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GLOBAL STRONG SOLUTIONS TO THE CAUCHY PROBLEM OF 1D COMPRESSIBLE MHD EQUATIONS WITH NO RESISTIVITY*

ZILAI LI†, HUAQIAO WANG‡, AND YULIN YE§

Abstract. We consider the Cauchy problem to the 1D non-resistive compressible magnetohydrodynamics (MHD) equations. We establish the global existence and uniqueness of strong solutions for large initial data and vacuum when the viscosity coefficient is assumed to be constant or density-dependent. The analysis is based on the full use of effective viscous flux and the Caffarelli-Kohn-Nirenberg weighted inequality to get the higher-order estimates of the solutions. This result could be viewed as the first one on the global well-posedness of strong solutions to the Cauchy problem of 1D non-resistive compressible MHD equations while the initial data may be arbitrarily large and permit vacuum.

Keywords. 1D compressible MHD equations; zero-resistivity; Cauchy problem; global strong solutions; vacuum.

AMS subject classifications. 35D35; 35Q35; 76N10; 76W05.

1. Introduction and main result

Compressible magnetohydrodynamics (MHD) equations with density-dependent viscosity in $\mathbb{R}^1$ can be described as:

$$\begin{cases} \rho_t+(\rho u)_x=0, \\ (\rho u)_t+(\rho u^2+P(\rho)+\frac{1}{2}b^2)_x=(\mu(\rho)u_x)_x, \\ b_t+(ub)_x=\nu b_{xx}, \end{cases} \tag{1.1}$$

where $\rho, u, P(\rho)$ and b denote the density, velocity, pressure and magnetic field, respectively. $\mu(\rho)=1+a\rho^\beta$ $(\beta\geq 0)$ is the viscosity, the constant $\nu>0$ is the resistivity coefficient acting as the magnetic diffusion coefficient of the magnetic field. In this paper, we consider the isentropic compressible MHD equations in which the equation of the state has the form

$$P(\rho)=R\rho^\gamma, \gamma>1.$$

For simplicity, we set $a=R=1$.

The compressible MHD equations are used to describe the macroscopic behavior of the electrically conducting fluid in a magnetic field. However, it is well known that the resistivity coefficient ν is inversely proportional to the electrical conductivity, therefore it is more reasonable to ignore the magnetic diffusion which means $\nu=0$, when the conducting fluid considered is of high conductivity, for example the ideal conductors. So instead of Equations (1.1), when there is no resistivity, the system reduces to the so-called compressible, isentropic, viscous and non-resistive MHD equations with the

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†School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, P.R. China (lizl@hpu.edu.cn).

‡College of Mathematics and Statistics, Chongqing University, Chongqing 401331, P.R. China (hqwang111@163.com).

§Corresponding author. School of Mathematics and Statistics, Henan University, Kaifeng 475004, P.R. China (ylye@vip.henu.edu.cn).

following form:

$$\begin{cases}\rho_t+(\rho u)_x=0,\\ (\rho u)_t+(\rho u^2+P(\rho)+\frac{1}{2}b^2)_x=(\mu(\rho)u_x)_x,\\ b_t+(ub)_x=0.\end{cases} \tag{1.2}$$

In this paper, we consider the Cauchy problem for (1.2) with (ρ,u,b) satisfying the initial conditions as follows:

$$(\rho,u,b)|_{t=0}=(\rho_0(x),u_0(x),b_0(x))\to(0,0,0),\ x\in\mathbb{R}. \tag{1.3}$$

There has been a huge amount of literature on the studies of the compressible MHD equations by physicists and mathematics due to its physical importance, complexity, rich phenomena, and mathematical challenges. Before stating our main theorem, we briefly recall some previously known results on compressible MHD equations. Firstly, we begin with the MHD equations with magnetic diffusion. For one-dimensional case, Vol'pert-Hudjaev [24] proved the local existence and uniqueness of strong solutions to the Cauchy problem and Kawashima-Okada [20] obtained the global smooth solutions with small initial data. For large initial data and the density that may contain vaccum, the authors Ye-Li [27] proved the global existence of strong solutions to the 1D Cauchy problem. When considering the full MHD equations and the heat conductivity depending on the temperature θ, Chen-Wang [3] studied the free boundary value problem and established the existence, uniqueness and Lipschitz dependence of strong solutions. Recently, Fan-Huang-Li [7] obtained the global strong solutions to the initial boundary value problem to the planner MHD equations with temperature-dependent heat conductivity. Later, with the effect of self-gravitation as well as the influence of radiation on the dynamics in high temperature regimes taken into account, Zhang-Xie [29] obtained the global strong solutions to the initial boundary value problem for the nonlinear planner MHD equations. For multi-dimensional MHD equations, Lv-Shi-Xu [22] considered the 2-D isentropic MHD equations and proved the global existence of classical solutions provided that the initial energy is small, in which the decay rates of the solutions were also obtained. By exploiting some L^p-L^q estimates of solutions for the heat equation and linearized Navier-Stokes system, Zhang-Zhao [30] obtained the time-asymptotic behavior of solutions to the 3D isentropic MHD equations when the initial data around a constant state is sufficiently small in H^3 and is bounded in L^p with any given $1\le p\le\frac{6}{5}$. Vol'pert-Hudjaev [24] and Fan-Yu [6] obtained the local classical solution to the 3-D compressible MHD equations where the initial density is strictly positive or could contain vacuum, respectively. Hu-Wang [9] derived the global weak solutions to the 3-D compressible MHD equations with large initial data. Recently, Li-Xu-Zhang [21] established the global existence of classical solution of 3-D MHD equations which are of small energy but possibly large oscillations. Later, the result was improved by Hong-Hou-Peng-Zhu [8] provided $((\gamma-1)^{\frac{1}{9}}+\nu^{-\frac{1}{4}})E_0$ is suitably small. When the resistivity is zero, then the magnetic equation is reduced from the heat-type equation to the hyperbolic-type equation, the problem becomes more challenging, hence the results are few. Kawashima [19] obtained the classical solutions to 3-D MHD equations when the initial data are of small perturbations in H^3 norm and away from vacuum. Xu-Zhang [25] proved a blow-up criterion of strong solutions for 3-D isentropic MHD equations with vacuum. Fan-Hu [5] established the global strong solutions to the initial boundary value problem of 1-D heat-conducting MHD equations with no resistivity. We note that some authors considered the Rayleigh-Taylor problem of the non-resistive MHD equations (see [4, 10–16] and references therein). With more general heat-conductivity, Zhang-Zhao [31] established the global strong solutions and also obtained the non-resistivity limits of the solutions in L^2 norm. Jiang-Zhang [17] considered the initial boundary value problem to 1-D isentropic MHD equations and obtained the global strong solutions, in which they also gave the "magnetic boundary layer" estimates when the resistivity $\nu\to 0$. Yu [28] obtained the global existence of strong solutions to the initial boundary value problem of 1-D isentropic non-resistive MHD equations. It is easy to see that the previous results concern the initial boundary value problem, however, for the Cauchy problem, the global strong solution with large initial data and vacuum is still unknown. The aim of this paper is to study the existence and uniqueness of global strong solutions to the Cauchy problem of 1-D non resistive MHD equations with large initial data and vacuum.

Notations: We denote the material derivative of u and effective viscous flux by

$$\dot u \triangleq u_t + uu_x \text{ and } F \triangleq \mu(\rho)u_x - P(\rho) - \frac{1}{2}b^2,\quad \int \cdot = \int_{\mathbb{R}} \cdot dx,$$
$$D^k = \{u \in L^1_{loc}(\mathbb{R}) |\ \|\partial_x^k u\|_{L^2} < \infty\},\quad \|u\|_{D^k} = \|\partial_x^k u\|_{L^2},\quad H^k = L^2 \cap D^k.$$

The main result of this paper can be stated as:

THEOREM 1.1. *Suppose that the initial data $(\rho_0, u_0, b_0)(x)$ satisfies*

$$\begin{aligned} &0 \le \rho_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R}),\ b_0 \in H^1(\mathbb{R}),\ u_0 \in H^2(\mathbb{R}),\\ &\sqrt{\rho_0}u_0(1+|x|^{\frac{\alpha}{2}}) \in L^2(\mathbb{R}),\quad |x|^{\frac{\alpha}{2}}u_{0x} \in L^2(\mathbb{R}),\quad (\rho_0^{\frac{\gamma}{2}}|x|^{\frac{\alpha}{2}}, b_0|x|^{\frac{\alpha}{2}}) \in L^2(\mathbb{R}), \end{aligned} \tag{1.4}$$

for $2 < \alpha < 1 + \frac{2}{\sqrt[3]{1+\sqrt[3]{4}}} (\approx 2.46)$, and the compatibility condition

$$\left(\mu(\rho_0)u_{0x} - P(\rho_0) - \frac{1}{2}b_0^2\right)_x = \sqrt{\rho_0}g(x),\quad x \in \mathbb{R}, \tag{1.5}$$

with some g satisfying $g(1+|x|^{\frac{\alpha}{2}}) \in L^2(\mathbb{R})$. Then for any $T > 0$ and $\beta \ge 1$ or $\beta = 0$, there exists a unique global strong solution (ρ, u, b) to the Cauchy problem (1.2)-(1.3) such that

$$\begin{aligned} &0 \le (\rho, b) \le C,\ (\rho, b) \in L^\infty(0,T;H^1(\mathbb{R})),\ (\rho_t, b_t) \in L^\infty(0,T;L^2(\mathbb{R})),\\ &\sqrt{\rho}u(1+|x|^{\frac{\alpha}{2}}),\ \sqrt{\rho}\dot u(1+|x|^{\frac{\alpha}{2}}),\ |x|^{\frac{\alpha}{2}}(u_x, \rho^{\frac{\gamma}{2}}, b) \in L^\infty(0,T;L^2(\mathbb{R})),\\ &u \in L^\infty(0,T;H^2(\mathbb{R})),\ \sqrt{\rho}u_t \in L^\infty(0,T;L^2(\mathbb{R})),\ u_t \in L^2(0,T;D^1(\mathbb{R})). \end{aligned}$$

REMARK 1.1. The restrictions $\beta \ge 1$ or $\beta = 0$ in Theorem 1.1 can be generalized to $\beta \ge 0$ if an additional condition $\rho_0^\beta \in H^1(\mathbb{R})$ is given.

REMARK 1.2. It is easy to see that our result holds for both the density-dependent viscosity ($\beta \ge 1$) case and constant viscosity ($\beta = 0$) case.

