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CAUCHY PROBLEM FOR THERMOELASTIC PLATE EQUATIONS WITH DIFFERENT DAMPING MECHANISMS*

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Abstract. In this paper we study the Cauchy problem for thermoelastic plate equations with friction or structural damping in $\mathbb{R}^n$, $n\geq 1$, where the heat conduction is modeled by Fourier's law. We explain some qualitative properties of solutions influenced by different damping mechanisms. We show which damping in the model has a dominant influence on smoothing effect, energy estimates, L^p-L^q estimates not necessary on the conjugate line, and on diffusion phenomena. Moreover, we derive asymptotic profiles of solutions in a framework of weighted L^1 data. In particular, sharp decay estimates for lower bounds and upper bounds of solutions in the $\dot{H}^s$ norm ($s\geq 0$) are shown.

Keywords. thermoelastic plate equations; Fourier's law; friction; structural damping; diffusion phenomena; asymptotic profiles.

AMS subject classifications. 35Q99; 35B40; 74F05.

1. Introduction

In recent years, thermoelastic plate equations have attracted a lot of attention. The Cauchy problem for linear thermoelastic plate equations are modeled by

$$\begin{cases} u_{tt}+\Delta^2 u+\Delta\theta=0, & t>0,\, x\in\mathbb{R}^n,\\ \theta_t-\Delta\theta-\Delta u_t=0, & t>0,\, x\in\mathbb{R}^n,\\ (u,u_t,\theta)(0,x)=(u_0,u_1,\theta_0)(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where the unknowns $u=u(t,x)$ and $\theta=\theta(t,x)$ denote the elongation of a plate and the temperature difference to the equilibrium state, respectively. The recent papers [28,33] investigated L^2-decay estimates of solutions to (1.1) by using energy methods in the Fourier space. Simultaneously, [28] proved the sharpness of the derived decay estimates by calculating explicit eigenvalues. Other studies on thermoelastic plate equations can be found in the literature. We refer the reader to [1,18–23,25,26] for the initial boundary value problem in bounded domains, [5–7,25,26] for the Cauchy problem or in general exterior domains.

In the real world and applications, due to some kinds of resistance in the elongation of a plate, we always model a plate equation with damping terms, for instance, plate equations with structural damping in [11,16]. When the thermal dissipation modeled by Fourier's law and the dissipation for the elongation of a plate appear at the same time, we model thermoelastic plate equations with an additional damping in the equation for u, for example, the thermoelastic plate equations with friction u_t presented in [35], with structural damping $-\Delta u_t$ presented in [9,38], with Kelvin-Voigt-type damping or viscoelastic damping $\Delta^2 u_t$ presented in [37]. For this reason, we may consider thermoelastic plate equations with different damping mechanisms in the present paper.

In this paper we are concerned with the following Cauchy problem for thermoelastic

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plate equations in $\mathbb{R}^n$, $n\geq 1$, where the heat conduction is modeled by Fourier's law:

$$\begin{cases} u_{tt}+\Delta^2 u+\Delta\theta+(-\Delta)^\sigma u_t=0, & t>0,\, x\in\mathbb{R}^n, \\ \theta_t-\Delta\theta-\Delta u_t=0, & t>0,\, x\in\mathbb{R}^n, \\ (u,u_t,\theta)(0,x)=(u_0,u_1,\theta_0)(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.2}$$

where $\sigma\in[0,2]$. To be more specific, $\sigma=0$ stands for the system with *friction or external damping*, $\sigma\in(0,2]$ stands for the system with *structural damping*, especially, $\sigma=2$ stands for the system with *Kelvin-Voigt-type damping*. The example of the model of thermoelastic plate equations with friction or structural damping (1.2) is a special case of $\alpha-\beta-\gamma$ systems, which have been introduced in [10], namely,

$$\begin{cases} u_{tt}+\mathcal{A}u-\mathcal{A}^\beta\theta+\mathcal{A}^\gamma u_t=0, \\ \theta_t+\mathcal{A}^\alpha\theta+\mathcal{A}^\beta u_t=0, \end{cases}$$

when we choose

$$\mathcal{A}=(-\Delta)^2 \text{ and } \alpha=\beta=\frac{1}{2},\ [0,1]\ni\gamma=\frac{1}{2}\sigma.$$

Nevertheless, the questions of influence of an additional damping in the equation for u in thermoelastic plate equations on some qualitative properties of solutions as L^p-L^q estimates, diffusion phenomena, asymptotic profiles of solutions are still open.

Our main purpose of this paper is to study various qualitative properties of solutions to the thermoelastic plate equations with different damping mechanisms. More specifically, we are interested in the following properties of solutions to (1.2):

(1) smoothing effect and L^2 well-posedness;

(2) energy estimates with different assumptions on initial data;

(3) L^p-L^q estimates not necessary on the conjugate line;

(4) diffusion phenomena;

(5) asymptotic profiles of solutions in a framework of weighted L^1 data.

Then, due to the fact that different kinds of damping (friction, structural damping, thermal damping) have a different influence on the model, we will analyze the dominant influence from the damping to various qualitative properties of solutions. In other words, there exists a competition between "friction or structural damping" and "thermal damping generated by Fourier's law". Our main new contributions in the present paper are to derive new thresholds for diffusion phenomena (see Theorems 5.1 and 5.3) and asymptotic profiles of solutions (see Theorems 6.1-6.3) for thermoelastic plate equations.

In order to study the above qualitative properties of solutions, especially, L^p-L^p estimates for $1\leq p\leq\infty$, diffusion phenomena and asymptotic profiles of solutions, we need to derive representations of solutions instead of using pointwise estimates in the Fourier space. However, because the fractional power operator $(-\Delta)^\sigma$ acts on u_t in the damping term, the method of asymptotic expansions of eigenprojections (c.f. [2, 12]) seems to be difficult to apply. Moreover, the method of asymptotic expansions of eigenvalues (c.f. [12, 28]) also seems to be not easy to apply to prove the sharpness for the derived estimates of solutions. To overcome these difficulties, we may derive representations of solutions by applying methods of WKB analysis. The main tool is the application of a multi-step diagonalization procedure, which was mainly proposed in [17, 31].

In the study of diffusion phenomena of solutions to (1.2), we may observe the corresponding reference system to (1.2), which consists of different evolution equations, e.g., heat equation, fractional heat equation, Schrödinger equation, fourth-order parabolic equation. The equations of such a reference system are determined by the value of σ in the damping term $(-\Delta)^\sigma u_t$. It will provide some opportunities for us to understand the model (1.2) in a more precise way.

For asymptotic profiles of solutions in a framework of weighted L^1 data in Section 6, by introducing

$$\begin{aligned} U(t,x) &:= \left(u_t+|D|^2u, u_t-|D|^2u, \theta\right)^{\mathrm{T}}(t,x),\\ U_0(x) &:= \left(u_1+|D|^2u_0, u_1-|D|^2u_0, \theta_0\right)^{\mathrm{T}}(x),\\ P_{U_0} &:= \int_{\mathbb{R}^n} U_0(x)dx \ \text{ with } \ |P_{U_0}|\neq 0, \end{aligned}$$

we will prove the following estimates for $t\gg 1$:

$$t^{-\frac{n+2s}{4\max\{2-\sigma;1\}}}|P_{U_0}|\lesssim \|U(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)}\lesssim t^{-\frac{n+2s}{4\max\{2-\sigma;1\}}}\|U_0\|_{H^s(\mathbb{R}^n)\cap L^{1,1}(\mathbb{R}^n)},$$

with $s\geq 0$, $\sigma\in[0,2]$. It immediately leads to sharp decay rate of the estimates for the $\dot{H}^s$ norm of solutions. To the best of the author's knowledge, sharp estimates for lower bound of solutions for dissipative elastic systems are unknown, although the estimates for upper bound of solutions in the L^2 norm have been extensively discussed in several kinds of elastic systems, see [2, 3, 29] for dissipative elastic waves, [17, 31, 36, 40, 41] for thermoelastic systems. We remark that our method can probably be applied to some other systems in elastic material (see Remark 7.1) too. Furthermore, due to the double damping, including friction or structural damping and thermal damping generated by Fourier's law, it is interesting to investigate which damping will give stronger effects on asymptotic profiles of solutions.

The paper is organized as follows. In Section 2, we prepare representations of solutions to (1.2) by employing WKB analysis. In Section 3, by using these representations of solutions we study smoothing effect of solutions and L^2 well-posedness of the Cauchy problem (1.2). In Section 4, we derive some estimates of solutions, including energy estimates with initial data taken from $H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)$ with $s\geq 0$, $m\in[1,2]$, and L^p-L^q estimates not necessary on the conjugate line. In Section 5, diffusion phenomena for linear thermoelastic plate equations with friction or structural damping are investigated. In Section 6, we derive long-time asymptotic profiles of solutions in a framework of weighted L^1 data. Finally, in Section 7 some concluding remarks complete the paper.

Finally, we now give some notations to be used in this paper. We denote the identity matrix of dimension $k\times k$ by I_k. $f\lesssim g$ means that there exists a positive constant C such that $f\leq Cg$. Moreover, $H^s_p(\mathbb{R}^n)$ and $\dot{H}^s_p(\mathbb{R}^n)$ with $s\geq 0$ and $1\leq p<\infty$, denote Bessel and Riesz potential spaces based on $L^p(\mathbb{R}^n)$, respectively. Here $\langle D\rangle^s$ and $|D|^s$ stand for the pseudo-differential operators with symbols $\langle\xi\rangle^s$ and $|\xi|^s$, respectively, where $\langle\xi\rangle=\sqrt{1+|\xi|^2}$.

Let us define the Gevrey spaces $\Gamma^\kappa(\mathbb{R}^n)$ for $\kappa\in[1,\infty)$ by (c.f. [32])

$$\begin{aligned}\Gamma^\kappa(\mathbb{R}^n):=\Big\{ & f\in L^2(\mathbb{R}^n): \text{ there exists a constant } c>0\\ & \text{such that } \exp\left(c\langle\xi\rangle^{\frac{1}{\kappa}}\right)\hat{f}\in L^2(\mathbb{R}^n)\Big\}.\end{aligned}$$

Let us define the weighted L^1 spaces $L^{1,\delta}(\mathbb{R}^n)$ for $\delta\in[0,\infty)$ by

$$L^{1,\delta}(\mathbb{R}^n):=\left\{f\in L^1(\mathbb{R}^n):\|f\|_{L^{1,\delta}(\mathbb{R}^n)}:=\int_{\mathbb{R}^n}(1+|x|)^\delta|f(x)|dx<\infty\right\}.$$

Particularly, we notice that $L^{1,0}(\mathbb{R}^n)=L^1(\mathbb{R}^n)$.

2. Asymptotic behavior of solutions

First of all, we apply the partial Fourier transformation with respect to spatial variables to (1.2) to get the following second-order ordinary differential system:

$$\begin{cases}\hat{u}_{tt}+|\xi|^4\hat{u}-|\xi|^2\hat{\theta}+|\xi|^{2\sigma}\hat{u}_t=0, & t>0,\ \xi\in\mathbb{R}^n,\\ \hat{\theta}_t+|\xi|^2\hat{\theta}+|\xi|^2\hat{u}_t=0, & t>0,\ \xi\in\mathbb{R}^n,\\ \left(\hat{u},\hat{u}_t,\hat{\theta}\right)(0,\xi)=\left(\hat{u}_0,\hat{u}_1,\hat{\theta}_0\right)(\xi), & \xi\in\mathbb{R}^n.\end{cases}\tag{2.1}$$

Introducing $w^{(0)}=w^{(0)}(t,\xi)$ by

$$w^{(0)}:=\left(\hat{u}_t+|\xi|^2\hat{u},\hat{u}_t-|\xi|^2\hat{u},\hat{\theta}\right)^{\mathrm{T}},$$

we obtain the first-order system as follows:

$$\begin{cases}w^{(0)}_t+\left(|\xi|^2A_0+|\xi|^{2\sigma}A_1\right)w^{(0)}=0, & t>0,\ \xi\in\mathbb{R}^n,\\ w^{(0)}(0,\xi)=w^{(0)}_0(\xi), & \xi\in\mathbb{R}^n,\end{cases}\tag{2.2}$$

where $w^{(0)}_0=w^{(0)}_0(\xi)$ is defined by

$$w^{(0)}_0:=\left(\hat{u}_1+|\xi|^2\hat{u}_0,\hat{u}_1-|\xi|^2\hat{u}_0,\hat{\theta}_0\right)^{\mathrm{T}},$$

and the coefficient matrices are given by

$$A_0=\frac{1}{2}\begin{pmatrix}0&-2&-2\\2&0&-2\\1&1&2\end{pmatrix}\ \text{and}\ A_1=\frac{1}{2}\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}.$$

Additionally, we denote the matrix

$$A(|\xi|;\sigma):=|\xi|^2A_0+|\xi|^{2\sigma}A_1,$$

and

$$U(t,x):=\left(u_t+|D|^2u,u_t-|D|^2u,\theta\right)^{\mathrm{T}}(t,x),\tag{2.3}$$

$$U_0(x):=\left(u_1+|D|^2u_0,u_1-|D|^2u_0,\theta_0\right)^{\mathrm{T}}(x).\tag{2.4}$$

It is clear that $\mathcal{F}_{x\to\xi}(U(t,x))=w^{(0)}(t,\xi)$ and $\mathcal{F}(U_0(x))=w^{(0)}_0(\xi)$.

2.1. Diagonalization schemes.

In the beginning, we divide the phase space into three regions

$$Z_{\mathrm{int}}(\varepsilon)=\{\xi\in\mathbb{R}^n:|\xi|\leq\varepsilon\ll1\},$$
$$Z_{\mathrm{mid}}(\varepsilon,N)=\{\xi\in\mathbb{R}^n:\varepsilon\leq|\xi|\leq N\},$$

$$Z_{\text{ext}}(N)=\{\xi\in\mathbb{R}^n:|\xi|\geq N\gg 1\},$$

for small, bounded and large frequencies. Later, we will diagonalize the principal part of the first-order system (2.2) in each region. Furthermore, let us define $\chi_{\text{int}}(\xi),\chi_{\text{mid}}(\xi),\chi_{\text{ext}}(\xi)\in\mathcal{C}^\infty(\mathbb{R}^n)$ having their supports in $Z_{\text{int}}(\varepsilon)$, $Z_{\text{mid}}(\varepsilon/2,2N)$ and $Z_{\text{ext}}(N)$, respectively, so that $\chi_{\text{mid}}(\xi)=1-\chi_{\text{int}}(\xi)-\chi_{\text{ext}}(\xi)$.

To understand the influence of the parameter ξ on the asymptotic behavior of solutions, we now distinguish between the next four cases.

