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QUALITATIVE ANALYSIS OF AN INTEGRATED CHEMOTAXIS-FLUID MODEL: GLOBAL EXISTENCE AND EXTENSIBILITY CRITERION*

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Abstract. In this paper, we study the global existence and extensibility criterion of large-amplitude solutions to a chemotaxis-fluid model in bounded domains. The model under consideration is an integrated version of several recently studied models in bio-fluids, which is a coupled system of partial differential equations with strong nonlinearities. The results obtained in this paper appear to be among the first ones for the integrated model.

Keywords. Chemotaxis; Navier-Stokes equations; global existence; extensibility criterion.

AMS subject classifications. 35D30; 35B40; 35Q30.

1. Introduction

As one of the contemporary research topics on the crossroads of mathematical biology and fluid dynamics, the investigation of the mutual influences of chemotaxis and hydrodynamics has been actively conducted for more than a decade. Among the first generation of mathematical models describing chemotaxis-fluid interactions, the Keller-Segel-Navier-Stokes system proposed by Tuval *et al.* [17] has been widely recognized by researchers in the area, its success being a consequence of its capability to describe some of the fundamental phenomena in biofluids, such as the formation of plumes [7]. In the simplest form, the original Keller-Segel-Navier-Stokes system reads:

$$\begin{cases} \partial_t \mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\pi=\Delta\mathbf{u}-n\nabla\phi, \\ \nabla\cdot\mathbf{u}=0, \\ \partial_t n+\mathbf{u}\cdot\nabla n=\Delta n-\nabla\cdot(n\chi(p)\nabla p), \\ \partial_t p+\mathbf{u}\cdot\nabla p=\Delta p-nf(p), \end{cases} \tag{1.1}$$

$\mathbf{x}\in\mathbb{R}^d$, $t>0$, where the unknown functions $\mathbf{u},\pi,n,p$ denote the fluid velocity field, scalar pressure, bacteria density, and oxygen concentration, respectively. The functions $\chi(p)$ and $f(p)$ are given smooth functions of p describing the so-called chemotactic sensitivity and oxygen consumption rate, respectively. In the derivation of the model, the Boussinesq approximation was applied to reflect the effect due to heavy bacteria, where the function ϕ denotes the potential function produced by various physical mechanisms, e.g., gravitational force ($\phi(\mathbf{x})=x_d$) or centrifugal force ($\phi(\mathbf{x})=g(|\mathbf{x}|)$).

Since its initiation in the early 2000s, the Keller-Segel-Navier-Stokes system (1.1) has been frequently utilized in the scientific computations of chemotaxis-fluid interactions. In the pioneering work [7], Chertock-Fellner-Kurganov-Lorz-Markowich successfully produced the formation of plumes in the numerical simulations of the model,

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which is consistent with the experimental results reported in [17], and inspired much of the later works dealing with the rigorous analysis of the model. We refer the readers to [5, 6, 9–11, 14, 15] and the references therein for a series of studies of the qualitative behavior of solutions to (1.1), such as existence, uniqueness, regularity, extensibility criteria and long-time behavior, under various initial and/or boundary conditions.

On the other hand, depending on specific biological / physical considerations, a family of variants of the model (1.1) has been proposed to account for particular mechanisms in chemotaxis-fluid interactions. Of relevance to this work, we would like to point out the following generalized versions of (1.1):

- The model with *porous medium-like diffusion* [8]:

$$\begin{cases}\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\pi=\Delta\mathbf{u}-n\nabla\phi,\\ \nabla\cdot\mathbf{u}=0,\\ \partial_t n+\mathbf{u}\cdot\nabla n=\Delta n^m-\nabla\cdot(n\chi(p)\nabla p),\\ \partial_t p+\mathbf{u}\cdot\nabla p=\Delta p-nf(p),\end{cases}\tag{1.2}$$

  accounting for the finite size of bacteria.

- The *self-consistent* model with porous medium-like diffusion [8]:

$$\begin{cases}\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\pi=\Delta\mathbf{u}-n\nabla\phi+n\chi(p)\nabla p,\\ \nabla\cdot\mathbf{u}=0,\\ \partial_t n+\mathbf{u}\cdot\nabla n=\Delta n^m-\nabla\cdot(n\chi(p)\nabla p)+\nabla\cdot(n\nabla\phi),\\ \partial_t p+\mathbf{u}\cdot\nabla p=\Delta p-nf(p),\end{cases}\tag{1.3}$$

  containing the effect of potential force on cells and the effect of chemotactic force on the fluid.

- The model with *general chemotactic sensitivity* [19]:

$$\begin{cases}\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\pi=\Delta\mathbf{u}-n\nabla\phi,\\ \nabla\cdot\mathbf{u}=0,\\ \partial_t n+\mathbf{u}\cdot\nabla n=\Delta n-\nabla\cdot(nS(\mathbf{x},n,q)\nabla q),\\ \partial_t q+\mathbf{u}\cdot\nabla q=\Delta q-q+n.\end{cases}\tag{1.4}$$

- The model with *double chemical signals* [13]:

$$\begin{cases}\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\pi=\Delta\mathbf{u}-n\mathbf{f},\\ \nabla\cdot\mathbf{u}=0,\\ \partial_t n+\mathbf{u}\cdot\nabla n=\Delta n-\nabla\cdot(n\nabla p)-\nabla\cdot(n\nabla q),\\ \partial_t p+\mathbf{u}\cdot\nabla p=\Delta p-np,\\ \partial_t q+\mathbf{u}\cdot\nabla q=\Delta q-q+n.\end{cases}\tag{1.5}$$

Along with the proposition of the models, the qualitative properties, such as global well-posedness and long-time behavior, of solutions under various initial and/or boundary conditions have been studied in the corresponding references.

In this paper, we consider the following coupled chemotaxis-fluid model:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi = \Delta\mathbf{u} + n\nabla\phi + nS\nabla p + nS\nabla q, & (1.6)\\ \nabla\cdot\mathbf{u} = 0, & (1.7)\\ \partial_t n + \mathbf{u}\cdot\nabla n = \Delta n^m - \nabla\cdot(nS\nabla p) - \nabla\cdot(nS\nabla q) - \nabla\cdot(n\nabla\phi), & (1.8)\\ \partial_t p + \mathbf{u}\cdot\nabla p = \Delta p - np, & (1.9)\\ \partial_t q + \mathbf{u}\cdot\nabla q = \Delta q - q + n, & (1.10) \end{cases}$$

$\mathbf{x}\in\mathbb{R}^d$, $t>0$, where $S=S(\mathbf{x},n,p,q)$ denotes the general chemotactic sensitivity, which is an integrated system incorporating the various mechanisms presented in (1.2)–(1.5), including nonlinear diffusion, self-consistency, general chemotactic sensitivity and double chemical signals.

From the point of view of physical / biological applications, the integrated model is capable of describing more realistic situations than those portrayed by (1.1)–(1.5). On the other hand, however, the additional nonlinearities (comparing with those in (1.1)–(1.5)) and stronger coupling between the solution components make the mathematical analysis of the integrated model a significant challenge. As a starting point of research, we devote this paper to the rigorous analysis of some of the fundamental qualitative behaviors of the integrated model, such as the existence, regularity and extensibility criterion of large-amplitude solutions subject to appropriate initial and boundary conditions.

Considering its physical / biological applications, we study an initial-boundary value problem of the model (1.6)–(1.10) supplemented with the following initial and boundary conditions:

$$(\mathbf{u},n,p,q)(\mathbf{x},0) = (\mathbf{u}_0,n_0,p_0,q_0)(\mathbf{x}), \qquad \mathbf{x}\in\overline{\Omega}, \qquad (1.11)$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \nabla n^m\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \nabla p\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \nabla q\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad t\geq 0, \qquad (1.12)$$

where $\Omega\subset\mathbb{R}^d$ is a bounded convex domain with smooth boundary $\partial\Omega$, and $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.

The objective of this paper is twofold. First, for the model with porous medium-like diffusion (i.e., $m>1$), we prove the global existence of large-amplitude strong solutions subject to the initial and boundary conditions, (1.11)–(1.12), when the space dimension is two and under appropriate growth conditions on the chemotactic sensitivity function $S(\mathbf{x},n,p,q)$. Second, for the model with linear diffusion ($m=1$), we establish an extensibility criterion of large-amplitude classical solutions in scaling invariant energy spaces for the three-dimensional case. We achieve the goals by utilizing L^r-based energy methods. To the authors' knowledge, these are among the first generation of analytical results concerning the qualitative behaviors of large-amplitude solutions to the integrated model (1.6)–(1.10).

Before stating the main results, we introduce some notations for convenience.

NOTATION 1.1. *Throughout this paper, $\|\cdot\|_{L^r}, \|\cdot\|_{L^\infty}$ and $\|\cdot\|_{W^{s,r}}$ denote respectively the norms of the usual Lebesgue measurable spaces $L^r(\Omega)$, $L^\infty(\Omega)$ and the usual Sobolev space $W^{s,r}(\Omega)$. When $r=2$, we denote the norm $\|\cdot\|_{W^{s,2}}$ by $\|\cdot\|_{H^s}$. Unless otherwise specified, C will denote a generic constant which is independent of the unknown functions, but may depend on Ω, T and initial data. The value of the constant may vary line by line according to the context.*

Concerning the global existence of large-amplitude solutions to (1.6)–(1.12), we have the following:

THEOREM 1.1. *Consider the initial-boundary value problem (1.6)–(1.12) and let $d=2$. Suppose that $S=S(\mathbf{x},n,p,q)\in C^2(\overline{\Omega}\times[0,\infty)^3)$ satisfies the following condition:*

$$|S(\mathbf{x},n,p,q)|\le S_0(1+n)^{-\alpha},\quad \forall(\mathbf{x},n,p,q)\in\overline{\Omega}\times[0,\infty)^3, \tag{1.13}$$

for some constants $S_0>0$ and $\alpha\ge 0$. Suppose also that

$$m+\alpha<2,\quad 1<m<\frac{3}{2},\quad \frac{1}{2}<\alpha<1 \tag{1.14}$$

or

$$m+\alpha>2,\quad \frac{3}{2}<m<2,\quad 0<\alpha<\frac{1}{2}, \tag{1.15}$$

where m is the coefficient of nonlinear diffusion in Equation (1.8). Let $\mathbf{u}_0\in H^2(\Omega)$, $(p_0,q_0)\in W^{2-\frac{2}{r},r}(\Omega)$ and $n_0\in C^k(\overline{\Omega})$ for some $4<r<\infty$ and $k>0$. Suppose that $\nabla\cdot\mathbf{u}_0=0$, $n_0\ge 0$, $p_0\ge 0$, $q_0\ge 0$ in Ω, and $\mathbf{u}_0=\mathbf{0}$, $\nabla n_0^m\cdot\mathbf{n}=0$, $\nabla p_0\cdot\mathbf{n}=0$ and $\nabla q_0\cdot\mathbf{n}=0$ on $\partial\Omega$. Suppose also that $\phi=\phi(\mathbf{x})$ is a smooth function satisfying $\nabla\phi\cdot\mathbf{n}=0$ on $\partial\Omega$. Then the initial-boundary value problem (1.6)–(1.12) has a strong solution satisfying

$$\begin{aligned}
&\|\mathbf{u}\|_{L^\infty(0,T;H^1\cap L^\infty)}+\|\mathbf{u}\|_{L^2(0,T;H^2)}+\|\partial_t\mathbf{u}\|_{L^2(0,T;L^2)}\le C,\\
&\|(p,q)\|_{L^\infty(0,T;W^{2-\frac{2}{r},r})}+\|(\partial_t p,\partial_t q)\|_{L^r(0,T;L^r)}+\|(p,q)\|_{L^r(0,T;W^{2,r})}\le C,\\
&\|n(\cdot,t)\|_{L^\infty(0,T;L^\infty)}\le C,
\end{aligned} \tag{1.16}$$

for any given $T>0$, where the constant is independent of the unknown functions.

REMARK 1.1. In the statement of Theorem 1.1, by a strong solution to the initial-boundary value problem (1.6)–(1.12), we mean a quadruple of functions $(\mathbf{u},n,p,q)$, which satisfies the system of equations (1.6)–(1.10) in the sense of distributions and the initial and boundary conditions (1.11)–(1.12) in the classical sense, and possess the regularities as stated in (1.16). Similar definition can be found in [18], and we omit the details to simplify the presentation.

For the three-dimensional case, we have the following extensibility criterion for large-amplitude classical solutions to the initial-boundary value problem (1.6)–(1.12) with linear diffusion.