REMARK 1.3. When there is no magnetic field, that is, $b = 0$, our result reduces to that of the 1-D isentropic compressible Navier-Stokes equations in [18], which means that we generalized the result in [18] to the MHD equations even with no magnetic diffusion.

We now give some comments on the analysis of this paper. Comparing with the previously known results relating to resistive MHD equations and the initial boundary value problems, some new difficulties will arise. Firstly, we can not get the integrability of the magnetic field b in $L^p(1<p<\infty)$ space directly due to the lack of magnetic diffusion, where the estimate $\int_0^T\|u_x\|_{L^\infty}dt$ is needed (see (3.19)); to overcome this difficulty, motivated by the Navier-Stokes equations in which the effective viscous flux has more regularity than the gradient of velocity, similarly we introduce the effective viscous flux F in MHD equations and estimate the integral of the effective viscous flux F instead of u_x (see (3.19)-(3.24)). Next, the 1-D non-resistive MHD equations look similar to the two-phase fluids (e.g. [26]), hence, when handling the MHD equations, usually it is technically assumed that the proportion between the magnetic field and density is bounded, that is $\rho,b\geq 0$ and $0\leq\frac{b}{\rho}<\infty$, which implies the magnetic field b is bounded provided the density ρ is bounded. However, this assumption is not physical and realistic in magnetohydrodynamics, so how to obtain the upper bound of the magnetic field is another main difficulty, our strategy is based on the full use of effective viscous flux, the material derivative and the structure of the equations. Finally, when deriving the high order derivatives of the solutions, the Poincáre-type inequality is no longer valid and we have no $L^p(1\leq p<\infty)$ norm of the velocity because the region considered is the whole space. To overcome this difficulty, we establish the weighted estimates on the solutions by using the Caffarelli-Kohn-Nirenberg weighted inequality and furthermore obtain some $L^p(1<p<\infty)$ and L^∞ norm of the velocity (see (3.37) and (3.38)).

The rest of the paper is organized as follows. In Section 2, we recall some preliminary lemmas which will be used later. Section 3 is devoted to proving our main results.

2. Preliminaries

In this section, we give some well-known inequalities which will be used frequently when deriving the global a priori estimates. The first one is the Gagliardo-Nirenberg inequality.

LEMMA 2.1 (**Gagliardo-Nirenberg inequality [23]**). *For any $f\in W^{1,m}(\mathbb{R})\cap L^r(\mathbb{R})$, there exists some generic constant $C>0$ which may depend on q,r such that*

$$\|f\|_{L^q}\leq C\|f\|_{L^r}^{1-\theta}\|\nabla f\|_{L^m}^{\theta}, \tag{2.1}$$

where $\theta=(\frac{1}{r}-\frac{1}{q})(\frac{1}{r}-\frac{1}{m}+1)^{-1}$, if $m\geq 1$, then $q\in[r,\infty]$.

The following Caffarelli-Kohn-Nirenberg weighted inequality is the key to deal with the Cauchy problem in this paper.

LEMMA 2.2 (**Caffarelli-Kohn-Nirenberg weighted inequality [1,2]**).

(1) *For any $h\in C_0^\infty(\mathbb{R})$, it holds that*

$$\||x|^\kappa h\|_r\leq C\||x|^\alpha|\partial_x h|\|_p^\theta\||x|^\beta h\|_q^{1-\theta}, \tag{2.2}$$

where $1\leq p,q<\infty, 0<r<\infty, 0\leq\theta\leq 1, \frac{1}{p}+\alpha>0, \frac{1}{q}+\beta>0, \frac{1}{r}+\kappa>0$ and satisfying

$$\frac{1}{r}+\kappa=\theta\left(\frac{1}{p}+\alpha-1\right)+(1-\theta)\left(\frac{1}{q}+\beta\right), \tag{2.3}$$

and

$$\kappa=\theta\sigma+(1-\theta)\beta.$$

with $0\leq\alpha-\sigma$ if $\theta>0$ and $0\leq\alpha-\sigma\leq 1$ if $\theta>0$ and $\frac{1}{p}+\alpha-1=\frac{1}{r}+\kappa$.

(2) (Best constant for Caffarelli-Kohn-Nirenberg weighted inequality.) *For any* $h \in C_0^\infty(\mathbb{R})$, *it holds that*

$$\||x|^b h\|_p \le C_{a,b}\||x|^a \partial_x h\|_2, \tag{2.4}$$

where $a > \frac{1}{2}, a-1 \le b \le a - \frac{1}{2}$ *and* $p = \frac{2}{2(a-b)-1}$. *If* $b = a-1$, *then* $p = 2$ *and the best constant in the inequality* (2.4) *is*

$$C_{a,b} = C_{a,a-1} = \left|\frac{2a-1}{2}\right|.$$

3. Proof of the Theorem 1.1

In this section, we get a global solution to (1.2)-(1.4) with initial density having lower bound $\delta e^{-|x|^2} > 0$ by using some a priori estimates of the solution based on the local existence. Theorem 1.1 would be obtained after we make some a priori estimates uniformly for δ and take $\delta \to 0^+$.

Denote $\rho_0^\delta = \rho_0 + \delta e^{-|x|^2} > 0$ for $\delta \in (0,1)$. Throughout this section, we denote C to be a generic constant depending on ρ_0, u_0, b_0, T and some other known constants but independent of δ for any $\delta \in (0,1)$. Clearly,

$$\rho_0^\delta \to \rho_0 \quad \text{in } L^1(\mathbb{R}) \cap H^1(\mathbb{R}), \quad \text{as } \delta \to 0^+.$$

To approximate the initial velocity, we define u_0^δ as

$$u_0^\delta = \begin{cases} \tilde{u}_0^\delta, & |x| < M+1, \\ u_0, & |x| \ge M+1, \end{cases} \tag{3.1}$$

where $\tilde{u}_0^\delta$ is the unique solution to the following elliptic equation:

$$\begin{cases} (\mu(\rho_0^\delta)\tilde{u}_{0x}^\delta)_x = P(\rho_0^\delta)_x + \frac{1}{2}(b_0^2)_x + \sqrt{\rho_0} g(x), \text{ in } \Omega_M := \{x | \ |x| < M+1\}, \\ \tilde{u}_0^\delta|_{|x|=M+1} = u_0. \end{cases} \tag{3.2}$$

Since $\rho_0^\delta = \rho_0 + \delta e^{-|x|^2} \in H^1(\mathbb{R})$, $P(\rho_0^\delta) \in H^1(\mathbb{R})$, $b_0 \in H^1(\mathbb{R})$ and $\sqrt{\rho_0} g \in L^2(\mathbb{R})$, by the elliptic theory, (1.5) and (3.2), we have

$$\begin{aligned} \|\tilde{u}_0^\delta\|_{H^2(\Omega_M)} &\le C\left(\|(\rho_0^\delta)_x^\beta\|_{L^2}\|\tilde{u}_{0x}^\delta\|_{L^\infty} + \|P(\rho_0^\delta)_x\|_{L^2} + \|(b_0^2)_x\|_{L^2} + \|\sqrt{\rho_0} g\|_{L^2}\right) \\ &\le C\left(\varepsilon\|\tilde{u}_0^\delta\|_{H^2(\Omega_M)} + 1\right), \end{aligned} \tag{3.3}$$

which implies

$$\|\tilde{u}_0^\delta\|_{H^2(\Omega_M)} \le C. \tag{3.4}$$

From the compatibility conditions (1.5) and (3.2), it follows that

$$\begin{cases} \left(\mu(\rho_0)(\tilde{u}_0^\delta - u_0)_x\right)_x = \left(\left(\mu(\rho_0) - \mu(\rho_0^\delta)\right)\tilde{u}_{0x}^\delta\right)_x + \left(P(\rho_0^\delta) - P(\rho_0)\right)_x, \text{ in } \Omega_M, \\ (\tilde{u}_0^\delta - u_0)|_{|x|=M+1} = 0, \end{cases} \tag{3.5}$$

which yields

$$\|\tilde{u}_0^\delta - u_0\|_{H^2(\Omega_M)}$$

$$\begin{aligned}
&\le C(\|\mu(\rho_0)_x\|_{L^2}\|(\tilde{u}_0^\delta-u_0)_x\|_{L^\infty}+\|\mu(\rho_0^\delta)-\mu(\rho_0)\|_{L^\infty}\|\tilde{u}_0^\delta\|_{H^2}\\
&\quad+\|(\mu(\rho_0^\delta)-\mu(\rho_0))_x\|_{L^2}\|\tilde{u}_{0x}^\delta\|_{L^\infty}+\|(P(\rho_0^\delta)-P(\rho_0))_x\|_{L^2})\\
&\le C(\|\mu(\rho_0)_x\|_{L^2}\|(\tilde{u}_0^\delta-u_0)_x\|_{L^2}^{\frac{1}{2}}\|(\tilde{u}_0^\delta-u_0)_{xx}\|_{L^2}^{\frac{1}{2}}+\|\mu(\rho_0^\delta)-\mu(\rho_0)\|_{L^\infty}\|\tilde{u}_0^\delta\|_{H^2}\\
&\quad+\|(\mu(\rho_0^\delta)-\mu(\rho_0))_x\|_{L^2}\|\tilde{u}_{0x}^\delta\|_{L^\infty}+\|(P(\rho_0^\delta)-P(\rho_0))_x\|_{L^2})\\
&\le \delta\|\tilde{u}_0^\delta-u_0\|_{H^2(\Omega_M)}+C(\|\mu(\rho_0)_x\|_{L^2}^2\|(\tilde{u}_0^\delta-u_0)_x\|_{L^2}+\|\mu(\rho_0^\delta)-\mu(\rho_0)\|_{L^\infty}\|\tilde{u}_0^\delta\|_{H^2}\\
&\quad+\|(\mu(\rho_0^\delta)-\mu(\rho_0))_x\|_{L^2}\|\tilde{u}_{0x}^\delta\|_{L^\infty}+\|(P(\rho_0^\delta)-P(\rho_0))_x\|_{L^2}).
\end{aligned}\tag{3.6}$$

