
\end{aligned} \tag{4.14}$$

where we have used the following identities due to the divergence-free property of u:

$$\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot u|u|^2dx=0,\quad \int_{\mathbb{R}^3}(u\cdot\nabla f)\cdot f|f|^2dx=0.$$

For the term I_1, by using Hölder's inequality and Young's inequality, we have

$$I_1\leq\|f\cdot\nabla f\|_{L^2}^2+c\|u\|_{L^4}^4+c\||u|^2\|_{L^4}^4. \tag{4.15}$$

Next, appealing to integration by parts and the divergence-free condition of u, the term I_2 can be estimated as

$$I_2 = 2\int_{\mathbb{R}^3} pu\cdot(u\cdot\nabla u)dx \le c\int_{\mathbb{R}^3} |pu|^2 dx + 2\|u\nabla u\|_{L^2}^2. \tag{4.16}$$

Lastly, for the term I_3, by integration by parts, the Hölder and Young's inequality, we deduce that

$$\begin{aligned} I_3 &= -\int_{\mathbb{R}^3} (u_i f_k)\cdot\partial_k(f_i|f|^2)dx \\ &\le c\|u\|_{L^4}^4 + c\||f|^2\|_{L^4}^4 + \|f\cdot\nabla f\|_{L^2}^2, \end{aligned} \tag{4.17}$$

where $f_k = \{f_{kj}\}_{j=1}^3$, $f_i = \{f_{ij}\}_{j=1}^3$ and $i,k = 1,2,3$.

Now, plugging (4.15)-(4.17) into (4.14), we conclude that

$$\begin{aligned} &\frac{d}{dt}(\|u\|_{L^4}^4 + \|f\|_{L^4}^4) + (\||u|\nabla u\|_{L^2}^2 + \||f|\nabla f\|_{L^2}^2) \\ \le& c\int_{\mathbb{R}^3} |pu|^2 dx + c\|u\|_{L^4}^4 + c\||u|^2\|_{L^4}^4 + c\||f|^2\|_{L^4}^4. \end{aligned} \tag{4.18}$$

To estimate the right-hand side of (4.18), we apply the divergence operator $\nabla\cdot$ to $(1.20)_1$, there holds

$$p = (-\Delta)^{-1}\sum_{i,j=1}^3 \frac{\partial^2}{\partial x_i\partial x_j}u_iu_j + (\Delta)^{-1}\sum_{i,j,k=1}^3 \frac{\partial^2}{\partial x_i\partial x_j} f_{ki}f_{kj} := \mathscr{P},$$

which implies, by the Calderón-Zygmund inequality,

$$\|p\|_{L^r} \le c\sum_{i,j=1}^3 \|u_iu_j\|_{L^r} + c\sum_{i,j,k=1}^3 \|f_{ki}f_{kj}\|_{L^r} + c\sup_{t>0}\|u\|_{L^2}, \quad 1<r<\infty. \tag{4.19}$$

We shall note that the pressure term p is uniquely determined by $\mathscr{P}$. In fact, based on the Theorem 2.6 in [21] (see also G.P. Galdi [2] (Th. III. 3.1, Th. III. 5.2.)), there exist unique functions p_0 and p_h ($\Delta p_h = 0$) such that $p = p_0 + p_h$ holds. In addition, the following a priori estimates hold

$$\begin{aligned} \|p_0\|_{L_t^rL_x^r} &\le c(\|u_iu_j\|_{L_t^rL_x^r} + \|f_{ki}f_{kj}\|_{L_t^rL_x^r}), \\ \|p_h\|_{L_t^\infty L_x^r} &\le c(\|u\|_{L_t^\infty L_x^2} + \|u_iu_j\|_{L_t^rL_x^r} + \|f_{ki}f_{kj}\|_{L_t^rL_x^r}). \end{aligned}$$