THEOREM 1.2. *Consider the initial-boundary value problem (1.6)–(1.12) with $m=1$ and let $d=3$. Suppose that $S=S(\mathbf{x},n,p,q)\in C^2(\overline{\Omega}\times[0,\infty)^3)$ satisfies the following condition:*

$$|S|+|\partial_{\mathbf{x}}S|+|\partial_n S|+|\partial_p S|+|\partial_q S|\le S_1,\quad \forall(\mathbf{x},n,p,q)\in\overline{\Omega}\times[0,\infty)^3, \tag{1.17}$$

for some positive constant S_1. Let $(\mathbf{u}_0,n_0,p_0,q_0)\in H^2(\Omega)$. Suppose that $\nabla\cdot\mathbf{u}_0=0$, $n_0\ge$, $p_0\ge 0$, and $q_0\ge 0$ in Ω, and $\mathbf{u}_0=\mathbf{0}$, $\nabla n_0\cdot\mathbf{n}=0$, $\nabla p_0\cdot\mathbf{n}=0$ and $\nabla q_0\cdot\mathbf{n}=0$ on $\partial\Omega$. Suppose also that $\phi=\phi(\mathbf{x})$ is a smooth function satisfying $\nabla\phi\cdot\mathbf{n}=0$ on $\partial\Omega$. Let $(\mathbf{u},n,p,q)\in L^\infty(0,T;H^2)\cap L^2(0,T;H^3)$ be a local classical solution to the initial-boundary value problem (1.6)–(1.12) for some $0<T<\infty$. If the following hold true:

$$\mathbf{u}\in L^2(0,T;BMO),\quad \nabla p\in L^{\frac{2r}{r-3}}(0,T;L^r),\quad \nabla q\in L^{\frac{2s}{s-3}}(0,T;L^s) \tag{1.18}$$

for some $3<r,s\le\infty$, then the solution $(\mathbf{u},n,p,q)$ can be extended beyond T.

Remark 1.2. In the statement of Theorem 1.2, BMO is the space of functions with bounded mean oscillation, whose norm is defined by

$$\|f\|_{BMO} \equiv \|f\|_{L^2} + [f]_{BMO},$$

with

$$[f]_{BMO} \equiv \sup_{\substack{\mathbf{x}\in\Omega \\ R\in(0,d)}} \frac{1}{|\Omega_R(\mathbf{x})|}\int_{\Omega_R(\mathbf{x})} |f(\mathbf{y}) - f_{\Omega_R(\mathbf{x})}|\mathrm{d}\mathbf{y},$$
$$f_{\Omega_R(\mathbf{x})} \equiv \frac{1}{|\Omega_R(\mathbf{x})|}\int_{\Omega_R(\mathbf{x})} f(\mathbf{y})\mathrm{d}\mathbf{y},$$

where $\Omega_R(\mathbf{x}) \equiv B_R(\mathbf{x}) \cap \Omega$, $B_R(\mathbf{x})$ is the ball with center $\mathbf{x}$ and radius R, and d is the diameter of Ω. Moreover, $|\Omega_R(\mathbf{x})|$ denotes the Lebesgue measure of $\Omega_R(\mathbf{x})$.

Remark 1.3. We observe that (1.18) is optimal from the point of view of scaling invariance. Indeed, when $m=1$ and neglecting the linear lower order term q in (1.10), it can be shown that (1.6)–(1.10) is invariant under the scaling transform:

$$\begin{aligned} &\mathbf{u} \to \mathbf{u}_\lambda \equiv \lambda\mathbf{u}(\lambda^2 t, \lambda\mathbf{x}), && \pi \to \pi_\lambda \equiv \lambda^2\pi(\lambda^2 t, \lambda\mathbf{x}), \\ &n \to n_\lambda \equiv \lambda^2 n(\lambda^2 t, \lambda\mathbf{x}), && p \to p_\lambda \equiv p(\lambda^2 t, \lambda\mathbf{x}), \\ &q \to q_\lambda \equiv q(\lambda^2 t, \lambda\mathbf{x}), && \phi \to \phi_\lambda \equiv \phi(\lambda^2 t, \lambda\mathbf{x}), \\ &S \to S_\lambda \equiv S(\lambda\mathbf{x}, n_\lambda, p_\lambda, q_\lambda), && \end{aligned} \tag{1.19}$$

which indicates that the functional spaces in (1.18) are energy critical for the corresponding variables.

We prove Theorems 1.1 and 1.2 by using L^r-based energy methods. Major difficulties in the proof of Theorem 1.1 come from the strong coupling between the chemotactic and hydrodynamic parts of the system, and the high order nonlinearities arising from the porous medium-like diffusion and chemotaxis-induced nonlinear diffusion. We overcome the difficulties by renovating classical energy methods based on the fundamental inequalities, such as the Hölder, Young and Gagliardo-Nirenberg inequalities, and applying standard theory of the heat equation. On the other hand, the BMO setting in the extensibility criterion, (1.18), raises a significant technical barrier for the proof of Theorem 1.2, since the BMO-norm of a function comes out as a penalty term when one improves the usual Sobolev embedding: $W^{1,\gamma} \hookrightarrow L^\infty$ $(3<\gamma<\infty)$ by a logarithmic correction, see Lemma 3.1. We survive the situation by carefully monitoring the energy bound within a small time interval near the end of the lifespan of the local solution, and deriving an iterative scheme based on the classical theory of the heat equation to close the overall energy estimates.

The rest of this paper is organized as follows. In Section 2 and Section 3, we prove Theorem 1.1 and Theorem 1.2, respectively, by deriving *a priori* estimates. The paper finishes with concluding remarks in Section 4.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. First of all, we note that the local existence of strong solutions to the initial-boundary value problem (1.6)–(1.12) can be established by applying a similar regularization procedure as in [18] (mainly to avoid possible singularities stemming from the porous medium-like diffusion) to construct a sequence of approximate solutions, then carrying out standard energy estimation and taking proper limits by standard compactness arguments. The heart of the matter is the *a priori* estimates of the solution. Hence, in what follows we shall focus on deriving the *a priori* estimates of the local solution, in order to extend it to be a global one. We divide the proof into several steps.

Step 1. First of all, since the initial functions n_0, p_0 and q_0 are non-negative, it follows from (1.8), (1.9) and (1.10) and similar arguments in [8] (see Section 3.1 of [8]) that

$$n(\mathbf{x},t)\geq 0, \quad p(\mathbf{x},t)\geq 0, \quad q(\mathbf{x},t)\geq 0, \quad \forall(\mathbf{x},t)\in\overline{\Omega}\times(0,T), \tag{2.1}$$

where $T>0$ denotes the lifespan of the local strong solution. As a result of the positivity of $n(\mathbf{x},t)$, by testing (1.9) with $p^{\beta-1}$ for any $\beta\geq 2$ and using (1.7) and the boundary conditions, we can show that

$$\frac{d}{dt}\left(\|p\|_{L^\beta}^\beta\right)+\beta(\beta-1)\int_\Omega |p|^{\beta-2}|\nabla p|^2 d\mathbf{x}+\beta n\|p\|_{L^\beta}^\beta=0,$$

which implies $\|p(\cdot,t)\|_{L^\beta}\leq\|p_0(\cdot)\|_{L^\beta}$. By letting $\beta\to\infty$, we obtain

$$\|p(t)\|_{L^\infty}\leq\|p_0\|_{L^\infty}.$$

Note that $p_0\in W^{2-\frac{2}{r},r}(\Omega)$ for some $4<r<\infty$, it follows from the Sobolev embedding $W^{2-\frac{2}{r},r}(\Omega)\hookrightarrow C^{1,\frac{r-4}{2r}}(\overline{\Omega})$ that

$$\|p(t)\|_{L^\infty}\leq C, \quad \forall t\in(0,T), \tag{2.2}$$

where the constant depends only on the initial data.

Next, we derive some relatively easy estimates from (1.8)–(1.10). By integrating (1.8) over $\Omega\times(0,t)$ and using (1.7), the boundary conditions for $\mathbf{u}$, n, p, q and the boundary condition for ϕ as specified in the statement of Theorem 1.1, we get

$$\int_\Omega n(\mathbf{x},t)\,\mathrm{d}\mathbf{x}=\int_\Omega n_0(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{2.3}$$

As a result of (2.3), by integrating (1.10) over Ω, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega q(\mathbf{x},t)\mathrm{d}\mathbf{x}+\int_\Omega q(\mathbf{x},t)\mathrm{d}\mathbf{x}=\int_\Omega n(\mathbf{x},t)\mathrm{d}\mathbf{x}=\int_\Omega n_0(\mathbf{x})\mathrm{d}\mathbf{x},$$

which gives

$$\int_\Omega q(\mathbf{x},t)\mathrm{d}\mathbf{x}\leq e^{-t}\int_\Omega q_0(\mathbf{x})\mathrm{d}\mathbf{x}+\left(1-e^{-t}\right)\int_\Omega n_0(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{2.4}$$

Moreover, testing (1.9) by p and using (1.7) and (2.1), we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p\|_{L^2}^2+\|\nabla p\|_{L^2}^2+\int_\Omega np^2\mathrm{d}\mathbf{x}=0,$$

which yields

$$\int_0^T\|\nabla p(t)\|_{L^2}^2\mathrm{d}t\leq C. \tag{2.5}$$

Step 2. Now we derive some nonlinear estimates for n. Testing (1.8) by n^{m-2} and using (1.7) and Young's inequality, we can show that

$$\begin{aligned}
&\frac{m}{m-1}\|\nabla n^{m-1}\|_{L^2}^2\\
&=\int_\Omega (S\nabla p+S\nabla q+\nabla\phi)\cdot\nabla n^{m-1}\mathrm{d}\mathbf{x}+\frac{1}{2-m}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^{m-1}\mathrm{d}\mathbf{x}\\
&\le \frac{m}{4(m-1)}\|\nabla n^{m-1}\|_{L^2}^2+\frac{3(m-1)}{m}\left(S_0^2\|\nabla p\|_{L^2}^2+S_0^2\|\nabla q\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right)\\
&\quad+\frac{1}{2-m}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^{m-1}\mathrm{d}\mathbf{x} \qquad (2.6)
\end{aligned}$$

where $1<m<2$. After rearranging terms, we obtain

$$\begin{aligned}
&\frac{3m}{4(m-1)}\|\nabla n^{m-1}\|_{L^2}^2\\
&\le \frac{3(m-1)}{m}\left(S_0^2\|\nabla p\|_{L^2}^2+S_0^2\|\nabla q\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right)+\frac{1}{2-m}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^{m-1}\mathrm{d}\mathbf{x}. \qquad (2.7)
\end{aligned}$$

Testing (1.10) by q and using (1.7), we infer that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_{L^2}^2+\|q\|_{H^1}^2=\int_\Omega nq\,\mathrm{d}\mathbf{x}. \qquad (2.8)$$

For the term on the right-hand side of (2.8), by using the Hölder and Gagliardo-Nirenberg inequalities, we can show that

$$\begin{aligned}
\int_\Omega nq\,\mathrm{d}\mathbf{x}&\le\|n\|_{L^\theta}\|q\|_{L^{\frac{\theta}{\theta-1}}}\\
&=\|n^{m-1}\|_{L^{\frac{\theta}{m-1}}}^{\frac{1}{m-1}}\|q\|_{L^{\frac{\theta}{\theta-1}}}\\
&\le C\left(\|n^{m-1}\|_{L^{\frac{1}{m-1}}}^{\frac{1}{\theta}}\|\nabla n^{m-1}\|_{L^2}^{1-\frac{1}{\theta}}+\|n^{m-1}\|_{L^{\frac{1}{m-1}}}\right)^{\frac{1}{m-1}}\|q\|_{L^{\frac{\theta}{\theta-1}}}, \qquad (2.9)
\end{aligned}$$

where $\theta>1$ is a constant to be determined. Note that

$$\|n^{m-1}\|_{L^{\frac{1}{m-1}}}=\left(\int_\Omega n(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\right)^{m-1}=\left(\int_\Omega n_0(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)^{m-1},$$

and $x^{\frac{1}{m-1}}$ is a convex function (since $1<m<2$). Then we update (2.9) as

$$\begin{aligned}
\int_\Omega nq\,\mathrm{d}\mathbf{x}&\le C\left(\|\nabla n^{m-1}\|_{L^2}^{\left(1-\frac{1}{\theta}\right)\frac{1}{m-1}}+1\right)\|q\|_{L^{\frac{\theta}{\theta-1}}}\\
&\le C\left(\|\nabla n^{m-1}\|_{L^2}^{\left(1-\frac{1}{\theta}\right)\frac{1}{m-1}}+1\right)\|q\|_{H^1}\\
&\le\frac{1}{2}\|q\|_{H^1}^2+C+C\|\nabla n^{m-1}\|_{L^2}^{\left(1-\frac{1}{\theta}\right)\frac{2}{m-1}}, \qquad (2.10)
\end{aligned}$$

where the Sobolev embedding: $H^1(\Omega)\hookrightarrow L^r(\Omega)$, $\Omega\subset\mathbb{R}^2$, $\forall 1<r<\infty$, is applied. By substituting (2.10) into (2.8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_{L^2}^2+\|q\|_{H^1}^2\le C+C\|\nabla n^{m-1}\|_{L^2}^{\left(1-\frac{1}{\theta}\right)\frac{2}{m-1}}. \qquad (2.11)$$

By multiplying (2.11) with $\frac{4(m-1)}{m}S_0^2$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4(m-1)}{m}S_0^2\|q\|_{L^2}^2\right)+\frac{4(m-1)}{m}S_0^2\|q\|_{H^1}^2\le C+C\|\nabla n^{m-1}\|_{L^2}^{\left(1-\frac{1}{\theta}\right)\frac{2}{m-1}}. \tag{2.12}$$

