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BLOW UP PHENOMENA AND GLOBAL EXISTENCE FOR THE NONLOCAL PERIODIC ROTATION-CAMASSA-HOLM SYSTEM*

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Abstract. Under consideration in the present paper is a mathematical model proposed as an equation of long-crested shallow-water waves propagating in one direction with the effect of Earth's rotation. The system is called Rotation-Camassa-Holm system(RCH2). The local well-posedness of the periodic Cauchy problem is then established by the linear transport theory. Then, wave-breaking phenomena is investigated based on the method of characteristics and the Riccati-type differential inequality with two different kinds of methods. Finally, the wave-breaking data are illustrated and the existence of global solutions is obtained in detail for the periodic RCH2 system.

Keywords. Rotation-Camassa-Holm system; blow up; wave breaking; persistence.

AMS subject classifications. 35B44; 35G25.

1. Introduction

It is known that various asymptotic systems have been proposed as approximations to the Euler equations under some particular physical regimes, like the celebrated Korteweg-de Vries (KdV) equation [31] and others. We often need simpler model equations because of the complexity appearing in the theoretical and numerical study of fluid dynamics. There are two important dimensionless parameters in water-wave theory: amplitude parameter $\varepsilon = a/h_0$ and shallowness parameter $\mu = h_0^2/\lambda^2$ where h_0 is the mean depth of water, a and λ are the typical amplitude and wavelength of the waves, respectively. When we say the shallow-water (or long-wave), it means there is a presumption of small depth (compared with wavelength), i.e. $\mu << 1$. Whereas there are at least two cases for the amplitude parameter $\varepsilon = a/h_0$: Boussinesq scaling (weakly nonlinear regime): $\mu << 1, \varepsilon = O(\mu)$; and the Camassa-Holm (CH) scaling (moderately nonlinear regime): $\mu << 1, \varepsilon = O(\sqrt{\mu})$.

Using the Boussinesq scaling in the asymptotic approximation to the incompressible and irrotational Euler equations yields the classical Boussinesq equation [16], KdV equation and Benjamin-Bona-Mahoney equations in the unidirectional case [2]. Some interesting phenomena like the solitary waves observed by Russell in 1834 can be shown through those equations. It is however observed that the models derived by the Boussinesq scaling can not capture those phenomena that usually occur in nature, for instance, wave breaking and waves of greatest height [1,37]. Wave breaking is an impressed phenomena which means the solution remains bounded while its slope becomes unbounded in finite time.

Fortunately, the Camassa-Holm model [5], by using the CH scaling, can describe these phenomena [9,10]. The Camassa-Holm equation has been rigorously justified as an approach to the governing equations for shallow-water waves [14]. The Degasperis-Procesi (DP) equation [14], which was derived by the CH scaling, also has wave breaking phenomena [19,20]. These equations have richer mathematical structures, capturing the

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multi-solitons consisting of a branch of peaked solitary waves, named peakons as well as breaking waves.

On the other hand, for some motions which are presented on an enormous range of spatial and temporal scales, such as massive and persistent oceanic current system, the Coriolis force, which emerges from the Earth's rotation, is not negligible. More specifically, for the larger-scale motion, the Coriolis force is critical. Here the larger-scale motion means that in this motion the Rossby number $R=U/\lambda\Omega\ll 1$, where Ω characterizes the constant rotational speed of the Earth, U is the typical horizontal vertical scale for the fluid motion. If we ignore the relative accelerations, it still has a movement, namely, geostrophic motion. The interplay of gravity and the Coriolis force gives rise to complex phenomena. In fact, the effect of solid-body rotation of the Earth produces two accelerations. One is called the Coriolis acceleration, and another is called the centripetal acceleration. Specifically, the centripetal acceleration is much smaller than the Coriolis acceleration and is usually ignored.

In the equatorial region, despite the Coriolis force being small due to the smallness of the variation of latitude in the equatorial region (Case 2), the Coriolis force still affects the motion of waves [12]. And also the vertical stratification of the ocean is greater in this region than anywhere else. Kelvin wave is a well-known example of equatorial waves. The work [8] gives the existence for the steady periodic rotational equatorial water wave. For example, the Rossby wave, a kind of the geophysical equatorial water wave, can be seen as a shallow-water wave whose wavelength can be 500 km. There have been some significant modifications which account for the geophysical effect related to the generalized Dullin-Gottwald-Holm system and two-component Camassa-Holm system [11].

In this paper, our work is to research the following rotation-two-component Camassa-Holm system from the incompressible and irrotational two-dimensional shallow water in the equatorial region under the CH scaling, namely RCH2 system

$$\begin{cases} m_t+\sigma(2mu_x+um_x)+3(1-\sigma)uu_x+\frac{1}{2}(\rho^2)_x+2\Omega(\rho^2u)_x-8\Omega(\rho u)_x+4\Omega u_x \\ \qquad +24\beta_1(u^2\rho(\rho-1))_x+4\beta_2(u^3)_x=0, \\ \rho_t+(\rho u)_x+2\Omega uu_x\rho+8\beta_1(u^3)_x\rho=0. \end{cases} \tag{1.1}$$

where $\beta_1=\frac{5c^2-11}{48c^2}, \beta_2=\frac{2c^4-2c^2+8}{8c^3}, c=\sqrt{\Omega^2+1}-\Omega$ and Ω is the constant Coriolis frequency due to the Earth's rotation. Here $u(t,x)$ is the average fluid velocity in the x-direction and $m=u-u_{xx}$ is the momentum density. The real dimensionless constant σ is a parameter which shows a balance between nonlinear steepening and amplification in fluid convection due to stretching. The function $\rho(t,x)$ is a variable which relates to the surface elevation from equilibrium. The boundary assumptions associated with the above equation are restricted to $u\to 0$ and $\rho\to 1$ as $|x|\to\infty$. It is also easy to see that the following function, associated with the system in the equation, is conserved: $I(u,\rho)=\int_{\mathbb{S}} u dx$.