- Case 2.1: $\sigma\in[0,1)$ with $\xi\in Z_{\text{int}}(\varepsilon)$ or $\sigma\in(1,2]$ with $\xi\in Z_{\text{ext}}(N)$;
- Case 2.2: $\sigma\in[0,1)$ with $\xi\in Z_{\text{ext}}(N)$ or $\sigma\in(1,2]$ with $\xi\in Z_{\text{int}}(\varepsilon)$;
- Case 2.3: $\sigma=1$ for all frequencies;
- Case 2.4: $\sigma\neq 1$ with $\xi\in Z_{\text{mid}}(\varepsilon,N)$.

For frequencies in the small zone or the large zone, i.e., Cases 2.1 and 2.2, the diagonalization procedure is available. This procedure, which is developed [17,31,39], allows us to derive representations of solutions. For Case 2.3, the matrix $A(|\xi|;1)$ may be understood as nonperturbed linear operator for all frequencies due to $A(|\xi|;1)=|\xi|^2(A_0+A_1)$. For this reason, we only need to calculate the eigenvalues of the matrix $A(|\xi|;1)$ directly in Case 2.3. For frequencies in the bounded zone and $\sigma\neq 1$, i.e., Case 2.4, we construct a contradiction to prove that the real parts of the characteristic roots have a fixed sign.

LEMMA 2.1 (Treatment for Case 2.1). *When $\sigma\in[0,1)$ with $\xi\in Z_{\text{int}}(\varepsilon)$, or $\sigma\in(1,2]$ with $\xi\in Z_{\text{ext}}(N)$, after ℓ steps of diagonalization procedure the starting system (2.2) is transformed to*

$$\begin{cases} w_t^{(\ell)}+(\Lambda_0+\cdots+\Lambda_\ell+R_{\ell+1})w^{(\ell)}=0, & t>0,\,\xi\in\mathbb{R}^n,\\ w^{(\ell)}(0,\xi)=w_0^{(\ell)}(\xi), & \xi\in\mathbb{R}^n,\end{cases}$$

with the diagonalized matrices $\Lambda_1,\dots,\Lambda_\ell$ and the remainder $R_{\ell+1}$. The asymptotic behavior of these matrices can be described as follows:

$$\Lambda_0=\mathcal{O}\big(|\xi|^{2\sigma}\big),\ \Lambda_j=\mathcal{O}\big(|\xi|^{2(1-\sigma)(j-1)+2}\big),\ R_{\ell+1}=\mathcal{O}\big(|\xi|^{2(1-\sigma)\ell+2}\big).$$

Moreover, the characteristic roots $\lambda_{\ell,j}=\lambda_{\ell,j}(|\xi|)$ with $j=1,2,3$, having the following asymptotic behavior:

$$\lambda_{\ell,1}=|\xi|^{4-2\sigma},\ \lambda_{\ell,2}=|\xi|^2+|\xi|^{4-2\sigma},\ \lambda_{\ell,3}=|\xi|^{2\sigma}-2|\xi|^{4-2\sigma},$$

modulo $\mathcal{O}\big(|\xi|^{6-4\sigma}\big)$.

Proof. To start the diagonalization procedure, the matrix $|\xi|^{2\sigma}A_1$ has a dominant influence in comparison with the matrix $|\xi|^2A_0$ in Case 2.1. As the consequence, we should diagonalize $|\xi|^{2\sigma}A_1$ in the first place. With the aid of variable change

$$w^{(1)}:=T_0^{-1}w^{(0)}:=\begin{pmatrix}-1&0&1\\1&0&1\\0&1&0\end{pmatrix}^{-1}w^{(0)},$$

we derive

$$w_t^{(1)}+(\Lambda_0+R_1)w^{(1)}=0,$$

where

$$\Lambda_0 = \operatorname{diag}\left(0,0,|\xi|^{2\sigma}\right) = \mathcal{O}\left(|\xi|^{2\sigma}\right)$$

and $R_1 = |\xi|^2 A_0^{(1)} = \mathcal{O}\left(|\xi|^2\right)$ with

$$A_0^{(1)} = T_0^{-1} A_0 T_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix} = \mathcal{O}(1).$$

Next, we define

$$w^{(2)} := T_1^{-1} w^{(1)}$$

with $T_1 := I_3 + N_1(|\xi|)$, where

$$N_1(|\xi|) := |\xi|^{2-2\sigma} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \mathcal{O}\left(|\xi|^{2-2\sigma}\right).$$

Thus, we have

$$w_t^{(2)} + \left(\Lambda_0 + T_1^{-1}\left(|\xi|^2 A_0^{(1)} - [N_1(|\xi|), \Lambda_0]\right) + |\xi|^2 T_1^{-1} A_0^{(1)} N_1(|\xi|)\right) w^{(2)} = 0, \tag{2.5}$$

where we use

$$[N_1(|\xi|), \Lambda_0] := N_1(|\xi|)\Lambda_0 - \Lambda_0 N_1(|\xi|) = |\xi|^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

By the relation $T_1^{-1} = I_3 - T_1^{-1} N_1(|\xi|)$, we transform (2.5) to the following first-order system:

$$w_t^{(2)} + (\Lambda_0 + \Lambda_1 + R_2) w^{(2)} = 0,$$

where

$$\Lambda_1 := \operatorname{diag}\left(0, |\xi|^2, 0\right) = \mathcal{O}\left(|\xi|^2\right)$$

and $R_2 = A_0^{(2)} - T_1^{-1} N_1(|\xi|) A_0^{(2)} = \mathcal{O}\left(|\xi|^{4-2\sigma}\right)$ with

$$A_0^{(2)} = -N_1(|\xi|)\Lambda_1 + |\xi|^2 A_0^{(1)} N_1(|\xi|) = |\xi|^{4-2\sigma} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix} = \mathcal{O}\left(|\xi|^{4-2\sigma}\right).$$

By a similar procedure, we introduce

$$w^{(3)} := T_2^{-1} T_{1\frac{1}{2}}^{-1} w^{(2)}$$

with $T_{1\frac{1}{2}} := I_3 + N_{1\frac{1}{2}}(|\xi|)$ and $T_2 := I_3 + N_2(|\xi|)$, where

$$N_{1\frac{1}{2}}(|\xi|) := |\xi|^{4-4\sigma} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \mathcal{O}\left(|\xi|^{4-4\sigma}\right),$$

$$N_2(|\xi|):=|\xi|^{2-2\sigma}\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=\mathcal{O}\big(|\xi|^{2-2\sigma}\big).$$

So, we derive the following system:

$$w_t^{(3)}+(\Lambda_0+\Lambda_1+\Lambda_2+R_3)w^{(3)}=0,$$

where

$$\Lambda_2=\mathrm{diag}\left(|\xi|^{4-2\sigma},|\xi|^{4-2\sigma},-2|\xi|^{4-2\sigma}\right)=\mathcal{O}\big(|\xi|^{4-2\sigma}\big)$$

and $R_3=\mathcal{O}\big(|\xi|^{6-4\sigma}\big)$. Notice that the matrices $T_{\sigma,\mathrm{int}}$ and $T_{\sigma,\mathrm{ext}}$, respectively, are defined by

$$\begin{aligned} T_{\sigma,\mathrm{int}}&:=T_0T_1T_{1\frac{1}{2}}T_2 \text{ if } \sigma\in[0,1),\\ T_{\sigma,\mathrm{ext}}&:=T_0T_1T_{1\frac{1}{2}}T_2 \text{ if } \sigma\in(1,2]. \end{aligned} \tag{2.6}$$

Then, we carry out further steps of diagonalization proposed in [30,39] to complete the proof. ∎

Lemma 2.2 (Treatment for Case 2.2). *When $\sigma\in[0,1)$ with $\xi\in Z_{\mathrm{ext}}(N)$, or $\sigma\in(1,2]$ with $\xi\in Z_{\mathrm{int}}(\varepsilon)$, after ℓ steps of diagonalization procedure the starting system (2.2) is transformed to*

$$\begin{cases} w_t^{(\ell)}+(\Lambda_0+\cdots+\Lambda_\ell+R_{\ell+1})w^{(\ell)}=0, & t>0,\ \xi\in\mathbb{R}^n,\\ w^{(\ell)}(0,\xi)=w_0^{(\ell)}(\xi), & \xi\in\mathbb{R}^n, \end{cases}$$

with the diagonalized matrices $\Lambda_1,\dots,\Lambda_\ell$ and the remainder $R_{\ell+1}$. The asymptotic behavior of these matrices can be described as follows:

$$\Lambda_0=\mathcal{O}\big(|\xi|^2\big),\ \Lambda_j=\mathcal{O}\big(|\xi|^{2(\sigma-1)(j-1)+2\sigma}\big),\ R_{\ell+1}=\mathcal{O}\big(|\xi|^{2(\sigma-1)\ell+2\sigma}\big).$$

Moreover, the characteristic roots $\lambda_{\ell,j}=\lambda_{\ell,j}(|\xi|)$ with $j=1,2,3$, having the following asymptotic behavior:

$$\lambda_{\ell,1}=y_1|\xi|^2,\ \lambda_{\ell,2}=y_2|\xi|^2,\ \lambda_{\ell,3}=y_3|\xi|^2,$$

modulo $\mathcal{O}\big(|\xi|^{2\sigma}\big)$, where the constants y_j for $j=1,2,3$ will be defined in (2.7) later.

Proof. In this part, the matrix $|\xi|^2A_0$ has a dominant influence in comparison with the matrix $|\xi|^{2\sigma}A_1$. Thus, we should diagonalize $|\xi|^2A_0$ firstly. After applying the substitution

$$w^{(1)}:=T_0^{-1}w^{(0)},$$

we arrive at the system

$$w_t^{(1)}+(\Lambda_0+R_1)w^{(1)}=0,$$

with the diagonal matrix

$$\Lambda_0=|\xi|^2T_0^{-1}A_0T_0=\mathrm{diag}\left(y_1|\xi|^2,y_2|\xi|^2,y_3|\xi|^2\right)=\mathcal{O}\big(|\xi|^2\big)$$

and the remainder $R_1 = |\xi|^{2\sigma} T_0^{-1} A_1 T_0 = \mathcal{O}\big(|\xi|^{2\sigma}\big)$. In the above, the values of y_j for $j = 1,2,3$ are the solutions to the cubic equation

$$y^3 - y^2 + 2y - 1 = 0.$$

Then, from direct calculations the values y_j for $j = 1,2,3$, are given by

$$y_1 = \frac{1}{3}(1 + z_1),\ y_2 = \frac{1}{3}\left(1 - \frac{1}{2}z_1 + \frac{\sqrt{3}}{2} i z_2\right),\ y_3 = \frac{1}{3}\left(1 - \frac{1}{2}z_1 - \frac{\sqrt{3}}{2} i z_2\right), \tag{2.7}$$

where

$$z_1 = \sqrt[3]{\frac{1}{2}\left(3\sqrt{69} + 11\right)} - \sqrt[3]{\frac{1}{2}\left(3\sqrt{69} - 11\right)},\ z_2 = \sqrt[3]{\frac{1}{2}\left(3\sqrt{69} + 11\right)} + \sqrt[3]{\frac{1}{2}\left(3\sqrt{69} - 11\right)}.$$

Note that $y_1 \neq y_2 \neq y_3$ and the real parts of y_j are positive for all $j = 1,2,3$. We now denote the matrices $T_{\sigma,\mathrm{int}}$ and $T_{\sigma,\mathrm{ext}}$, respectively, by

$$\begin{aligned} T_{\sigma,\mathrm{int}} &:= T_0 \text{ if } \sigma \in (1,2], \\ T_{\sigma,\mathrm{ext}} &:= T_0 \text{ if } \sigma \in [0,1). \end{aligned} \tag{2.8}$$

Finally, one may apply further steps of diagonalization proposed in [30, 39] to complete the proof. ∎

Lemma 2.3 (Treatment for Case 2.3). *When $\sigma = 1$ with $\xi \in \mathbb{R}^n$, the starting system (2.2) can be transformed to*

$$\begin{cases} w_t^{(1)} + \Lambda_0 w^{(1)} = 0, & t > 0,\ \xi \in \mathbb{R}^n, \\ w^{(1)}(0,\xi) = w_0^{(1)}(\xi), & \xi \in \mathbb{R}^n, \end{cases}$$

with the diagonalized matrix $\Lambda_0 = \mathrm{diag}\left(y_4|\xi|^2, y_5|\xi|^2, y_6|\xi|^2\right)$, where the constants y_j for $j = 4,5,6$ will be defined in (2.9) later.

Proof. Here the matrices $|\xi|^2 A_0$ and $|\xi|^{2\sigma} A_1$ with $\sigma = 1$ have the same influence on the principal part. For this reason, the following system is obtained:

$$w_t^{(0)} + A(|\xi|;1) w^{(0)} = 0.$$

From direct calculation, we get

$$\begin{aligned} 0 = \det\left(A(|\xi|;1) - \lambda I_3\right) &= \begin{vmatrix} \frac{1}{2}|\xi|^2 - \lambda & \frac{1}{2}|\xi|^2 - |\xi|^2 & -|\xi|^2 \\ \frac{1}{2}|\xi|^2 + |\xi|^2 & \frac{1}{2}|\xi|^2 - \lambda & -|\xi|^2 \\ \frac{1}{2}|\xi|^2 & \frac{1}{2}|\xi|^2 & |\xi|^2 - \lambda \end{vmatrix} \\ &= -\lambda^3 + 2|\xi|^2\lambda^2 - 3|\xi|^4\lambda + |\xi|^6 \\ &= -|\xi|^6 \left(\left(\frac{\lambda}{|\xi|^2}\right)^3 - 2\left(\frac{\lambda}{|\xi|^2}\right)^2 + 3\left(\frac{\lambda}{|\xi|^2}\right) - 1 \right). \end{aligned}$$

In other words, we only need to study the solution to the cubic equation

$$y^3 - 2y^2 + 3y - 1 = 0.$$

By a simple calculation, we find the solution to above cubic equation given by

$$y_4=z_3-\frac{5}{9z_3}+\frac{2}{3},\ y_5=z_4-\frac{5}{9z_4}+\frac{2}{3},\ y_6=z_5-\frac{5}{9z_5}+\frac{2}{3},\tag{2.9}$$

where

$$z_3=\frac{1}{3}\sqrt[3]{-\frac{11}{2}+\frac{3}{2}\sqrt{69}},\ z_4=\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)z_3,\ z_5=\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)z_3.$$

Note that $y_4\neq y_5\neq y_6$ with $\mathrm{Re}\ y_j>0$ for $j=4,5,6$.