By adding (2.12) to (2.7), we infer that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4(m-1)}{m}S_0^2\|q\|_{L^2}^2\right)+\frac{(m-1)}{m}S_0^2\|q\|_{H^1}^2+\frac{3m}{4(m-1)}\|\nabla n^{m-1}\|_{L^2}^2\\ &\le C+C\|\nabla n^{m-1}\|_{L^2}^{\left(1-\frac{1}{\theta}\right)\frac{2}{m-1}}+\frac{3(m-1)}{m}\left(S_0^2\|\nabla p\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right)\\ &\quad+\frac{1}{2-m}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^{m-1}\mathrm{d}\mathbf{x}.\end{aligned} \tag{2.13}$$

By choosing

$$\theta>1 \quad and \quad 1-\frac{1}{\theta}<m-1 \quad \left(note\ that\ 1<m<2\Longleftrightarrow\frac{1}{2-m}>1\right), \tag{2.14}$$

then applying Young's inequality to the second term on the right-hand side of (2.13), we can show that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4(m-1)}{m}S_0^2\|q\|_{L^2}^2\right)+\frac{(m-1)}{m}S_0^2\|q\|_{H^1}^2+\frac{3m}{4(m-1)}\|\nabla n^{m-1}\|_{L^2}^2\\ &\le C+\frac{3(m-1)}{m}\left(S_0^2\|\nabla p\|_{L^2}^2+\|\nabla\phi\|_{L^2}^2\right)+\frac{1}{2-m}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^{m-1}\mathrm{d}\mathbf{x}.\end{aligned} \tag{2.15}$$

By integrating (2.15) with respect to time, we obtain

$$\begin{aligned}&\frac{4(m-1)}{m}S_0^2\|q\|_{L^2}^2+\int_0^t\left(\frac{(m-1)}{m}S_0^2\|q\|_{H^1}^2+\frac{3m}{4(m-1)}\|\nabla n^{m-1}\|_{L^2}^2\right)d\tau\\ &\le C(t+1)+\frac{1}{2-m}\int_\Omega n^{m-1}\mathrm{d}\mathbf{x}.\end{aligned} \tag{2.16}$$

Note that $1<m<2$, and

$$\int_\Omega n^{m-1}\mathrm{d}\mathbf{x}\le\left(\int_\Omega n(\mathbf{x},t)\mathrm{d}\mathbf{x}\right)^{m-1}|\Omega|^{2-m}=\left(\int_\Omega n_0(\mathbf{x})\mathrm{d}\mathbf{x}\right)^{m-1}|\Omega|^{2-m}.$$

Hence, we obtain from (2.16) that

$$\|q\|_{L^2}^2+\int_0^T\left(\|q\|_{H^1}^2+\|\nabla n^{m-1}\|_{L^2}^2\right)\mathrm{d}t\le C. \tag{2.17}$$

Step 3. Now we turn to the estimate of the velocity field. Testing (1.6) by $\mathbf{u}$ and using (1.7), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2=\int_\Omega n\nabla\phi\cdot\mathbf{u}\mathrm{d}\mathbf{x}+\int_\Omega nS\nabla(p+q)\cdot\mathbf{u}\mathrm{d}\mathbf{x}\equiv I_1+I_2. \tag{2.18}$$

For I_1, by using the smoothness of ϕ and similar arguments as those in (2.9) and (2.10), we can show that

$$\begin{aligned}|I_1|\le C\left|\int_\Omega n|\mathbf{u}|\mathrm{d}\mathbf{x}\right|&\le C\left(\|\nabla n^{m-1}\|_{L^2}^{\left(1-\frac{1}{\theta}\right)\frac{1}{m-1}}+1\right)\|\mathbf{u}\|_{H^1}\\ &\le C\left(\|\nabla n^{m-1}\|_{L^2}^{\left(1-\frac{1}{\theta}\right)\frac{1}{m-1}}+1\right)\|\nabla\mathbf{u}\|_{L^2}\\ &\le\frac{1}{4}\|\nabla\mathbf{u}\|_{L^2}^2+C+C\|\nabla n^{m-1}\|_{L^2}^{\left(1-\frac{1}{\theta}\right)\frac{2}{m-1}},\end{aligned}$$

where the Poincaré inequality is applied (due to the no-flow boundary condition for $\mathbf{u}$). By choosing $\theta = \frac{1}{2-m}$, we get

$$|I_1| \le \frac{1}{4}\|\nabla \mathbf{u}\|_{L^2}^2 + C + C\|\nabla n^{m-1}\|_{L^2}^2. \tag{2.19}$$

For I_2, by using the assumption on the chemotactic sensitivity $S(\mathbf{x},n,p,q)$ (cf. (1.13)) and Gagliardo-Nirenberg and Poincaré inequalities, we can show that for any $2 < s < \infty$,

$$\begin{aligned} |I_2| &\le S_0 \int_\Omega |\nabla(p+q)| \frac{n}{(1+n)^\alpha} |\mathbf{u}|\,\mathrm{d}\mathbf{x} \\ &\le S_0 \int_\Omega |\nabla(p+q)| n^{1-\alpha} |\mathbf{u}|\,\mathrm{d}\mathbf{x} \\ &\le S_0 \|\nabla(p+q)\|_{L^2} \|n^{1-\alpha}\|_{L^s} \|\mathbf{u}\|_{L^{\frac{2s}{s-2}}} \\ &\le C\|\nabla(p+q)\|_{L^2} \|n^{1-\alpha}\|_{L^s} \|\mathbf{u}\|_{L^2}^{1-\frac{2}{s}} \|\nabla \mathbf{u}\|_{L^2}^{\frac{2}{s}}. \end{aligned} \tag{2.20}$$

By further applying the Young inequality, we can show that

$$\begin{aligned} & C\|\nabla(p+q)\|_{L^2} \|n^{1-\alpha}\|_{L^s} \|\mathbf{u}\|_{L^2}^{1-\frac{2}{s}} \|\nabla \mathbf{u}\|_{L^2}^{\frac{2}{s}} \\ \le\ & \frac{1}{4}\|\nabla \mathbf{u}\|_{L^2}^2 + C\|\nabla(p+q)\|_{L^2}^{\frac{s}{s-1}} \|n^{1-\alpha}\|_{L^s}^{\frac{s}{s-1}} \|\mathbf{u}\|_{L^2}^{\frac{s-2}{s-1}} \\ \le\ & \frac{1}{4}\|\nabla \mathbf{u}\|_{L^2}^2 + C\|\nabla(p+q)\|_{L^2}^2 + C\|n^{1-\alpha}\|_{L^s}^{\frac{2s}{s-2}} \|\mathbf{u}\|_{L^2}^2. \end{aligned} \tag{2.21}$$

Note that the temporal integral of the second term on the right-hand side of (2.21) is finite. For the third term on the right-hand side of (2.21), by using Gagliardo-Nirenberg interpolation inequality and (2.3), we can show that for any $s > \frac{1}{1-\alpha}$ (note that $0 < \alpha < 1$),

$$\begin{aligned} \|n^{1-\alpha}\|_{L^s}^{\frac{2s}{s-2}} &= \|n^{m-1}\|_{L^{\frac{s(1-\alpha)}{m-1}}}^{\frac{s(1-\alpha)}{m-1}\cdot\frac{2}{s-2}} \\ &\le C\left(\|n^{m-1}\|_{L^{\frac{1}{m-1}}}^{\frac{1}{s(1-\alpha)}} \|\nabla n^{m-1}\|_{L^2}^{1-\frac{1}{s(1-\alpha)}} + \|n^{m-1}\|_{L^{\frac{1}{m-1}}} \right)^{\frac{s(1-\alpha)}{m-1}\cdot\frac{2}{s-2}} \\ &\le C\left(\|\nabla n^{m-1}\|_{L^2}^{1-\frac{1}{s(1-\alpha)}} + 1 \right)^{\frac{s(1-\alpha)}{m-1}\cdot\frac{2}{s-2}}. \end{aligned} \tag{2.22}$$

By taking $s = \frac{3-2m}{2-m-\alpha}$, we obtain

$$C\left(\|\nabla n^{m-1}\|_{L^2}^{1-\frac{1}{s(1-\alpha)}} + 1 \right)^{\frac{s(1-\alpha)}{m-1}\cdot\frac{2}{s-2}} \le C + C\|\nabla n^{m-1}\|_{L^2}^2. \tag{2.23}$$

Note that under the conditions (1.14) or (1.15), it holds that $s = \frac{3-2m}{2-m-\alpha} > \frac{1}{1-\alpha}$. By substituting (2.23) and (2.22) into (2.21), then substituting the result into (2.20), we obtain

$$|I_2| \le \frac{1}{4}\|\nabla \mathbf{u}\|_{L^2}^2 + C\|\nabla(p+q)\|_{L^2}^2 + C\|\mathbf{u}\|_{L^2}^2 + C\|\nabla n^{m-1}\|_{L^2}^2 \|\mathbf{u}\|_{L^2}^2. \tag{2.24}$$

By substituting (2.19) and (2.24) into (2.18), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_{L^2}^2 + \frac{1}{2}\|\nabla \mathbf{u}\|_{L^2}^2 \le C\left(\|\nabla n^{m-1}\|_{L^2}^2 + 1\right)\|\mathbf{u}\|_{L^2}^2 + C\left(\|\nabla(p+q)\|_{L^2}^2 + \|\nabla n^{m-1}\|_{L^2}^2 + 1\right). \tag{2.25}$$

By applying the Grönwall inequality to (2.25) and using (2.5) and (2.17), we have

$$\|\mathbf{u}(t)\|_{L^2}^2+\int_0^T\|\nabla\mathbf{u}(t)\|_{L^2}^2\,\mathrm{d}t\le C. \tag{2.26}$$

Step 4. Now we continue to the higher order estimates of q. Testing (1.10) by q^{s-1} $(s>2)$, denoting $v\equiv q^{\frac{s}{2}}$ and using (1.7) and (2.17), we have

$$\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L^2}^2+\frac{4(s-1)}{s^2}\|\nabla v\|_{L^2}^2+\|v\|_{L^2}^2=\int_\Omega nv^{\frac{2(s-1)}{s}}\mathrm{d}\mathbf{x}. \tag{2.27}$$

For the right-hand side of (2.27), we can show that for any $1<\theta<s$,

$$\begin{aligned}\int_\Omega nv^{\frac{2(s-1)}{s}}\mathrm{d}\mathbf{x}&\le\|n\|_{L^\theta}\|v^{\frac{2(s-1)}{s}}\|_{L^{\frac{\theta}{\theta-1}}}\\&=\|n\|_{L^\theta}\|v\|_{L^{\frac{2(s-1)}{s}\cdot\frac{\theta}{\theta-1}}}^{\frac{2(s-1)}{s}}\\&\le C\|n\|_{L^\theta}\left(\|v\|_{L^2}^{\frac{s}{s-1}\cdot\frac{\theta-1}{\theta}}\|v\|_{H^1}^{1-\frac{s}{s-1}\cdot\frac{\theta-1}{\theta}}\right)^{\frac{2(s-1)}{s}}\\&=C\|n\|_{L^\theta}\|v\|_{L^2}^{\frac{2(\theta-1)}{\theta}}\|v\|_{H^1}^{\frac{2(s-\theta)}{s\theta}}.\end{aligned} \tag{2.28}$$

By applying the Young inequality, we can show that

$$C\|n\|_{L^\theta}\|v\|_{L^2}^{\frac{2(\theta-1)}{\theta}}\|v\|_{H^1}^{\frac{2(s-\theta)}{s\theta}}\le\frac{2(s-1)}{s^2}\|v\|_{H^1}^2+C\|n\|_{L^\theta}^{\frac{s\theta}{s\theta-s+\theta}}\|v\|_{L^2}^{\frac{2s(\theta-1)}{s\theta-s+\theta}}. \tag{2.29}$$

For the second term on the right-hand side of (2.29), we can show that

$$\begin{aligned}\|n\|_{L^\theta}^{\frac{s\theta}{s\theta-s+\theta}}&=\|n^{m-1}\|_{L^{\frac{\theta}{m-1}}}^{\frac{1}{m-1}\cdot\frac{s\theta}{s\theta-s+\theta}}\\&\le C\left(\|n^{m-1}\|_{L^{\frac{1}{m-1}}}^{\frac{1}{\theta}}\|\nabla n^{m-1}\|_{L^2}^{1-\frac{1}{\theta}}+\|n^{m-1}\|_{L^{\frac{1}{m-1}}}\right)^{\frac{1}{m-1}\cdot\frac{s\theta}{s\theta-s+\theta}}\\&\le C\left(\|\nabla n^{m-1}\|_{L^2}^{1-\frac{1}{\theta}}+1\right)^{\frac{1}{m-1}\cdot\frac{s\theta}{s\theta-s+\theta}},\end{aligned} \tag{2.30}$$

where $\frac{1}{m-1}\cdot\frac{s\theta}{s\theta-s+\theta}>1$. Note that when $1<m<\frac{3}{2}$, by taking $\theta=\frac{3-2m}{3-2m-\frac{2(m-1)}{s}}$ and $s>\frac{2(m-1)}{3-2m}$, it is straightforward to verify that

$$\left(1-\frac{1}{\theta}\right)\frac{1}{m-1}\cdot\frac{s\theta}{s\theta-s+\theta}=2,$$

which implies

$$\|n\|_{L^\theta}^{\frac{s\theta}{s\theta-s+\theta}}\le C\left(\|\nabla n^{m-1}\|_{L^2}^2+1\right). \tag{2.31}$$