In particular, when $\Omega=0$, i.e. without considering the effect of the Earth's rotation, we have $\beta_1=-\frac{1}{8}$ and $\beta_2=1$. Then (1.1) becomes

$$\begin{cases} m_t+\sigma(2mu_x+um_x)+3(1-\sigma)uu_x+\frac{1}{2}(\rho^2)_x-3(u^2\rho(\rho-1))_x+4(u^3)_x=0, \\ \rho_t+(\rho u)_x-(u^3)_x\rho=0 \end{cases} \tag{1.2}$$

where $m=u-u_{xx}$. The following energy is independent of time t,

$$E(u,\rho)=\int_{\mathbb{S}}(u^2+u_x^2+(\rho-1)^2)dx.$$

Actually, system has significant relationship with the models describing the motion of waves at the free surface of shallow water under the influence of gravity. If we consider the system in (1.1) without the effect of the earth's rotation, i.e., $\Omega=0$ and under the KdV scaling, it becomes the standard generalized two-component Camassa-Holm system,

$$\begin{cases} m_t+\sigma(2mu_x+um_x)+3(1-\sigma)uu_x+\frac{1}{2}(\rho^2)_x=0, \\ \rho_t+(\rho u)_x=0. \end{cases} \tag{1.3}$$

When $\sigma=1$ it is a classical two-component Camassa-Holm system

$$\begin{cases} m_t+2mu_x+um_x+\frac{1}{2}(\rho^2)_x=0, \\ \rho_t+(\rho u)_x=0. \end{cases} \tag{1.4}$$

This system (1.4) is completely integrable and can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter ξ, that is

$$\begin{aligned} \Psi_{xx}&=(-\sigma\xi^2\rho^2+\xi m+\frac{\sigma_1}{4})\Psi, \\ \Psi_t&=(\frac{1}{2\xi}-u)\Psi_x+\frac{1}{2}u_x\Psi. \end{aligned} \tag{1.5}$$

It has a bi-Hamiltonian structure corresponding to the Hamiltonians

$$H_1=\frac{1}{2}\int(mu+(\rho-1)^2)dx.$$

$$H_2=\frac{1}{2}\int(u(\rho-1)^2+2u(\rho-1)+u^3+uu_x^2)dx.$$

There are also two Casimirs, i.e. $\int(\rho-1)dx$ and $\int mdx$ with boundary conditions, $u\to 0$ and $\rho\to 1$ as $|x|\to\infty$. It can be rewritten as the Hamiltonian system for $(u,\rho)^T$, where J is a closed skew symmetric operator

$$\begin{pmatrix} -\partial_x(1-\partial_x^2)^{-1} & 0 \\ 0 & -\partial_x \end{pmatrix}, \tag{1.6}$$

It was shown in [11] that all solitary waves for the two-component Camassa-Holm system (1.4) are smooth, symmetric, and monotonic away from the crest, and decay exponentially far out. Moreover, the system can be reduced to the Camassa-Holm equation when $\rho=0$:

$$u_t+2\omega u_x-u_{txx}+3uu_x=2u_xu_{xx}+uu_{xxx}$$

where $\omega=\frac{A}{2}$ is a constant related to the critical shallow water wave speed. The local well-posedness in H^s, with $s>\frac{3}{2}$ for the Cauchy non-periodic problem was elaborated in [35], and [36] for the Cauchy periodic problem. With respect to blow-up criteria on the line, we refer to [25], and for the unit torus, to [36].

From the rotation-Green-Naghdi equations, we introduce a new variable ρ in the spirit of Constantin and Ivanov's work [11] and use Camassa-Holm scaling to derive our system. The goal is to study some properties of this model, such as wave breaking phenomenon and permanent waves. Wave breaking is a remarkable property for the Camassa-Holm equation, that is, the solution remains bounded while its slope becomes unbounded in finite time [5].

We first provide the blow up criteria utilizing the theory of the transportation equation and show that if finite-time blow-up of solution occurs, u_x must be unbounded from below along the characteristic, while the solution u itself is always bounded. For the condition of the initial data which can guarantee the wave breaking in finite tine, we choose the free surface ρ to vanish at some point x_0. It makes ρ vanished all the time at this point and we only need consider the wave breaking near x_0. Finally, we obtain the Riccati-type differential inequality of u_x and ρ. It is a powerful tool which takes the form of $\frac{df}{dt} \lesssim -kf^2 + C^2$ with the function f of t, and the constants k and C. The blow-up time is then obtained by solving this Riccati-type differential inequality. It is observed that under the CH scaling our system has more higher order nonlocal nonlinear terms compared to the classical Camassa-Holm system. That makes the blow-up issue more subtle and the some refined estimates are then needed.