By introducing

$$w^{(1)}:=T_{1,0}^{-1}w^{(0)},$$

we obtain the following system:

$$w_t^{(1)}+\Lambda_0 w^{(1)}=0,$$

with the diagonal matrix

$$\Lambda_0=T_{1,0}^{-1}A(|\xi|;1)T_{1,0}=\mathrm{diag}\left(y_4|\xi|^2,y_5|\xi|^2,y_6|\xi|^2\right)=\mathcal{O}\left(|\xi|^2\right).$$

Then, the proof of this lemma is completed. □

Lastly, we derive an exponential decay result for frequencies in the bounded zone $Z_{\mathrm{mid}}(\varepsilon,N)$ to guarantee the stability of solutions to (2.2) for $\sigma\in[0,1)\cup(1,2]$.

LEMMA 2.4 (Treatment for Case 2.4). *The solution $w^{(0)}=w^{(0)}(t,\xi)$ to the Cauchy problem (2.2) with $\sigma\in[0,1)\cup(1,2]$ satisfies*

$$\left|w^{(0)}(t,\xi)\right|\lesssim e^{-ct}\left|w_0^{(0)}(\xi)\right|,$$

for $\xi\in Z_{\mathrm{mid}}(\varepsilon,N)$, where c is a positive constant.

Proof. The following considerations help us obtain an a priori estimate for the characteristic roots for frequencies in the bounded zone $Z_{\mathrm{mid}}(\varepsilon,N)$. We assume that there is a purely imaginary eigenvalue $\lambda=ia$ with $a\in\mathbb{R}\backslash\{0\}$ of the coefficient matrix $A(|\xi|;\sigma)$ for $\xi\neq0$. The eigenvalue λ satisfies the following cubic equation:

$$\begin{aligned}0=\det(A(|\xi|;\sigma)-\lambda I_3)&=\begin{vmatrix}\frac{1}{2}|\xi|^{2\sigma}-\lambda & \frac{1}{2}|\xi|^{2\sigma}-|\xi|^2 & -|\xi|^2\\ \frac{1}{2}|\xi|^{2\sigma}+|\xi|^2 & \frac{1}{2}|\xi|^{2\sigma}-\lambda & -|\xi|^2\\ \frac{1}{2}|\xi|^2 & \frac{1}{2}|\xi|^2 & |\xi|^2-\lambda\end{vmatrix}\\ &=-\lambda^3+\left(|\xi|^{2\sigma}+|\xi|^2\right)\lambda^2-\left(2|\xi|^4+|\xi|^{2+2\sigma}\right)\lambda+|\xi|^6.\end{aligned}\tag{2.10}$$

Plugging $\lambda=ia$ in (2.10) and considering the real and imaginary parts of the coefficient of a, we conclude the following two equations, respectively:

$$\begin{cases}-a^2\left(|\xi|^{2\sigma}+|\xi|^2\right)+|\xi|^6=0,\\ a\left(a^2-2|\xi|^4-|\xi|^{2+2\sigma}\right)=0,\end{cases}\quad\Rightarrow\quad\begin{cases}a^2=\dfrac{|\xi|^6}{|\xi|^{2\sigma}+|\xi|^2},\\ a^2=2|\xi|^4+|\xi|^{2+2\sigma},\end{cases}$$

where we use $a\neq0$. They lead to a contradiction immediately because $\xi\in Z_{\mathrm{mid}}(\varepsilon,N)$. Then, no purely imaginary characteristic roots of $A(|\xi|;\sigma)$ for all $\sigma\in[0,1)\cup(1,2]$ can exist for frequencies in the bounded zone. Consequently, due to the compactness of the bounded zone $Z_{\mathrm{mid}}(\varepsilon,N)$ and the continuity of $\mathrm{Re}\ \lambda_j(|\xi|)$ together with $\mathrm{Re}\ \lambda_j(|\xi|)>0$, $j=1,2,3$, for $|\xi|=\varepsilon$ and $|\xi|=N$, we complete the proof immediately. □

2.2. Representations of solutions. From Lemmas 2.1 and 2.2, we know that when $\sigma\in[0,1)\cup(1,2]$ with small frequencies or large frequencies the uniform invertibility of $T_{\sigma,\text{int}}$ and $T_{\sigma,\text{ext}}$ hold. Thus, we have the next theorems for the representations of solutions. Their proofs are based on Lemmas 2.1 and 2.2.

THEOREM 2.1. *There exists a matrix $T_{\sigma,\text{int}}$ for $\sigma\in[0,1)\cup(1,2]$, which is uniformly invertible for small frequencies such that the following representation formula for the Cauchy problem (2.2) with $\sigma\in[0,1)\cup(1,2]$ holds:*

$$\chi_{\text{int}}(\xi)w^{(0)}(t,\xi)=\chi_{\text{int}}(\xi)T_{\sigma,\text{int}}\,\text{diag}\left(e^{-\lambda_1(|\xi|)t},e^{-\lambda_2(|\xi|)t},e^{-\lambda_3(|\xi|)t}\right)T^{-1}_{\sigma,\text{int}}w_0^{(0)}(\xi),$$

where the characteristic roots $\lambda_j(|\xi|)$ for $j=1,2,3$, have the following asymptotic behavior:

- *if $\sigma\in[0,1)$, then we have*
$$\lambda_1(|\xi|)=|\xi|^{4-2\sigma},\ \lambda_2(|\xi|)=|\xi|^2+|\xi|^{4-2\sigma},\ \lambda_3(|\xi|)=|\xi|^{2\sigma}-2|\xi|^{4-2\sigma},$$
modulo $\mathcal{O}\big(|\xi|^{6-4\sigma}\big)$;
- *if $\sigma\in(1,2]$, then we have*
$$\lambda_1(|\xi|)=y_1|\xi|^2,\ \lambda_2(|\xi|)=y_2|\xi|^2,\ \lambda_3(|\xi|)=y_3|\xi|^2,$$
modulo $\mathcal{O}\big(|\xi|^{2\sigma}\big)$, where y_1,y_2,y_3 are determined in (2.7).

THEOREM 2.2. *There exists a matrix $T_{\sigma,\text{ext}}$ for $\sigma\in[0,1)\cup(1,2]$, which is uniformly invertible for large frequencies such that the following representation formula for the Cauchy problem (2.2) with $\sigma\in[0,1)\cup(1,2]$ holds:*

$$\chi_{\text{ext}}(\xi)w^{(0)}(t,\xi)=\chi_{\text{ext}}(\xi)T_{\sigma,\text{ext}}\,\text{diag}\left(e^{-\lambda_1(|\xi|)t},e^{-\lambda_2(|\xi|)t},e^{-\lambda_3(|\xi|)t}\right)T^{-1}_{\sigma,\text{ext}}w_0^{(0)}(\xi),$$

where the characteristic roots $\lambda_j(|\xi|)$ for $j=1,2,3$, have the following asymptotic behavior:

- *if $\sigma\in[0,1)$, then we have*
$$\lambda_1(|\xi|)=y_1|\xi|^2,\ \lambda_2(|\xi|)=y_2|\xi|^2,\ \lambda_3(|\xi|)=y_3|\xi|^2,$$
modulo $\mathcal{O}\big(|\xi|^{2\sigma}\big)$, where y_1,y_2,y_3 are determined in (2.7);
- *if $\sigma\in(1,2]$, then we have*
$$\lambda_1(|\xi|)=|\xi|^{4-2\sigma},\ \lambda_2(|\xi|)=|\xi|^2+|\xi|^{4-2\sigma},\ \lambda_3(|\xi|)=|\xi|^{2\sigma}-2|\xi|^{4-2\sigma},$$
modulo $\mathcal{O}\big(|\xi|^{6-4\sigma}\big)$.

Lastly, considering (2.2) with $\sigma=1$, from Lemma 2.3 we can derive the explicit representation of solutions in the following statement.

THEOREM 2.3. *There exists a matrix $T_{1,0}$, which is uniformly invertible for all frequencies such that the following representation formula for the Cauchy problem (2.2) with $\sigma=1$ holds:*

$$w^{(0)}(t,\xi)=T_{1,0}\,\text{diag}\left(e^{-\lambda_1(|\xi|)t},e^{-\lambda_2(|\xi|)t},e^{-\lambda_3(|\xi|)t}\right)T^{-1}_{1,0}w_0^{(0)}(\xi),$$

where the characteristic roots $\lambda_j(|\xi|)$ have the following explicit expressions:

$$\lambda_1(|\xi|)=y_4|\xi|^2,\ \lambda_2(|\xi|)=y_5|\xi|^2,\ \lambda_3(|\xi|)=y_6|\xi|^2,$$

where y_4,y_5,y_6 are determined in (2.9).

3. Some qualitative properties of solutions

In this section we derive smoothing effect of solutions and L^2 well-posedness of the Cauchy problem for linear thermoelastic plate equations with friction or structural damping.

Let us study smoothing effect of solutions initially.

THEOREM 3.1. *Let us assume $\left(|D|^2u_0,u_1,\theta_0\right)\in L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Then, the solution to the Cauchy problem (1.2) with $\sigma\in[0,2)$ belongs to Gevrey spaces such that*

$$\left(|D|^{s+2}u,|D|^s u_t,|D|^s\theta\right)(t,\cdot)\in\Gamma^\kappa(\mathbb{R}^n)\times\Gamma^\kappa(\mathbb{R}^n)\times\Gamma^\kappa(\mathbb{R}^n)\text{ for any } t>0,$$

with $s\geq 0$, where the parameter $\kappa=1$ when $\sigma\in\left[0,\frac{3}{2}\right]$, and $\kappa=\frac{1}{4-2\sigma}$ when $\sigma\in\left(\frac{3}{2},2\right)$.

Proof. To understand Gevrey smoothing of the solution, we only need to study the regularity properties of the solution for frequencies in the large zone $Z_{\text{ext}}(N)$. From Theorem 2.2 we may estimate

$$\chi_{\text{ext}}(\xi)|\xi|^s\left|w^{(0)}(t,\xi)\right|\lesssim\begin{cases}\chi_{\text{ext}}(\xi)|\xi|^s e^{-|\xi|^2t}\left|w_0^{(0)}(\xi)\right| & \text{if } \sigma\in[0,1],\\ \chi_{\text{ext}}(\xi)|\xi|^s e^{-|\xi|^{4-2\sigma}t}\left|w_0^{(0)}(\xi)\right| & \text{if } \sigma\in(1,2).\end{cases}$$

When we take the parameter κ in Gevrey spaces $\Gamma^\kappa(\mathbb{R}^n)$ such that $\kappa=1$ if $\sigma\in\left[0,\frac{3}{2}\right]$, and $\kappa=\frac{1}{4-2\sigma}$ if $\sigma\in\left(\frac{3}{2},2\right)$, they lead to

$$\mathcal{F}^{-1}_{\xi\to x}\left(|\xi|^s w^{(0)}(t,\xi)\right)(t,\cdot)\in\Gamma^\kappa(\mathbb{R}^n)\text{ for any } t>0.$$

Then, we can get

$$|D|^sU(t,\cdot)\in\Gamma^\kappa(\mathbb{R}^n)\text{ for any } t>0.$$

So, according to (2.3), we complete the proof. □

REMARK 3.1. We notice that for the Cauchy problem (1.2) with $\sigma\in\left[0,\frac{3}{2}\right]$, the solution belongs to the Gevrey space $\Gamma^1(\mathbb{R}^n)$, which means analytic smoothing of the solution.

REMARK 3.2. Let us consider the Cauchy problem (1.2) with $\sigma=2$. The representation of solutions for large frequencies from Theorem 2.2 implies for $s\geq 0$ that

$$\chi_{\text{ext}}(\xi)|\xi|^s\left|w^{(0)}(t,\xi)\right|\lesssim\chi_{\text{ext}}(\xi)e^{-t}|\xi|^s\left|w_0^{(0)}(\xi)\right|.$$

Thus, the solution does not belong to Gevrey spaces $\Gamma^\kappa(\mathbb{R}^n)$ with $\kappa\in[1,\infty)$.

REMARK 3.3. The statement of Theorem 3.1 tells us that the threshold of Gevrey smoothing of solutions is $\sigma=\frac{3}{2}$. The observation of this threshold is as follows. In the case when $\sigma\in[0,1]$, the thermal damping generated by Fourier's law plays a dominant role of smoothing effect. Thus, we may observe analytic smoothing. In the case when $\sigma\in(1,2)$, the structural damping has dominant influence. Then, we may consider the smoothing effect of solutions to the following structurally damped plate equation:

$$\begin{cases}u_{tt}+\Delta^2u+(-\Delta)^\sigma u_t=0, & t>0,\ x\in\mathbb{R}^n,\\ (u,u_t)(0,x)=(u_0,u_1)(x), & x\in\mathbb{R}^n,\end{cases}\tag{3.1}$$

with $\sigma\in(1,2)$. From the proof of Proposition 22 in the paper [27], concerning smoothing effect of solutions to (3.1), we may immediately obtain analytic smoothing if $1<\sigma\leq$

$\frac{3}{2}$, and Gevrey smoothing $\Gamma^{\frac{1}{2(2-\sigma)}}$ if $\frac{3}{2}<\sigma<2$. By this way, we may expect that the threshold for smoothing effect for (1.2) is $\sigma=\frac{3}{2}$.

After applying the representations of solutions from Theorems 2.1, 2.2 and 2.3, we immediately prove the following L^2 well-posedness for the Cauchy problem (1.2).

THEOREM 3.2. *Let us assume $\left(|D|^2u_0,u_1,\theta_0\right)\in L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Then, there exists a uniquely determined solution to the Cauchy problem (1.2) with $\sigma\in[0,2]$, which satisfies*

$$u\in\mathcal{C}\left([0,\infty),\dot{H}^2(\mathbb{R}^n)\right),\quad u_t\in\mathcal{C}\left([0,\infty),L^2(\mathbb{R}^n)\right),\quad \theta\in\mathcal{C}\left([0,\infty),L^2(\mathbb{R}^n)\right).$$

REMARK 3.4. One also can derive H^s well-posedness for the Cauchy problem (1.2) for all $s\in\mathbb{R}$. In other words, there exists a uniquely determined solution to the Cauchy problem (1.2) with $\sigma\in[0,2]$, which fulfills

$$\left(|D|^2u,u_t,\theta\right)\in\mathcal{C}\left([0,\infty),H^s(\mathbb{R}^n)\right)\times\mathcal{C}\left([0,\infty),H^s(\mathbb{R}^n)\right)\times\mathcal{C}\left([0,\infty),H^s(\mathbb{R}^n)\right),$$

if we assume $\left(|D|^2u_0,u_1,\theta_0\right)\in H^s(\mathbb{R}^n)\times H^s(\mathbb{R}^n)\times H^s(\mathbb{R}^n)$ for $s\in\mathbb{R}$.