On the other hand, when $\frac{3}{2}<m<2$, it is easy to verify that

$$\left(1-\frac{1}{\theta}\right)\frac{1}{m-1}\cdot\frac{s\theta}{s\theta-s+\theta}<2.$$

In this case, by applying the Young inequality to (2.30), we can show that

$$\|n\|_{L^\theta}^{\frac{s\theta}{s\theta-s+\theta}} \leq C\left(\|\nabla n^{m-1}\|_{L^2}^2+1\right). \tag{2.32}$$

Further notice that $\frac{2s(\theta-1)}{s\theta-s+\theta}<2$. Hence, we update (2.29) as

$$C\|n\|_{L^\theta}\|v\|_{L^2}^{\frac{2(\theta-1)}{\theta}}\|v\|_{H^1}^{\frac{2(s-\theta)}{s\theta}} \leq \frac{2(s-1)}{s^2}\|v\|_{H^1}^2+C\left(\|\nabla n^{m-1}\|_{L^2}^2+1\right)\left(\|v\|_{L^2}^2+1\right). \tag{2.33}$$

By substituting (2.33) into (2.27), we obtain

$$\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L^2}^2+\frac{2(s-1)}{s^2}\|\nabla v\|_{L^2}^2+\frac{(s-1)^2+1}{s^2}\|v\|_{L^2}^2 \leq C\left(\|\nabla n^{m-1}\|_{L^2}^2+1\right)\left(\|v\|_{L^2}^2+1\right). \tag{2.34}$$

By applying the Grönwall inequality to (2.34) and using (2.17), we can show that

$$\|v(t)\|_{L^2}^2+\int_0^T\|v(t)\|_{H^1}^2 dt \leq C,$$

which, together with the definition of $v=q^{\frac{s}{2}}$, implies

$$\|q\|_{L^\infty(0,T;L^s)}\leq C, \quad \begin{cases} \forall \max\left\{\dfrac{2(m-1)}{3-2m},2\right\}<s<\infty, & 1<m<\dfrac{3}{2}, \\ 2<s<\infty, & \dfrac{3}{2}<m<2. \end{cases} \tag{2.35}$$

Step 5. Testing (1.9) by $-\Delta p$ and using (1.7), we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla p\|_{L^2}^2+\|\Delta p\|_{L^2}^2 &= -\int_\Omega\sum_{i,j=1}^2(\partial_{x_j}u_i)(\partial_{x_i}p)(\partial_{x_j}p)\mathrm{d}x+\int_\Omega np\Delta p\mathrm{d}\mathbf{x} \\ &\leq \|\nabla\mathbf{u}\|_{L^2}\|\nabla p\|_{L^4}^2+\|p\|_{L^\infty}\|n\|_{L^2}\|\Delta p\|_{L^2} \\ &\leq \|\nabla\mathbf{u}\|_{L^2}\|\nabla p\|_{L^4}^2+C\|n\|_{L^2}\|\Delta p\|_{L^2}, \end{aligned} \tag{2.36}$$

where (2.2) is applied. By applying Gagliardo-Nirenberg interpolation inequality, we can show that

$$\|\nabla p\|_{L^4}^2 \leq C\left(\|p\|_{L^\infty}\|\Delta p\|_{L^2}+\|\nabla p\|_{L^2}^2\right).$$

Since $\nabla p\cdot\mathbf{n}|_{\partial\Omega}=0$, we deduce that

$$\|\nabla p\|_{L^2}^2=-\int_\Omega p\Delta p\mathrm{d}\mathbf{x}\leq\|p\|_{L^2}\|\Delta p\|_{L^2}\leq\|p\|_{L^\infty}|\Omega|^{\frac{1}{2}}\|\Delta p\|_{L^2},$$

which implies

$$\|\nabla p\|_{L^4}^2\leq C\|p\|_{L^\infty}\|\Delta p\|_{L^2}.$$

So we update (2.36) as

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla p\|_{L^2}^2+\|\Delta p\|_{L^2}^2 &\leq C\|\nabla\mathbf{u}\|_{L^2}\|p\|_{L^\infty}\|\Delta p\|_{L^2}+C\|n\|_{L^2}\|\Delta p\|_{L^2} \\ &\leq \frac{1}{4}\|\Delta p\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2+C\|n\|_{L^2}^2, \end{aligned} \tag{2.37}$$

which implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla p\|_{L^2}^2+\frac{3}{4}\|\Delta p\|_{L^2}^2\le C\|\nabla \mathbf{u}\|_{L^2}^2+C\|n\|_{L^2}^2, \tag{2.38}$$

Testing (1.10) by $-\Delta q$ and using (1.7) and Gagliardo-Nirenberg inequality, we can show that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q\|_{L^2}^2+\|\Delta q\|_{L^2}^2+\|\nabla q\|_{L^2}^2\\
=&-\int_\Omega n\Delta q\mathrm{d}\mathbf{x}+\int_\Omega(\mathbf{u}\cdot\nabla q)\Delta q\mathrm{d}\mathbf{x}\\
\le&\ \|n\|_{L^2}\|\Delta q\|_{L^2}+\|\mathbf{u}\|_{L^4}\|\nabla q\|_{L^4}\|\Delta q\|_{L^2}\\
\le&\ \|n\|_{L^2}\|\Delta q\|_{L^2}+C\|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\left(\|\nabla q\|_{L^2}^{\frac{1}{2}}\|\Delta q\|_{L^2}^{\frac{1}{2}}+\|\nabla q\|_{L^2}\right)\|\Delta q\|_{L^2}\\
\le&\ \frac{1}{4}\|\Delta q\|_{L^2}^2+C\|n\|_{L^2}^2+C\|\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla q\|_{L^2}^2+C\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\|\nabla q\|_{L^2}^2\\
\le&\ \frac{1}{4}\|\Delta q\|_{L^2}^2+C\|n\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla q\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla q\|_{L^2}^2,
\end{aligned}\tag{2.39}$$

where (2.26) and Poincaré inequality are applied. After rearranging terms, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q\|_{L^2}^2+\frac{3}{4}\|\Delta q\|_{L^2}^2+\|\nabla q\|_{L^2}^2\le C\|n\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla q\|_{L^2}^2. \tag{2.40}$$

Testing (1.8) by n^{m-1} and using (1.7), we deduce that

$$\begin{aligned}
&\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^m\mathrm{d}\mathbf{x}+m(m-1)\int_\Omega n^{2m-3}|\nabla n|^2\mathrm{d}\mathbf{x}\\
=&\ (m-1)\int_\Omega n^{m-1}S\nabla(p+q)\cdot\nabla n\mathrm{d}\mathbf{x}+(m-1)\int_\Omega n^{m-1}\nabla\phi\cdot\nabla n\mathrm{d}\mathbf{x}\\
\le&\ S_0(m-1)\int_\Omega n^{m-1}|\nabla(p+q)||\nabla n|\mathrm{d}\mathbf{x}+C(m-1)\int_\Omega n^{m-1}|\nabla n|\mathrm{d}\mathbf{x}\\
\le&\ \frac{m(m-1)}{4}\int_\Omega n^{2m-3}|\nabla n|^2\mathrm{d}\mathbf{x}+C\int_\Omega n\mathrm{d}\mathbf{x}+C\int_\Omega n|\nabla(p+q)|^2\mathrm{d}\mathbf{x}.
\end{aligned}$$

After rearranging terms, we have

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^m\mathrm{d}\mathbf{x}+\frac{3m(m-1)}{4}\int_\Omega n^{2m-3}|\nabla n|^2\mathrm{d}\mathbf{x}\le C\int_\Omega n_0\mathrm{d}\mathbf{x}+C\int_\Omega n|\nabla(p+q)|^2\mathrm{d}\mathbf{x}, \tag{2.41}$$

where (2.3) is applied. For the second term on the right-hand side of (2.41), we can show that

$$\int_\Omega n|\nabla(p+q)|^2\mathrm{d}\mathbf{x}\le\eta\int_\Omega n^{2m}\mathrm{d}\mathbf{x}+C\int_\Omega|\nabla(p+q)|^{\frac{4m}{2m-1}}\mathrm{d}\mathbf{x}, \tag{2.42}$$

where $\eta>0$ is a constant to be determined. By using the Gagliardo-Nirenberg inequality, we deduce that

$$\int_\Omega n^{2m}\mathrm{d}\mathbf{x}=\left\|n^{\frac{2m-1}{2}}\right\|_{L^{\frac{4m}{2m-1}}}^{\frac{4m}{2m-1}}$$

$$\le C\left(\left\|n^{\frac{2m-1}{2}}\right\|_{L^{\frac{2}{2m-1}}}^{\frac{1}{2m}}\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^{1-\frac{1}{2m}}+\left\|n^{\frac{2m-1}{2}}\right\|_{L^{\frac{2}{2m-1}}}\right)^{\frac{4m}{2m-1}}$$
$$\le C\left(\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^{1-\frac{1}{2m}}+1\right)^{\frac{4m}{2m-1}}$$
$$\le C\left(\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^{2}+1\right). \tag{2.43}$$

In a similar fashion, we can show that

$$\int_\Omega |\nabla(p+q)|^{\frac{4m}{2m-1}}\mathrm{d}\mathbf{x}\le C\left(\|\Delta(p+q)\|_{L^2}^{\xi}\|p+q\|_{L^\ell}^{1-\xi}+\|p+q\|_{L^\ell}\right)^{\frac{4m}{2m-1}}, \tag{2.44}$$

where ℓ is a positive number to be determined and

$$\xi=\xi(\ell)\equiv\frac{\frac{1}{\ell}+\frac{1}{4m}}{\frac{1}{\ell}+\frac{1}{2}}.$$

We observe that since $1<m<2$, $\xi(\ell)$ is strictly decreasing for $\ell\in(0,\infty)$, and $\xi\left(\frac{1}{m-1}\right)=\frac{2m-1}{2m}$. Hence, for some fixed number ℓ_0 such that

$$\max\left\{\frac{1}{m-1},\frac{2(m-1)}{3-2m},2\right\}<\ell_0<\infty,$$

by using (2.2) and (2.35) we deduce that

$$C\left(\|\Delta(p+q)\|_{L^2}^{\xi}\|p+q\|_{L^{\ell_0}}^{1-\xi}+\|p+q\|_{L^{\ell_0}}\right)^{\frac{4m}{2m-1}}\le C\left(\|\Delta(p+q)\|_{L^2}^{\xi}+1\right)^{\frac{4m}{2m-1}}$$
$$\le C\|\Delta(p+q)\|_{L^2}^{\xi\frac{4m}{2m-1}}+C$$
$$\le \eta\|\Delta(p+q)\|_{L^2}^{2}+C, \tag{2.45}$$

where the Young inequality is applied and $\eta>0$ is a constant to be determined. By substituting (2.45) into (2.44), we have

$$\int_\Omega |\nabla(p+q)|^{\frac{4m}{2m-1}}\mathrm{d}\mathbf{x}\le \eta\|\Delta(p+q)\|_{L^2}^{2}+C. \tag{2.46}$$

By substituting (2.43) and (2.46) into (2.42), we obtain

$$\int_\Omega n|\nabla(p+q)|^2\mathrm{d}\mathbf{x}\le \eta C\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^{2}+\eta C\|\Delta(p+q)\|_{L^2}^{2}+C(\eta+1), \tag{2.47}$$

By substituting (2.47) into (2.41), we have

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^m\mathrm{d}\mathbf{x}+\frac{3m(m-1)}{4}\int_\Omega n^{2m-3}|\nabla n|^2\mathrm{d}\mathbf{x}$$
$$\le C\int_\Omega n_0\,\mathrm{d}\mathbf{x}+\eta C\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^{2}+\eta C\|\Delta(p+q)\|_{L^2}^{2}+C(\eta+1). \tag{2.48}$$

Note that

$$\int_\Omega n^{2m-3}|\nabla n|^2\mathrm{d}\mathbf{x}=\frac{4}{(2m-1)^2}\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^{2}.$$

So we update (2.48) as

$$\begin{aligned}&\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}n^m\mathrm{d}\mathbf{x}+\frac{3m(m-1)}{(2m-1)^2}\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^2\\&\leq C\int_{\Omega}n_0\,\mathrm{d}\mathbf{x}+\eta C\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^2+\eta C\|\Delta(p+q)\|_{L^2}^2+C(\eta+1).\end{aligned}\tag{2.49}$$

By adding up (2.38), (2.40) and (2.49), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla p\|_{L^2}^2+\frac{1}{2}\|\nabla q\|_{L^2}^2+\frac{1}{m}\int_{\Omega}n^m\mathrm{d}\mathbf{x}\right)+\frac{3}{4}\|\Delta p\|_{L^2}^2\\&\qquad+\frac{3}{4}\|\Delta q\|_{L^2}^2+\|\nabla q\|_{L^2}^2+\frac{3m(m-1)}{(2m-1)^2}\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^2\\&\leq C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla q\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2+C\|n\|_{L^2}^2+C\int_{\Omega}n_0\,\mathrm{d}\mathbf{x}\\&\qquad+\eta C\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^2+\eta C\|\Delta(p+q)\|_{L^2}^2+C(\eta+1).\end{aligned}\tag{2.50}$$