To deal with the second term on the right-hand side of (3.6), we multiply the Equation $(3.5)_1$ by $(\tilde{u}_0^\delta-u_0)$ and integrate over Ω_M to get

$$\begin{aligned}
&\int_{\Omega_M}\mu(\rho_0)\left((\tilde{u}_0^\delta-u_0)_x\right)^2dx\\
&=\int_{\Omega_M}(\mu(\rho_0)-\mu(\rho_0^\delta))\tilde{u}_{0x}^\delta(\tilde{u}_0^\delta-u_0)_xdx+\int_{\Omega_M}(P(\rho_0^\delta)-P(\rho_0))(\tilde{u}_0^\delta-u_0)_xdx\\
&\le\delta\int_{\Omega_M}\left((\tilde{u}_0^\delta-u_0)_x\right)^2dx+\int_{\Omega_M}(\mu(\rho_0)-\mu(\rho_0^\delta))^2(\tilde{u}_{0x}^\delta)^2dx+\int_{\Omega_M}(P(\rho_0^\delta)-P(\rho_0))^2dx\\
&\le\delta\int_{\Omega_M}\left((\tilde{u}_0^\delta-u_0)_x\right)^2dx+\|\mu(\rho_0^\delta)-\mu(\rho_0)\|_{L^2(\Omega_M)}^2\|\tilde{u}_{0x}^\delta\|_{L^\infty(\Omega_M)}^2+\|P(\rho_0^\delta-P(\rho_0)\|_{L^2(\Omega_M)}^2,
\end{aligned}\tag{3.7}$$

which together with $\mu(\rho_0)=1+\rho_0^\beta$ gives

$$\|(\tilde{u}_0^\delta-u_0)_x\|_{L^2}\le\|\mu(\rho_0^\delta)-\mu(\rho_0)\|_{L^2}\|\tilde{u}_{0x}^\delta\|_{L^\infty}+\|P(\rho_0^\delta)-P(\rho_0)\|_{L^2}.\tag{3.8}$$

It follows from (3.6) and (3.8) that

$$\begin{aligned}
&\|\tilde{u}_0^\delta-u_0\|_{H^2(\Omega_M)}\\
&\le\|\mu(\rho_0)_x\|_{L^2(\Omega_M)}^2\left(\|\mu(\rho_0^\delta)-\mu(\rho_0)\|_{L^2(\Omega_M)}\|\tilde{u}_0^\delta\|_{H^2(\Omega_M)}+\|P(\rho_0^\delta)-P(\rho_0)\|_{L^2(\Omega_M)}\right)\\
&\quad+\|\mu(\rho_0^\delta)-\mu(\rho_0)\|_{L^\infty}\|\tilde{u}_0^\delta\|_{H^2(\Omega_M)}+\|(\mu(\rho_0^\delta)-\mu(\rho_0))_x\|_{L^2(\Omega_M)}\|\tilde{u}_0^\delta\|_{H^2(\Omega_M)}\\
&\quad+\|(P(\rho_0^\delta)-P(\rho_0))_x\|_{L^2(\Omega_M)}\\
&\le C\delta\to0,\quad\text{as }\delta\to0^+.
\end{aligned}\tag{3.9}$$

Then, by using (3.1) and (3.9), we get

$$u_0^\delta\to u_0,\ \text{in } H^2(\mathbb{R})\ \text{ as }\delta\to0^+.\tag{3.10}$$

Furthermore

$$\sqrt{\rho_0^\delta}u_0^\delta(1+|x|^{\frac{\alpha}{2}})\to\sqrt{\rho_0}u_0(1+|x|^{\frac{\alpha}{2}})\ \text{ in } L^2(\mathbb{R}),\ \text{ as }\delta\to0^+,$$

and

$$(\rho_0^\delta)^{\frac{\gamma}{2}}|x|^{\frac{\alpha}{2}}\to\rho_0^{\frac{\gamma}{2}}|x|^{\frac{\alpha}{2}},\ \ u_{0x}^\delta|x|^{\frac{\alpha}{2}}\to u_{0x}|x|^{\frac{\alpha}{2}}\ \text{ in } L^2(\mathbb{R}),\ \text{ as }\delta\to0^+.$$

Before proving Theorem 1.1, we need the following auxiliary theorem.

THEOREM 3.1. *Under the same assumption as in Theorem 1.1, for any $\delta\in(0,1)$, there exists a unique global solution $(\rho^\delta,u^\delta,b^\delta)$ to (1.2)-(1.4) with the initial data replaced by $(\rho_0^\delta,u_0^\delta,b_0)$ such that for any $T>0$ and $\beta\ge1$ or $\beta=0$*

$$0<\delta e^{(-|x|^2-C(T))}\le\rho^\delta\le C,\quad 0\le b^\delta\le C,$$

$$(\rho^\delta, b^\delta) \in L^\infty(0,T;H^1(\mathbb{R})),\ (\rho_t^\delta, b_t^\delta) \in L^\infty(0,T;L^2(\mathbb{R})),$$

$$(\sqrt{\rho^\delta}u^\delta, \sqrt{\rho^\delta}\dot{u}^\delta)(1+|x|^{\frac{\alpha}{2}}),\ |x|^{\frac{\alpha}{2}}\left((\rho^\delta)^{\frac{\gamma}{2}}, b^\delta, u_x^\delta\right) \in L^\infty(0,T;L^2(\mathbb{R})),$$

$$u^\delta \in L^\infty(0,T;H^2(\mathbb{R})),\ \sqrt{\rho^\delta}u_t^\delta \in L^\infty(0,T;L^2(\mathbb{R})),\ u_t^\delta \in L^2(0,T;D^1(\mathbb{R})).$$

Proof. The local existence and uniqueness of the strong solutions in Theorem 3.1 is standard by the fixed point theorem, and we omit it here for brevity. Based on it, Theorem 3.1 can be proved by some global a priori estimates.

For any $T\in(0,\infty)$, let $(\rho^\delta,u^\delta,b^\delta)$ be the solution to (1.2)-(1.4) as in Theorem 3.1, without confusion, we still denote the solution by (ρ,u,b) instead of $(\rho^\delta,u^\delta,b^\delta)$ to simplify the presentation. First, we establish the upper bounds of the density ρ and magnetic field b. ◻

3.1. Pointwise bounds on ρ and b.

LEMMA 3.1. *Let (ρ,u,b) be a smooth solution to (1.2)-(1.4). Then for any $T>0$ and $\beta\geq 0$, it holds that*

$$\int_{\mathbb{R}} \left(\frac{1}{2}\rho u^2 + \frac{1}{\gamma-1}\rho^\gamma + \frac{1}{2}b^2\right)dx + \int_0^T\int_{\mathbb{R}} \mu(\rho)(u_x)^2 dxdt \leq C. \tag{3.11}$$

The upper bound of the density is stated as follows.

LEMMA 3.2. *Suppose that (ρ,u,b) is a smooth solution to (1.2)-(1.4). Then for any $T>0$, there exists an absolute constant $C>0$ which depends on the initial data and $\beta\geq 0$ such that*

$$\rho(x,t)\leq C,\ (x,t)\in\mathbb{R}\times(0,T].$$

Proof. Let

$$\xi = \int_{-\infty}^x \rho u(y)dy.$$

Using the momentum Equation $(1.2)_2$, we have

$$\xi_{tx} + \left(\rho u^2 + P(\rho) + \frac{1}{2}b^2\right)_x = (\mu(\rho)u_x)_x.$$

Integrating with respect to x over $(-\infty,x)$ yields

$$\xi_t + \rho u^2 + P(\rho) + \frac{1}{2}b^2 - \mu(\rho)u_x = 0. \tag{3.12}$$

By using the mass Equation $(1.2)_1$, we rewrite (3.12) as

$$\xi_t + \rho u^2 + \mu(\rho)\frac{\rho_t + u\rho_x}{\rho} + P(\rho) + \frac{1}{2}b^2 = 0. \tag{3.13}$$

Let $X(t,x)$ be the particle trajectory defined by

$$\begin{cases} \frac{dX(t,x)}{dt} = u(X(t,x),t), \\ X(0,x) = x. \end{cases} \tag{3.14}$$

Then

$$\frac{d\xi}{dt}(X(t,x),t)=\xi_t+u\xi_x=\xi_t+\rho u^2. \tag{3.15}$$

Denote

$$\eta(\rho)=\int_1^\rho \frac{\mu(s)}{s}ds=\begin{cases}\ln\rho+\frac{1}{\beta}(\rho^\beta-1), & \text{if } \beta>0,\\ 2\ln\rho, & \text{if } \beta=0.\end{cases}$$

It follows from (3.13) and (3.15) that

$$\frac{d}{dt}(\xi+\eta)(X(t,x),t)+P(X(t,x),t)+\frac{1}{2}b^2(X(t,x),t)=0, \tag{3.16}$$

which together with $P(X(t,x),t)+\frac{1}{2}b^2(X(t,x),t)\geq 0$ gives

$$\frac{d}{dt}(\xi+\eta)(X(t,x),t)\leq 0. \tag{3.17}$$

Integrating (3.17) over $(0,t)$, we have

$$(\xi+\eta)(X(t,x),t)\leq \xi(X(0,x),0)+\eta(X(0,x),0).$$

Since

$$\xi(X(0,x),0)=\int_{-\infty}^{X((0,x),0)}\rho_0u_0(y)dy\leq\left|\int\rho_0u_0dy\right|\leq\|\sqrt{\rho_0}u_0\|_{L^2}\|\rho_0\|_{L^1}^{1/2}\leq C,$$

and

$$\begin{aligned}\eta(X(0,x),0)=\int_1^{\rho_0}\frac{\mu(s)}{s}ds&=\begin{cases}\ln\rho_0+\frac{1}{\beta}(\rho_0^\beta-1), & \text{if } \beta>0,\\ 2\ln\rho_0, & \text{if } \beta=0.\end{cases}\\ &\leq\begin{cases}\rho_0+\frac{1}{\beta}(\rho_0^\beta-1), & \text{if } \beta>0,\\ 2\rho_0, & \text{if } \beta=0.\end{cases},\end{aligned}$$

we get

$$\xi(x,t)+\eta(x,t)\leq C,$$

which implies

$$\ln\rho+\frac{1}{\beta}(\rho^\beta-1)\leq C-\int_{-\infty}^x\rho u dx\leq C+\int|\rho u|dx\leq C+\|\sqrt{\rho}u\|_{L^2}\|\rho\|_{L^1}^{\frac{1}{2}}\leq C,\ \beta>0,$$

or

$$2\ln\rho\leq C-\int_{-\infty}^x\rho u dx\leq C+\int|\rho u|dx\leq C+\|\sqrt{\rho}u\|_{L^2}\|\rho\|_{L^1}^{\frac{1}{2}}\leq C,\ \beta=0.$$

Thus we can obtain

$$\ln\rho\leq\begin{cases}C+\frac{1}{\beta}, & \text{if } \beta>0,\\ C, & \text{if } \beta=0.\end{cases}$$

Consequently,

$$\rho(x,t)\leq C,\ \beta\geq 0.$$

The proof of Lemma 3.2 is completed. □

LEMMA 3.3. *Let (ρ,u,b) be a smooth solution to (1.2)-(1.4). Then for any $T>0$, it holds that*

$$\begin{aligned}&\|b\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}\leq C,\\&\|u_x\|_{L^\infty(0,T;L^2(\mathbb{R}))}\leq C,\\&\|\sqrt{\rho}\dot{u}\|_{L^2(0,T;L^2(\mathbb{R}))}\leq C.\end{aligned}\tag{3.18}$$

Proof. The proof of Lemma 3.3 will be divided into three steps.