If the free surface ρ never vanishes at any point x_0, there may exist permanent waves. The proper Lyapunov functional is not easy to find since the system has more higher order nonlocal nonlinear terms. It is worth remarking that the first equation in (1.1) is a transport equation for the free surface ρ along characteristics which illustrates, when the parameter $\Omega = 0$, that ρ is increasing when u_x is decreasing with respect of the time t. We are then motivated from this fact to analyze the competition between the slope of velocity u_x and the free surface ρ. The observation we have is that increase of the free surface ρ is always faster than decrease of u_x. That makes the ratio of ρ and u_x to have a limitation at T^* if there exists a blow-up at time T^*. Meanwhile, this limitation also implies the boundedness of ρ during the time $t \in [0, T^*]$ by using the structure of transport equation. Then we know that u_x can not go to infinity as $t \to T^*$ which is a contradiction, i.e., there is no wave breaking.

The rest of the paper is organized as follows. In Section 2, some preliminaries and dynamics along the characteristics are given. Then, in Sections 3 and 4, the wave-breaking data are illustrated. Finally, in Section 5, the existence of global solutions is obtained.

2. Preliminaries and dynamics along the characteristics

In order to research the wave breaking phenomenon of the periodic RCH2 equation, we have the following well-posedness theory for the periodic problem. We use the new variable $\zeta = \rho - 1$ and consider the initial-value problem for the (1.1) in the weak form

$$\begin{cases} u_t + \sigma u u_x = -G_x * \Big(\dfrac{3-\sigma}{2}u^2 + \dfrac{\sigma}{2}u_x^2 + \dfrac{1}{2}\zeta^2 + \zeta + 2\Omega(\zeta^2 u - 2\zeta u - u) \\ \qquad\qquad + 24\beta_1(u^2\zeta^2 + u^2\zeta) + 4\beta_2 u^3\Big), \\ \zeta_t + u\zeta_x = -\zeta u_x - u_x - 8\beta_1 (u^3)_x \zeta - 8\beta_1 (u^3)_x - \Omega(u^2)_x \zeta - \Omega(u^2)_x, \\ u(0,x) = u_0(x), \zeta(0,x) = \zeta_0(x) = \rho_0(x) - 1, \\ u(t,0) = u(t,1), \zeta(t,0) = \zeta(t,1), \end{cases} \tag{2.1}$$

where $\beta_1 = \frac{5c^2-11}{48c^2}, \beta_2 = \frac{2c^4-2c^2+8}{8c^3}$ and $G(x) = \dfrac{\cosh(x - [x] - \frac{1}{2})}{2\sinh(\frac{1}{2})}$, which is the fundamental solution of $(1-\partial_x^2)^{-1}$ on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, that is for any $x \in \mathbb{S}$,

$$\begin{aligned} (1-\partial_x^2)^{-1} f(x) = (G * f)(x) &= \int_0^1 \frac{\cosh\big((x-y) - [x-y] - \frac{1}{2}\big)}{2\sinh\big(\frac{1}{2}\big)} f(y)\,dy \\ &= \int_0^x \frac{\cosh\big(x - y - \frac{1}{2}\big)}{2\sinh(\frac{1}{2})} f(y)\,dy + \int_x^1 \frac{\cosh\big(x - y + \frac{1}{2}\big)}{2\sinh(\frac{1}{2})} f(y)\,dy. \end{aligned} \tag{2.2}$$

Now we are in a position to state the local well-posedness result of the periodic RCH2 system (2.1), which may be similarly obtained as in [14] (up to a slight modification).

THEOREM 2.1. *If initial data* $(u_0,\zeta_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$ *with* $s>\frac{3}{2}$*, there exists a time* T *such that the initial-value problem* (2.1) *has a unique solution*

$$(u,\zeta)\in (C([0,T];H^s(\mathbb{S})\bigcap C^1([0,T];H^{s-1}(\mathbb{S}))\times (C([0,T];H^{s-1}(\mathbb{S}))\bigcap C^1([0,T];H^{s-2}(\mathbb{S})).$$

LEMMA 2.1 ([36]). *For every* $f\in H^1(\mathbb{S})$*, we have*

$$\max_{x\in[0,1]} f^2(x)\leq \kappa\int_{\mathbb{S}}(f^2+\epsilon^2 f_x^2)dx,$$

where

$$\kappa=\frac{\cosh(\frac{1}{2\epsilon})}{2\epsilon\sinh(\frac{1}{2\epsilon})},$$

and κ *is the optimal constant which is obtained by the associated Green function defined by*

$$G_1=\frac{\cosh(\frac{x}{\epsilon}-\frac{[x]}{\epsilon}-\frac{1}{2\epsilon})}{2\epsilon\sinh(\frac{1}{2\epsilon})}.$$

In particular, when the parameter $\epsilon=1$*, the constant* $\kappa=\dfrac{e+1}{2(e-1)}$ *is sharp.*

LEMMA 2.2 ([9]). *Let* g *be a monotone function on* $[a,b]$*, and* f *be a real continuous function on* $[a,b]$*. Then there exists a* $\xi\in[a,b]$ *such that*

$$\int_a^b f(s)g(s)ds=g(a)\int_a^{\xi} f(s)ds+g(b)\int_{\xi}^b f(s)ds.$$

Moreover, we present the following useful results obtained in [4]. For any real constant γ, define $I(\gamma)\geq-\infty$ by

$$I(\gamma)=\inf\{\int_0^1 (G+\gamma G_x)*(2u^2+u_x^2)dx|u\in H^1(\mathbb{S}^1),u(0)=u(1)=1\},$$

and the quantity $\gamma^*\in[0,+\infty)$ by

$$\gamma^*=\inf\{\gamma\in(0,+\infty)|\gamma^2+I(\gamma)-2\geq 0\}$$

with the usual convention that $\gamma^*=+\infty$ if the infimum is taken on the empty set.