Hence, by Hölder's inequality, (4.19) and Lemma 2.2, the first term on the right-hand side of (4.18) can be estimated as

$$\begin{aligned} c\int_{\mathbb{R}^3}|pu|^2dx &\le c\|p\|_{L^4}^2\|u\|_{L^4}^2 \\ &\le c\left(\sum_{i,j=1}^3\|u_iu_j\|_{L^4}^2 + \sum_{i,j,k=1}^3\|f_{ki}f_{kj}\|_{L^4}^2 + \sup_{t>0}\|u\|_{L^2}^2\right)\|u\|_{L^4}^2 \\ &\le c(\|u\|_{BMO}^2\|u\|_{L^4}^4 + \|f\|_{BMO}^2\|u\|_{L^4}^2\|f\|_{L^4}^2 + \sup_{t>0}\|u\|_{L^2}^2\|u\|_{L^4}^2). \end{aligned} \tag{4.20}$$

Inserting (4.20) into (4.18), from Lemma 2.2, we arrive at

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{L^4}^4+\|f\|_{L^4}^4)+(\||u|\nabla u\|_{L^2}^2+\||f|\nabla f\|_{L^2}^2)\\
\leq& c(\|u\|_{BMO}^2\|u\|_{L^4}^4+\|f\|_{BMO}^2\|u\|_{L^4}^2\|f\|_{L^4}^2+\sup_{t>0}\|u\|_{L^2}^2\|u\|_{L^4}^2)+c\|u\|_{L^4}^4\\
&+c\|u\|_{L^4}^4\|u\|_{BMO}^4+c\|f\|_{L^4}^4\|f\|_{BMO}^4\\
\leq& c(1+\|u\|_{BMO}^2+\|u\|_{BMO}^4+\|f\|_{BMO}^2+\|f\|_{BMO}^4)(\|u\|_{L^4}^4+\|f\|_{L^4}^4)+\|u\|_{L_t^\infty L_x^2}^4.
\end{aligned}$$

By Gronwall's inequality, this implies the desired estimate

$$\begin{aligned}
&\sup_{0\leq t<T}(\|u(t)\|_{L^4}^4+\|f(t)\|_{L^4}^4)+\int_0^T(\||u|\nabla u\|_{L^2}^2+\||f|\nabla f\|_{L^2}^2)dt\\
\leq& c(\|u_0\|_{L^4}^4+\|f_0\|_{L^4}^4+\|u\|_{L_t^\infty L_x^2}^4T)exp\int_0^T(1+\|u\|_{BMO}^2+\|u\|_{BMO}^4+\|f\|_{BMO}^2\\
&+\|f\|_{BMO}^4)dt.
\end{aligned}$$

The remaining proof is similar to the proof of Theorem 2.1 in [20]. Thus, we have completed the proof of Theorem 1.4. ∎

Remark 4.1. From the proof of Theorem 1.3 and Theorem 1.4, one can verify that the precondition $\mathrm{div} f=0$ can be removed under the conditions (1.21), (1.23)-(1.24). Indeed, in the estimation of (4.4), (4.12) we have not used integration by parts, which prevents us from using the property $\mathrm{div} f=0$. On the other hand, without the assumption $\mathrm{div} f=0$, then I_1 in (4.14)-(4.15) may be rewritten as

$$\begin{aligned}
I_1=&\int_{\mathbb{R}^3}\nabla\cdot(ff)\cdot u|u|^2dx\\
=&\int_{\mathbb{R}^3}(f\cdot\nabla f+f\nabla\cdot f)\cdot u|u|^2dx\\
\leq& 2\||f||\nabla f|\|_{L^2}^2+c\|u\|_{L^4}^4+c\||u|^2\|_{L^4}^4,
\end{aligned}$$

and I_3 in the estimation of (4.17), can be estimated as

$$\begin{aligned}
I_3=&-\int_{\mathbb{R}^3}(u_i\partial_kf_k)\cdot(f_i|f|^2)dx-\int_{\mathbb{R}^3}(u_if_k)\cdot\partial_k(f_i|f|^2)dx\\
\leq& c\|u\|_{L^4}^4+c\||f|^2\|_{L^4}^4+\|f\cdot\nabla f\|_{L^2}^2,
\end{aligned}$$

with $i,k=1,2,3$. Thus, we have the same conclusion as above. Finally, we aim to prove the regularity criteria provided the Assumption (1.25) holds.