Step 1. Multiplying the equation$(1.2)_3$ by $2nb^{2n-1}$ and integrating over $\mathbb{R}$, we have

$$\begin{aligned}\frac{d}{dt}\int b^{2n}dx&=-\int u\left(b^{2n}\right)_x dx-2n\int u_x b^{2n}dx\\&=-(2n-1)\int b^{2n}u_x dx=-(2n-1)\int b^{2n}\frac{F+P+\frac{1}{2}b^2}{\mu(\rho)}dx\\&\leq-(2n-1)\int b^{2n}\frac{F}{\mu(\rho)}dx\\&\leq(2n-1)\|F\|_{L^\infty}\|b\|_{L^{2n}}^{2n}\\&\leq C(2n-1)(1+\|u_x\|_{L^2}+\|b\|_{L^4}^2)^{\frac{1}{2}}\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{1}{2}}\|b\|_{L^{2n}}^{2n},\end{aligned}\tag{3.19}$$

where $F=\mu(\rho)u_x-P(\rho)-\frac{1}{2}b^2$ and we have used the following inequality:

$$\begin{aligned}\|F\|_{L^\infty}\leq\|F\|_{L^2}^{\frac{1}{2}}\|F_x\|_{L^2}^{\frac{1}{2}}&\leq\|\mu(\rho)u_x-P(\rho)-\frac{1}{2}b^2\|_{L^2}^{\frac{1}{2}}\|\rho\dot{u}\|_{L^2}^{\frac{1}{2}}\\&\leq C(1+\|u_x\|_{L^2}+\|b\|_{L^4}^2)^{\frac{1}{2}}\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{1}{2}}.\end{aligned}\tag{3.20}$$

Step 2. Next, we estimate the term $\|u_x\|_{L^2}$ on the right-hand side of (3.19). By multiplying the equation $(1.2)_2$ by $\dot{u}$, integrating the resulting equality over $\mathbb{R}$ and using $(1.2)_1$, $(1.2)_3$ and effective fluid flux $F=\mu(\rho)u_x-P(\rho)-\frac{1}{2}b^2$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\mu(\rho)(u_x)^2dx+\int\rho\dot{u}^2dx\\&=\int\frac{1}{2}(\rho_t^\beta+u\rho_x^\beta)(u_x)^2dx-\frac{1}{2}\int\mu(\rho)(u_x)^3dx\\&\quad+\frac{d}{dt}\int\left(P+\frac{1}{2}b^2\right)u_x dx-\int\left(\left(P+\frac{1}{2}b^2\right)_t+\left(P+\frac{1}{2}b^2\right)_x u\right)u_x dx\\&=\frac{d}{dt}\int\left(P+\frac{1}{2}b^2\right)u_x dx-\frac{1}{2}\int(1+(\beta+1)\rho^\beta)(u_x)^3dx+\int\left(\gamma\rho^\gamma+b^2\right)(u_x)^2dx\\&\leq\frac{d}{dt}\int\left(P+\frac{1}{2}b^2\right)u_x dx+C(\|F+P+\frac{1}{2}b^2\|_{L^3}^3+\|b\|_{L^6}^6+1)\\&\leq\frac{d}{dt}\int\left(P+\frac{1}{2}b^2\right)u_x dx+C\big((1+\|u_x\|_{L^2}+\|b\|_{L^4}^2)^{\frac{5}{2}}\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{1}{2}}+\|b\|_{L^6}^6+1\big)\\&\leq\frac{d}{dt}\int\left(P+\frac{1}{2}b^2\right)u_x dx+\varepsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C(\|u_x\|_{L^2}^{\frac{10}{3}}+\|b\|_{L^4}^{\frac{20}{3}}+\|b\|_{L^6}^6+1)\end{aligned}$$

$$\le \frac{d}{dt}\int\left(P+\frac{1}{2}b^2\right)u_x dx+\varepsilon\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C_1(\|u_x\|_{L^2}^4+\|b\|_{L^6}^6+1), \tag{3.21}$$

where in the fourth and last inequalities we have used the following inequalities, respectively,

$$\begin{aligned}\|F\|_{L^3}^3 &\le \|F\|_{L^2}^{\frac{5}{2}}\|F_x\|_{L^2}^{\frac{1}{2}} \le \|\mu(\rho)u_x-P-\frac{1}{2}b^2\|_{L^2}^{\frac{5}{2}}\|\rho\dot{u}\|_{L^2}^{\frac{1}{2}}\\ &\le C(1+\|u_x\|_{L^2}+\|b\|_{L^4}^2)^{\frac{5}{2}}\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{1}{2}},\end{aligned} \tag{3.22}$$

and

$$\|b\|_{L^4}\le \|b\|_{L^2}^{\frac{1}{4}}\|b\|_{L^6}^{\frac{3}{4}}\le C\|b\|_{L^6}^{\frac{3}{4}}.$$

Choosing $\varepsilon>0$ suitably small and integrating (3.21) over $[0,T]$, we have

$$\begin{aligned}&\int u_x^2dx+\int_0^T\int\rho\dot{u}^2dxdt\\ &\le \int\left(P+\frac{1}{2}b^2\right)u_xdx+C_1\int_0^T\|b\|_{L^6}^6dt+C\left(\int_0^T\|u_x\|_{L^2}^4dt+1\right)\\ &\le \varepsilon\int(u_x)^2dx+C_2\int b^4dx+C_1\int_0^T\|b\|_{L^6}^6dt+C\left(\int_0^T\|u_x\|_{L^2}^4dt+1\right).\end{aligned} \tag{3.23}$$

Step 3. To estimate the term $\|b\|_{L^6}$ on the right-hand side of (3.23), we use the fact that $u_x=\frac{F+P+\frac{1}{2}b^2}{\mu(\rho)}$ and multiply the Equation $(1.2)_3$ by $4b^3$ and then integrate over $\mathbb{R}$ to get

$$\begin{aligned}&\frac{d}{dt}\int b^4dx+3\int\left(\frac{Pb^4}{\mu(\rho)}+\frac{1}{2}\frac{b^6}{\mu(\rho)}\right)dx=-3\int\frac{Fb^4}{\mu(\rho)}dx\\ &\le C\|F\|_{L^3}\|b\|_{L^6}^4\\ &\le \frac{\varepsilon}{2}\|b\|_{L^6}^6+C\|F\|_{L^3}^3\\ &\le \frac{\varepsilon}{2}\|b\|_{L^6}^6+C(1+\|u_x\|_{L^2}+\|b\|_{L^6}^{\frac{3}{2}})^{\frac{5}{2}}\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{1}{2}}\\ &\le \frac{\varepsilon}{2}(\|b\|_{L^6}^6+\|\sqrt{\rho}\dot{u}\|_{L^2}^2)+C(1+\|u_x\|_{L^2}+\|b\|_{L^6}^{\frac{3}{2}})^{\frac{10}{3}}\\ &\le \varepsilon(\|b\|_{L^6}^6+\|\sqrt{\rho}\dot{u}\|_{L^2}^2)+C(1+\|u_x\|_{L^2}^4).\end{aligned} \tag{3.24}$$

Choosing $\varepsilon>0$ suitably small and integrating (3.24) over $[0,T]$, we have

$$\int b^4dx+\int_0^T\int b^6dxdt\le\varepsilon\int_0^T\|\sqrt{\rho}\dot{u}\|_{L^2}^2dt+C\left(1+\int_0^T\|u_x\|_{L^2}^4dt\right). \tag{3.25}$$

Adding (3.25) multiplied by large enough C_1+C_2+1 to (3.23), and then taking ε suitably small, we obtain

$$\int(u_x^2+b^4)\,dx+\int_0^T\int\rho\dot{u}^2dxdt+\int_0^T\int b^6dxdt\le C\int_0^T\|u_x\|_{L^2}^2\|u_x\|_{L^2}^2dt+C,$$

which together with Grönwall's inequality and (3.11) yields

$$\int (u_x^2+b^4)\,dx+\int_0^T\int \rho\dot{u}^2dxdt+\int_0^T\int b^6dxdt\le C. \tag{3.26}$$

The combination of (3.26) with (3.19) leads to

$$\frac{d}{dt}\int b^{2n}dx\le C(2n-1)\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{1}{2}}\|b\|_{L^{2n}}^{2n},$$

which together with Grönwall's inequality and (3.26) gives

$$\|b\|_{L^{2n}}\le \exp\left\{C\int_0^T\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{1}{2}}dt\right\}\|b_0\|_{L^{2n}}\le C.$$

Let $n\to+\infty$, we have

$$\|b\|_{L^\infty((0,T)\times\mathbb{R})}\le C.$$

Then the proof of Lemma 3.3 is completed. □

COROLLARY 3.1. *Let (ρ,u,b) be a smooth solution to (1.2)-(1.4). For any $T>0$, we have*

$$\rho\ge\delta e^{\left(-|x|^2-C(T)\right)}>0,$$
$$\|(F,u_x)\|_{L^4(0,T;L^\infty(\mathbb{R}))}\le C.$$

Proof. First, we deduce the positive lower bound of the density as follows.

By using $\rho\le C(T)$ in Lemma 3.2, $b\le C(T)$ in Lemma 3.3 and integrating (3.16) over $[0,T]$, we have

$$\begin{aligned}&\eta(X(x,t),t)-\eta(X(x,0),0)\\&=-\xi(X(x,t),t)+\xi(X(x,0),0)-\int_0^T(P+\frac{1}{2}b^2)(X(x,s),s)ds\\&\le C(T),\end{aligned} \tag{3.27}$$

which implies

$$|\eta(X(x,t),t)-\eta(X(x,0),0)|\le C(T)\Rightarrow \eta(X(x,t),t)-\eta(X(x,0),0)\ge -C(T).$$

Consequently, we get

$$\ln\rho\ge\ln\rho_0-C(T)\Rightarrow\rho\ge\rho_0e^{-C(T)}>\delta e^{\left(-|x|^2-C(T)\right)}>0,$$

where we have used $\rho_0^\delta=\rho_0+\delta e^{-|x|^2}$ and $\rho_0\ge0$.