In [4], the authors proved that $I(\gamma)$ is even with respect to the variable $\gamma\in\mathbb{R}$ and $I(\gamma)>-\infty$ if and only if

$$-\frac{e+1}{e-1}\leq\gamma\leq\frac{e+1}{e-1}.$$

Especially, if $|\gamma|<\frac{e+1}{e-1}$, then $I(\gamma)$ is in fact a minimum with only one minimizer $u\in H^1(\mathbb{S}^1)$ with $u(0)=u(1)=1$. In addition, γ^* was computed numerically as the zero point of the function $\gamma\to\gamma^2+I(\gamma)-2$ by

$$\gamma^*=0.513...... \tag{2.3}$$

More importantly, they established the following convolution estimates, which is a very important technical issue for the blow-up analysis.

LEMMA 2.3. *[[4]] For any $\gamma\in\mathbb{R}$ and all $u\in H^1(\mathbb{S})$, the following convolution estimate holds*

$$(G\pm\gamma G_x)*(2u^2+u_x^2)(x)\geq I(\gamma)u^2(x),\quad \forall x\in\mathbb{S},$$

and $I(\gamma)$ is the best possible constant.

Moreover, we can obtain the following lemma regarding $\|\rho(t)\|_{L^\infty}$, which is very important to research the wave breaking phenomena about the RCH2 equations.

LEMMA 2.4. *For the transport equation*

$$\rho_t+u\rho_x=-\rho u_x-8\beta_1(u^3)_x\rho-2\Omega uu_x\rho, \tag{2.4}$$

with initial data $u_0\in H^s(s>\frac{3}{2})$, there is

$$\|\rho(t)\|_{L^\infty}\leq(1+\|\zeta(0)\|_{L^\infty})e^{\int_0^t\|u_x+8\beta_1(u^3)_x+2\Omega uu_x\|_{L^\infty}}d\tau.$$

Proof. We have the following associated Lagrangian scale

$$\begin{cases}\dfrac{dq(t,x)}{dt}=u(t,q(t,x)), & x\in\mathbb{S},\quad t\in[0,T).\\ q(0,x)=x, & \end{cases} \tag{2.5}$$

Then we have that

$$q_x(t,x)=e^{\int_0^t(u_x(\tau,q(\tau,x))+8\beta_1(u^3(\tau,q(\tau,x)))_x+2\Omega u(\tau,q(\tau,x))u_x(\tau,q(\tau,x))}>0,$$

with $(t,x)\in[0,T)\times\mathbb{S}$. Moreover

$$\|\rho(t)\|_{L^\infty}\leq\|\rho_0\|_{L^\infty}+C\int_0^t\|u_x+8\beta_1(u^3)_x+2\Omega uu_x\|_{L^\infty}\|\rho(t)\|_{L^\infty}d\tau.$$

Applying Grönwall's inequality, there is

$$\|\rho(t)\|_{L^\infty}\leq\|\rho_0\|_{L^\infty}e^{\int_0^t\|u_x+8\beta_1(u^3)_x+2\Omega uu_x\|_{L^\infty}}d\tau.$$

We know that $q(t,\cdot):\mathbb{S}\to\mathbb{S}$ is a diffeomorphism of the line for every $t\in[0,T)$, i.e.

$$\|\rho(t,\cdot)\|_{L^\infty(\mathbb{S})}=\|\rho(t,q(t,\cdot))\|_{L^\infty(\mathbb{S})}.$$

Then

$$\|\rho(t)\|_{L^\infty}\leq(1+\|\zeta(0)\|_{L^\infty})e^{\int_0^t\|u_x+8\beta_1(u^3)_x+2\Omega uu_x\|_{L^\infty}d\tau},$$

which is the result. ∎

We can also obtain the following blow-up criteria lemma, which may be similarly obtained as in [29].

LEMMA 2.5. *Consider the solution of the RCH2 system (1.1) with initial data $(u_0,\rho_0-1)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$ with $s>\frac{3}{2}$, and let T be the maximal time of existence. Then*

$$T<\infty\Rightarrow\int_0^T\|u_x(\tau)\|_{L^\infty}d\tau=\infty.$$

3. Wave-breaking criterion

We now focus on the following periodic RCH2 equations with $\Omega=0$ and $\sigma=1$ in (1.2)

$$\begin{cases} m_t+2mu_x+um_x+\dfrac{1}{2}(\rho^2)_x+4(u^3)_x-3(u^2\rho(\rho-1))_x=0,\\ \rho_t+(\rho u)_x-(u^3)_x\rho=0,\\ m_0=m(x,0),\rho_0=\rho(x,0),\\ m(t,0)=m(t,1),\rho(t,0)=\rho(t,1), \end{cases} \tag{3.1}$$

where $m=u-u_{xx}$. Moreover, we have the following conservation law

$$E(u,\rho)=\int_{\mathbb{S}}(u^2+u_x^2+(\rho-1)^2)dx.$$

Equations (3.1) can be written as the following RCH2 equations,

$$\begin{cases} u_t-u_{xxt}+3uu_x-2u_xu_{xx}-uu_{xxx}+\dfrac{1}{2}(\rho^2)_x+4(u^3)_x-3(u^2\rho(\rho-1))_x=0,\\ \rho_t=-(\rho u)_x+(u^3)_x\rho,\\ u_0=u(x,0),\rho_0=\rho(x,0),\\ u(t,0)=u(t,1),\rho(t,0)=\rho(t,1). \end{cases} \tag{3.2}$$

Then we convert the above equations into the following weak form,

$$\begin{cases} u_t+uu_x=-\partial_xG*(u^2+\dfrac{1}{2}u_x^2+\dfrac{1}{2}\rho^2+4u^3-3u^2\rho(\rho-1)),\\ \rho_t=-(\rho u)_x+(u^3)_x\rho,\\ u_0=u(x,0),\rho_0=\rho(x,0),\\ u(t,0)=u(t,1),\rho(t,0)=\rho(t,1). \end{cases} \tag{3.3}$$

Now, we prove the following accurate blow-up criteria.