4. Estimates for solutions

This section mainly develops some estimates for solutions to linear thermoelastic plate equations with different damping mechanisms in $\mathbb{R}^n$, $n\geq1$. The section is organized as follows. First of all, by using phase space analysis and the representations of solutions stated in Theorems 2.1, 2.2, 2.3, we derive estimates of solutions to (1.2) with initial data taken from $H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)$ for $s\geq0$ and $m\in[1,2]$. Moreover, inspired by [11,13,15], we investigate estimates of solutions to (1.2) with initial data taken from wighted L^1 spaces, i.e., $H^s(\mathbb{R}^n)\cap L^{1,\delta}(\mathbb{R}^n)$ for $s\geq0$ and $\delta\in(0,1]$. Eventually, we study L^p-L^q estimates not necessary on the conjugate line with the aid of some applications of L^r estimates for oscillating integrals.

4.1. Energy estimates. Before stating our main results, let us denote the parameters for $n\geq1$, $s\geq0$ and $m\in[1,2]$ by the following way:

$$\gamma(\sigma,n,m,s):=\begin{cases}\dfrac{(2-m)n+2ms}{4m(2-\sigma)} & \text{if } \sigma\in[0,1),\\[2mm] \dfrac{(2-m)n+2ms}{4m} & \text{if } \sigma\in[1,2].\end{cases}$$

It will be used to describe the decay rate of the energy estimates later.

Additionally, we define the function spaces $\mathcal{A}_{m,s}(\mathbb{R}^n)$ for $s\geq0$ and $m\in[1,2]$

$$\mathcal{A}_{m,s}(\mathbb{R}^n):=\left(H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\right)\times\left(H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\right)\times\left(H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)\right),$$

and the function spaces $\mathcal{B}_{\delta,s}(\mathbb{R}^n)$ for $s\geq0$ and $\delta\in[0,1]$

$$\mathcal{B}_{\delta,s}(\mathbb{R}^n):=\left(H^s(\mathbb{R}^n)\cap L^{1,\delta}(\mathbb{R}^n)\right)\times\left(H^s(\mathbb{R}^n)\cap L^{1,\delta}(\mathbb{R}^n)\right)\times\left(H^s(\mathbb{R}^n)\cap L^{1,\delta}(\mathbb{R}^n)\right),$$

carrying their corresponding norms.

THEOREM 4.1. *Let us assume $\left(|D|^2u_0,u_1,\theta_0\right)\in\mathcal{A}_{2,s}(\mathbb{R}^n)$. Then, the solution to the Cauchy problem (1.2) with $\sigma\in[0,2]$ satisfies the following estimates:*

$$\left\||D|^2u(t,\cdot)\right\|_{\dot{H}^s(\mathbb{R}^n)}+\|u_t(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)}+\|\theta(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)}$$

$$\lesssim (1+t)^{-\gamma(\sigma,n,2,s)} \left\| \left(|D|^2 u_0, u_1, \theta_0\right) \right\|_{\mathcal{A}_{2,s}(\mathbb{R}^n)}.$$

Proof. For one thing, considering small frequencies, we have

$$\begin{aligned}
&\left\| \chi_{\rm int}(\xi)|\xi|^s w^{(0)}(t,\xi) \right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \begin{cases} \left\| \chi_{\rm int}(\xi)|\xi|^s e^{-c|\xi|^{4-2\sigma}t} \right\|_{L^\infty(\mathbb{R}^n)} \left\| \mathcal{F}^{-1}\left(w_0^{(0)}\right)\right\|_{L^2(\mathbb{R}^n)} & \text{if } \sigma \in [0,1), \\ \left\| \chi_{\rm int}(\xi)|\xi|^s e^{-c|\xi|^{2}t} \right\|_{L^\infty(\mathbb{R}^n)} \left\| \mathcal{F}^{-1}\left(w_0^{(0)}\right)\right\|_{L^2(\mathbb{R}^n)} & \text{if } \sigma \in [1,2], \end{cases} \\
&\lesssim \begin{cases} (1+t)^{-\frac{s}{2(2-\sigma)}} \left\| \mathcal{F}^{-1}\left(w_0^{(0)}\right)\right\|_{L^2(\mathbb{R}^n)} & \text{if } \sigma \in [0,1), \\ (1+t)^{-\frac{s}{2}} \left\| \mathcal{F}^{-1}\left(w_0^{(0)}\right)\right\|_{L^2(\mathbb{R}^n)} & \text{if } \sigma \in [1,2]. \end{cases}
\end{aligned}$$

For another, considering bounded frequencies and large frequencies, we may immediately obtain exponential decay estimates for initial data taken from H^s spaces.

Then, we can complete the proof of Theorem 4.1 by applying the Parseval-Plancherel theorem. ☐

REMARK 4.1. Let us assume $\left(|D|^2u_0,u_1,\theta_0\right) \in \dot{H}^s(\mathbb{R}^n)\times\dot{H}^s(\mathbb{R}^n)\times\dot{H}^s(\mathbb{R}^n)$ for $s\geq 0$. Then, the solution satisfies the following bounded estimates:

$$\begin{aligned}
&\left\| |D|^2 u(t,\cdot)\right\|_{\dot{H}^s(\mathbb{R}^n)} + \|u_t(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} + \|\theta(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} \\
&\lesssim \left\| \left(|D|^2 u_0, u_1, \theta_0\right) \right\|_{\dot{H}^s(\mathbb{R}^n)\times\dot{H}^s(\mathbb{R}^n)\times\dot{H}^s(\mathbb{R}^n)}.
\end{aligned}$$

Next, we consider initial data taken from H^s with additional regularity L^m, $m\in[1,2)$, which implies an additional decay in the corresponding estimates.

THEOREM 4.2. *Let us assume* $\left(|D|^2u_0,u_1,\theta_0\right)\in\mathcal{A}_{m,s}(\mathbb{R}^n)$, *where* $s\geq 0$ *and* $m\in[1,2)$. *Then, the solution to the Cauchy problem* (1.2) *with* $\sigma\in[0,2]$ *satisfies the next estimates:*

$$\begin{aligned}
&\left\| |D|^2 u(t,\cdot)\right\|_{\dot{H}^s(\mathbb{R}^n)} + \|u_t(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} + \|\theta(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} \\
&\lesssim (1+t)^{-\gamma(\sigma,n,m,s)} \left\| \left(|D|^2 u_0, u_1, \theta_0\right) \right\|_{\mathcal{A}_{m,s}(\mathbb{R}^n)}.
\end{aligned}$$

Proof. For frequencies in the small zone, we apply Hölder's inequality and the Hausdorff-Young inequality to get the following estimates:

$$\begin{aligned}
&\left\| \chi_{\rm int}(\xi)|\xi|^s w^{(0)}(t,\xi) \right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \begin{cases} \left\| \chi_{\rm int}(\xi)|\xi|^s e^{-c|\xi|^{4-2\sigma}t} \right\|_{L^{\frac{2m}{2-m}}(\mathbb{R}^n)} \left\| \mathcal{F}^{-1}\left(w_0^{(0)}\right)\right\|_{L^m(\mathbb{R}^n)} & \text{if } \sigma \in [0,1), \\ \left\| \chi_{\rm int}(\xi)|\xi|^s e^{-c|\xi|^{2}t} \right\|_{L^{\frac{2m}{2-m}}(\mathbb{R}^n)} \left\| \mathcal{F}^{-1}\left(w_0^{(0)}\right)\right\|_{L^m(\mathbb{R}^n)} & \text{if } \sigma \in [1,2], \end{cases} \\
&\lesssim \begin{cases} (1+t)^{-\frac{(2-m)n+2ms}{4m(2-\sigma)}} \left\| \mathcal{F}^{-1}\left(w_0^{(0)}\right)\right\|_{L^m(\mathbb{R}^n)} & \text{if } \sigma \in [0,1), \\ (1+t)^{-\frac{(2-m)n+2ms}{4m}} \left\| \mathcal{F}^{-1}\left(w_0^{(0)}\right)\right\|_{L^m(\mathbb{R}^n)} & \text{if } \sigma \in [1,2], \end{cases}
\end{aligned}$$

where we use the following facts for $\bar{m}\in[1,\infty)$, $\alpha_0>0$ and $s\geq 0$:

$$\left\| \chi_{\rm int}(\xi)|\xi|^s e^{-c|\xi|^{\alpha_0}t} \right\|^{\bar{m}}_{L^{\bar{m}}(\mathbb{R}^n)} \lesssim 1 \quad \text{if } 0\leq t\leq 1,$$

and

$$\begin{aligned}\left\|\chi_{\rm int}(\xi)|\xi|^s e^{-c|\xi|^{\alpha_0}t}\right\|^{\bar m}_{L^{\bar m}(\mathbb{R}^n)} &= \int_0^{\varepsilon} r^{s\bar m+n-1}e^{-c\bar m r^{\alpha_0}t}dr\\ &= \frac{1}{\alpha_0}t^{-\frac{1}{\alpha_0}(s\bar m+n)}\int_0^{\varepsilon^{\alpha_0}t}\tau^{\frac{1}{\alpha_0}(s\bar m+n)-1}e^{-c\bar m\tau^{\alpha_0}}d\tau\\ &\lesssim t^{-\frac{\bar m}{\alpha_0}\left(s+\frac{n}{\bar m}\right)} \quad \text{if } 1\le t.\end{aligned}$$

For frequencies in the bounded zone and the large zone, we obtain an exponential decay estimate

$$\left\|(\chi_{\rm mid}(\xi)+\chi_{\rm ext}(\xi))\,|\xi|^s w^{(0)}(t,\xi)\right\|_{L^2(\mathbb{R}^n)} \lesssim e^{-ct}\left\|\mathcal{F}^{-1}\left(w_0^{(0)}\right)\right\|_{H^s(\mathbb{R}^n)},$$

where the constant $c>0$. Finally, combining with the Parseval-Plancherel theorem, the proof of Theorem 4.2 is complete. □

REMARK 4.2. Concerning the sharpness of the derived energy estimates in Theorem 4.2, we point out that the estimates for $\left\||\xi|^s w^{(0)}(t,\xi)\right\|_{L^2(\mathbb{R}^n)}$ seem to be sharp because diagonalization procedure is used in deriving representations of solutions.

Next, we discuss energy estimates with initial data taken from the weighted spaces $L^{1,\delta}$ for $\delta\in(0,1]$ (see Notation in Section 1). Before stating our result, we recall the following useful lemma, which was introduced in Lemma 2.1 in the paper [13].

LEMMA 4.1. *Let $\delta\in(0,1]$ and $f\in L^{1,\delta}(\mathbb{R}^n)$. Then, the following estimate holds:*

$$|\hat f(\xi)|\le C_\delta|\xi|^\delta\|f\|_{L^{1,\delta}(\mathbb{R}^n)}+\left|\int_{\mathbb{R}^n}f(x)dx\right|,$$

with some constant $C_\delta>0$.

THEOREM 4.3. *Let us assume $\left(|D|^2u_0,u_1,\theta_0\right)\in\mathcal{B}_{\delta,s}(\mathbb{R}^n)$, where $s\ge0$ and $\delta\in(0,1]$. Then, the solution to the Cauchy problem (1.2) with $\sigma\in[0,2]$ satisfies the following estimates:*

$$\begin{aligned}&\left\||D|^2u(t,\cdot)\right\|_{\dot H^s(\mathbb{R}^n)}+\|u_t(t,\cdot)\|_{\dot H^s(\mathbb{R}^n)}+\|\theta(t,\cdot)\|_{\dot H^s(\mathbb{R}^n)}\\ &\lesssim(1+t)^{-\gamma(\sigma,n,1,s+\delta)}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{\mathcal{B}_{\delta,s}(\mathbb{R}^n)}+(1+t)^{-\gamma(\sigma,n,1,s)}\left|\int_{\mathbb{R}^n}U_0(x)dx\right|,\end{aligned}$$

where initial data $U_0(x)$ is defined in (2.4).

Proof. Here we only need to modify estimates for small frequencies in the proof of Theorem 4.2. We apply Lemma 4.1 to get

$$\begin{aligned}\chi_{\rm int}(\xi)|\xi|^s\left|w^{(0)}(t,\xi)\right| &\lesssim \begin{cases}\chi_{\rm int}(\xi)|\xi|^s e^{-c|\xi|^{4-2\sigma}t}\left|w_0^{(0)}(\xi)\right| & \text{if } \sigma\in[0,1),\\ \chi_{\rm int}(\xi)|\xi|^s e^{-c|\xi|^{2}t}\left|w_0^{(0)}(\xi)\right| & \text{if } \sigma\in[1,2],\end{cases}\\ &\lesssim \begin{cases}\chi_{\rm int}(\xi)|\xi|^s e^{-c|\xi|^{4-2\sigma}t}\left(|\xi|^\delta\|U_0\|_{L^{1,\delta}(\mathbb{R}^n)}+\left|\int_{\mathbb{R}^n}U_0(x)dx\right|\right) & \text{if } \sigma\in[0,1),\\ \chi_{\rm int}(\xi)|\xi|^s e^{-c|\xi|^{2}t}\left(|\xi|^\delta\|U_0\|_{L^{1,\delta}(\mathbb{R}^n)}+\left|\int_{\mathbb{R}^n}U_0(x)dx\right|\right) & \text{if } \sigma\in[1,2].\end{cases}\end{aligned}$$

Next, we may estimate for $\delta\in(0,1]$

$$\left\|\chi_{\rm int}(\xi)|\xi|^{s+\delta}e^{-c|\xi|^{4-2\sigma}t}\right\|_{L^2(\mathbb{R}^n)}\lesssim(1+t)^{-\frac{n+2(s+\delta)}{4(2-\sigma)}},$$
$$\left\|\chi_{\rm int}(\xi)|\xi|^{s+\delta}e^{-c|\xi|^{2}t}\right\|_{L^2(\mathbb{R}^n)}\lesssim(1+t)^{-\frac{n+2(s+\delta)}{4}}.$$

Then, combining with the estimates

$$\left\|\chi_{\rm int}(\xi)|\xi|^{s}e^{-c|\xi|^{4-2\sigma}t}\right\|_{L^2(\mathbb{R}^n)}\lesssim(1+t)^{-\frac{n+2s}{4(2-\sigma)}},$$
$$\left\|\chi_{\rm int}(\xi)|\xi|^{s}e^{-c|\xi|^{2}t}\right\|_{L^2(\mathbb{R}^n)}\lesssim(1+t)^{-\frac{n+2s}{4}},$$

we immediately complete the proof. □

Remark 4.3. By restricting

$$\left|\int_{\mathbb{R}^n}U_0(x)dx\right|=0, \tag{4.1}$$

and comparing with derived estimates in Theorem 4.3, the decay rate given in Theorem 4.2 when $m=1$ can be improved by $(1+t)^{-\frac{\delta}{2}}$ for $\delta\in(0,1]$. We need to point out that the additional condition (4.1) holds when $U_0(x)$ is an odd function with respect to x_n, in other words,

$$U_0(x_1,\dots,x_{n-1},-x_n)=-U_0(x_1,\dots,x_{n-1},x_n).$$

Remark 4.4. The statements of Theorems 4.1, 4.2 and 4.3 tell us that the thermal dissipation generated by Fourier's law has a dominant influence on energy estimates in comparison with friction and structural damping only if $\sigma\in[1,2]$.