Next, using (3.20), (3.26) and $1\le\mu(\rho)\le C$, we obtain

$$\int_0^T\|F\|_{L^\infty}^4dt\le C\int_0^T\|\sqrt{\rho}\dot{u}\|_{L^2}^2dt\le C. \tag{3.28}$$

It follows from Lemma 3.2 and $(3.18)_1$ that

$$\|u_x\|_{L^\infty}\le\|\mu(\rho)u_x\|_{L^\infty}=\|F+P+\frac{1}{2}b^2\|_{L^\infty}\le\|F\|_{L^\infty}+C\le\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{1}{2}}+C, \tag{3.29}$$

which together with Lemma 3.3 gives

$$\int_0^T\|u_x\|_{L^\infty}^4dt\le C\int_0^T\|\sqrt{\rho}\dot{u}\|_{L^2}^2dt\le C.$$

Then the proof of Corollary 3.1 is completed. □

3.2. Higher order estimates on ρ, u and b.

LEMMA 3.4. *Let (ρ,u,b) be a smooth solution to (1.2)-(1.4). Then for any $T>0$ and $\beta\geq 1$ or $\beta=0$, it holds that*

$$\int(\rho_x^2+b_x^2)dx+\int_0^T\int u_{xx}^2dxdt\leq C.$$

Proof. Differentiating the mass Equation $(1.2)_1$ with respect to x, multiplying the resulting equation by ρ_x and integrating over $\mathbb{R}$, we have

$$\begin{aligned}\frac{d}{dt}\int\frac{1}{2}\rho_x^2dx&=-\frac{3}{2}\int u_x\rho_x^2dx-\int u_{xx}\rho\rho_xdx\\&\leq C\|u_x\|_{L^\infty}\|\rho_x\|_{L^2}^2+\|u_{xx}\|_{L^2}\|\rho_x\|_{L^2}\\&\leq\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C(1+\|u_x\|_{L^\infty})(\|\rho_x\|_{L^2}^2+\|b_x\|_{L^2}^2),\end{aligned}\tag{3.30}$$

where we have used the following inequality

$$\begin{aligned}\|u_{xx}\|_{L^2}\leq\|\mu(\rho)u_{xx}\|_{L^2}&=\|\rho\dot{u}+P_x+(\frac{1}{2}b^2)_x-\rho_x^\beta u_x\|_{L^2}\\&\leq\|\sqrt{\rho}\dot{u}\|_{L^2}+\|\rho_x\|_{L^2}+\|b_x\|_{L^2}+\|u_x\|_{L^\infty}\|\rho_x\|_{L^2}.\end{aligned}\tag{3.31}$$

Similar to the argument of (3.30), we have

$$\begin{aligned}\frac{d}{dt}\int\frac{1}{2}b_x^2dx&=-\frac{3}{2}\int u_xb_x^2dx-\int u_{xx}bb_xdx\\&\leq C\|u_x\|_{L^\infty}\|b_x\|_{L^2}^2+\|u_{xx}\|_{L^2}\|b_x\|_{L^2}\\&\leq\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C(1+\|u_x\|_{L^\infty})(\|b_x\|_{L^2}^2+\|\rho_x\|_{L^2}^2).\end{aligned}\tag{3.32}$$

Adding (3.32) to (3.30), we obtain

$$\frac{d}{dt}\int\frac{1}{2}(b_x^2+\rho_x^2)dx\leq\|\sqrt{\rho}\dot{u}\|_{L^2}^2+C(1+\|u_x\|_{L^\infty})(\|\rho_x\|_{L^2}^2+\|b_x\|_{L^2}^2),\tag{3.33}$$

which together with Grönwall's inequality, Corollary 3.1 and $(3.18)_3$ yields

$$\int(b_x^2+\rho_x^2)dx\leq C(T).\tag{3.34}$$

It follows from (3.31), (3.34), (3.29) and Cauchy's inequality that

$$\|u_{xx}\|_{L^2}\leq\|\sqrt{\rho}\dot{u}\|_{L^2}+C.$$

This together with $(3.18)_3$ yields that

$$\int_0^T\int|u_{xx}|^2dxdt\leq C.$$

Then we complete the proof of Lemma 3.4. ∎

To obtain the estimate of $\|u\|_{L^p(\mathbb{R})}$, (for some $p\in(1,+\infty)$), we need the following weighted energy estimates.

LEMMA 3.5. *Let (ρ,u,b) be a smooth solution to (1.2)-(1.4). Then for any $T>0$ and $\beta\geq 1$ or $\beta=0$, if $2<\alpha<1+\frac{2}{\sqrt[3]{1+\sqrt[3]{4}}}$, it holds that*

$$\int |x|^{\alpha}\left(\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}P+\frac{1}{2}b^2\right)dx+\int_0^T\int |x|^{\alpha}(1+\rho^{\beta})u_x^2dxdt\leq C(T),$$

moreover, there exists a fixed constant $\frac{2}{\alpha-1}<\tilde{p}<2$ such that

$$\int_0^T \|u\|_{L^{\tilde{p}}}^2dxdt\leq C(T).$$

Proof. To prove Lemma 3.5, we divide it into two cases.

(i) Case 1: $\beta\geq 1$. Multiplying $(1.2)_1,(1.2)_2,(1.2)_3$ by $\frac{\gamma}{\gamma-1}|x|^{\alpha}\rho^{\gamma-1}$, $|x|^{\alpha}u$, $|x|^{\alpha}b$, respectively and integrating with respect to x over $\mathbb{R}$, we get

$$\begin{aligned}
&\frac{d}{dt}\int\left(\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}\rho^{\gamma}+\frac{1}{2}b^2\right)|x|^{\alpha}dx+\int(1+\rho^{\beta})|x|^{\alpha}u_x^2dx\\
&=\frac{\alpha}{2}\int\rho u^3|x|^{\alpha-2}xdx-\int\alpha|x|^{\alpha-2}x(1+\rho^{\beta})uu_xdx+\int\left(\frac{\gamma}{\gamma-1}\rho^{\gamma}+b^2\right)\alpha|x|^{\alpha-2}xudx\\
&=:I_1+I_2+I_3.
\end{aligned}\tag{3.35}$$

Now, we estimate $I_i(\,i=1,2,3)$ in the following way. First, for the term I_1, it follows from Hölder's inequality and Lemma 3.1 that

$$\begin{aligned}
I_1&\leq C\int\rho u^3|x|^{\alpha-1}dx\leq C\int(\rho u^2|x|^{\alpha})^{\frac{\alpha-1}{\alpha}}(\rho u^2)^{\frac{1}{\alpha}}udx\\
&\leq C\|u\|_{L^{\infty}}\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_{L^2}^{2(1-\frac{1}{\alpha})}\|\sqrt{\rho}u\|_{L^2}^{\frac{2}{\alpha}}\\
&\leq C\|u\|_{L^{\infty}}\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_{L^2}^{2(1-\frac{1}{\alpha})}.
\end{aligned}\tag{3.36}$$

To estimate the terms $\|u\|_{L^{\infty}}$ on the right-hand side of (3.36), firstly, using the Caffarelli-Kohn-Nirenberg weighted inequality (2.4) gives

$$\|u\|_{L^{\tilde{p}}}\leq C\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^{1-\theta}\|u_x\|_{L^2}^{\theta},\tag{3.37}$$

where the indexes $\tilde{p},\alpha,\theta$ satisfy

$$\frac{1}{\tilde{p}}=(\frac{1}{2}+\frac{\alpha}{2}-1)(1-\theta)+(\frac{1}{2}-1)\theta\Rightarrow\theta=\frac{\tilde{p}\alpha-\tilde{p}-2}{\tilde{p}\alpha}\in(0,1),$$

for some fixed large constant $\tilde{p}>\frac{2}{\alpha-1}$. Moreover, we have $u\in L^{\tilde{p}}(\mathbb{R})$, $\tilde{p}<2$ provided $\alpha>2$. Then, by the Gagliardo-Nirenberg inequality (2.1) and (3.37), we have

$$\begin{aligned}
\|u\|_{L^{\infty}}&\leq C\|u\|_{L^{\tilde{p}}}^{\frac{\tilde{p}}{\tilde{p}+2}}\|u_x\|_{L^2}^{\frac{2}{\tilde{p}+2}}\\
&\leq C(\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^{\frac{\tilde{p}+2}{\tilde{p}\alpha}}\|u_x\|_{L^2}^{\frac{\tilde{p}\alpha-\tilde{p}-2}{\tilde{p}\alpha}})^{\frac{\tilde{p}}{\tilde{p}+2}}\|u_x\|_{L^2}^{\frac{2}{\tilde{p}+2}}\\
&\leq C\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^{\frac{1}{\alpha}}\|u_x\|_{L^2}^{1-\frac{1}{\alpha}}.
\end{aligned}\tag{3.38}$$

Putting (3.38) into (3.36) and then using Cauchy's inequality gives

$$I_1\leq C\|u\|_{L^{\infty}}\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_{L^2}^{2(1-\frac{1}{\alpha})}$$

$$
\begin{aligned}
&\le C\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^{\frac{1}{\alpha}}\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_{L^2}^{2(1-\frac{1}{\alpha})}\\
&\le C(\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}+\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_{L^2}^2)\\
&\le \varepsilon\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2+C(1+\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_{L^2}^2).
\end{aligned}
\tag{3.39}
$$