LEMMA 3.1. *Suppose $E(0)\le\frac{1}{3\kappa}$ with the constant $\kappa=\frac{e+1}{2(e-1)}$ and (u,ρ) is the solution of equation in (3.1) with initial data $(u_0,\rho_0)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})(s>\frac{3}{2})$. Let $T>0$ be the maximal time of existence of the solution (u,ρ). Then (u,ρ) blows up in finite time $T<\infty$ if and only if*

$$\lim_{t\to T}\inf_{x\in\mathbb{S}}u_x(t,x)=-\infty. \tag{3.4}$$

Proof. Take derivative with respect to x to the first equation in (3.3) to get the following form

$$\begin{cases} u_{t,x}+uu_{x,x}=u^2-\dfrac{1}{2}u_x^2+\dfrac{1}{2}\rho^2+4u^3-3u^2\rho(\rho-1)\\ \qquad\qquad\qquad -G*(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2+4u^3-3u^2\rho(\rho-1)),\\ \rho_t=-(\rho u)_x+(u^3)_x\rho,\\ u_0=u(x,0),\rho_0=\rho(x,0),\\ u(t,0)=u(t,1),\rho(t,0)=\rho(t,1). \end{cases} \tag{3.5}$$

Let

$$M(t)=u_x(t,q(t,x)),\gamma(t)=\rho(t,q(t,x)),t\in[0,T),$$

where $q(t,x)$ is the characteristic as defined in (2.5). Using these notations, (3.5) can be rewritten respectively as

$$\begin{cases} M'(t) = -\frac{1}{2}M^2(t) + \frac{1}{2}\gamma^2(t) + f(t,q(t,x)) \\ \gamma'(t) = -\gamma M(1-3u^2), \end{cases} \quad x \in \mathbb{S}, t \in [0,T). \tag{3.6}$$

for $t \in [0,T)$, where

$$f = u^2 + 4u^3 - 3u^2\rho(\rho-1) - G*(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + 4u^3 - 3u^2\rho(\rho-1)).$$

Suppose that $T < \infty$ and (3.4) is not valid. Then there exists a positive number $M_1 > 0$, such that

$$\inf_{x\in\mathbb{S}} u_x(t,x) \geq -M_1, \qquad \forall t \in [0,T).$$

For each $x \in \mathbb{S}$, we have

$$\rho(t,q) = \gamma(t) = \gamma(0)e^{\int_0^t -M(t)(1-3u^2)dt}.$$

Using the assumption that $E(0) \leq \frac{1}{3\kappa}$, there is

$$1 - 3u^2 \geq 1 - 3\kappa E(0) \geq 0.$$

It means that

$$\|\rho(t,\cdot)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{M_1 C_1 t},$$

where $C_1 = 1 - 3\kappa E(0)$. Now we will estimate the upper bound of the function f by

$$\begin{aligned} f =& u^2 + 4u^3 - 3u^2\rho(\rho-1) - G*(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + 4u^3 - 3u^2\rho(\rho-1)) \\ \leq& u^2 + 4|u^3| + 3|u^2\rho(\rho-1)| + |G*(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2)| + 4|G*u^3| + 3|G*u^2(\rho-1)|. \end{aligned}$$

On the other hand, it is found that

$$u^2 \leq \kappa \int_{\mathbb{S}} (u^2 + u_x^2)dx = \kappa\|u\|_{H^1}^2 < \kappa E(0), \tag{3.7}$$

$$|u^3| \leq (\sqrt{\kappa})^3 E^{\frac{3}{2}}(0), \tag{3.8}$$

$$\begin{aligned} &G*(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2) = G*(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}(\rho-1+1)^2) \\ \leq& G*(u^2 + \frac{1}{2}u_x^2 + (\rho-1)^2 + 1) \leq G*(u^2 + \frac{1}{2}u_x^2 + (\rho-1)^2) + G*1 \\ \leq& \frac{e+1}{2(e-1)}E(0) + \frac{e+1}{2(e-1)}. \end{aligned}$$

$$|u^2\rho(\rho-1)| \leq |u\|_{L^\infty}^2 \|\rho\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \|\rho\|_{L^\infty}$$

$$\leq\frac{1}{2}(\|u\|_{L^\infty}^4+\|\rho\|_{L^\infty}^4)+\frac{1}{2}(\|u\|_{L^\infty}^4+\|\rho\|_{L^\infty}^2)=\|u\|_{L^\infty}^4+\frac{1}{2}\|\rho\|_{L^\infty}^4+\frac{1}{2}\|\rho\|_{L^\infty}^2$$
$$=(\sqrt{\kappa}E^{\frac{1}{2}}(0))^4+\frac{1}{2}\|\rho\|_{L^\infty}^4+\frac{1}{2}\|\rho\|_{L^\infty}^2=\kappa^2E^2(0)+\frac{1}{2}\|\rho\|_{L^\infty}^4+\frac{1}{2}\|\rho\|_{L^\infty}^2. \tag{3.9}$$