4.2. L^p-L^q estimates not necessary on the conjugate line. In the beginning, let us introduce the parameters to depict the decay rate

$$\mu(\sigma,n,p,q,s):=\begin{cases}\dfrac{s}{4-2\sigma}+\dfrac{n}{4-2\sigma}\left(\dfrac{1}{p}-\dfrac{1}{q}\right) & \text{if } \sigma\in[0,1),\\ \dfrac{s}{2}+\dfrac{n}{2}\left(\dfrac{1}{p}-\dfrac{1}{q}\right) & \text{if } \sigma\in[1,2],\end{cases} \tag{4.2}$$

where $s\geq0$ and $1\leq p\leq q\leq\infty$.

Moreover, we define the parameter to depict the regularity for initial data

$$M_{n,s,p,q}>s+n\left(\frac{1}{p}-\frac{1}{q}\right), \tag{4.3}$$

where $s\geq0$ and $1\leq p\leq2\leq q\leq\infty$.

4.2.1. L^p-L^q estimates for the model with $\sigma\in[0,1)\cup(1,2]$. Before starting our main theorem, we prove the following useful lemma first.

Lemma 4.2. *Let $f\in\mathcal{S}(\mathbb{R}^n)$ and $\kappa_1>0$, $\kappa_2\geq0$, $s\geq0$. Then, the next estimates hold:*

$$\left\|\mathcal{F}^{-1}_{\xi\to x}\left(\chi_{\rm int}(\xi)|\xi|^{s}e^{-c|\xi|^{\kappa_1}t}\hat{f}(\xi)\right)\right\|_{L^q(\mathbb{R}^n)}\lesssim(1+t)^{-\frac{s}{\kappa_1}-\frac{n}{\kappa_1}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\mathbb{R}^n)}, \tag{4.4}$$

$$\left\|\mathcal{F}^{-1}_{\xi\to x}\left(\chi_{\text{ext}}(\xi)|\xi|^s e^{-c|\xi|^{\kappa_2}t}\hat{f}(\xi)\right)\right\|_{L^q(\mathbb{R}^n)} \lesssim e^{-ct}\left\|\langle D\rangle^{M_{n,s,p,q}} f\right\|_{L^p(\mathbb{R}^n)}, \tag{4.5}$$

where $c>0$, $1\le p\le 2\le q\le\infty$ and $M_{n,s,p,q}$ is chosen as in (4.3).

Proof. Let us prove (4.4) first. Applying the Hausdorff-Young inequality yields

$$\left\|\mathcal{F}^{-1}_{\xi\to x}\left(\chi_{\text{int}}(\xi)|\xi|^s e^{-c|\xi|^{\kappa_1}t}\hat{f}(\xi)\right)\right\|_{L^q(\mathbb{R}^n)} \lesssim \left\|\chi_{\text{int}}(\xi)|\xi|^s e^{-c|\xi|^{\kappa_1}t}\hat{f}(\xi)\right\|_{L^{q'}(\mathbb{R}^n)}. \tag{4.6}$$

Here $\frac{1}{q}+\frac{1}{q'}=1$ with $2\le q\le\infty$. By using Hölder's inequality, the estimate holds

$$\left\|\chi_{\text{int}}(\xi)|\xi|^s e^{-c|\xi|^{\kappa_1}t}\hat{f}(\xi)\right\|_{L^{q'}(\mathbb{R}^n)} \lesssim \left\|\chi_{\text{int}}(\xi)|\xi|^s e^{-c|\xi|^{\kappa_1}t}\right\|_{L^{\tilde{p}}(\mathbb{R}^n)}\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)}, \tag{4.7}$$

where $\frac{1}{q'}=\frac{1}{\tilde{p}}+\frac{1}{p'}$ with $2\le p'\le\infty$.

Finally, combining with (4.6), (4.7) and the Hausdorff-Young inequality leads to

$$\left\|\mathcal{F}^{-1}_{\xi\to x}\left(\chi_{\text{int}}(\xi)|\xi|^s e^{-c|\xi|^{\kappa_1}t}\hat{f}(\xi)\right)\right\|_{L^q(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{s}{\kappa_1}-\frac{n}{\kappa_1}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\mathbb{R}^n)}.$$

Next, we begin with proving (4.5). For $0\le t\le 1$, by a similar approach we have

$$\begin{aligned}\left\|\mathcal{F}^{-1}_{\xi\to x}\left(\chi_{\text{ext}}(\xi)|\xi|^s e^{-c|\xi|^{\kappa_2}t}\hat{f}(\xi)\right)\right\|_{L^q(\mathbb{R}^n)} &\lesssim \left\|\chi_{\text{ext}}(\xi)\langle\xi\rangle^s\hat{f}(\xi)\right\|_{L^{q'}(\mathbb{R}^n)}\\ \lesssim \left\|\chi_{\text{ext}}(\xi)\langle\xi\rangle^{-n\left(\frac{1}{p}-\frac{1}{q}\right)-\epsilon}\right\|_{L^{\tilde{p}}(\mathbb{R}^n)}&\left\|\chi_{\text{ext}}(\xi)\langle\xi\rangle^{s+n\left(\frac{1}{p}-\frac{1}{q}\right)+\epsilon}\hat{f}(\xi)\right\|_{L^{p'}(\mathbb{R}^n)},\end{aligned} \tag{4.8}$$

where $\frac{1}{q}+\frac{1}{q'}=1$, $\frac{1}{q'}=\frac{1}{\tilde{p}}+\frac{1}{p'}$ with $2\le q\le\infty$, $2\le p'\le\infty$ and $\epsilon>0$.

The following fact holds:

$$\begin{aligned}\left\|\chi_{\text{ext}}(\xi)\langle\xi\rangle^{-n\left(\frac{1}{p}-\frac{1}{q}\right)-\epsilon}\right\|^{\tilde{p}}_{L^{\tilde{p}}(\mathbb{R}^n)} &= \int_N^\infty \langle r\rangle^{-n\left(\frac{1}{p}-\frac{1}{q}\right)\tilde{p}-\epsilon\tilde{p}+n-1}dr\\ &= \int_N^\infty \langle r\rangle^{-\epsilon\tilde{p}-1}dr<\infty.\end{aligned} \tag{4.9}$$

Summarizing (4.8), (4.9) and using the Hausdorff-Young inequality we derive

$$\left\|\mathcal{F}^{-1}_{\xi\to x}\left(\chi_{\text{ext}}(\xi)|\xi|^s e^{-c|\xi|^{\kappa_2}t}\hat{f}(\xi)\right)\right\|_{L^q(\mathbb{R}^n)} \lesssim \left\|\langle D\rangle^{s+n\left(\frac{1}{p}-\frac{1}{q}\right)+\epsilon}f\right\|_{L^p(\mathbb{R}^n)}$$

for $1\le p\le 2\le q\le\infty$ and $0\le t\le 1$.

For the case $t\ge 1$, according to $|\xi|\ge N$ we may obtain

$$\left\|\mathcal{F}^{-1}_{\xi\to x}\left(\chi_{\text{ext}}(\xi)|\xi|^s e^{-c|\xi|^{\kappa_2}t}\hat{f}(\xi)\right)\right\|_{L^q(\mathbb{R}^n)} \lesssim e^{-ct}\left\|\langle D\rangle^{s+n\left(\frac{1}{p}-\frac{1}{q}\right)+\epsilon}f\right\|_{L^p(\mathbb{R}^n)}.$$

Hence, the proof of Lemma 4.2 is completed. ∎

Now, let us derive L^p-L^q estimates of solutions to the Cauchy problem (1.2) with $\sigma\in[0,1)\cup(1,2]$, where $1\le p\le 2\le q\le\infty$.

Theorem 4.4. *Let us assume $\left(|D|^2u_0,u_1,\theta_0\right)\in\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)$. Then, the solution to the Cauchy problem (1.2) with $\sigma\in[0,1)\cup(1,2]$ satisfies the following estimates:*

$$\left\||D|^2u(t,\cdot)\right\|_{\dot{H}^s_q(\mathbb{R}^n)}+\|u_t(t,\cdot)\|_{\dot{H}^s_q(\mathbb{R}^n)}+\|\theta(t,\cdot)\|_{\dot{H}^s_q(\mathbb{R}^n)}$$

$$\lesssim (1+t)^{-\mu(\sigma,n,p,q,s)} \left\| \left(|D|^2 u_0, u_1, \theta_0\right) \right\|_{H_p^{M_{n,s,p,q}}(\mathbb{R}^n)\times H_p^{M_{n,s,p,q}}(\mathbb{R}^n)\times H_p^{M_{n,s,p,q}}(\mathbb{R}^n)},$$

with $s\geq 0$, $1\leq p\leq 2\leq q\leq\infty$ *and* $M_{n,s,p,q} > s+n\left(\frac{1}{p}-\frac{1}{q}\right)$.

REMARK 4.5. If one is interested in the case $p\in(1,2]$ only, then we can choose $M_{n,s,p,q} = s+n\left(\frac{1}{p}-\frac{1}{q}\right)$.

Proof. From Theorems 2.1, 2.3, 2.3 and Lemma 2.4 we obtain

$$\begin{aligned}
&\left\| |D|^s \mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}\right)(t,\cdot)\right\|_{L^q(\mathbb{R}^n)} \lesssim \left\| |\xi|^s w^{(0)}(t,\xi)\right\|_{L^{q'}(\mathbb{R}^n)} \\
&\lesssim \left\| \chi_{\text{int}}(\xi)|\xi|^s w^{(0)}(t,\xi)\right\|_{L^{q'}(\mathbb{R}^n)} + \left\| \chi_{\text{mid}}(\xi)|\xi|^s w^{(0)}(t,\xi)\right\|_{L^{q'}(\mathbb{R}^n)} \\
&\quad + \left\| \chi_{\text{ext}}(\xi)|\xi|^s w^{(0)}(t,\xi)\right\|_{L^{q'}(\mathbb{R}^n)},
\end{aligned}$$

where $\frac{1}{q}+\frac{1}{q'}=1$ with $2\leq q\leq\infty$.

Following all steps from Lemma 4.2 we immediately complete the proof. ∎

4.2.2. $L^p - L^q$ estimates for the model with $\sigma=1$. Due to the treatment in Lemma 2.3, it allows us to obtain explicit representations of solutions. Therefore, it is helpful for us to derive $L^p - L^q$ estimates of solutions to the Cauchy problem (1.2), where $1\leq p\leq q\leq\infty$. To do this, let us introduce some results in L^p estimates for some oscillating integral by using modified Bessel functions (c.f. [8, 27]).

LEMMA 4.3. *Let* $p\in[1,\infty]$ *and* $c_1>0$, $c_2\neq 0$. *Then, the following estimates hold for any* $t>0$:

$$\left\| \mathcal{F}^{-1}_{\xi\to x}\left(|\xi|^s e^{-c_1|\xi|^2 t}\sin\left(c_2|\xi|^2 t\right)\right)(t,\cdot)\right\|_{L^p(\mathbb{R}^n)} \lesssim t^{-\frac{s}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)}, \tag{4.10}$$

$$\left\| \mathcal{F}^{-1}_{\xi\to x}\left(|\xi|^s e^{-c_1|\xi|^2 t}\cos\left(c_2|\xi|^2 t\right)\right)(t,\cdot)\right\|_{L^p(\mathbb{R}^n)} \lesssim t^{-\frac{s}{2}-\frac{n}{2}\left(1-\frac{1}{p}\right)}, \tag{4.11}$$

where $s\geq 0$ *and* $n\geq 1$.

Proof. For the proof of (4.11), one can see Proposition 12 in [27]. One can prove (4.10) by some minor modifications of the proof of Proposition 12 in [27]. ∎

THEOREM 4.5. *Let us assume* $\left(|D|^2 u_0, u_1, \theta_0\right) \in L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n)$, *where* $p\geq 1$. *Then, the solution to the Cauchy problem* (1.2) *with* $\sigma=1$ *satisfies the next estimates:*

$$\begin{aligned}
&\left\| |D|^2 u(t,\cdot)\right\|_{\dot H^s_q(\mathbb{R}^n)} + \|u_t(t,\cdot)\|_{\dot H^s_q(\mathbb{R}^n)} + \|\theta(t,\cdot)\|_{\dot H^s_q(\mathbb{R}^n)} \\
&\lesssim t^{-\mu(1,n,p,q,s)} \left\| \left(|D|^2 u_0, u_1, \theta_0\right)\right\|_{L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n)},
\end{aligned}$$

where $s\geq 0$ *and* $1\leq p\leq q\leq\infty$.

Proof. From Theorem 2.3, the solutions to (1.2) can be explicitly represented by the following way:

$$\begin{aligned}
&\left(u_t+|D|^2 u, u_t-|D|^2 u, \theta\right)^{\mathrm{T}}(t,x) \\
&= \left(\sum_{j,k=1}^{3} c_{jkl}\mathcal{F}^{-1}_{\xi\to x}\left(e^{-\mathrm{Re}\ y_{j+3}|\xi|^2 t - i\mathrm{Im}\ y_{j+3}|\xi|^2 t}\right) *_{(x)} U_{0,k}(x)\right)_{l=1}^{3}
\end{aligned}$$

$$= \left(\sum_{j,k=1}^{3} c_{jkl} \left(K_0^{(j)}(t,x) + K_1^{(j)}(t,x) \right) *_{(x)} U_{0,k}(x) \right)_{l=1}^{3}, \tag{4.12}$$

where c_{jkl} are constants and the kernels are

$$K_0^{(j)} := \mathcal{F}^{-1}_{\xi\to x} \left(-i \sin\left(\text{Im } y_{j+3}|\xi|^2 t\right) e^{-\text{Re } y_{j+3}|\xi|^2 t} \right), \tag{4.13}$$

$$K_1^{(j)} := \mathcal{F}^{-1}_{\xi\to x} \left(\cos\left(\text{Im } y_{j+3}|\xi|^2 t\right) e^{-\text{Re } y_{j+3}|\xi|^2 t} \right). \tag{4.14}$$

By applying Lemma 4.3 we get

$$\sum_{j=1}^{3} \left\| |D|^s K_0^{(j)}(t,\cdot) \right\|_{L^r(\mathbb{R}^n)} + \sum_{j=1}^{3} \left\| |D|^s K_1^{(j)}(t,\cdot) \right\|_{L^r(\mathbb{R}^n)} \lesssim t^{-\frac{s}{2} - \frac{n}{2}\left(1-\frac{1}{r}\right)}$$

for all $r \in [1,\infty]$. Then, we directly apply Young's inequality in (4.12) to complete the proof. □

Taking $p = q$ in Theorem 4.5 and supposing the higher regularity for initial data, the singularity will disappear as $t \to +0$. So, we have the next result.