Next, for the term I_2, by Hölder's inequality, the Caffarelli-Kohn-Nirenberg weighted inequalities (2.2) and (2.4), Young's inequality and $(3.18)_2$, we deduce that for any $\varepsilon>0$

$$
\begin{aligned}
I_2=&-\alpha\int(1+\rho^\beta)uu_x|x|^{\alpha-2}xdx\\
=&-\frac{\alpha}{2}\int(u^2)_x|x|^{\alpha-2}xdx-\int\alpha\rho^\beta|x|^{\alpha-2}xuu_xdx\\
\le&\frac{\alpha}{2}\int u^2\left((\alpha-2)|x|^{\alpha-3}\frac{x}{|x|}x+|x|^{\alpha-2}\right)+\alpha\|\sqrt{\rho^\beta}|x|^{\frac{\alpha}{2}}u_x\|_{L^2}\|\sqrt{\rho^\beta}\|_{L^{p_1}}\||x|^{\frac{\alpha}{2}-1}u\|_{L^{p_2}}\\
\le&\frac{\alpha(\alpha-1)}{2}\||x|^{\frac{\alpha}{2}-1}u\|_{L^2}^2+\alpha\|\sqrt{\rho^\beta}|x|^{\frac{\alpha}{2}}u_x\|_{L^2}\|u_x\|_{L^2}^{\theta}\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^{1-\theta}\\
\le&\left(\frac{\alpha(\alpha-1)^3}{8}+\varepsilon\right)\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2+C,
\end{aligned}
\tag{3.40}
$$

where the indexes $\alpha>2, p_1>2, p_2>2, \theta\in(0,1)$, satisfy

$$
\begin{aligned}
&\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{2},\ \frac{1}{p_2}+\frac{\alpha}{2}-1=\left(\frac{1}{2}-1\right)\theta+\left(\frac{1}{2}+\frac{\alpha}{2}-1\right)(1-\theta),\\
&\frac{\alpha(\alpha-1)^3}{8}<1,
\end{aligned}
\tag{3.41}
$$

which implies

$$
\frac{1}{1-\theta}<\alpha<\frac{1}{\theta}\Rightarrow\theta<\frac{1}{2}.
$$

By (3.41) and $\alpha>2$, we first choose α as

$$
(\alpha-1)^3<\frac{8}{\alpha}<4\Longrightarrow\alpha<1+\sqrt[3]{4}.
$$

Then, to guarantee $(3.41)_2$, we impose

$$
\frac{\alpha(\alpha-1)^3}{8}<\frac{(1+\sqrt[3]{4})(\alpha-1)^3}{8}<1,
$$

which implies that

$$
(\alpha-1)^3<\frac{8}{1+\sqrt[3]{4}}\Longrightarrow\alpha<1+\frac{2}{\sqrt[3]{1+\sqrt[3]{4}}}.
\tag{3.42}
$$

Combining $\alpha>2$ and (3.42), the index α is chosen to satisfy

$$
2<\alpha<1+\frac{2}{\sqrt[3]{1+\sqrt[3]{4}}}.
\tag{3.43}
$$

Finally, for the term I_3, it follows from Hölder's inequality, the Caffarelli-Kohn-Nirenberg weighted inequality (2.4) and Young's inequality that

$$\begin{aligned}I_3=&\int\left(\frac{\gamma}{\gamma-1}\rho^\gamma+b^2\right)\alpha|x|^{\alpha-2}xudx\\ \le& C\|\sqrt{\rho^\gamma}|x|^{\frac{\alpha}{2}}\|_{L^2}\||x|^{\frac{\alpha}{2}-1}u\|_{L^2}+C\||x|^{\frac{\alpha}{2}}b\|_{L^2}\|b\|_{L^\infty}\||x|^{\frac{\alpha}{2}-1}u\|_{L^2}\\ \le& C\|\sqrt{\rho^\gamma}|x|^{\frac{\alpha}{2}}\|_{L^2}\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}+C\||x|^{\frac{\alpha}{2}}b\|_{L^2}\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}\\ \le& \varepsilon\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2+C(\|\sqrt{\rho^\gamma}|x|^{\frac{\alpha}{2}}\|_{L^2}^2+\||x|^{\frac{\alpha}{2}}b\|_{L^2}^2).\end{aligned}\tag{3.44}$$

Substituting (3.39)-(3.44) into (3.35), we have

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}\rho^\gamma+\frac{1}{2}b^2\right)|x|^\alpha dx+\int_0^T\int\mu(\rho)u_x^2|x|^\alpha dxdt\\ \le&\left(\frac{\alpha(\alpha-1)^3}{8}+3\varepsilon\right)\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2+C(\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_{L^2}^2+\|\sqrt{\rho^\gamma}|x|^{\frac{\alpha}{2}}\|_{L^2}^2+\||x|^{\frac{\alpha}{2}}b\|_{L^2}^2+1).\end{aligned}\tag{3.45}$$

Taking $0<\varepsilon\ll1$, $2<\alpha<1+\frac{2}{\sqrt[3]{1+\sqrt[3]{4}}}$ in (3.45) and using Grönwall's inequality, we have

$$\int\left(\frac{1}{2}\rho u^2+\rho^\gamma+b^2\right)|x|^\alpha dx+\int_0^T\int\mu(\rho)u_x^2|x|^\alpha dxdt\le C.\tag{3.46}$$

(ii) Case 2: $\beta=0$. Taking $\beta=0$ in (3.35), one has

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}\rho^\gamma+\frac{1}{2}b^2\right)|x|^\alpha+2\int|x|^\alpha u_x^2dx\\ =&\frac{\alpha}{2}\int\rho u^3|x|^{\alpha-2}xdx-2\alpha\int uu_x|x|^{\alpha-2}xdx+\int\left(\frac{\gamma}{\gamma-1}\rho^\gamma+b^2\right)u\alpha|x|^{\alpha-2}xdx\\ =:&I_1'+I_2'+I_3'.\end{aligned}\tag{3.47}$$

The estimates of I_1' and I_3' are same as those of I_1 and I_3. Thus, we only need to focus on the estimates of I_2'. For the term I_2', by Hölder's inequality, we can obtain

$$\begin{aligned}I_2'=&-2\alpha\int uu_x|x|^{\alpha-2}xdx\\ \le&2\cdot\frac{\alpha}{2}\int u^2\left((\alpha-2)|x|^{\alpha-3}\frac{x}{|x|}x+|x|^{\alpha-2}\right)\\ \le&2\cdot\frac{\alpha(\alpha-1)}{2}\||x|^{\frac{\alpha}{2}-1}u\|_{L^2}^2\\ \le&2\cdot\frac{\alpha(\alpha-1)^3}{8}\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2.\end{aligned}\tag{3.48}$$

Substituting (3.39), (3.44) and (3.48) into (3.47), we have

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}\rho^\gamma+\frac{1}{2}b^2\right)|x|^\alpha dx+2\int|x|^\alpha u_x^2dx\\ \le&\left(2\cdot\frac{\alpha(\alpha-1)^3}{8}+2\varepsilon\right)\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2+C(\|\sqrt{\rho}u|x|^{\frac{\alpha}{2}}\|_{L^2}^2+\|\sqrt{\rho^\gamma}|x|^{\frac{\alpha}{2}}\|_{L^2}^2+\||x|^{\frac{\alpha}{2}}b\|_{L^2}^2+1).\end{aligned}$$

Similarly, to guarantee $2\cdot\frac{\alpha(\alpha-1)^3}{8}<2$, we choose $2<\alpha<1+\frac{2}{\sqrt[3]{1+\sqrt[3]{4}}}$ and $0<\varepsilon\ll 1$ and then use Grönwall's inequality to infer that

$$\int(\rho u^2+\rho^\gamma+b^2)|x|^\alpha dx+\int_0^T\int|x|^\alpha u_x^2 dxdt\le C(T).$$

Then, the proof of Lemma 3.5 is completed. □

LEMMA 3.6. *Let (ρ,u,b) be a smooth solution to (1.2)-(1.4). For any $T>0$ and $\beta\ge 1$ or $\beta=0$, it holds that*

$$\|\rho_t\|_{L^2(0,T;L^2(\mathbb{R}))}+\|b_t\|_{L^2(0,T;L^2(\mathbb{R}))}\le C(T).$$

Proof. It follows from $(1.2)_1$, $(1.2)_3$, $(3.18)_2$, Lemma 3.4 and (3.38) that

$$\begin{aligned}\|\rho_t\|_{L^2(\mathbb{R})}&\le\|u\rho_x\|_{L^2(\mathbb{R})}+\|\rho u_x\|_{L^2(\mathbb{R})}\\&\le\|u\|_{L^\infty(\mathbb{R})}\|\rho_x\|_{L^2(\mathbb{R})}+C\|u_x\|_{L^2(\mathbb{R})}\\&\le C\|u\|_{L^\infty(\mathbb{R})}+C\\&\le C\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^{\frac{1}{\alpha}}+C.\end{aligned}$$

This together with Lemma 3.5 gives

$$\|\rho_t\|_{L^2(0,T;L^2(\mathbb{R}))}\le C(T).$$

Similarly, we can get $\|b_t\|_{L^2(0,T;L^2(\mathbb{R}))}\le C(T)$. Hence, we complete the proof of Lemma 3.6. □

LEMMA 3.7. *Let (ρ,u,b) be a smooth solution to (1.2)-(1.4). For any $T>0$ and $\beta\ge 1$ or $\beta=0$, we have*

$$\int\rho\dot{u}^2dx+\int_0^T\int\mu(\rho)\dot{u}_x^2dxdt\le C(T),$$

and

$$\|u_{xx}\|_{L^\infty(0,T;L^2(\mathbb{R}))}+\|u_x\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}\le C(T),$$

where $\dot{u}=u_t+uu_x$ is the material derivative.

Proof. Applying the operator $\partial_t+\partial_x(u\cdot)$ to Equation $(1.2)_2$, we have

$$\rho\dot{u}_t+\rho u\dot{u}_x-(\mu(\rho)\dot{u}_x)_x=[(\gamma P+b^2)u_x-(1+(1+\beta)\rho^\beta)u_x^2]_x. \tag{3.49}$$

Multiplying (3.49) by $\dot{u}$ and integrating over $\mathbb{R}$, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\rho\dot{u}^2+\int\mu(\rho)\dot{u}_x^2dx&=\int[(1+(1+\beta)\rho^\beta)u_x^2-(\gamma P+b^2)u_x]\dot{u}_x\\&\le\varepsilon\|\dot{u}_x\|_{L^2}^2+C(\|u_x\|_{L^2}^2\|u_x\|_{L^\infty}^2+\|\gamma P+b^2\|_{L^\infty}^2\|u_x\|_{L^2}^2)\\&\le\varepsilon\|\dot{u}_x\|_{L^2}^2+C(\|u_x\|_{L^\infty}^2+1)\\&\le\varepsilon\|\dot{u}_x\|_{L^2}^2+C(\|\sqrt{\rho}\dot{u}\|_{L^2}+1),\end{aligned} \tag{3.50}$$

where we have used (3.29), Lemma 3.2, Lemma 3.3 and Young's inequality. By using Grönwall's inequality, we get

$$\int \rho \dot{u}^2 + \int_0^T \int \mu(\rho) \dot{u}_x^2 dx dt \le (\|\sqrt{\rho}\dot{u}(\cdot,0)\|_{L^2} + C(T)) \exp\{C(T)\}. \tag{3.51}$$