$$4|G*u^3|\leq\frac{2(e+1)}{e-1}\|u\|_{L^\infty}^3\leq\frac{2(e+1)}{e-1}\kappa^{\frac{3}{2}}E^{\frac{3}{2}}(0), \tag{3.10}$$

$$\begin{aligned}|G*u^2(\rho-1)| &\leq\frac{e+1}{2(e-1)}\|u\|_{L^\infty}(\|u\|_{L^2}^2+\|\rho-1\|_{L^2}^2)\leq\frac{e+1}{2(e-1)}\kappa^{\frac{1}{2}}E^{\frac{1}{2}}(0)E(0)\\ &=\frac{\kappa^{\frac{1}{2}}(e+1)}{2(e-1)}E^{\frac{3}{2}}(0).\end{aligned} \tag{3.11}$$

Consequently, the upper bound of f is obtained by

$$f=C_2^2+\frac{3}{2}\|\rho_0\|_{L^\infty}^4e^{4M_1C_1t}+\frac{3}{2}\|\rho_0\|_{L^\infty}^2e^{2M_1C_1t}, \tag{3.12}$$

where

$$C_2^2=(\kappa+\frac{e+1}{2(e-1)})E(0)+3\kappa^2E^2(0)+(4\kappa^{\frac{3}{2}}+\frac{2(e+1)}{e-1}\kappa^{\frac{3}{2}}+\frac{3\kappa^{\frac{1}{2}}(e+1)}{2(e-1)})E(0)^{\frac{3}{2}}+\frac{e+1}{2(e-1)}.$$

Given any $x\in\mathbb{S}$ and $t\in[0,t)$, we introduce a new C^1-differentiable function:

$$P(t)=M(t)-\|u_{0,x}\|_{L^\infty}-2C_2-\sqrt{3}\|\rho_0\|_{L^\infty}^2e^{2M_1C_1t}-2\|\rho_0\|_{L^\infty}e^{M_1C_1t}, \tag{3.13}$$

and it satisfies

$$P(0)=u_{0,x}-\|u_{0,x}\|_{L^\infty}-2C_2-\sqrt{3}\|\rho_0\|_{L^\infty}^2-2\|\rho_0\|_{L^\infty}\leq 0.$$

We now claim that $P(t)\leq 0(\forall t\in[0,t)$. If not, there is some $t_0\in[0,t)$, such that $P(t)>0$. Let

$$t_1=\max\{t<t_0;P(t)\leq 0\}.$$

Then $P(t_1)=0$ and $P'(t_1)\geq 0$, or equivalently,

$$M(t_1)=\|u_{0,x}\|_{L^\infty}+2C_2+\sqrt{3}\|\rho_0\|_{L^\infty}^2e^{2M_1C_1t_1}+2\|\rho_0\|_{L^\infty}e^{M_1C_1t_1},$$

and

$$M'(t_1)\geq 2\sqrt{3}M_1C_1\|\rho_0\|_{L^\infty}^2e^{2M_1C_1t_1}+2M_1C_1\|\rho_0\|_{L^\infty}e^{M_1C_1t_1}\geq 0, \tag{3.14}$$

While we have

$$\begin{aligned}M'(t_1) &=-\frac{1}{2}M^2(t_1)+\frac{1}{2}\gamma^2(t_1)+f(t_1,q(t,x))\\ &\leq-\frac{1}{2}(\|u_{0,x}\|_{L^\infty}+2C_2+\sqrt{3}\|\rho_0\|_{L^\infty}^2e^{2M_1C_1t_1}+2\|\rho\|_{L^\infty}e^{M_1C_1t_1})^2\\ &\quad+\frac{1}{2}\|\rho_0\|_{L^\infty}^2e^{2M_1C_1t_1}+C_2^2+\frac{3}{2}\|\rho_0\|_{L^\infty}^4e^{4M_1C_1t_1}+\frac{3}{2}\|\rho_0\|_{L^\infty}^2e^{2M_1C_1t_1}\leq 0.\end{aligned} \tag{3.15}$$

which contradicts (3.14). Hence we have

$$P(t) \leq 0 \quad \forall t \in [0,T).$$

Therefore, the arbitrary choice of x implies

$$\sup_{x\in\mathbb{S}} u_x \leq \|u_{0,x}\|_{L^\infty} + 2C_2 + \sqrt{3}\|\rho_0\|_{L^\infty}^2 e^{2M_1C_1t} + 2\|\rho_0\|_{L^\infty} e^{M_1C_1t}.$$

Recall that

$$\inf_{x\in\mathbb{S}} u_x(t,x) \geq -M_1, \quad \forall t \in [0,T).$$

This thus implies that $|u_x| < \infty$ which is a contradiction to our assumption $T < \infty$. This completes the proof of the lemma. ◻

REMARK 3.1. With the aid of

$$\inf_{x\in\mathbb{S}} u_x(t,q(t,x)) = \inf_{x\in\mathbb{S}} u_x(t,x),$$

the wave-breaking criterion

$$\liminf_{t\to T}\inf_{x\in\mathbb{S}} u_x(t,x) = -\infty,$$

can be replaced by

$$\lim_{t\to T}\inf_{x\in\mathbb{S}} u_x(t,q(t,x)) = -\infty.$$

REMARK 3.2. Actually, if we use the assumption $\rho_0(x_0) = 0$, the embedding theorem gives that

$$\|\rho_0 - 1\|_{H^1} \geq \frac{1}{\kappa}\|\rho_0 - 1\|_{L^\infty} \geq \frac{1}{\kappa}|\rho_0 - 1| = \frac{1}{\kappa}.$$

This implies that the initial energy satisfies

$$\|(u_0, \rho_0 - 1)\|_{H^s\times H^{s-1}}^2 \geq \|\rho_0 - 1\|_{H^1}^2 \geq \frac{1}{\kappa}.$$

for $s > \frac{3}{2}$. This estimate tells us that even the assumption $E(0) \leq \frac{1}{3\kappa}$ could imply that $(u_0, \rho_0 - 1)$ might be small under the $H^1 \times L^2$-norm, but it is unnecessary to be small with the higher $H^s \times H^{s-1}$-norm.