Corollary 4.1. *Let us assume $\left(|D|^2 u_0, u_1, \theta_0\right) \in \dot{H}^s_p(\mathbb{R}^n) \times \dot{H}^s_p(\mathbb{R}^n) \times \dot{H}^s_p(\mathbb{R}^n)$, where $p \geq 1$ and $s \geq 0$. Then, the solution to the Cauchy problem (1.2) with $\sigma = 1$ satisfies the following bounded estimates:*

$$\begin{aligned} &\left\| |D|^2 u(t,\cdot) \right\|_{\dot{H}^s_p(\mathbb{R}^n)} + \|u_t(t,\cdot)\|_{\dot{H}^s_p(\mathbb{R}^n)} + \|\theta(t,\cdot)\|_{\dot{H}^s_p(\mathbb{R}^n)} \\ &\lesssim \left\| \left(|D|^2 u_0, u_1, \theta_0\right) \right\|_{\dot{H}^s_p(\mathbb{R}^n) \times \dot{H}^s_p(\mathbb{R}^n) \times \dot{H}^s_p(\mathbb{R}^n)}. \end{aligned}$$

Remark 4.6. One can apply Theorem 4.5 for $t > t_0 \gg 1$ and Corollary 4.1 for $0 \leq t \leq t_0$ to obtain decay estimates for

$$\left\| |D|^2 u(t,\cdot) \right\|_{\dot{H}^s_q(\mathbb{R}^n)} + \|u_t(t,\cdot)\|_{\dot{H}^s_q(\mathbb{R}^n)} + \|\theta(t,\cdot)\|_{\dot{H}^s_q(\mathbb{R}^n)}$$

with decay rate $(1+t)^{-\mu(1,n,p,q,s)}$. At this time, initial data should belong to the function spaces $\dot{H}^s_q(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, where $n \geq 1$, $1 \leq p \leq q \leq \infty$ and $s \geq 0$.

Remark 4.7. The statements of Theorems 4.4 and 4.5 indicate that the thermal dissipation generated by Fourier's law has a dominant influence on $L^p - L^q$ estimates away from the conjugate line in comparison with friction and structural damping only if $\sigma \in [1,2]$.

5. Diffusion phenomena

It is well known that diffusion phenomena allow one to bridge the decay behavior of the solution to (1.2) with the solution for the corresponding evolution. It also provides a tool to tackle the asymptotic profiles of solutions. In this section we study diffusion phenomena of solutions to the Cauchy problem (1.2) for $\sigma \in [0,1) \cup (1,2]$ with initial data taken from different assumptions on the regularity.

In the view of the derived estimates of solutions in Section 4, we find that the decay rate of estimates of solutions are determined by the behavior of the characteristic roots for $\xi \in Z_{\text{int}}(\varepsilon)$ only. For $\xi \in Z_{\text{mid}}(\varepsilon,N) \cup Z_{\text{ext}}(N)$, the solutions satisfy an exponential decay when we assume initial data taken with suitable regularities. For this reason, we explain diffusion phenomena of solutions for small frequencies in this section.

REMARK 5.1. Considering $\sigma=1$ in the system (2.2), we find that $e^{-y_j|\xi|^2t}$ with $y_j\in\mathbb{C}$ for $j=4,5,6$, plays a dominant role in the explicit representation of $w^{(0)}(t,\xi)$ from Theorem 2.3. Then, there is no improvement in the decay estimates for the difference between the solutions to the system (2.2) with $\sigma=1$ and the solutions to its reference system. Hence, we explain diffusion phenomena for $\sigma\in[0,1)\cup(1,2]$ only.

5.1. Diffusion phenomena for the model with $\sigma\in[0,1)$. To describe diffusion phenomena of the solutions to the Cauchy problem (2.2) with $\sigma\in[0,1)$, we consider the following reference system:

$$\begin{cases}\tilde{u}_t+\operatorname{diag}\left((-\Delta)^{2-\sigma},(-\Delta),(-\Delta)^{\sigma}\right)\tilde{u}=0, & t>0,\, x\in\mathbb{R}^n,\\ \tilde{u}(0,x)=\mathcal{F}^{-1}\left(T_{1\frac{1}{2}}^{-1}T_1^{-1}T_0^{-1}w_0^{(0)}(\xi)\right)(x), & x\in\mathbb{R}^n,\end{cases}\tag{5.1}$$

where $\tilde{u}=\left(\tilde{u}^{(1)},\tilde{u}^{(2)},\tilde{u}^{(3)}\right)^{\mathrm{T}}$ and $T_0,T_1,T_{1\frac{1}{2}}$ are defined in Lemma 2.1. By applying the partial Fourier transform $\tilde{w}(t,\xi)=\mathcal{F}_{x\to\xi}(\tilde{u}(t,x))$, (5.1) can be transformed to

$$\begin{cases}\tilde{w}_t+\operatorname{diag}\left(|\xi|^{4-2\sigma},|\xi|^2,|\xi|^{2\sigma}\right)\tilde{w}=0, & t>0,\, \xi\in\mathbb{R}^n,\\ \tilde{w}(0,\xi)=T_{1\frac{1}{2}}^{-1}T_1^{-1}T_0^{-1}w_0^{(0)}(\xi), & \xi\in\mathbb{R}^n.\end{cases}\tag{5.2}$$

We know that the solution $\tilde{w}=\tilde{w}(t,\xi)$ to (5.2) can be explicitly represented by

$$\tilde{w}(t,\xi)=\operatorname{diag}\left(e^{-|\xi|^{4-2\sigma}t},e^{-|\xi|^2t},e^{-|\xi|^{2\sigma}t}\right)T_{1\frac{1}{2}}^{-1}T_1^{-1}T_0^{-1}w_0^{(0)}(\xi).\tag{5.3}$$

REMARK 5.2. According to the evolution system (5.1) with $\sigma=0$, we find that the reference system consists of two different evolution equations such that

$$\begin{aligned}&\text{fourth-order parabolic equation:} && \tilde{u}_t^{(1)}+\Delta^2\tilde{u}^{(1)}=0,\\ &\text{heat equation:} && \tilde{u}_t^{(2)}-\Delta\tilde{u}^{(2)}=0.\end{aligned}$$

Therefore, we obtain double diffusion phenomena of solution to (1.2) with $\sigma=0$. The effect of double diffusion phenomena was introduced in the recent papers [3,4].

REMARK 5.3. Let us consider (1.2) with $\sigma\in(0,1)$. Inspired by the dominant asymptotic behavior of eigenvalues such that

$$\lambda_1(|\xi|)=\mathcal{O}\left(|\xi|^{4-2\sigma}\right),\ \lambda_2(|\xi|)=\mathcal{O}\left(|\xi|^2\right),\ \lambda_3(|\xi|)=\mathcal{O}\left(|\xi|^{2\sigma}\right)$$

for $\xi\in Z_{\mathrm{int}}(\varepsilon)$, we observe that the evolution system (5.1) consists of three different evolution equations, which are

$$\begin{aligned}&\text{fractional heat equation 1:} && \tilde{u}_t^{(1)}+(-\Delta)^{2-\sigma}\tilde{u}^{(1)}=0,\\ &\text{heat equation:} && \tilde{u}_t^{(2)}-\Delta\tilde{u}^{(2)}=0,\\ &\text{fractional heat equation 2:} && \tilde{u}_t^{(3)}+(-\Delta)^{\sigma}\tilde{u}^{(3)}=0.\end{aligned}$$

We may interpret this effect as triple diffusion phenomena, which is a natural generalization of the effect of double diffusion phenomena.

THEOREM 5.1. *Let us consider the Cauchy problem* (2.2) *with* $\sigma\in[0,1)$. *We assume* $\left(|D|^2u_0,u_1,\theta_0\right)\in L^m(\mathbb{R}^n)\times L^m(\mathbb{R}^n)\times L^m(\mathbb{R}^n)$ *with* $m\in[1,2]$. *Then, we have the following refinement estimates:*

$$\begin{aligned}&\left\|\chi_{\rm int}(D)\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}-T_0T_1T_{1\frac{1}{2}}\tilde{w}\right)(t,\cdot)\right\|_{\dot{H}^s(\mathbb{R}^n)}\\&\lesssim(1+t)^{-\frac{(2-m)n+2ms}{4m(2-\sigma)}-\frac{1-\sigma}{2-\sigma}}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{L^m(\mathbb{R}^n)\times L^m(\mathbb{R}^n)\times L^m(\mathbb{R}^n)},\end{aligned}$$

where $T_0,T_1,T_{1\frac{1}{2}}$ *are defined in Lemma 2.1.*

Proof. According to the representations of solutions for small frequencies in Theorem 2.1 and the definition of the matrices in (2.6), we may obtain

$$\chi_{\rm int}(\xi)|\xi|^s\left(w^{(0)}-T_0T_1T_{1\frac{1}{2}}\tilde{w}\right)(t,\xi)=\chi_{\rm int}(\xi)|\xi|^s\left(J_1(t,|\xi|)+J_2(t,|\xi|)+J_3(t,|\xi|)\right),$$

where

$$\begin{aligned}&J_0(t,|\xi|)={\rm diag}\left(e^{-\lambda_1(|\xi|)t}-e^{-|\xi|^{4-2\sigma}t},e^{-\lambda_2(|\xi|)t}-e^{-|\xi|^2t},e^{-\lambda_3(|\xi|)t}-e^{-|\xi|^{2\sigma}t}\right),\\&J_1(t,|\xi|)=T_0T_1T_{1\frac{1}{2}}J_0(t,|\xi|)T^{-1}_{1\frac{1}{2}}T_1^{-1}T_0^{-1}w_0^{(0)}(\xi),\\&J_2(t,|\xi|)=T_0T_1T_{1\frac{1}{2}}N_2(|\xi|)\,{\rm diag}\left(e^{-\lambda_1(|\xi|)t},e^{-\lambda_2(|\xi|)t},e^{-\lambda_3(|\xi|)t}\right)T_2^{-1}T_1^{-1}T_0^{-1}w_0^{(0)}(\xi),\\&J_3(t,|\xi|)=-T_0T_1T_{1\frac{1}{2}}T_2\,{\rm diag}\left(e^{-\lambda_1(|\xi|)t},e^{-\lambda_2(|\xi|)t},e^{-\lambda_3(|\xi|)t}\right)T_2^{-1}N_2(|\xi|)T_1^{-1}T_0^{-1}w_0^{(0)}(\xi),\end{aligned}$$

with $N_2(|\xi|)=\mathcal{O}\left(|\xi|^{2-2\sigma}\right)$ for $\sigma\in[0,1)$.

Let us define

$$\begin{aligned}&h_1(|\xi|)=|\xi|^{4-2\sigma}, && h_2(|\xi|)=|\xi|^2, && h_3(|\xi|)=|\xi|^{2\sigma},\\&g_1(|\xi|)=\lambda_1(|\xi|)-h_1(|\xi|), && g_2(|\xi|)=\lambda_2(|\xi|)-h_2(|\xi|), && g_3(|\xi|)=\lambda_3(|\xi|)-h_3(|\xi|).\end{aligned}$$

Applying the following formula for $j=1,2,3$:

$$e^{-h_j(|\xi|)t-g_j(|\xi|)t}-e^{-h_j(|\xi|)t}=-g_j(|\xi|)te^{-h_j(|\xi|)t}\int_0^1e^{-g_j(|\xi|)t\tau}d\tau,$$

we can get

$$\begin{aligned}&\left\|\chi_{\rm int}(\xi)|\xi|^s\left(w^{(0)}-T_0T_1T_{1\frac{1}{2}}\tilde{w}\right)(t,\xi)\right\|_{L^2(\mathbb{R}^n)}\\&=\left\|\chi_{\rm int}(\xi)|\xi|^s\left(J_1(t,|\xi|)+J_2(t,|\xi|)+J_3(t,|\xi|)\right)\right\|_{L^2(\mathbb{R}^n)}\\&\lesssim\left\|\chi_{\rm int}(\xi)|\xi|^{s+2-2\sigma}e^{-|\xi|^{4-2\sigma}t}\right\|_{L^{\frac{2m}{2-m}}(\mathbb{R}^n)}\left\|\mathcal{F}^{-1}\left(w_0^{(0)}\right)\right\|_{L^m(\mathbb{R}^n)}\\&\lesssim(1+t)^{-\gamma(\sigma,n,m,s)-\frac{1-\sigma}{2-\sigma}}\left\|\mathcal{F}^{-1}\left(w_0^{(0)}\right)\right\|_{L^m(\mathbb{R}^n)}.\end{aligned}$$

Thus, the proof is complete. ∎

THEOREM 5.2. *Let us consider the Cauchy problem* (2.2) *with* $\sigma\in[0,1)$. *We assume* $\left(|D|^2u_0,u_1,\theta_0\right)\in\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)$. *Then, we have the following refinement estimates:*

$$\left\|\chi_{\rm int}(D)\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}-T_0T_1T_{1\frac{1}{2}}\tilde{w}\right)(t,\cdot)\right\|_{\dot{H}^s_q(\mathbb{R}^n)}$$

$$\lesssim (1+t)^{-\frac{s}{4-2\sigma}-\frac{n}{4-2\sigma}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1-\sigma}{2-\sigma}}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n)},$$

where $T_0,T_1,T_{1\frac{1}{2}}$ *are defined in Lemma 2.1 with* $s\geq 0$, $1\leq p\leq 2\leq q\leq\infty$.

Proof. We may prove this result immediately by using Lemma 4.2. ∎

Remark 5.4. From the statements of Theorems 5.1 and 5.2, we know when $\sigma\in[0,1)$ the thermal dissipation generated by Fourier's law and friction or structural damping have an influence on the reference system at the same time. However, the friction and structural damping have a dominant influence on the decay rate of the estimates.