Proof. **(Proof of Theorem 1.5.)** By the divergence-free property of u and f, and integration by parts, we have

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2)+2(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)\\
=&-2\int_{\mathbb{R}^3}(f\cdot\nabla f)\cdot\Delta udx+2\int_{\mathbb{R}^3}(u\cdot\nabla u)\cdot\Delta udx\\
&-2\int_{\mathbb{R}^3}(f\cdot\nabla u)\cdot\Delta fdx+2\int_{\mathbb{R}^3}(u\cdot\nabla f)\cdot\Delta fdx
\end{aligned}$$

$$
\begin{aligned}
&=2\int_{\mathbb{R}^3}(\partial_i f_k\partial_k f)\cdot\partial_i u dx-2\int_{\mathbb{R}^3}(\partial_i u\cdot\nabla u)\cdot\partial_i u dx\\
&\quad+2\int_{\mathbb{R}^3}(\partial_i f_k\partial_k u)\cdot\partial_i f dx-2\int_{\mathbb{R}^3}(\partial_i u\cdot\nabla f)\cdot\partial_i f dx\\
&\leq 2\int_{\mathbb{R}^3}|\nabla u||\nabla u|^2dx+6\int_{\mathbb{R}^3}|\nabla u||\nabla f|^2dx,
\end{aligned}\tag{4.21}
$$

where $i,k=1,2,3$. Making use of the homogeneous Littlewood-Paley decomposition, the right-hand side of (4.21) can be estimated as

$$
\begin{aligned}
&2\int_{\mathbb{R}^3}|\nabla u||\nabla u|^2dx+6\int_{\mathbb{R}^3}|\nabla u||\nabla f|^2dx\\
=&2\sum_{-\infty<j<\infty}\int_{\mathbb{R}^3}\left|\dot{\Delta}_j\nabla u\right||\nabla u|^2dx+6\sum_{-\infty<j<\infty}\int_{\mathbb{R}^3}\left|\dot{\Delta}_j\nabla u\right||\nabla f|^2dx\\
\leq& c\sum_{-\infty<j<\infty}\|\dot{\Delta}_j\nabla u\|_{L^3}(\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}+\|\nabla f\|_{L^2}\|\nabla f\|_{L^6})\\
\leq& c\|\nabla u\|_{\dot{B}^0_{3,1}}(\|\nabla u\|_{L^2}+\|\nabla f\|_{L^2})(\|\Delta u\|_{L^2}+\|\Delta f\|_{L^2}).
\end{aligned}\tag{4.22}
$$

Collecting (4.21) and (4.22), by Young's inequality, it holds that

$$
\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2)+(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)\leq c\|\nabla u\|_{\dot{B}^0_{3,1}}^2(\|\nabla u\|_{L^2}^2+\|\nabla f\|_{L^2}^2).
$$

Hence, appealing to Gronwall's inequality, we conclude that

$$
\begin{aligned}
&\sup_{0\leq t<T}(\|\nabla u(t)\|_{L^2}^2+\|\nabla f(t)\|_{L^2}^2)+\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta f\|_{L^2}^2)dt\\
\leq& exp\left\{c\int_0^T\|\nabla u\|_{\dot{B}^0_{3,1}}^2dt\right\}(\|\nabla u_0\|_{L^2}^2+\|\nabla f_0\|_{L^2}^2).
\end{aligned}
$$

Thus, we have completed the proof of Theorem 1.5. □

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