4. Wave-breaking phenomena

Our attention in this section is now turned to searching certain wave-breaking data for the system with (3.3). Using the characteristics method outlined, we establish the following wave-breaking result with certain initial data.

THEOREM 4.1. *Suppose that (u,ρ) is the solution of (3.3) with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, with $s > \frac{3}{2}$. Let T_1 be the maximal time of existence. Assume there exists a point $x_0 \in \mathbb{S}$ such that*

$$\rho_0(x_0) = 0, \tag{4.1}$$

and

$$u_{0,x}(x_0) < -\gamma^*|u_0(x_0)| - \sqrt{2}C_3. \tag{4.2}$$

Then the corresponding solution (u,ρ) *to system* (3.1) *blows up in finite time* T_1 *with*

$$T_1 \leq \frac{2}{\sqrt{2}C_3 - \sqrt{u_{0,x}^2(x_0) - {\gamma^*}^2 u_0^2(x_0)}} < \infty. \tag{4.3}$$

where

$$C_1 = 1 - 3\kappa E(0), \quad \inf_{x\in S} u_x(t,x) \geq -M_1 \ (M_1 > 0 \textit{ is a positive number}).$$

and

$$\begin{aligned} C_3^2 = & 4\kappa^{\frac{3}{2}} E^{\frac{3}{2}}(0) + \frac{1}{2}\Big(\frac{e+1}{2(e-1)} + \frac{|\gamma^*|}{2}\Big)(4\|\rho_0\|_{L^\infty}^2 e^{2M_1C_1t} + 8\kappa^{\frac{3}{2}} E^{\frac{3}{2}}(0) \\ & + 6\kappa^2 E^2(0) + 3\|\rho_0\|_{L^\infty}^4 e^{4M_1C_1t}). \end{aligned}$$

Proof. A simple computation reveals that u and u_x satisfy along the characteristics

$$u_t + uu_x = -G_x * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + 4u^3 - 3u^2\rho(\rho-1)),$$

and

$$u_{tx} + uu_{xx} = u^2 - \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + 4u^3 - 3u^2\rho(\rho-1) - G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + 4u^3 - 3u^2\rho(\rho-1)).$$

On account of the assumption in (4.1), we have $\rho(0) = \rho_0(x_0) = 0$. It is then inferred from Lemma 2.4 that

$$\rho(t) = 0, \quad t \in [0,T). \tag{4.4}$$

Then, we have

$$\frac{du(t,q)}{dt} = -G_x * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + 4u^3 - 3u^2\rho(\rho-1)), \tag{4.5}$$

and

$$\frac{du_x(t,q)}{dt} = u^2 - \frac{1}{2}u_x^2 + 4u^3 - G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + 4u^3 - 3u^2\rho(\rho-1)). \tag{4.6}$$

Let $F_1 = \gamma^* u - u_x$ and $F_2 = \gamma^* u + u_x$. Then

$$\begin{aligned} \frac{dF_1}{dt} = & \frac{1}{2}u_x^2 - u^2 - 4u^3 + G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + 4u^3 - 3u^2\rho(\rho-1)) \\ & - \gamma^* G_x * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + 4u^3 - 3u^2\rho(\rho-1)), \end{aligned} \tag{4.7}$$

and

$$\frac{dF_2}{dt} = -\frac{1}{2}u_x^2 + u^2 + 4u^3 - G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + 4u^3 - 3u^2\rho(\rho-1))$$

$$-\gamma^* G_x * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 + 4u^3 - 3u^2\rho(\rho-1)). \tag{4.8}$$

Moreover, using Lemma 2.3,

$$\begin{aligned}
\frac{dF_1}{dt} =& \frac{1}{2}(u_x^2 - 2u^2 - 8u^3) + \frac{1}{2}G * (2u^2 + u_x^2 + \rho^2 + 8u^3 - 6u^2\rho(\rho-1)) \\
&- \frac{1}{2}\gamma^* G_x * (2u^2 + u_x^2 + \rho^2 + 8u^3 - 6u^2\rho(\rho-1)), \\
\geq& \frac{1}{2}(u_x^2 - 2u^2 - 8u^3) + \frac{1}{2}I(\gamma^*)u^2 + \frac{1}{2}(G - \gamma^* G_x) * (\rho^2 + 8u^3 - 6u^2\rho(\rho-1)), \\
=& \frac{1}{2}(u_x^2 - 2u^2 + I(\gamma^*)u^2 - 8u^3) + \frac{1}{2}(G - \gamma^* G_x) * (\rho^2 + 8u^3 - 6u^2\rho(\rho-1)).
\end{aligned} \tag{4.9}$$