From $(1.2)_1$ and $(1.2)_2$, one has

$$\rho \dot{u} + P_x + \frac{1}{2}(b^2)_x = (\mu(\rho) u_x)_x. \tag{3.52}$$

Multiplying (3.52) by $\frac{1}{\sqrt{\rho}}$, taking $t \to 0^+$, and using (3.2) and (3.10), we have

$$|\sqrt{\rho^\delta}\dot{u}^\delta(\cdot,0)| \le \frac{|(\mu(\rho_0^\delta)u_{0x}^\delta)_x - p(\rho_0^\delta)_x - \frac{1}{2}(b_0^2)_x|}{\sqrt{\rho_0^\delta}} = \frac{|\sqrt{\rho_0}g|}{\sqrt{\rho_0^\delta}} \le |g|, \tag{3.53}$$

which implies

$$\|\sqrt{\rho}\dot{u}(\cdot,0)\|_{L^2} \le \|g(1+|x|^{\frac{\alpha}{2}})\|_{L^2} \le C. \tag{3.54}$$

This together with (3.51) yields

$$\int \rho \dot{u}^2 + \int_0^T \int \mu(\rho) \dot{u}_x^2 dx dt \le C(T). \tag{3.55}$$

Due to (3.29) and (3.31), it follows from (3.55), Lemmas 3.3 and 3.4 that

$$\|u_x\|_{L^\infty(\mathbb{R})} \le \|\sqrt{\rho}\dot{u}\|_{L^2(\mathbb{R})}^{\frac{1}{2}} + C \le C,$$

and

$$\|u_{xx}\|_{L^2} \le \|\sqrt{\rho}\dot{u}\|_{L^2} + \|\rho_x\|_{L^2} + \|b_x\|_{L^2} + \|u_x\|_{L^\infty}\|\rho_x\|_{L^2} \le C.$$

Then the proof of Lemma 3.7 is completed. □

To obtain the estimate of $\|\dot{u}\|_{L^p(\mathbb{R})}$ (for some $p \in (1,+\infty)$), we need the following weighted estimates of the material derivative.

Lemma 3.8. *Let (ρ,u,b) be a smooth solution to (1.2)-(1.4). Then for any $T>0$ and $\beta \ge 1$ or $\beta = 0$, if $2 < \alpha < 1 + \frac{2}{\sqrt[3]{1+\sqrt[3]{4}}}$, it holds that*

$$\int \rho \dot{u} |x|^\alpha dx + \int_0^T \int \mu(\rho) |x|^\alpha \dot{u}_x^2 dx dt \le C(T),$$

moreover, there exists a fixed constant $\frac{2}{\alpha-1} < \tilde{p} < 2$ such that

$$\int_0^T \|\dot{u}\|_{L^{\tilde{p}}}^2 dx dt \le C(T).$$

Proof. When $\beta \ge 1$, multiplying (3.49) by $|x|^\alpha \dot{u}$ and integrating over $\mathbb{R}$, we have

$$\frac{1}{2}\frac{d}{dt}\int \rho \dot{u}^2 |x|^\alpha + \int \mu(\rho)|x|^\alpha \dot{u}_x^2 = \frac{1}{2}\int \rho u \dot{u}^2 \alpha |x|^{\alpha-2} x - \int \mu(\rho) \alpha |x|^{\alpha-2} x \dot{u} \dot{u}_x$$

$$
\begin{aligned}
&+\int[(1+(1+\beta)\rho^\beta)u_x^2-(\gamma P+b^2)u_x](\dot{u}_x|x|^\alpha+\alpha|x|^{\alpha-2}x\dot{u})\\
&=: J_1+J_2+J_3. \qquad (3.56)
\end{aligned}
$$

Now, we estimate the terms J_1-J_3 as follows:

$$
\begin{aligned}
J_1=\frac{1}{2}\int\rho u\dot{u}^2\alpha|x|^{\alpha-2}x &\le C\|u\|_{L^\infty}\|\sqrt{\rho}\dot{u}|x|^{\frac{\alpha}{2}}\|_{L^2}\||x|^{\frac{\alpha}{2}-1}\dot{u}\|_{L^2}\\
&\le C\|u\|_{L^\infty}\|\sqrt{\rho}\dot{u}|x|^{\frac{\alpha}{2}}\|_{L^2}\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}\\
&\le \varepsilon\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\sqrt{\rho}\dot{u}|x|^{\frac{\alpha}{2}}\|_{L^2}^2, \qquad (3.57)
\end{aligned}
$$

and

$$
\begin{aligned}
J_2&=-\alpha\int|x|^{\alpha-2}x\dot{u}\dot{u}_x-\int\rho^\beta\alpha|x|^{\alpha-2}x\dot{u}\dot{u}_x\\
&\le -\frac{\alpha}{2}\int(\dot{u}^2)_x|x|^{\alpha-2}x+C\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}\|\rho^\beta\|_{L^{p_3}}\||x|^{\frac{\alpha}{2}-1}\dot{u}\|_{L^{q_3}}\\
&\le \frac{\alpha}{2}\int\dot{u}^2\left((\alpha-2)|x|^{\alpha-3}\frac{x}{|x|}x+|x|^{\alpha-2}\right)+C\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}\|\dot{u}_x\|_{L^2}^\theta\||x|^\kappa\dot{u}\|_{L^p}^{1-\theta}\\
&\le \frac{\alpha(\alpha-1)}{2}\||x|^{\frac{\alpha-1}{2}}\dot{u}\|_{L^2}^2+C\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}\|\dot{u}_x\|_{L^2}^\theta\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}^{1-\theta}\\
&\le \left(\frac{\alpha(\alpha-1)^3}{8}+\varepsilon\right)\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}^2+C\|\dot{u}_x\|_{L^2}^2, \qquad (3.58)
\end{aligned}
$$

where we have used Hölder's inequality, Young's inequality, the Caffarelli-Kohn-Nirenberg weighted inequality from Lemma 2.2 and the indexes $p_3>2$, $q_3>2$, $\kappa\ge 0$, $p>1$, $\theta\in(0,1)$ are chosen to satisfy

$$
\begin{aligned}
&\frac{1}{p_3}+\frac{1}{q_3}=\frac{1}{2},\ \frac{1}{q_3}+\frac{\alpha}{2}-1=\left(\frac{1}{2}-1\right)\theta+\left(\frac{1}{p}+\kappa\right)(1-\theta),\\
&\frac{1}{p}+\kappa=\frac{1}{2}+\frac{\alpha}{2}-1,\ \frac{\alpha}{2}-1\le\kappa\le\frac{\alpha-1}{2},
\end{aligned}
$$

which implies

$$
q_3=\frac{2}{1-\alpha\theta}>2\Longrightarrow 0<\alpha\theta<1.
$$

From Lemma 3.7 and the Caffarelli-Kohn-Nirenberg weighted inequality from Lemma 2.2, we can obtain

$$
\begin{aligned}
J_3&=\int[(1+(1+\beta)\rho^\beta)u_x^2-(\gamma P+b^2)u_x](\dot{u}_x|x|^\alpha+\alpha|x|^{\alpha-2}x\dot{u})\\
&\le C(\|u_x\|_{L^\infty}\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}+\||x|^{\frac{\alpha}{2}}u_x\|_{L^2})(\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}+\||x|^{\frac{\alpha}{2}-1}\dot{u}\|_{L^2})\\
&\le C(1+\|u_x\|_{L^\infty})\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}\\
&\le \varepsilon\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}^2+C\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2. \qquad (3.59)
\end{aligned}
$$

Finally, substituting (3.57)-(3.59) into (3.56), we get

$$
\frac{1}{2}\frac{d}{dt}\int\rho\dot{u}^2|x|^\alpha dx+\int\mu(\rho)|x|^\alpha\dot{u}_x^2dxdt
$$

$$\leq \left(\frac{\alpha(\alpha-1)^3}{8}+3\varepsilon\right)\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\sqrt{\rho}\dot{u}|x|^{\frac{\alpha}{2}}\|_{L^2}^2+C(\|\dot{u}_x\|_{L^2}^2+\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2). \tag{3.60}$$

By choosing $2<\alpha<1+\frac{2}{\sqrt[3]{1+\sqrt[3]{4}}}$ to guarantee $\frac{\alpha(\alpha-1)^3}{8}<1$, $0<\varepsilon\ll 1$ and then applying Grönwall's inequality, Lemma 3.5 and Lemma 3.7, we have

$$\int\rho\dot{u}|x|^\alpha dx+\int_0^T\int\mu(\rho)|x|^\alpha\dot{u}_x^2dxdt\leq C(T)\left(\int\rho\dot{u}|x|^\alpha(x,0)dx+C(T)\right).$$

Similar to the arguments of (3.53) and (3.54), one has

$$\int\rho\dot{u}^2|x|^\alpha(x,0)dx\leq C.$$

Thus

$$\int\rho\dot{u}|x|^\alpha dx+\int_0^T\int\mu(\rho)|x|^\alpha\dot{u}_x^2dxdt\leq C(T).$$

When $\beta=0$ ($\mu(\rho)=1+\rho^\beta=2$), the term J_2 can be estimated as follows:

$$J_2=-2\alpha\int|x|^{\alpha-2}x\dot{u}\dot{u}_x\leq 2\cdot\frac{\alpha(\alpha-1)}{2}\||x|^{\frac{\alpha}{2}-1}\dot{u}\|_{L^2}^2\leq 2\cdot\frac{\alpha(\alpha-1)^3}{8}\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}^2. \tag{3.61}$$

Similar to the argument of (3.60), choosing $2<\alpha<1+\frac{2}{\sqrt[3]{1+\sqrt[3]{4}}}$ to guarantee $2\cdot\frac{\alpha(\alpha-1)^3}{8}<2$, $0<\varepsilon\ll 1$ and applying Grönwall's inequality, we have

$$\int\rho\dot{u}|x|^\alpha dx+\int_0^T\int|x|^\alpha\dot{u}_x^2dxdt\leq C(T).$$

Then similar to the argument of (3.37), we use Lemma 3.7 to get

$$\begin{aligned}\|\dot{u}\|_{L^{\tilde{p}}}&\leq C\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}^{1-\theta}\|\dot{u}_x\|_{L^2}^\theta\\&\leq C(\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}+\|\dot{u}_x\|_{L^2})\in L^2(0,T),\end{aligned} \tag{3.62}$$

where the constant $\frac{2}{\alpha-1}<\tilde{p}<2$. Then the proof of Lemma 3.8 is completed. ∎

LEMMA 3.9. *Let (ρ,u,b) be a smooth solution to (1.2)-(1.4). Then for any $T>0$ and $\beta\geq 1$ or $\beta=0$, it holds that*

$$\|(\rho_t,b_t)\|_{L^\infty(0,T;L^2(\mathbb{R}))}\leq C(T).$$

Proof. Differentiating the mass Equation $(1.2)_1$ and $(1.2)_3$ with respect to t, multiplying the resulting equation by ρ_t and b_t and integrating over $\mathbb{R}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int(\rho_t^2+b_t^2)dx&=-\frac{1}{2}\int(\rho_t^2+b_t^2)u_xdx-\int(\rho\rho_t+bb_t)u_{xt}-\int u_t(\rho_x\rho_t+b_xb_t)dx\\&\leq C\|u_x\|_{L^\infty}(\|\rho_t\|_{L^2}^2+\|b_t\|_{L^2}^2)+C\|u_{xt}\|_{L^2}(\|\rho_t\|_{L^2}+\|b_t\|_{L^2})\\&\quad+C\|u_t\|_{L^\infty}(\|\rho_t\|_{L^2}\|\rho_x\|_{L^2}+\|b_t\|_{L^2}\|b_x\|_{L^2})\\&\leq C(\|\rho_t\|_{L^2}^2+\|b_t\|_{L^2}^2+1)(1+\|u_{xt}\|_{L^2}^2+\|u_t\|_{L^\infty}^2),\end{aligned} \tag{3.63}$$

where we have used Lemma 3.4 and Lemma 3.7.