Note that since $m>1$, by Hölder's inequality and (2.43), we can show that

$$\begin{aligned}\|n\|_{L^2}^2=\int_{\Omega}n^2\,\mathrm{d}\mathbf{x}\leq\left(\int_{\Omega}n^{2m}\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{m}}|\Omega|^{\frac{m-1}{m}}&\leq C\left(\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^2+1\right)^{\frac{1}{m}}\\&\leq\eta\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^2+C,\end{aligned}$$

by which we update (2.50) as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla p\|_{L^2}^2+\frac{1}{2}\|\nabla q\|_{L^2}^2+\frac{1}{m}\int_{\Omega}n^m\mathrm{d}\mathbf{x}\right)+\frac{3}{4}\|\Delta p\|_{L^2}^2\\&\qquad+\frac{3}{4}\|\Delta q\|_{L^2}^2+\|\nabla q\|_{L^2}^2+\frac{3m(m-1)}{(2m-1)^2}\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^2\\&\leq C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla q\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2+C\int_{\Omega}n_0\,\mathrm{d}\mathbf{x}\\&\qquad+\eta C\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^2+\eta C\|\Delta(p+q)\|_{L^2}^2+C(\eta+1).\end{aligned}\tag{2.51}$$

By choosing $\eta>0$ to be sufficiently small, we can further update (2.51) as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla p\|_{L^2}^2+\frac{1}{2}\|\nabla q\|_{L^2}^2+\frac{1}{m}\int_{\Omega}n^m\mathrm{d}\mathbf{x}\right)+\frac{1}{2}\|\Delta p\|_{L^2}^2\\&\qquad+\frac{1}{2}\|\Delta q\|_{L^2}^2+\|\nabla q\|_{L^2}^2+\frac{3m(m-1)}{2(2m-1)^2}\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^2\\&\leq C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla q\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^2+C\int_{\Omega}n_0\,\mathrm{d}\mathbf{x}+C.\end{aligned}\tag{2.52}$$

By applying the Grönwall inequality to (2.52) and using (2.26), we can show that

$$\|\nabla p\|_{L^2}^2+\|\nabla q\|_{L^2}^2+\int_{\Omega}n^m\mathrm{d}\mathbf{x}+\int_0^T\left(\|\Delta p\|_{L^2}^2+\|\Delta q\|_{L^2}^2+\left\|\nabla n^{\frac{2m-1}{2}}\right\|_{L^2}^2\right)\mathrm{d}t\leq C,$$

which further implies

$$\int_0^T\int_\Omega n^{2m}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\le C, \tag{2.53}$$

according to (2.43), and

$$\|(p,q)\|_{L^\infty(0,T;H^1)}+\|(p,q)\|_{L^2(0,T;H^2)}+\|(\partial_t p,\partial_t q)\|_{L^2(0,T;L^2)}\le C, \tag{2.54}$$

by virtue of (1.9) and (1.10).

Step 6. Testing (1.8) by n^j $(2\le j<\infty)$ and using (1.7), we compute

$$\begin{aligned}&\frac{1}{j+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^{j+1}\mathrm{d}\mathbf{x}+\frac{4jm}{(j+m)^2}\int_\Omega|\nabla n^{\frac{j+m}{2}}|^2\mathrm{d}\mathbf{x}\\&=\frac{2j}{j+m}\int_\Omega Sn^{\frac{j-m+2}{2}}\nabla(p+q)\cdot\nabla n^{\frac{j+m}{2}}\mathrm{d}\mathbf{x}+\frac{2j}{j+m}\int_\Omega n^{\frac{j-m+2}{2}}\nabla\phi\cdot\nabla n^{\frac{j+m}{2}}\mathrm{d}\mathbf{x}\\&\le\frac{jm}{(j+m)^2}\left\|\nabla n^{\frac{j+m}{2}}\right\|_{L^2}^2+C\left\|n^{\frac{j-m+2-2\alpha}{2}}\right\|_{L^k}^2\|\nabla(p+q)\|^2_{L^{\frac{2k}{k-2}}}+C\int_\Omega n^{j-m+2}\mathrm{d}\mathbf{x},\end{aligned} \tag{2.55}$$

where (1.13) and the smoothness of ϕ are applied and $2<k<\infty$ is to be determined. After rearranging terms, we have

$$\begin{aligned}&\frac{1}{j+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^{j+1}\mathrm{d}\mathbf{x}+\frac{3jm}{(j+m)^2}\int_\Omega|\nabla n^{\frac{j+m}{2}}|^2\mathrm{d}\mathbf{x}\\&\le C\left\|n^{\frac{j-m+2-2\alpha}{2}}\right\|_{L^k}^2\|\nabla(p+q)\|^2_{L^{\frac{2k}{k-2}}}+C\int_\Omega n^{j-m+2}\mathrm{d}\mathbf{x}.\end{aligned} \tag{2.56}$$

For the first term on the right-hand side of (2.56), by using the Gagliardo-Nirenberg and Young inequalities, we have

$$\begin{aligned}\|\nabla(p+q)\|^2_{L^{\frac{2k}{k-2}}}&\le C\left(\|\Delta(p+q)\|_{L^2}^{\frac{4}{k}}\|\nabla(p+q)\|_{L^2}^{2-\frac{4}{k}}+\|\nabla(p+q)\|_{L^2}^2\right)\\&\le C\left(\|\Delta(p+q)\|_{L^2}^2+\|\nabla(p+q)\|_{L^2}^2\right),\quad\forall 2<k<\infty.\end{aligned}$$

Note that according to (1.14) or (1.15), it holds that $2<m+2\alpha<3$, which implies

$$\frac{(j-1)k}{2}<\frac{(j-m+2-2\alpha)k}{2}<\frac{jk}{2}.$$

Since $j\ge2$, there exists $k_0>2$, such that $jk_0\le2(j+1)$. Hence, we can show that

$$\left\|n^{\frac{j-m+2-2\alpha}{2}}\right\|_{L^{k_0}}^2\le C\int_\Omega n^{j+1}\mathrm{d}\mathbf{x}+C.$$

Moreover, since $1<m<2$, it holds that $j<j-m+2<j+1$. Hence, we can show that

$$\int_\Omega n^{j-m+2}\mathrm{d}\mathbf{x}\le C\int_\Omega n^{j+1}\mathrm{d}\mathbf{x}+C.$$

By using the above estimates, we update (2.56) as

$$\frac{1}{j+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^{j+1}\mathrm{d}\mathbf{x}+\frac{3jm}{(j+m)^2}\int_\Omega|\nabla n^{\frac{j+m}{2}}|^2\mathrm{d}\mathbf{x}$$

$$\leq C\left(\int_\Omega n^{j+1}\mathrm{d}\mathbf{x}\right)\left(\|(p,q)\|_{H^2}^2+1\right)+\left(\|(p,q)\|_{H^2}^2+1\right). \tag{2.57}$$

By applying the Grönwall inequality to (2.57) and using (2.54), we can show that

$$\int_\Omega n^{j+1}\mathrm{d}\mathbf{x}+\int_0^T\int_\Omega |\nabla n^{\frac{j+m}{2}}|^2\mathrm{d}\mathbf{x}\mathrm{d}t\leq C, \quad \forall 2<j<\infty. \tag{2.58}$$

Step 7. Testing (1.6) by $\nabla\pi-\Delta\mathbf{u}$ and using (1.7), we derive

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\pi-\Delta\mathbf{u}\|_{L^2}^2\\
=&\int_\Omega (n\nabla\phi+nS\nabla p+nS\nabla q-\mathbf{u}\cdot\nabla\mathbf{u})\cdot(\nabla\pi-\Delta\mathbf{u})\mathrm{d}\mathbf{x}\\
\leq& C\left(\|n\|_{L^2}+\|n\|_{L^4}\|\nabla p\|_{L^4}+\|n\|_{L^4}\|\nabla q\|_{L^4}+\|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\right)\|\nabla\pi-\Delta\mathbf{u}\|_{L^2}\\
\leq& C\left(1+\|\nabla p\|_{L^4}+\|\nabla q\|_{L^4}+\|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}\|\Delta\mathbf{u}\|_{L^2}^{\frac{1}{2}}\right)\|\nabla\pi-\Delta\mathbf{u}\|_{L^2}\\
\leq& \frac{1}{2}\|\nabla\pi-\Delta\mathbf{u}\|_{L^2}^2+C+C\|\nabla p\|_{L^4}^2+C\|\nabla q\|_{L^4}^2+C\|\nabla\mathbf{u}\|_{L^2}^4,
\end{aligned} \tag{2.59}$$

where the smoothness of ϕ, (2.58), (2.26) and the well-known H^2-estimate of the Stokes system:

$$\|\mathbf{u}\|_{H^2}\leq C\|\nabla\pi-\Delta\mathbf{u}\|_{L^2},$$

are applied. After rearranging terms, we get from (2.59) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{u}\|_{L^2}^2+\frac{1}{2}\|\nabla\pi-\Delta\mathbf{u}\|_{L^2}^2\leq C+C\|\nabla p\|_{L^4}^2+C\|\nabla q\|_{L^4}^2+C\|\nabla\mathbf{u}\|_{L^2}^4. \tag{2.60}$$

By applying the Grönwall inequality to (2.60) and using (2.26) and (2.54), we can show that

$$\|\mathbf{u}\|_{L^\infty(0,T;H^1)}+\|\mathbf{u}\|_{L^2(0,T;H^2)}+\|\partial_t\mathbf{u}\|_{L^2(0,T;L^2)}\leq C. \tag{2.61}$$

Step 8. With the *a priori* estimates established in the previous steps, we now apply standard theory for the heat equation to derive the desired estimates of the solution as stated in Theorem 1.1.

First, by the standard L^∞-estimate of the heat equation (cf. [2]), it follows from (1.10), (2.58), and (2.61) that

$$\|q\|_{L^\infty(0,T;L^\infty)}\leq C. \tag{2.62}$$

Next, we decompose p as $p\equiv p_1+p_2$, where p_1 and p_2 satisfy

$$\begin{cases}
\partial_t p_1-\Delta p_1=-\nabla\cdot(\mathbf{u}p), & (\mathbf{x},t)\in\Omega\times(0,T),\\
\nabla p_1\cdot\mathbf{n}=0, & (\mathbf{x},t)\in\partial\Omega\times(0,T),\\
p_1(\mathbf{x},0)=0, & \mathbf{x}\in\Omega,
\end{cases} \tag{2.63}$$

and

$$\begin{cases}
\partial_t p_2-\Delta p_2=-np, & (\mathbf{x},t)\in\Omega\times(0,T),\\
\nabla p_2\cdot\mathbf{n}=0, & (\mathbf{x},t)\in\partial\Omega\times(0,T),\\
p_2(\cdot,0)=p_0, & \mathbf{x}\in\Omega,
\end{cases} \tag{2.64}$$

respectively. By using (2.2), (2.58), (2.61) and the classical theory for the heat equation (cf. [2]), it can be shown that

$$\begin{aligned}\|\nabla p_1\|_{L^r(0,T;L^r)} &\le C\|\mathbf{u}p\|_{L^r(0,T;L^r)} \\ &\le C\|\mathbf{u}\|_{L^r(0,T;L^r)} \le C\|\mathbf{u}\|_{L^\infty(0,T;H^1)} \le C,\end{aligned} \tag{2.65}$$

and

$$\|\partial_t p_2\|_{L^r(0,T;L^r)} + \|p_2\|_{L^r(0,T;W^{2,r})} \le C\left(\|p_0\|_{W^{2-\frac{2}{r},r}} + \|np\|_{L^r(0,T;L^r)}\right) \le C \tag{2.66}$$

for any $2 \le r < \infty$. Hence, it holds that

$$\|\nabla p\|_{L^r(0,T;L^r)} \le C. \tag{2.67}$$

In a similar fashion, we can show that

$$\|\nabla q\|_{L^r(0,T;L^r)} \le C. \tag{2.68}$$

Now, we observe that the Equation (1.9) can be rewritten as

$$\partial_t p - \Delta p = f \equiv -\mathbf{u}\cdot\nabla p - np.$$

Note that according to (2.61), (2.67), (2.2) and (2.58), it holds that $f \in L^r(0,T;L^r)$, and thus we have

$$\|p\|_{L^\infty(0,T;W^{2-\frac{2}{r},r})} + \|\partial_t p\|_{L^r(0,T;L^r)} + \|p\|_{L^r(0,T;W^{2,r})} \le C. \tag{2.69}$$

Similarly, we can show that

$$\|q\|_{L^\infty(0,T;W^{2-\frac{2}{r},r})} + \|\partial_t q\|_{L^r(0,T;L^r)} + \|q\|_{L^r(0,T;W^{2,r})} \le C. \tag{2.70}$$

Lastly, by applying the same estimates as those in [12, 20], one can show that

$$\|\mathbf{u}\|_{L^\infty(0,T;L^\infty)} + \|n\|_{L^\infty(0,T;L^\infty)} \le C \tag{2.71}$$

The combination of (2.61), (2.69), (2.70) and (2.71) gives the desired estimates of the solution as recorded in (1.16). This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2. For the reader's convenience, we recall the system of equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi - \Delta\mathbf{u} = n\nabla\phi + nS\nabla p + nS\nabla q, & (3.1)\\ \nabla\cdot\mathbf{u} = 0, & (3.2)\\ \partial_t n + \mathbf{u}\cdot\nabla n - \Delta n = -\nabla\cdot(nS\nabla p) - \nabla\cdot(nS\nabla q) - \nabla\cdot(n\nabla\phi), & (3.3)\\ \partial_t p + \mathbf{u}\cdot\nabla p - \Delta p = -np, & (3.4)\\ \partial_t q + \mathbf{u}\cdot\nabla q - \Delta q + q = n, & (3.5)\end{cases}$$

$\mathbf{x} \in \mathbb{R}^3$, $t > 0$, which is supplemented with the initial and boundary conditions:

$$\begin{aligned} &(\mathbf{u}, n, p, q)(\mathbf{x}, 0) = (\mathbf{u}_0, n_0, p_0, q_0)(\mathbf{x}), && \mathbf{x} \in \overline{\Omega}, && (3.6)\\ &\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \nabla n^m \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \nabla p \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \nabla q \cdot \mathbf{n}|_{\partial\Omega} = 0, && t \ge 0, && (3.7)\end{aligned}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded convex domain with smooth boundary $\partial\Omega$, and $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$. Moreover, the chemotactic sensitivity $S(\mathbf{x},n,p,q)\in C^2(\overline{\Omega}\times[0,\infty)^3)$ is assumed to satisfy

$$|S|+|\partial_{\mathbf{x}}S|+|\partial_n S|+|\partial_p S|+|\partial_q S|\le S_1,\quad \forall(\mathbf{x},n,p,q)\in\overline{\Omega}\times[0,\infty)^3, \tag{3.8}$$

for some positive constant S_1.