5.2. Diffusion phenomena for the model with $\sigma\in(1,2]$. Now, we describe diffusion phenomena of the solutions to the Cauchy problem (2.2) with $\sigma\in(1,2]$ by the reference system as follows:

$$\begin{cases}\tilde{u}_t-\text{diag}(y_1,y_2,y_3)\Delta\tilde{u}=0, & t>0,\, x\in\mathbb{R}^n,\\ \tilde{u}(0,x)=\mathcal{F}^{-1}\left(T_0^{-1}w_0^{(0)}(\xi)\right)(x), & x\in\mathbb{R}^n,\end{cases}\tag{5.4}$$

where $y_1,y_2,y_3\in\mathbb{C}$ are determined in (2.7) and T_0 are defined in Lemma 2.2. Applying the partial Fourier transform $\tilde{w}(t,\xi)=\mathcal{F}_{x\to\xi}(\tilde{u}(t,x))$ implies

$$\begin{cases}\tilde{w}_t+\text{diag}(y_1,y_2,y_3)|\xi|^2\tilde{w}=0, & t>0,\, \xi\in\mathbb{R}^n,\\ \tilde{w}(0,\xi)=T_0^{-1}w_0^{(0)}(\xi), & \xi\in\mathbb{R}^n.\end{cases}\tag{5.5}$$

The solution to (5.5) is explicitly given by

$$\tilde{w}(t,\xi)=\text{diag}\left(e^{-y_1|\xi|^2t},e^{-y_2|\xi|^2t},e^{-y_3|\xi|^2t}\right)T_0^{-1}w_0^{(0)}(\xi).\tag{5.6}$$

Remark 5.5. From (5.4), the reference system consists of evolution equations

$$\tilde{u}_t^{(j)}-\text{Re }y_j\Delta\tilde{u}^{(j)}-i\text{Im }y_j\Delta\tilde{u}^{(j)}=0,$$

for $j=1,2,3$. Here we interpret this effect as a classical diffusion phenomenon. Moreover, we have to point out that the reference system (5.4) consists of heat systems and Schrödinger systems due to the fact that Re $y_j>0$ and Im $y_j\neq 0$ for all $j=1,2,3$.

Remark 5.6. If one considers the reference system as the following heat system only:

$$\tilde{u}_t-\text{diag}(\text{Re }y_1,\text{Re }y_2,\text{Re }y_3)\Delta\tilde{u}=0,\tag{5.7}$$

or the following Schrödinger system only:

$$\tilde{u}_t-i\,\text{diag}(\text{Im }y_1,\text{Im }y_2,\text{Im }y_3)\Delta\tilde{u}=0,\tag{5.8}$$

we cannot observe any diffusion structure for $\sigma\in(1,2]$. In other words, comparing with Theorems 5.1 and 5.2, respectively, we observe that there is no improvement in the decay estimates for the difference between the solutions to the system (2.2) with $\sigma\in(1,2]$ and the solutions to the reference systems (5.7) or (5.8).

Similar as in the last subsection, one can prove the following results.

Theorem 5.3. *Let us consider the Cauchy problem (2.2) with $\sigma\in(1,2]$. We assume* $\left(|D|^2u_0,u_1,\theta_0\right)\in L^m(\mathbb{R}^n)\times L^m(\mathbb{R}^n)\times L^m(\mathbb{R}^n)$ *with* $m\in[1,2]$. *Then, we have the following refinement estimates:*

$$\left\|\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}-T_0\tilde{w}\right)(t,\cdot)\right\|_{\dot{H}^s(\mathbb{R}^n)}$$

$$\lesssim (1+t)^{-\frac{(2-m)n+2ms}{4m}-(\sigma-1)} \left\| \left(|D|^2 u_0, u_1, \theta_0\right) \right\|_{L^m(\mathbb{R}^n)\times L^m(\mathbb{R}^n)\times L^m(\mathbb{R}^n)},$$

where T_0 *are defined in Lemma 2.2.*

THEOREM 5.4. *Let us consider the Cauchy problem (2.2) with* $\sigma\in(1,2]$. *We assume* $\left(|D|^2 u_0, u_1, \theta_0\right)\in\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)$. *Then, we have the next refinement estimates:*

$$\begin{aligned}&\left\| \chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}-T_0\tilde{w}\right)(t,\cdot)\right\|_{\dot{H}^s_q(\mathbb{R}^n)}\\ &\lesssim (1+t)^{-\frac{s}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-(\sigma-1)} \left\| \left(|D|^2 u_0, u_1, \theta_0\right) \right\|_{L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n)\times L^p(\mathbb{R}^n)},\end{aligned}$$

where T_0 *are defined in Lemma 2.2 with* $s\geq 0$, $1\leq p\leq 2\leq q\leq\infty$.

REMARK 5.7. From Theorems 5.1, 5.2, 5.3 and 5.4, the diffusion structure appears for the Cauchy problem (1.2) with $\sigma\in[0,1)\cup(1,2]$. More precisely, comparing Theorems 4.2 and 4.4 with Theorems 5.1 and 5.2, respectively, we observe that the decay rate can be improved by $-\frac{1-\sigma}{2-\sigma}$ if $\sigma\in[0,1)$ as $t\to\infty$. In addition, comparing Theorems 4.2 and 4.4 with Theorems 5.3 and 5.4, respectively, we observe that the decay rate can be improved by $-(\sigma-1)$ if $\sigma\in(1,2]$ as $t\to\infty$.

REMARK 5.8. According to Theorems 5.3 and 5.4, the thermal dissipation generated by Fourier's law has a dominant influence on diffusion phenomena in comparison with structural damping when $\sigma\in(1,2]$.

6. Asymptotic profiles of solutions

Our main purpose in this section is to give asymptotic profiles of solutions to the Cauchy problem (1.2) in a framework of the weighted L^1 data. The idea is motivated by [14, 15].

In Section 4 we derived the following estimates for upper bounds of solutions with weighted L^1 data:

$$\|U(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} \lesssim (1+t)^{-\gamma(\sigma,n,1,s+1)}\|U_0\|_{H^s(\mathbb{R}^n)\cap L^{1,1}(\mathbb{R}^n)} + (1+t)^{-\gamma(\sigma,n,1,s)}\left|\int_{\mathbb{R}^n} U_0(x)dx\right|,$$

where $\sigma\in[0,2]$, $n\geq 1$, $s\geq 0$. Here the solution $U(t,x)$ and data $U_0(x)$ are defined in (2.3) and (2.4), respectively.

The natural questions are as follows. What is the estimate for the lower bounds of $\|U(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)}$ in a framework of weighted L^1 data? Is this estimate sharp? To answer these questions, we show some useful lemmas initially. Here Lemmas 6.1 and 6.2 have been proved in the papers [11, 13].

LEMMA 6.1. *Let* $f\in L^1(\mathbb{R}^n)$. *Then, we can expand* $\hat{f}(\xi)$ *by*

$$\hat{f}(\xi)=A_f(\xi)-iB_f(\xi)+P_f \textit{ for all } \xi\in\mathbb{R}^n,$$

where

$$\begin{aligned}A_f(\xi)&:=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}(\cos(x\cdot\xi)-1)f(x)dx,\\ B_f(\xi)&:=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\sin(x\cdot\xi)f(x)dx,\\ P_f&:=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}f(x)dx.\end{aligned}$$

LEMMA 6.2. Let us consider $A_f(\xi)$ and $B_f(\xi)$ defined in Lemma 6.1. Then, we have the following estimates for them:

$$|A_f(\xi)|\lesssim|\xi|\|f\|_{L^{1,1}(\mathbb{R}^n)},$$
$$|B_f(\xi)|\lesssim|\xi|\|f\|_{L^{1,1}(\mathbb{R}^n)}.$$

LEMMA 6.3. *Let us consider $s\geq 0$ and $\alpha_2>0$. Then, the following estimate holds for large-time $t\gg 1$:*

$$\left\|\chi_{\text{int}}(\xi)|\xi|^s e^{-c|\xi|^{\alpha_2}t}\right\|_{L^2(\mathbb{R}^n)}\gtrsim t^{-\frac{2s+n}{2\alpha_2}},$$

where the constant $c>0$.

Proof. By direct calculation, we obtain

$$\begin{aligned}\left\|\chi_{\text{int}}(\xi)|\xi|^s e^{-c|\xi|^{\alpha_2}t}\right\|^2_{L^2(\mathbb{R}^n)}&=\int_{\mathbb{R}^n}\chi^2_{\text{int}}(\xi)|\xi|^{2s}e^{-2c|\xi|^{\alpha_2}t}d\xi\\&=\int_0^{\varepsilon}\int_{|\xi|=r}r^{2s}e^{-2cr^{\alpha_2}t}dS_\xi dr\\&=\omega_n\int_0^{\varepsilon}r^{2s+n-1}e^{-2cr^{\alpha_2}t}dr,\end{aligned}$$

where $\omega_n=\int_{|\omega|=1}d\omega=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. By using the ansatz $r^{\alpha_2}t=\tau$, we complete the proof of the lemma. □

THEOREM 6.1. *Let us assume $U_0\in H^s(\mathbb{R}^n)\cap L^{1,1}(\mathbb{R}^n)$ with $|P_{U_0}|\neq 0$, where $s\geq 0$. Then, the solution $U=U(t,x)$ to the Cauchy problem (1.2) with $\sigma\in[0,1)$ satisfies the following estimates for $t\gg 1$:*

$$t^{-\frac{n+2s}{4(2-\sigma)}}|P_{U_0}|\lesssim\|U(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)}\lesssim t^{-\frac{n+2s}{4(2-\sigma)}}\|U_0\|_{H^s(\mathbb{R}^n)\cap L^{1,1}(\mathbb{R}^n)}.$$

Proof. To begin with, let us define

$$J_4(t,|\xi|):=T_0T_1T_{1\frac{1}{2}}\operatorname{diag}\left(e^{-|\xi|^{4-2\sigma}t},e^{-|\xi|^2t},e^{-|\xi|^{2\sigma}t}\right)T^{-1}_{1\frac{1}{2}}T_1^{-1}T_0^{-1}.$$

We use Lemmas 6.1, 6.2 and Theorem 5.1 to get

$$\begin{aligned}&\left\|\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}\right)-\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}(J_4(t,|\xi|))P_{U_0}\right\|_{\dot{H}^s(\mathbb{R}^n)}\\&=\Big\|\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}-T_0T_1T_{1\frac{1}{2}}\tilde{w}\right)\\&\qquad+\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(J_4(t,|\xi|)\left(A_{U_0}(\xi)-iB_{U_0}(\xi)\right)\right)\Big\|_{\dot{H}^s(\mathbb{R}^n)}\\&\lesssim(1+t)^{-\gamma(\sigma,n,1,s)-\frac{1-\sigma}{2-\sigma}}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)}\\&\qquad+\left\|\chi_{\text{int}}(\xi)|\xi|^{s+1}e^{-|\xi|^{4-2\sigma}t}\right\|_{L^2(\mathbb{R}^n)}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{L^{1,1}(\mathbb{R}^n)\times L^{1,1}(\mathbb{R}^n)\times L^{1,1}(\mathbb{R}^n)}\\&\lesssim(1+t)^{-\gamma(\sigma,n,1,s)-\frac{1-\sigma}{2-\sigma}}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)}\\&\qquad+(1+t)^{-\gamma(\sigma,n,1,s+1)}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{L^{1,1}(\mathbb{R}^n)\times L^{1,1}(\mathbb{R}^n)\times L^{1,1}(\mathbb{R}^n)}.\end{aligned}$$

To get the lower bounds estimates, we apply the Minkowski inequality to obtain

$$\begin{aligned}
&\left\|\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}\right)\right\|_{\dot{H}^s(\mathbb{R}^n)}\\
&\geq\left\|\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(J_4(t,|\xi|)\right)P_{U_0}\right\|_{\dot{H}^s(\mathbb{R}^n)}\\
&\qquad-\left\|\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}\right)-\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(J_4(t,|\xi|)\right)P_{U_0}\right\|_{\dot{H}^s(\mathbb{R}^n)}\\
&\gtrsim\left\|\chi_{\text{int}}(\xi)|\xi|^s\left(e^{-|\xi|^{4-2\sigma}t}+e^{-|\xi|^2t}+e^{-|\xi|^{2\sigma}t}\right)\right\|_{L^2(\mathbb{R}^n)}|P_{U_0}|\\
&\qquad-(1+t)^{-\gamma(\sigma,n,1,s)-\frac{1-\sigma}{2-\sigma}}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)}\\
&\qquad-(1+t)^{-\gamma(\sigma,n,1,s+1)}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{L^{1,1}(\mathbb{R}^n)\times L^{1,1}(\mathbb{R}^n)\times L^{1,1}(\mathbb{R}^n)}.
\end{aligned}$$

In conclusion, for $t\gg1$ the following estimate holds:

$$\left\|\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}\right)\right\|_{\dot{H}^s(\mathbb{R}^n)}\gtrsim t^{-\gamma(\sigma,n,1,s)}|P_{U_0}|.$$

Combining with the upper bounds estimate for $t\gg1$ such that

$$\left\|\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}\right)\right\|_{\dot{H}^s(\mathbb{R}^n)}\lesssim t^{-\gamma(\sigma,n,1,s)}\|U_0\|_{H^s(\mathbb{R}^n)\cap L^{1,1}(\mathbb{R}^n)},$$

and

$$\left\|\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}\right)\right\|_{\dot{H}^s(\mathbb{R}^n)}\lesssim\left\|\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}\right)\right\|_{\dot{H}^s(\mathbb{R}^n)},$$

we complete the proof. □

THEOREM 6.2. *Let us assume $U_0\in H^s(\mathbb{R}^n)\cap L^{1,1}(\mathbb{R}^n)$ with $|P_{U_0}|\neq0$, where $s\geq0$. Then, the solution $U=U(t,x)$ to the Cauchy problem (1.2) with $\sigma\in(1,2]$ satisfies the following estimates for $t\gg1$:*

$$t^{-\frac{n+2s}{4}}|P_{U_0}|\lesssim\|U(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)}\lesssim t^{-\frac{n+2s}{4}}\|U_0\|_{H^s(\mathbb{R}^n)\cap L^{1,1}(\mathbb{R}^n)}.$$

Proof. Following the proof of Theorem 6.1 one can complete this proof. □

Finally, we derive asymptotic profiles of solutions to the Cauchy problem (1.2) for the case $\sigma=1$.