Notice that

$$\begin{aligned}
\|u_{xt}\|_{L^2} &= \|\dot{u}_x - (uu_x)_x\|_{L^2} = \|\dot{u}_x - u_x^2 - uu_{xx}\|_{L^2} \\
&\leq C(\|\dot{u}_x\|_{L^2} + \|u_x^2\|_{L^2} + \|u\|_{L^\infty}\|u_{xx}\|_{L^2}) \\
&\leq C(\|\dot{u}_x\|_{L^2} + \|u_x\|_{L^\infty}\|u_x\|_{L^2} + \||x|^{\frac{\alpha}{2}}u_x\|_{L^2} + 1) \\
&\leq C(\|\dot{u}_x\|_{L^2} + \||x|^{\frac{\alpha}{2}}u_x\|_{L^2} + 1),
\end{aligned} \tag{3.64}$$

and

$$\begin{aligned}
\|u_t\|_{L^\infty} &= \|\dot{u} - uu_x\|_{L^\infty} \leq C(\|\dot{u}\|_{L^\infty} + \|u\|_{L^\infty}) \\
&\leq C(\|\dot{u}_x\|_{L^2}^{1-\frac{1}{\alpha}}\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}^{\frac{1}{\alpha}} + \|u_x\|_{L^2}^{1-\frac{1}{\alpha}}\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^{\frac{1}{\alpha}}) \\
&\leq C(\|\dot{u}_x\|_{L^2}^{1-\frac{1}{\alpha}}\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}^{\frac{1}{\alpha}} + \||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^{\frac{1}{\alpha}}) \\
&\leq C(\|\dot{u}_x\|_{L^2} + \||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2} + \||x|^{\frac{\alpha}{2}}u_x\|_{L^2} + 1),
\end{aligned} \tag{3.65}$$

where in the second inequalities we have used the Caffarelli-Kohn-Nirenberg weighted inequality from Lemma 2.2, (3.38) and $(3.18)_2$. Substituting (3.64)-(3.65) into (3.63), we have

$$\frac{1}{2}\frac{d}{dt}\int(\rho_t^2 + b_t^2)dx \leq C(\|\rho_t\|_{L^2}^2 + \|b_t\|_{L^2}^2 + 1)(\|\dot{u}_x\|_{L^2} + \||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2} + \||x|^{\frac{\alpha}{2}}u_x\|_{L^2} + 1).$$

Then by Grönwall's inequality, Lemma 3.5, Lemma 3.7 and Lemma 3.8, we obtain

$$\|(\rho_t, b_t)\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C(T).$$

The proof of Lemma 3.9 is completed. ∎

To obtain the estimate of $\|u\|_{L^\infty(0,T;L^2(\mathbb{R}))}$, due to (3.37), we need the following weighted estimate of $\||x|^{\frac{\alpha}{2}}u_x\|_{L^\infty(0,T;L^2(\mathbb{R}))}$.

Lemma 3.10. *Let (ρ,u,b) be a smooth solution to (1.2)-(1.4). Then for any $T>0$ and $\beta\geq 1$ or $\beta=0$, if $2<\alpha<1+\frac{2}{\sqrt[3]{1+\sqrt[3]{4}}}$, it holds that*

$$\||x|^{\frac{\alpha}{2}}u_x\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|\sqrt{\rho}\dot{u}|x|^{\frac{\alpha}{2}}\|_{L^2(0,T;L^2(\mathbb{R}))} + \|u\|_{L^\infty(0,T;L^{\tilde{p}}(\mathbb{R}))} \leq C(T),$$

where $\frac{2}{\alpha-1}<\tilde{p}<2$.

Proof. Multiplying the Equation $(1.2)_2$ by $|x|^\alpha\dot{u}$ and integrating over $\mathbb{R}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\mu(\rho)|x|^\alpha u_x^2 dx + \int\rho\dot{u}^2|x|^\alpha \\
&= \int\left(P(\rho) + \frac{1}{2}b^2\right)[\alpha|x|^{\alpha-2}x\dot{u} + |x|^\alpha\dot{u}_x] - \int\mu(\rho)\alpha|x|^{\alpha-2}xu_x\dot{u} \\
&\quad + \frac{1}{2}\int|x|^\alpha u_x^2\left(\rho_t^\beta + u\rho_x^\beta - \frac{1}{2}\mu(\rho)u_x\right) + \frac{1}{2}\int\mu(\rho)\alpha|x|^{\alpha-2}xuu_x^2 \\
&=: K_1 + K_2 + K_3 + K_4.
\end{aligned} \tag{3.66}$$

First, by the Caffarelli-Kohn-Nirenberg weighted inequality from Lemma 2.2 and Lemma 3.5, we have

$$K_1 = \int(P(\rho) + \frac{1}{2}b^2)[\alpha|x|^{\alpha-2}x\dot{u} + |x|^\alpha\dot{u}_x]$$

$$
\begin{aligned}
&\leq C\||x|^{\frac{\alpha}{2}-1}\dot{u}\|_{L^2}\||x|^{\frac{\alpha}{2}}(P+\frac{1}{2}b^2)\|_{L^2}+C\||x|^{\frac{\alpha}{2}}(P+\frac{1}{2}b^2)\|_{L^2}\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}\\
&\leq C\||x|^{\frac{\alpha}{2}}(P+\frac{1}{2}b^2)\|_{L^2}\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}\\
&\leq C\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}. \qquad (3.67)
\end{aligned}
$$

For the term K_2, using Hölder's inequality and the Caffarelli-Kohn-Nirenberg weighted inequality yields

$$
\begin{aligned}
K_2&=-\int\mu(\rho)\alpha|x|^{\alpha-2}xu_x\dot{u}\\
&\leq C\||x|^{\frac{\alpha-1}{2}}\dot{u}\|_{L^2}\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}\\
&\leq C\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}, \qquad (3.68)
\end{aligned}
$$

and K_3 can be estimated as

$$
\begin{aligned}
K_3&=\frac{1}{2}\int|x|^{\alpha}u_x^2(\rho_t^{\beta}+u\rho_x^{\beta}-\frac{1}{2}\mu(\rho)u_x)\\
&=-\frac{1}{2}\int|x|^{\alpha}u_x^3(\beta\rho^{\beta}+\frac{1}{2}\mu(\rho))\\
&\leq C\|u_x\|_{L^\infty}\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2\\
&\leq C(\|u_x\|_{L^2}+\|u_{xx}\|_{L^2})\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2\\
&\leq C\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2, \qquad (3.69)
\end{aligned}
$$

where we have used the following equation:

$$\rho_t^{\beta}+u\rho_x^{\beta}+\beta\rho^{\beta}u_x=0.$$

Similarly, we deal with the term K_4 as

$$
\begin{aligned}
K_4&=\frac{1}{2}\int\mu(\rho)\alpha|x|^{\alpha-2}xuu_x^2\\
&\leq C\|u_x\|_{L^\infty}\||x|^{\frac{\alpha-1}{2}}u\|_{L^2}\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}\\
&\leq C\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2. \qquad (3.70)
\end{aligned}
$$

Then substituting (3.67)-(3.70) into (3.66), we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int\mu(\rho)|x|^{\alpha}u_x^2+\int\rho\dot{u}^2|x|^{\alpha}&\leq C\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}(1+\||x|^{\frac{\alpha}{2}}u_x\|_{L^2})+C\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2\\
&\leq C\||x|^{\frac{\alpha}{2}}u_x\|_{L^2}^2+C(1+\||x|^{\frac{\alpha}{2}}\dot{u}_x\|_{L^2}^2). \qquad (3.71)
\end{aligned}
$$

Then by Grönwall's inequality and Lemma 3.8, we can obtain

$$\int\mu(\rho)|x|^{\alpha}u_x^2dx+\int_0^T\int\rho\dot{u}^2|x|^{\alpha}dxdt\leq C(T).$$

Moreover, combining the above result with (3.37) and (3.38), we have

$$\|u\|_{L^\infty(0,T;L^{\tilde{p}}(\mathbb{R}))}\leq C(T),\quad \tilde{p}>\frac{2}{\alpha-1}\Rightarrow\sqrt[3]{1+\sqrt[3]{4}}<\tilde{p}<2,$$

and

$$\|u\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \le C(T),$$

$$\|u\|_{L^\infty(0,T;L^2(\mathbb{R}))} \le C(T).$$

which together with Lemma 3.7 yields $\|u\|_{L^\infty(0,T;H^2(\mathbb{R}))} \le C(T)$. The proof of Lemma 3.10 is completed. □

Proof. **(Proof of Theorem 1.1.)** We study (1.2)-(1.4) with the initial data replaced by $(\rho_0^\delta, u_0^\delta, b_0)$. From Theorem 3.1, we know that there exists a unique solution $(\rho^\delta, u^\delta, b^\delta)$, such that Lemmas 3.1-3.9 are valid when we replace (ρ, u, b) by $(\rho^\delta, u^\delta, b^\delta)$. With the uniform estimates for δ, we let $\delta \to 0^+$ (take subsequence if necessary) to get a solution to (1.2)-(1.4) still denoted by (ρ, u, b) which satisfies Lemmas 3.1-3.9 by the lower semicontinuity of the norms. This yields the existence of the solutions as in Theorem 3.1. The uniqueness of the solutions can be proved by the standard L^2 energy method. Here we omit the details for brevity. Hence, the proof of Theorem 1.1 is completed. □

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