In what follows, we will show that a local solution $(\mathbf{u},n,p,q)\in L^\infty(0,T;H^2)\cap L^2(0,T;H^3)$ can be extended beyond $0<T<\infty$, provided that the solution satisfies the extensibility criterion:

$$\mathbf{u}\in L^2(0,T;BMO),\quad \nabla p\in L^{\frac{2r}{r-3}}(0,T;L^r),\quad \nabla q\in L^{\frac{2s}{s-3}}(0,T;L^s) \tag{3.9}$$

for some $3<r,s\le\infty$, where BMO stands for the space of functions with bounded mean oscillation. As usual, in order to prove that the local classical solution can be extended beyond the local lifespan under the additional regularity assumption (3.9), we need to show that the $L^\infty(0,T;H^2)$ norm of the solution is finite, and in particular, $\|(\mathbf{u},n,p,q)\|_{H^2}(T^-)<\infty$. In the proof of the main result, we shall utilize the following two lemmas.

Lemma 3.1. *Let Ω be a domain in $\mathbb{R}^3$. Then it holds that*

$$\|f\|_{L^\infty(\Omega)}\le C\left(1+\|f\|_{BMO(\Omega)}\sqrt{\log\left(e+\|f\|_{W^{1,\gamma}(\Omega)}\right)}\right) \tag{3.10}$$

for $\forall f\in W_0^{1,\gamma}(\Omega)$ with $3<\gamma<\infty$, where the constant depends only on Ω and γ.

Proof. When $\Omega=\mathbb{R}^3$, (3.10) is proved by Ogawa [16]. For a bounded domain $\Omega\subset\mathbb{R}^3$ with smooth boundary, we define

$$\tilde{f}\equiv\begin{cases} f & \mathbf{x}\in\Omega,\\ 0 & \mathbf{x}\in\Omega^c\equiv\mathbb{R}^3\setminus\Omega.\end{cases}$$

Then we have (cf. [1, p.71])

$$\|\tilde{f}\|_{W^{1,\gamma}(\mathbb{R}^3)}=\|f\|_{W^{1,\gamma}(\Omega)},$$

and it is obvious that

$$\|\tilde{f}\|_{L^\infty(\mathbb{R}^3)}=\|f\|_{L^\infty(\Omega)},\quad \textit{and}\quad \|\tilde{f}\|_{BMO(\mathbb{R}^3)}\le C\|f\|_{BMO(\Omega)}.$$

Thus (3.10) is proved. □

Lemma 3.2 ([3]). *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with smooth boundary. Then it holds that*

$$\|f\|_{L^4(\Omega)}^2\le C\|f\|_{L^2(\Omega)}\|f\|_{BMO(\Omega)}. \tag{3.11}$$

For the proof of Theorem 1.2, first of all, we note that the estimates (2.1)–(2.5) are still valid in the three-dimensional case. We divide the subsequent proof into several steps.

Step 1. Testing (3.3) by n^{i-1} $(i\geq 2)$, using (3.2), (3.8) and denoting $w\equiv n^{\frac{i}{2}}$, we infer that

$$\begin{aligned}&\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2+\frac{4(i-1)}{i^2}\|\nabla w\|_{L^2}^2\\&=\frac{2(i-1)}{i}\int_\Omega w(\nabla p+\nabla q)\cdot\nabla w\,\mathrm{d}\mathbf{x}+\frac{2(i-1)}{i}\int_\Omega w\nabla\phi\cdot\nabla w\,\mathrm{d}\mathbf{x},\end{aligned}$$

which implies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2+\frac{4(i-1)}{i}\|\nabla w\|_{L^2}^2\\&=2(i-1)\int_\Omega w(\nabla p+\nabla q)\cdot\nabla w\,\mathrm{d}\mathbf{x}+2(i-1)\int_\Omega w\nabla\phi\cdot\nabla w\,\mathrm{d}\mathbf{x}.\end{aligned}\tag{3.12}$$

For the first term on the right-hand side of (3.12), by Hölder's inequality, we have

$$\begin{aligned}&2(i-1)\left|\int_\Omega w(\nabla p+\nabla q)\cdot\nabla w\,\mathrm{d}\mathbf{x}\right|\\&\leq 2(i-1)\left(\|w\nabla p\|_{L^2}+\|w\nabla q\|_{L^2}\right)\|\nabla w\|_{L^2}\\&\leq 2(i-1)\left(\|\nabla p\|_{L^r}\|w\|_{L^{\frac{2r}{r-2}}}+\|\nabla q\|_{L^s}\|w\|_{L^{\frac{2s}{s-2}}}\right)\|\nabla w\|_{L^2}\end{aligned}\tag{3.13}$$

for some $3<r,s\leq\infty$. By applying the Gagliardo-Nirenberg interpolation inequality, we can show that

$$\begin{aligned}&\|\nabla p\|_{L^r}\|w\|_{L^{\frac{2r}{r-2}}}\leq C\|\nabla p\|_{L^r}\left(\|w\|_{L^2}^{1-\frac{3}{r}}\|\nabla w\|_{L^2}^{\frac{3}{r}}+\|w\|_{L^2}\right),\\&\|\nabla q\|_{L^s}\|w\|_{L^{\frac{2s}{s-2}}}\leq C\|\nabla q\|_{L^s}\left(\|w\|_{L^2}^{1-\frac{3}{s}}\|\nabla w\|_{L^2}^{\frac{3}{s}}+\|w\|_{L^2}\right).\end{aligned}$$

So we update (3.13) as

$$\begin{aligned}&2(i-1)\left|\int_\Omega w(\nabla p+\nabla q)\cdot\nabla w\,\mathrm{d}\mathbf{x}\right|\\&\leq C\|\nabla p\|_{L^r}\left(\|w\|_{L^2}^{1-\frac{3}{r}}\|\nabla w\|_{L^2}^{1+\frac{3}{r}}+\|w\|_{L^2}\|\nabla w\|_{L^2}\right)\\&\quad+C\|\nabla q\|_{L^s}\left(\|w\|_{L^2}^{1-\frac{3}{s}}\|\nabla w\|_{L^2}^{1+\frac{3}{s}}+\|w\|_{L^2}\|\nabla w\|_{L^2}\right)\\&\leq\frac{(i-1)}{i}\|\nabla w\|_{L^2}^2+C\left(\|\nabla p\|_{L^r}^{\frac{2r}{r-3}}+\|\nabla q\|_{L^s}^{\frac{2s}{s-3}}\right)\|w\|_{L^2}^2+C\|w\|_{L^2}^2,\end{aligned}\tag{3.14}$$

where the Young inequality is applied. Moreover, for the second term on the right-hand side of (3.12), by using the smoothness of ϕ, we can show that

$$2(i-1)\left|\int_\Omega w\nabla\phi\cdot\nabla w\,\mathrm{d}\mathbf{x}\right|\leq C\|w\|_{L^2}\|\nabla w\|_{L^2}\leq\frac{(i-1)}{i}\|\nabla w\|_{L^2}^2+C\|w\|_{L^2}^2.\tag{3.15}$$

By substituting (3.14) and (3.15) into (3.12), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2+\frac{2(i-1)}{i}\|\nabla w\|_{L^2}^2\leq C\left(\|\nabla p\|_{L^r}^{\frac{2r}{r-3}}+\|\nabla q\|_{L^s}^{\frac{2s}{s-3}}\right)\|w\|_{L^2}^2+C\|w\|_{L^2}^2.\tag{3.16}$$

By applying the Grönwall inequality to (3.16) and using (3.9), we infer that

$$\|n\|_{L^\infty(0,T;L^i)}+\|n\|_{L^2(0,T;H^1)}\le C,\quad \forall 2\le i<\infty. \tag{3.17}$$

Step 2. Testing (3.5) by q and using (3.2), we compute

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_{L^2}^2+\|\nabla q\|_{L^2}^2+\|q\|_{L^2}^2\le\|n\|_{L^2}\|q\|_{L^2}\le\frac{1}{2}\|n\|_{L^2}^2+\frac{1}{2}\|q\|_{L^2}^2,$$

which implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_{L^2}^2+\|\nabla q\|_{L^2}^2+\frac{1}{2}\|q\|_{L^2}^2\le\frac{1}{2}\|n\|_{L^2}^2. \tag{3.18}$$

By integrating (3.18) with respect to t and using (3.17), we have

$$\|q(t)\|_{L^2}^2+\int_0^T\|q(t)\|_{H^1}^2\mathrm{d}t\le C,\quad\forall t\in(0,T]. \tag{3.19}$$

It follows from (3.5), (3.2), (3.17), and the standard L^∞-estimate of the heat equation that

$$\|q\|_{L^\infty(0,T;L^\infty)}\le C. \tag{3.20}$$

Step 3. Testing (3.1) by $\mathbf{u}$ and using (3.2), (3.8) and (3.17), we derive

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2&=\int_\Omega(nS\nabla p+nS\nabla q+n\nabla\phi)\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\\&\le C\left(\|\nabla p\|_{L^r}\|n\|_{L^{\frac{2r}{r-2}}}+\|\nabla q\|_{L^s}\|n\|_{L^{\frac{2s}{s-2}}}+\|n\|_{L^2}\right)\|\mathbf{u}\|_{L^2}\\&\le C(\|\nabla p\|_{L^r}+\|\nabla q\|_{L^s}+1)\|\mathbf{u}\|_{L^2}\\&\le\frac{1}{2}\|\mathbf{u}\|_{L^2}^2+C\left(\|\nabla p\|_{L^r}^2+\|\nabla q\|_{L^s}^2+1\right)\\&\le\frac{1}{2}\|\mathbf{u}\|_{L^2}^2+C\left(\|\nabla p\|_{L^r}^{\frac{2r}{r-3}}+\|\nabla q\|_{L^s}^{\frac{2s}{s-3}}+1\right),\end{aligned} \tag{3.21}$$

where $3<r,s\le\infty$. By applying the Grönwall inequality to (3.21) and using (3.9), we infer that

$$\|\mathbf{u}\|_{L^\infty(0,T;L^2)}+\|\mathbf{u}\|_{L^2(0,T;H^1)}\le C. \tag{3.22}$$

Next, we shall move on to the estimation of the higher frequencies of the velocity field. As the classical situation encountered in the three-dimensional incompressible Navier-Stokes equations, the main difficulty in building up the higher order regularity of the velocity field comes from the nonlinear convection term in (3.1), which essentially leads to the Beale-Kato-Majda blowup criterion [4]. In our case, since we are to utilize the temporal cumulation of the BMO norm of the velocity field as an extensibility (blowup) criterion, the situation is even worse than that in [4], due to, as was mentioned in the Introduction, the BMO-norm of a function being a penalty term as one replaces the usual Sobolev embedding: $W^{1,\gamma}\hookrightarrow L^\infty$ $(3<\gamma<\infty)$ by a logarithmic correction, see (3.10). In what follows, we overcome the difficulty by performing a somewhat "microanalysis" by monitoring the growth of the energy bound of the solution within a small time interval near the end of the lifespan of the local solution, and using an iterative argument based on the classical theory of the heat equation to close the energy estimates.