THEOREM 6.3. *Let us assume $U_0\in H^s(\mathbb{R}^n)\cap L^{1,1}(\mathbb{R}^n)$ with $|P_{U_0}|\neq0$, where $s\geq0$. Then, the solution $U=U(t,x)$ to the Cauchy problem (1.2) with $\sigma=1$ satisfies the following estimates for $t\gg1$:*

$$t^{-\frac{n+2s}{4}}|P_{U_0}|\lesssim\|U(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)}\lesssim t^{-\frac{n+2s}{4}}\|U_0\|_{H^s(\mathbb{R}^n)\cap L^{1,1}(\mathbb{R}^n)}.$$

Proof. Let us define

$$J_5(t,|\xi|):=T_{1,0}\,\text{diag}\left(e^{-y_4|\xi|^2t},e^{-y_5|\xi|^2t},e^{-y_6|\xi|^2t}\right)T^{-1}_{1,0}.$$

From Theorem 2.3, we know

$$\left\|\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}\right)-\chi_{\text{int}}(D)\mathcal{F}^{-1}_{\xi\to x}\left(J_5(t,|\xi|)\right)P_{U_0}\right\|_{\dot{H}^s(\mathbb{R}^n)}$$

$$= \left\| \chi_{\text{int}}(D) \mathcal{F}^{-1}_{\xi \to x} (J_5(t,|\xi|)(A_{U_0}(\xi) - iB_{U_0}(\xi))) \right\|_{\dot{H}^s(\mathbb{R}^n)}$$
$$\lesssim (1+t)^{-\frac{n+2s}{4} - \frac{1}{2}} \|U_0\|_{L^{1,1}(\mathbb{R}^n)},$$

where $T_{0,1}$, y_4, y_5, y_6 are defined in Lemma 2.3.
Then, repeating the procedure of the proof of Theorem 6.1 we derive for $t \gg 1$

$$t^{-\frac{n+2s}{4}} |P_{U_0}| \lesssim \left\| \chi_{\text{int}}(D) \mathcal{F}^{-1}_{\xi \to x} \left(w^{(0)} \right) \right\|_{\dot{H}^s(\mathbb{R}^n)} \lesssim \left\| \mathcal{F}^{-1}_{\xi \to x} \left(w^{(0)} \right) \right\|_{\dot{H}^s(\mathbb{R}^n)},$$

and the proof of Theorem 6.3 is complete. □

Remark 6.1. According to Theorems 6.1, 6.2 and 6.3, the thermal dissipation generated by Fourier's law has a dominant influence on long-time asymptotic profiles of solutions in comparison with structural damping when $\sigma \in [1,2]$.

7. Concluding remarks

Remark 7.1. In general, our method to derive sharp asymptotic profiles of solutions in the framework of $L^{1,1}$ can be probably applied to the Cauchy problem for other systems in elastic materials including elastic waves with different damping mechanisms, thermoelastic systems, thermodiffusion systems.

In detail, for elastic waves with friction or structural damping [3, 29], elastic waves with Kelvin-Voigt damping [2], thermoelastic systems [17,31,36,40,41] and thermodiffusion systems [24], the authors applied diagonalization procedures or asymptotic expansions of eigenvalues/eigenprojections to derive representations of solutions. By these representations of solutions, one may obtain diffusion phenomena with weighted L^1 data. Then, one can follow the method in Section 6 to derive the sharp estimates for lower bounds and upper bounds of solutions in a framework of weighted L^1 data.

7.1. Summary.

In the following we will collect results for the Cauchy problem for thermoelastic plate equations with friction or structural damping (1.2).

In the paper we first derive Gevrey smoothing of solutions (see Table 7.1) and L^2 well-posedness for the Cauchy problem (1.2) such that

$$U \in \mathcal{C}\big([0,\infty), L^2(\mathbb{R}^n)\big) \text{ if we assume } U_0 \in L^2(\mathbb{R}^n).$$

Next, we obtain several decay estimates of solutions. On one hand, we derive the following energy estimates:

$$\|U(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{(2-m)n+2ms}{2mK}} \|U_0\|_{H^s(\mathbb{R}^n)\cap L^m(\mathbb{R}^n)},$$

where $s \geq 0$, $m \in [1,2]$, and

$$\|U(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{n+2(s+\delta)}{2K}} \|U_0\|_{H^s(\mathbb{R}^n)\cap L^{1,\delta}(\mathbb{R}^n)} + (1+t)^{-\frac{n+2s}{2K}} |P_{U_0}|,$$

where $s \geq 0$, $\delta \in (0,1]$. Here some numbers K are specified in the table below. On the other hand, there are $L^p - L^q$ estimates not necessary on the conjugate line of the form

$$\|U(t,\cdot)\|_{\dot{H}^s_q(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{s}{K} - \frac{n}{K}\left(\frac{1}{p} - \frac{1}{q}\right)} \|U_0\|,$$

for suitable p,q and some numbers K (specified in the table below). Here $\|U_0\|$ corresponds to initial data measured in an appropriate norm, which is based on L^p.

	$\sigma=0$	$\sigma\in(0,1)$	$\sigma=1$	$\sigma\in(1,3/2]$	$\sigma\in(3/2,2)$	$\sigma=2$
Gevrey smoothing	$\Gamma^1(\mathbb{R}^n)$ (analytic smoothing)				$\Gamma^{\frac{1}{4-2\sigma}}(\mathbb{R}^n)$	–
Energy estimates	$K=4-2\sigma$		$K=2$			
L^p-L^q estimates	$K=4-2\sigma$ and $1\leq p\leq 2\leq q\leq\infty$		$K=2$ and $1\leq p\leq q\leq\infty$	$K=2$ and $1\leq p\leq 2\leq q\leq\infty$		
Diffusion phenomena (dif. phe.)	double dif. phe.	triple dif. phe.	–	single dif. phe.		
Asymptotic profiles	$K=4-2\sigma$		$K=2$			

TABLE 7.1. *Summary for qualitative properties of solutions*

Finally, we derive diffusion phenomena with data taken from different function spaces (see Table 7.1), and asymptotic profiles of solutions with weighted L^1 data

$$t^{-\frac{n+2s}{2K}}|P_{U_0}|\lesssim\|U(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)}\lesssim t^{-\frac{n+2s}{2K}}\|U_0\|_{H^s(\mathbb{R}^n)\cap L^{1,1}(\mathbb{R}^n)}$$

for $t\gg1$, where $|P_{U_0}|\neq0$, $s\geq0$ and some numbers K are chosen in Table 7.1. We should point out that when $K=4$, friction has a dominant influence in the corresponding decay estimates; when $K=4-2\sigma$, structural damping has a dominant influence in the corresponding decay estimates; when $K=2$, thermal dissipation generated by Fourier's law has a dominant influence in the corresponding decay estimates.

7.2. Estimates for the solution itself. Throughout this paper, we apply diagonalization procedure to get the representations of solutions

$$U(t,x)=\left(u_t+|D|^2u,u_t-|D|^2u,\theta\right)^{\mathrm{T}}(t,x) \tag{7.1}$$

and study some qualitative properties of solutions to the Cauchy problem (1.2).

Nevertheless, up to now, concerning the qualitative properties of the solution $u=u(t,x)$ to the Cauchy problem (1.2), we did not derive any estimate for the solution itself. In this section, we will show some strategies to derive estimates for the solution itself. We propose three different strategies.

Strategy 1. Estimates of u by using the Riesz potential theory.

We formally define the Riesz potential in $\mathbb{R}^n$ by its action on a measurable function $f=f(x)$ by convolution, that is

$$\left(\tilde{I}_{2\kappa}f\right)(x)\equiv\left(\tilde{I}_{2\kappa}*f\right)(x):=\mathcal{F}^{-1}\left(|\xi|^{-2\kappa}\hat{f}(\xi)\right)(x)\equiv C_{n,\kappa}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2\kappa}}dy,$$

where $\kappa\in\left(0,\frac{n}{2}\right)$.

The study of the following mapping properties to $\tilde{I}_{2\kappa}$ was initiated by [34].

LEMMA 7.1. *Let us assume $f\in L^p(\mathbb{R}^n)$ for $p\in\left(1,\frac{n}{2\kappa}\right)$. Then, $\tilde{I}_{2\kappa}f\in L^{p^*}(\mathbb{R}^n)$, where*

$$\left\|\tilde{I}_{2\kappa}f\right\|_{L^{p^*}(\mathbb{R}^n)}\lesssim\|f\|_{L^p(\mathbb{R}^n)}\ \text{with}\ \frac{1}{p}-\frac{1}{p^*}=\frac{2\kappa}{n}.$$

Then, we may estimate the solution itself by

$$\begin{aligned}\|u(t,\cdot)\|_{L^q(\mathbb{R}^n)} &\lesssim \left\|\mathcal{F}^{-1}_{\xi\to x}\left(|\xi|^{-2}w^{(0)}\right)(t,\cdot)\right\|_{L^q(\mathbb{R}^n)} = \left\|\tilde{I}_2\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}\right)(t,\cdot)\right\|_{L^q(\mathbb{R}^n)}\\ &\lesssim \left\|\mathcal{F}^{-1}_{\xi\to x}\left(w^{(0)}\right)(t,\cdot)\right\|_{L^{\frac{nq}{2q+n}}(\mathbb{R}^n)},\end{aligned}$$

where $0<\frac{1}{q}<1-\frac{2}{n}$. Next, following a similar procedure of Section 4, one may complete estimates of the solution itself.

Strategy 2. Estimates of u by using the integral formula.

For the case $\frac{1}{q}\in[0,1]\setminus\left(0,1-\frac{2}{n}\right)$, we cannot apply *Strategy 1*. Therefore, by the integral formula

$$u(t,x)-u(0,x)=\int_0^t u_\tau(\tau,x)d\tau,$$

we obtain

$$\|u(t,\cdot)\|_{L^q(\mathbb{R}^n)}\lesssim\|u_0\|_{L^q(\mathbb{R}^n)}+\int_0^t\|u_\tau(\tau,\cdot)\|_{L^q(\mathbb{R}^n)}d\tau.$$

Next, we apply the estimates of $u_\tau(\tau,\cdot)$ in the L^q norm to complete estimates of the solution itself. We should remark that to apply this strategy, we need to take an additional assumption on the first data such that $u_0\in L^q(\mathbb{R}^n)$.

Strategy 3. Estimates of u by using the representation of the solution.

By some direct calculations, we may transfer the Cauchy problem (1.2) to the following Cauchy problem for third-order equation:

$$\begin{cases}u_{ttt}+(-\Delta)^\sigma u_{tt}-\Delta u_{tt}+2\Delta^2u_t+(-\Delta)^{\sigma+1}u_t-\Delta^3u=0, & t>0,\ x\in\mathbb{R}^n,\\ (u,u_t,u_{tt})(0,x)=(u_0,u_1,u_2)(x), & x\in\mathbb{R}^n,\end{cases}\tag{7.2}$$

where $\sigma\in[0,2]$ and

$$u_2(x):=-\Delta^2u_0(x)-(-\Delta)^\sigma u_1(x)-\Delta\theta_0(x).$$

Applying the partial Fourier transformation with respect to spatial variables to (7.2), we obtain an ordinary differential equation depending on the parameter $|\xi|$

$$\begin{cases}\hat{u}_{ttt}+\left(|\xi|^{2\sigma}+|\xi|^2\right)\hat{u}_{tt}+\left(2|\xi|^4+|\xi|^{2\sigma+2}\right)\hat{u}_t+|\xi|^6\hat{u}=0, & t>0,\ \xi\in\mathbb{R}^n,\\ (\hat{u},\hat{u}_t,\hat{u}_{tt})(0,\xi)=(\hat{u}_0,\hat{u}_1,\hat{u}_2)(\xi), & \xi\in\mathbb{R}^n.\end{cases}\tag{7.3}$$

The characteristic roots $\lambda_j=\lambda_j(|\xi|)$, $j=1,2,3$, for the equation of (7.3) satisfy the parameter-dependent cubic equation

$$\lambda^3+\left(|\xi|^{2\sigma}+|\xi|^2\right)\lambda^2+\left(2|\xi|^4+|\xi|^{2\sigma+2}\right)\lambda+|\xi|^6=0.\tag{7.4}$$

We may find the exact solution of (7.4) as follows:

$$\lambda_j=r_{3,j}-\frac{r_1}{3r_{3,j}}-\frac{|\xi|^{2\sigma}+|\xi|^2}{3},\tag{7.5}$$

where

$$\begin{aligned} r_1 &= \frac{1}{3}\left(5|\xi|^4+|\xi|^{2\sigma+2}-|\xi|^{4\sigma}\right),\\ r_2 &= \frac{1}{27}\left(11|\xi|^6+2|\xi|^{6\sigma}-3|\xi|^{4\sigma+2}-2|\xi|^{2\sigma+4}\right),\\ r_3^3 &= \frac{1}{2}\left(-r_2\pm\sqrt{r_2^2+\frac{4}{27}r_1^3}\right), \text{ where } r_{3,1},r_{3,2},r_{3,3} \text{ are its complex solutions.} \end{aligned}$$

It provides an opportunity for us to derive explicit representations of solutions to (7.3) such that

$$\hat{u}(t,\xi)=c_1(\xi)e^{\lambda_1(|\xi|)t}+c_2(\xi)e^{\lambda_2(|\xi|)t}+c_3(\xi)e^{\lambda_3(|\xi|)t},$$

where $\lambda_j(|\xi|)$ are given by (7.5) and the coefficients $c_j(\xi)$ are given by

$$\begin{aligned} c_1(\xi) &= \frac{\lambda_2(|\xi|)\lambda_3(|\xi|)\hat{u}_0(\xi)-\lambda_2(|\xi|)\hat{u}_1(\xi)-\lambda_3(|\xi|)\hat{u}_1(\xi)-|\xi|^4\hat{u}_0(\xi)-|\xi|^{2\sigma}\hat{u}_1(\xi)+|\xi|^2\hat{\theta}_0(\xi)}{\lambda_1^2(|\xi|)-\lambda_1(|\xi|)\lambda_2(|\xi|)-\lambda_1(|\xi|)\lambda_3(|\xi|)+\lambda_2(|\xi|)\lambda_3(|\xi|)},\\ c_2(\xi) &= \frac{\lambda_1(|\xi|)\lambda_3(|\xi|)\hat{u}_0(\xi)-\lambda_1(|\xi|)\hat{u}_1(\xi)-\lambda_3(|\xi|)\hat{u}_1(\xi)-|\xi|^4\hat{u}_0(\xi)-|\xi|^{2\sigma}\hat{u}_1(\xi)+|\xi|^2\hat{\theta}_0(\xi)}{\lambda_2^2(|\xi|)-\lambda_1(|\xi|)\lambda_2(|\xi|)+\lambda_1(|\xi|)\lambda_3(|\xi|)-\lambda_2(|\xi|)\lambda_3(|\xi|)},\\ c_3(\xi) &= \frac{\lambda_1(|\xi|)\lambda_2(|\xi|)\hat{u}_0(\xi)-\lambda_1(|\xi|)\hat{u}_1(\xi)-\lambda_2(|\xi|)\hat{u}_1(\xi)-|\xi|^4\hat{u}_0(\xi)-|\xi|^{2\sigma}\hat{u}_1(\xi)+|\xi|^2\hat{\theta}_0(\xi)}{\lambda_3^2(|\xi|)+\lambda_1(|\xi|)\lambda_2(|\xi|)-\lambda_1(|\xi|)\lambda_3(|\xi|)-\lambda_2(|\xi|)\lambda_3(|\xi|)}. \end{aligned}$$

It is possible to derive estimates for $u=u(t,x)$ by applying these representations.

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