Step 4. Testing (3.1) by $\partial_t \mathbf{u}$ and using (3.2), (3.8) and (3.10), we compute

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{u}\|_{L^2}^2+\|\partial_t\mathbf{u}\|_{L^2}^2\\
&=\int_\Omega (n\nabla\phi+nS\nabla p+nS\nabla q)\cdot\partial_t\mathbf{u}\mathrm{d}\mathbf{x}-\int_\Omega(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_t\mathbf{u}\mathrm{d}\mathbf{x}\\
&\le C\left(\|n\|_{L^2}+\|n\|_{L^{\frac{2r}{r-2}}}\|\nabla p\|_{L^r}+\|n\|_{L^{\frac{2s}{s-2}}}\|\nabla q\|_{L^s}\right)\|\partial_t\mathbf{u}\|_{L^2}\\
&\quad+\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\partial_t\mathbf{u}\|_{L^2}\\
&\le\frac{1}{2}\|\partial_t\mathbf{u}\|_{L^2}^2+C\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2+C\left(1+\|\nabla p\|_{L^r}^2+\|\nabla q\|_{L^s}^2\right)\\
&\le\frac{1}{2}\|\partial_t\mathbf{u}\|_{L^2}^2+C\left[1+\|\mathbf{u}\|_{BMO}^2\log(e+\|\mathbf{u}\|_{W^{1,\gamma}})\right]\|\nabla\mathbf{u}\|_{L^2}^2\\
&\quad+C\left(1+\|\nabla p\|_{L^r}^{\frac{2r}{r-3}}+\|\nabla q\|_{L^s}^{\frac{2s}{s-3}}\right),
\end{aligned}$$

where $3<r,s\le\infty$. After rearranging terms, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{u}\|_{L^2}^2+\frac{1}{2}\|\partial_t\mathbf{u}\|_{L^2}^2\\
&\le C\left[1+\|\mathbf{u}\|_{BMO}^2\log(e+\|\mathbf{u}\|_{W^{1,\gamma}})\right]\|\nabla\mathbf{u}\|_{L^2}^2+C\left(1+\|\nabla p\|_{L^r}^{\frac{2r}{r-3}}+\|\nabla q\|_{L^s}^{\frac{2s}{s-3}}\right). \quad (3.23)
\end{aligned}$$

By applying the Grönwall inequality to (3.23) and using (3.9), we deduce that for any $0<t_0<t\le T$,

$$\begin{aligned}
\|\nabla\mathbf{u}(t)\|_{L^2}^2&\le\left[\|\nabla\mathbf{u}(t_0)\|_{L^2}^2+C\int_{t_0}^t\left(1+\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla p\|_{L^r}^{\frac{2r}{r-3}}+\|\nabla q\|_{L^s}^{\frac{2s}{s-3}}\right)\mathrm{d}\tau\right]\\
&\qquad\times\exp\left\{\log[e+y(t)]\int_{t_0}^t\|\mathbf{u}\|_{BMO}^2\mathrm{d}\tau\right\}\\
&\le C\exp\left\{\log[e+y(t)]\int_{t_0}^t\|\mathbf{u}\|_{BMO}^2\mathrm{d}\tau\right\}, \qquad (3.24)
\end{aligned}$$

where

$$y(t)\equiv\sup_{[t_0,t]}\|\mathbf{u}(s)\|_{W^{1,\gamma}},\quad 3<\gamma\le 6. \qquad (3.25)$$

Since $\mathbf{u}\in L^2(0,T;BMO)$, for any $\epsilon<1$, there exists $t_0<T$, such that

$$\int_{t_0}^T\|\mathbf{u}\|_{BMO}^2\mathrm{d}\tau\le\epsilon. \qquad (3.26)$$

Hence, we update (3.24) as

$$\|\nabla\mathbf{u}(t)\|_{L^2}^2\le C[e+y(t)]^\epsilon,\quad \forall t_0<t\le T, \qquad (3.27)$$

where the constant C depends on $\|\mathbf{u}(t_0)\|_{L^2}^2$. Later, we will choose an appropriate value of ϵ to close the energy estimates, by which t_0 will be fixed. By substituting (3.27) into (3.23), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{u}\|_{L^2}^2+\frac{1}{2}\|\partial_t\mathbf{u}\|_{L^2}^2$$

$$
\begin{aligned}
&\le C\left[1+\|\mathbf{u}\|_{BMO}^2\log\left(e+\|\mathbf{u}\|_{W^{1,\gamma}}\right)\right]\left[e+y(t)\right]^{\epsilon}+C\left(1+\|\nabla p\|_{L^r}^{\frac{2r}{r-3}}+\|\nabla q\|_{L^s}^{\frac{2s}{s-3}}\right)\\
&\le C\left[1+\|\mathbf{u}\|_{BMO}^2\log\left(e+y(T)\right)\right]\left[e+y(T)\right]^{\epsilon}+C\left(1+\|\nabla p\|_{L^r}^{\frac{2r}{r-3}}+\|\nabla q\|_{L^s}^{\frac{2s}{s-3}}\right). \qquad (3.28)
\end{aligned}
$$

Integrating (3.28) over (t_0,T) and using (3.26), we infer that

$$
\begin{aligned}
&\int_{t_0}^{T}\|\partial_t\mathbf{u}\|_{L^2}^2\mathrm{d}\tau\\
&\le \|\nabla\mathbf{u}(t_0)\|_{L^2}^2+C\left[e+y(T)\right]^{\epsilon}+C\left[e+y(T)\right]^{\epsilon}\log\left(e+y(T)\right)\int_{t_0}^{T}\|\mathbf{u}\|_{BMO}^2\mathrm{d}\tau+C\\
&\le C+C\left[e+y(T)\right]^{\epsilon}+C\left[e+y(T)\right]^{\epsilon}\log\left(e+y(T)\right)^{\epsilon}\\
&\le C[e+y(T)]^{2\epsilon}. \qquad (3.29)
\end{aligned}
$$

Moreover, by using (3.27), (3.29), and (3.17), we can show that

$$
\begin{aligned}
&\|\mathbf{u}\|_{H^2}\\
&\le C\|\nabla\pi-\Delta\mathbf{u}\|_{L^2}\\
&= C\|\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-n\nabla\phi-nS\nabla p-nS\nabla q\|_{L^2}\\
&\le C\|\partial_t\mathbf{u}\|_{L^2}+C\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}+C\|n\|_{L^2}+C\|n\|_{L^{\frac{2r}{r-2}}}\|\nabla p\|_{L^r}+C\|n\|_{L^{\frac{2s}{s-2}}}\|\nabla q\|_{L^s}\\
&\le C\|\partial_t\mathbf{u}\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^2}^3+\frac{1}{2}\|\mathbf{u}\|_{H^2}+C+C\|\nabla p\|_{L^r}+C\|\nabla q\|_{L^s}, \qquad (3.30)
\end{aligned}
$$

where we have applied the following interpolation inequalities in $\mathbb{R}^3$:

$$
\begin{aligned}
&\|\mathbf{u}\|_{L^6}\le C\|\nabla\mathbf{u}\|_{L^2},\\
&\|\nabla\mathbf{u}\|_{L^3}\le C\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
$$

After rearranging terms and squaring on both sides, we get from (3.30) that

$$
\|\mathbf{u}\|_{H^2}^2\le C\|\partial_t\mathbf{u}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^6+C+C\|\nabla p\|_{L^r}^2+C\|\nabla q\|_{L^s}^2. \qquad (3.31)
$$

Integrating (3.31) over (t_0,T) and using (3.27) and (3.29), we infer that

$$
\int_{t_0}^{T}\|\mathbf{u}\|_{H^2}^2\mathrm{d}\tau\le C[e+y(T)]^{3\epsilon}. \qquad (3.32)
$$

Step 5. Now we decompose p as $p\equiv p_1+p_2$, where p_1 and p_2 satisfy

$$
\begin{cases}
\partial_t p_1-\Delta p_1=-\nabla\cdot(\mathbf{u}p), & (\mathbf{x},t)\in\Omega\times(0,T),\\
\nabla p_1\cdot\mathbf{n}=0, & (\mathbf{x},t)\in\partial\Omega\times(0,T),\\
p_1(\mathbf{x},0)=0, & \mathbf{x}\in\Omega,
\end{cases} \qquad (3.33)
$$

and

$$
\begin{cases}
\partial_t p_2-\Delta p_2=-np, & (\mathbf{x},t)\in\Omega\times(0,T),\\
\nabla p_2\cdot\mathbf{n}=0, & (\mathbf{x},t)\in\partial\Omega\times(0,T),\\
p_2(\cdot,0)=p_0, & \mathbf{x}\in\Omega,
\end{cases} \qquad (3.34)
$$

respectively. It follows from the classical theory for heat equation [2] and (2.2) that

$$\|\nabla p_1\|_{L^6(t_0,T;L^6)} \le C\|\mathbf{u}p\|_{L^6(t_0,T;L^6)} \le C\|\mathbf{u}\|_{L^6(t_0,T;L^6)}.$$

By using Gagliardo-Nirenberg inequality and (3.27), we can show that

$$\int_{t_0}^{T} \|\mathbf{u}\|_{L^6}^6 \,d\tau \le C\int_{t_0}^{T} \|\nabla\mathbf{u}\|_{L^2}^6 \,d\tau \le C(T-t_0)[e+y(T)]^{3\epsilon} \le C[e+y(T)]^{3\epsilon},$$

which implies

$$\|\nabla p_1\|_{L^6(t_0,T;L^6)} \le C[e+y(T)]^{\frac{1}{2}\epsilon}. \tag{3.35}$$

Similarly, by using (3.17) and (2.2), one can show that

$$\|\nabla p_2\|_{L^6(t_0,T;L^6)} \le C\|p_2\|_{L^6(t_0,T;W^{2,6})} \le C\left(\|p_0\|_{W^{2-\frac{1}{3},6}} + \|np\|_{L^6(t_0,T;L^6)}\right) \le C, \tag{3.36}$$

whence

$$\|\nabla p\|_{L^6(t_0,T;L^6)} \le C[e+y(T)]^{\frac{1}{2}\epsilon}. \tag{3.37}$$

In a similar fashion, we can show that

$$\|\nabla q\|_{L^6(t_0,T;L^6)} \le C[e+y(T)]^{\frac{1}{2}\epsilon}. \tag{3.38}$$

Step 6. We note that the Equation (3.3) can be rewritten as

$$\partial_t n - \Delta n = -\nabla\cdot(\mathbf{u}n + nS\nabla p + nS\nabla q + n\nabla\phi). \tag{3.39}$$

By using (3.27), (3.37), (3.38) and the standard L^∞-estimate of the heat equation, one can show that

$$\|n\|_{L^\infty(t_0,T;L^\infty)} \le C[e+y(T)]^{\frac{1}{2}\epsilon}. \tag{3.40}$$

It then follows from (3.40) and the previous estimates that

$$\|\nabla n\|_{L^6(t_0,T;L^6)} \le C[e+y(T)]^{\epsilon}. \tag{3.41}$$

Observe that the Equation (3.4) can be rewritten as

$$\partial_t p - \Delta p = f \equiv -\mathbf{u}\cdot\nabla p - np \quad \textit{with} \quad \|f\|_{L^3(t_0,T;L^3)} \le C[e+y(T)]^{\epsilon}, \tag{3.42}$$

and thus we have

$$\|\partial_t p\|_{L^3(t_0,T;L^3)} + \|p\|_{L^3(t_0,T;W^{2,3})} \le C[e+y(T)]^{\epsilon}. \tag{3.43}$$

Similarly, we have

$$\|\partial_t q\|_{L^3(t_0,T;L^3)} + \|q\|_{L^3(t_0,T;W^{2,3})} \le C[e+y(T)]^{\epsilon}. \tag{3.44}$$

It then follows from (3.27), (3.37), (3.38), (3.40), (3.41), (3.8) and the smoothness of ϕ that the right-hand side of (3.39) satisfies the estimate: $\|g\|_{L^3(t_0,T;L^3)} \le C[e+y(T)]^{2\epsilon}$, where $g \equiv -\nabla\cdot(\mathbf{u}n + nS\nabla p + nS\nabla q + n\nabla\phi)$. Therefore

$$\|\partial_t n\|_{L^3(t_0,T;L^3)} + \|n\|_{L^3(t_0,T;W^{2,3})} \le C[e+y(T)]^{2\epsilon}. \tag{3.45}$$

Step 7. Applying ∂_t to (3.1), testing by $\partial_t \mathbf{u}$ and using (3.2), we observe that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\mathbf{u}\|_{L^2}^2+\|\nabla\partial_t\mathbf{u}\|_{L^2}^2\\&=-\int_\Omega(\partial_t\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_t\mathbf{u}\mathrm{d}\mathbf{x}+\int_\Omega\partial_t n\nabla\phi\cdot\partial_t\mathbf{u}\mathrm{d}\mathbf{x}+\int_\Omega\partial_t(nS)\nabla(p+q)\cdot\partial_t\mathbf{u}\mathrm{d}\mathbf{x}\\&\quad+\int_\Omega nS\nabla\partial_t(p+q)\cdot\partial_t\mathbf{u}\mathrm{d}\mathbf{x}\ \equiv\ I_1+I_2+I_3+I_4.\end{aligned}\tag{3.46}$$

In what follows, we shall estimate $I_1,...I_4$ within the time interval $[t_0,T]$. To bound I_1, by using the Gagliardo-Nirenberg inequality and (3.27), we can show that

$$\begin{aligned}|I_1|\le\|\nabla\mathbf{u}\|_{L^2}\|\partial_t\mathbf{u}\|_{L^4}^2\le C\|\nabla\mathbf{u}\|_{L^2}\|\partial_t\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_t\mathbf{u}\|_{L^2}^{\frac{3}{2}}&\\\le\frac{1}{2}\|\nabla\partial_t\mathbf{u}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^2}^4\|\partial_t\mathbf{u}\|_{L^2}^2&\\\le\frac{1}{2}\|\nabla\partial_t\mathbf{u}\|_{L^2}^2+C[e+y(T)]^{2\epsilon}\|\partial_t\mathbf{u}\|_{L^2}^2.&\end{aligned}\tag{3.47}$$

For I_2, by using the smoothness of ϕ, we have

$$\begin{aligned}|I_2|\le\|\nabla\phi\|_{L^\infty}\|\partial_t n\|_{L^2}\|\partial_t\mathbf{u}\|_{L^2}\le\|\partial_t\mathbf{u}\|_{L^2}^2+C\|\partial_t n\|_{L^2}^2&\\\le\|\partial_t\mathbf{u}\|_{L^2}^2+C\|\partial_t n\|_{L^3}^2&\\\le\|\partial_t\mathbf{u}\|_{L^2}^2+C\|\partial_t n\|_{L^3}^3+C.&\end{aligned}\tag{3.48}$$

By using (3.8), we estimate I_3 as

$$\begin{aligned}|I_3|&=\left|\int_\Omega\left[S\partial_t n+n\left(\frac{\partial S}{\partial n}\partial_t n+\frac{\partial S}{\partial p}\partial_t p+\frac{\partial S}{\partial q}\partial_t q\right)\right]\nabla(p+q)\cdot\partial_t\mathbf{u}\mathrm{d}\mathbf{x}\right|\\&\le C\|(\partial_t n,\partial_t p,\partial_t q)\|_{L^3}\|\nabla(p+q)\|_{L^6}\|\partial_t\mathbf{u}\|_{L^2}(1+\|n\|_{L^\infty})\\&\le\|\partial_t\mathbf{u}\|_{L^2}^2(1+\|n\|_{L^\infty}^2)+C\left(\|(\partial_t n,\partial_t p,\partial_t q)\|_{L^3}^3+\|(\nabla p,\nabla q)\|_{L^6}^6\right)\\&\le C[e+y(T)]^\epsilon\|\partial_t\mathbf{u}\|_{L^2}^2+C\left(\|(\partial_t n,\partial_t p,\partial_t q)\|_{L^3}^3+\|(\nabla p,\nabla q)\|_{L^6}^6\right),\end{aligned}\tag{3.49}$$

where (3.40) is applied. Lastly, for I_4, we can show that

$$\begin{aligned}|I_4|&=\left|\int_\Omega\partial_t(p+q)\partial_t\mathbf{u}\cdot\nabla(nS)\mathrm{d}\mathbf{x}\right|\\&\le C\|\partial_t(p+q)\|_{L^3}\|\partial_t\mathbf{u}\|_{L^2}(\|n\|_{L^\infty}\|(\partial_\mathbf{x}S,\nabla n,\nabla p,\nabla q)\|_{L^6}+\|\nabla n\|_{L^6})\\&\le\|\partial_t\mathbf{u}\|_{L^2}^2\left(\|n\|_{L^\infty}^2+1\right)+C\left(\|(\partial_t p,\partial_t q)\|_{L^3}^3+\|(\nabla n,\nabla p,\nabla q)\|_{L^6}^6+1\right)\\&\le C[e+y(T)]^\epsilon\|\partial_t\mathbf{u}\|_{L^2}^2+C\left(\|(\partial_t p,\partial_t q)\|_{L^3}^3+\|(\nabla n,\nabla p,\nabla q)\|_{L^6}^6+1\right).\end{aligned}\tag{3.50}$$

By substituting (3.47)–(3.50) into (3.46), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t\mathbf{u}\|_{L^2}^2+\frac{1}{2}\|\nabla\partial_t\mathbf{u}\|_{L^2}^2\\&\le C[e+y(T)]^{2\epsilon}\|\partial_t\mathbf{u}\|_{L^2}^2+C\left(\|(\partial_t n,\partial_t p,\partial_t q)\|_{L^3}^3+\|(\nabla n,\nabla p,\nabla q)\|_{L^6}^6+1\right).\end{aligned}\tag{3.51}$$

By integrating (3.51) over (t_0,T), and using (3.29), (3.43), (3.44), (3.45), (3.37), (3.38) and (3.41), we can show that

$$\|\partial_t\mathbf{u}\|_{L^2}^2+\int_{t_0}^T\|\nabla\partial_t\mathbf{u}\|_{L^2}^2\mathrm{d}\tau\le C[e+y(T)]^{6\epsilon}.\tag{3.52}$$

Step 8. Testing (3.4) by $-\Delta p$ and using (3.11), (2.2), (3.17), and (3.22), we can show that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla p\|_{L^2}^2+\|\Delta p\|_{L^2}^2 &= \int_\Omega (\mathbf{u}\cdot\nabla p+np)\Delta p\,\mathrm{d}\mathbf{x}\\ &\le (\|\mathbf{u}\|_{L^4}\|\nabla p\|_{L^4}+\|n\|_{L^2}\|p\|_{L^\infty})\|\Delta p\|_{L^2}\\ &\le C(\|\mathbf{u}\|_{L^2}^{\frac12}\|\mathbf{u}\|_{BMO}^{\frac12}\|p\|_{L^\infty}^{\frac12}\|\Delta p\|_{L^2}^{\frac12}+1)\|\Delta p\|_{L^2}\\ &\le \frac12\|\Delta p\|_{L^2}^2+C\|\mathbf{u}\|_{BMO}^2+C,\end{aligned}\tag{3.53}$$

which implies, by (3.9),

$$\|p\|_{L^\infty(0,T;H^1)}+\|p\|_{L^2(0,T;H^2)}\le C.\tag{3.54}$$

In a similar fashion, we can show that

$$\|q\|_{L^\infty(0,T;H^1)}+\|q\|_{L^2(0,T;H^2)}\le C.\tag{3.55}$$

By using similar arguments in (3.30), we can show that

$$\begin{aligned}\|\mathbf{u}\|_{H^2}&\le C\|\partial_t\mathbf{u}\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^2}^3+C\|n\|_{L^2}+C\|n\|_{L^\infty}\|\nabla p\|_{L^2}+C\|n\|_{L^\infty}\|\nabla q\|_{L^2}\\ &\le C[e+y(T)]^{3\epsilon},\quad \forall t\in[t_0,T],\end{aligned}$$

where (3.52), (3.27), (3.17), (3.40), (3.54) and (3.55) are applied. By using the definition of $y(t)$, we infer that

$$\|\mathbf{u}(t)\|_{H^2}\le C\Big(e+\sup_{[t_0,T]}\|\mathbf{u}(t)\|_{W^{1,\gamma}}\Big)^{3\epsilon},\quad \forall t\in[t_0,T],\quad 3<\gamma\le 6.$$

According to the Sobolev embedding: $H^2\hookrightarrow W^{1,\gamma}$ $(3<\gamma\le 6)$, we see from above that

$$\sup_{[t_0,T]}\|\mathbf{u}(t)\|_{H^2}\le C\Big(e+C\sup_{[t_0,T]}\|\mathbf{u}(t)\|_{H^2}\Big)^{3\epsilon}.$$

By choosing $\epsilon=\frac16$, we obtain

$$\|\mathbf{u}\|_{L^\infty(t_0,T;H^2)}\le C.\tag{3.56}$$

At this point, we see that when the value of ϵ is fixed, one can fix the value of $t_0<T$, such that (3.26) is fulfilled, and therefore all the constants (depending on the energy of the local solution at $t=t_0$) appearing in the energy estimates following (3.26) are finite. Consequently, it follows from the definition of $y(t)$ and (3.56) that $y(T)\le C$, which in turn implies that

$$\begin{aligned}&\|\partial_t\mathbf{u}\|_{L^\infty(t_0,T;L^2)}+\|\nabla\partial_t\mathbf{u}\|_{L^2(t_0,T;L^2)}\le C,\\ &\|(\nabla n,\nabla p,\nabla q)\|_{L^6(t_0,T;L^6)}+\|(\partial_t n,\partial_t p,\partial_t q)\|_{L^3(t_0,T;L^3)}+\|(n,p,q)\|_{L^3(t_0,T;W^{2,3})}\le C,\\ &\|n\|_{L^\infty(t_0,T;L^\infty)}\le C,\end{aligned}$$

in view of (3.52), (3.37), (3.38), (3.41), (3.43), (3.44), (3.45) and (3.40). Since $t_0<T$ is fixed, according to the local existence theory we can update the above estimates as

$$\begin{aligned}&\|\partial_t\mathbf{u}\|_{L^\infty(0,T;L^2)}+\|\partial_t\mathbf{u}\|_{L^2(0,T;H^1)}\le C,\\ &\|(\nabla n,\nabla p,\nabla q)\|_{L^6(0,T;L^6)}+\|(\partial_t n,\partial_t p,\partial_t q)\|_{L^3(0,T;L^3)}+\|(n,p,q)\|_{L^3(0,T;W^{2,3})}\le C,\\ &\|n\|_{L^\infty(0,T;L^\infty)}\le C.\end{aligned}\tag{3.57}$$

Moreover, by applying (3.57) and classical estimates of the Stokes' system, we can improve the estimate of the velocity field as

$$\|\mathbf{u}\|_{L^2(0,T;H^3)} \leq C. \tag{3.58}$$

Step 9. Applying ∂_t to (3.4), testing by $\partial_t p$, using (3.2), (3.12), (3.57), and (3.58), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t p\|_{L^2}^2 + \|\nabla\partial_t p\|_{L^2}^2 + \int_\Omega n(\partial_t p)^2 \mathrm{d}\mathbf{x} \\
&= -\int_\Omega (\partial_t \mathbf{u}\cdot\nabla p)\,\partial_t p\,\mathrm{d}\mathbf{x} - \int_\Omega (\partial_t n)\,p\,(\partial_t p)\,\mathrm{d}\mathbf{x} \\
&\leq \|\partial_t \mathbf{u}\|_{L^3}\|\nabla p\|_{L^6}\|\partial_t p\|_{L^2} + \|p\|_{L^\infty}\|\partial_t n\|_{L^2}\|\partial_t p\|_{L^2} \\
&\leq C\|\partial_t \mathbf{u}\|_{H^1}\|\nabla p\|_{L^6}\|\partial_t p\|_{L^2} + C\|p\|_{H^2}\|\partial_t n\|_{L^2}\|\partial_t p\|_{L^2} \\
&\leq \|\partial_t p\|_{L^2}^2\left(\|\partial_t \mathbf{u}\|_{H^1}^2 + \|p\|_{H^2}^2\right) + C\left(\|\nabla p\|_{L^6} + \|\partial_t n\|_{L^2}\right).
\end{aligned} \tag{3.59}$$

By applying the Grönwall inequality to (3.59) and using (3.57), we can show that

$$\|\partial_t p\|_{L^\infty(0,T;L^2)} + \|\partial_t p\|_{L^2(0,T;H^1)} \leq C. \tag{3.60}$$

In a completely similar fashion, by applying ∂_t to (3.5) and (3.3) and using the previous estimates, we can show that

$$\begin{aligned}
&\|\partial_t q\|_{L^\infty(0,T;L^2)} + \|\partial_t q\|_{L^2(0,T;H^1)} \leq C, \\
&\|\partial_t n\|_{L^\infty(0,T;L^2)} + \|\partial_t n\|_{L^2(0,T;H^1)} \leq C.
\end{aligned}$$

Then by further iterating the estimates by using classical theory of the heat equation, we can show that

$$\begin{aligned}
&\|p\|_{L^\infty(0,T;H^2)} + \|p\|_{L^2(0,T;H^3)} \leq C, \\
&\|q\|_{L^\infty(0,T;H^2)} + \|q\|_{L^2(0,T;H^3)} \leq C, \\
&\|n\|_{L^\infty(0,T;H^2)} + \|n\|_{L^2(0,T;H^3)} \leq C.
\end{aligned}$$

Since the proof is standard, we omit the technical details to simplify the presentation. This completes the proof of Theorem 1.2.

4. Conclusion and looking ahead

We have studied the qualitative behavior of large-amplitude solutions to an integrated chemotaxis-fluid model, (1.6)–(1.10), under the initial and Dirichlet-Neumann-type boundary conditions, (1.11)–(1.12). In particular, it is shown that for the model with porous medium-like diffusion in the two-dimensional space, large-amplitude strong solutions exist globally in time for naturally prepared initial data under appropriate conditions on the generalized chemotactic sensitivity function, see Theorem 1.1. For the model with linear diffusion in the three-dimensional space, it is shown that local classical solutions can be extended beyond their lifespans provided that certain solution components belong to some critical energy spaces involving the BMO and L^r norms of the local solutions, see Theorem 1.2. These appear to be among the first analytical results for the coupled chemotaxis-fluid model (1.6)–(1.10).

On the other hand, many questions concerning the qualitative behavior of the integrated model are widely open, whose resolution may require more in-depth analyses than the ones presented in this paper. We leave the investigation to future works. These questions include, but are not limited to:

- long-time behavior of large-amplitude solutions to the initial-boundary value problems of (1.6)–(1.10) under the conditions of Theorem 1.1;
- global existence and long-time behavior of large-amplitude solutions to the initial-boundary value problems of (1.6)–(1.10) under the conditions different from those of Theorem 1.1, especially (1.13);
- extensibility criteria of large-amplitude solutions to the initial-boundary value problems of (1.6)–(1.10) in different functional spaces from (1.18).

We hope that the research of these topics will offer future opportunities in this area.

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