Because $I(\gamma^*) = 2 - (\gamma^*)^2$, we have

$$\begin{aligned}
\frac{dF_1}{dt} &= \frac{1}{2}(u_x^2 - (\gamma^*)^2 u^2) - 4u^3 + \frac{1}{2}(G - \gamma^* G_x) * (\rho^2 + 8u^3 - 6u^2\rho(\rho-1)), \\
&= -\frac{1}{2}(\gamma^* u + u_x)(\gamma^* u - u_x) - 4u^3 + \frac{1}{2}(G - \gamma^* G_x) * (\rho^2 + 8u^3 - 6u^2\rho(\rho-1)).
\end{aligned} \tag{4.10}$$

It then follows from the estimates in Lemma 2.4 that

$$\begin{aligned}
&|-4u^3 + \frac{1}{2}(G - \gamma^* G_x) * (\rho^2 + 8u^3 - 6u^2\rho(\rho-1))| \\
&\leq 4\|u\|_{L^\infty}^3 + \frac{1}{2}(\frac{e+1}{2(e-1)} + \frac{|\gamma^*|}{2})(\|\rho\|_{L^\infty}^2 + 8\|u\|_{L^\infty}^3 + 6\|u^2\rho(\rho-1)\|_{L^\infty}) \\
&\leq 4\kappa^{\frac{3}{2}} E^{\frac{3}{2}}(0) + \frac{1}{2}(\frac{e+1}{2(e-1)} + \frac{|\gamma^*|}{2})(4\|\rho_0\|_{L^\infty}^2 e^{2M_1C_1t} + 8\kappa^{\frac{3}{2}} E^{\frac{3}{2}}(0) \\
&+ 6\kappa^2 E^2(0) + 3\|\rho_0\|_{L^\infty}^4 e^{4M_1C_1t}) \\
&= C_3^2.
\end{aligned} \tag{4.11}$$

It is then inferred from above that

$$\frac{dF_1}{dt} \geq -\frac{1}{2}F_1F_2 - C_3^2. \tag{4.12}$$

With the same approach, we have

$$\frac{dF_2}{dt} \leq \frac{1}{2}F_1F_2 + C_3^2. \tag{4.13}$$

By the assumption on $u_0(x_0)$, it is easy to see that

$$F_1(0) = \gamma^* u_0(x_0) - u_{0,x}(x_0) > 0, \quad F_2(0) = \gamma^* u_0(x_0) + u_{0,x}(x_0) < 0,$$

and

$$\frac{1}{2}F_1(0)F_2(0) + C_3^2 < 0.$$

By the continuity of $F_1(t)$ and $F_2(t)$, it is then ensured that

$$\frac{dF_1}{dt} > 0, \quad \frac{dF_2}{dt} < 0, \quad \forall t \in [0, T).$$

This in turn implies that

$$F_1(t) > F_1(0) > 0, \quad F_2(t) < F_2(0) < 0, \quad \forall t \in [0,T).$$

Let $h(t) = \sqrt{-F_1(t)F_2(t)}$. It then follows that

$$\begin{aligned}\frac{dh}{dt} &= \frac{-F_1'(t)F_2(t) - F_1(t)F_2'(t)}{2h} \geq \frac{(-\frac{1}{2}F_1F_2 - C_3^2)(-F_2) - F_1(\frac{1}{2}F_1F_2 + C_3^2)}{2h} \\ &= \frac{F_1 - F_2}{2h}(-\frac{1}{2}F_1F_2 - C_3^2).\end{aligned}$$

Using the estimate $\dfrac{F_1 - F_2}{2h} \geq 1$ and the fact that $h + \sqrt{2}C_3 > h - \sqrt{2}C_3 > 0$, we obtain the following differential inequalities

$$\frac{dh}{dt} \geq -\frac{1}{2}F_1F_2 - C_3^2 = \frac{1}{2}(h - \sqrt{2}C_3)(h + \sqrt{2}C_3) \geq \frac{1}{2}(h - \sqrt{2}C_3)^2.$$

Hence, solving this inequality implies that the wave breaks

$$\lim_{t \to T_1^-} h(t, q(t, x_0)) = \infty$$

at the wave breaking time

$$T_1 \leq \frac{2}{\sqrt{2}C_3 - \sqrt{u_{0,x}^2(x_0) - {\gamma^*}^2 u_0^2(x_0)}} < \infty.$$

Using the fact that $-u_x(t, q(t, x_0)) = \dfrac{1}{2}(F_1 - F_2) \geq h(t, q(t, x_0))$, this in turn implies that there exists $T_1 < \infty$, such that

$$\lim_{t \to T_1^-} \inf_{x \in \mathbb{S}} \{u_x(t,x)\} = -\infty.$$

The proof of the theorem is thus complete. $\square$

Moreover, we can obtain the following wave breaking phenomena with the second method. We set

$$m_1(t) := \min_{x \in \mathbb{S}} (u_x(t,x)), \qquad m_2(t) := \max_{x \in \mathbb{S}} (u_x(t,x)).$$

THEOREM 4.2. *Let $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{3}{2}$ satisfy*

$$m_1(0) + m_2(0) < -4\kappa E(0)\frac{\cosh(1/2) - 1}{\sinh(1/2)} - 2\sqrt{2}C_4, \tag{4.14}$$

where

$$\begin{aligned}C_4^2 = &\kappa E^2(0) + \frac{\cosh(1/2)}{2\sinh(1/2)}\big(E(0) + 1 + 3\kappa^2 E^2(0) + \frac{3}{2}\|\rho_0\|_{L^\infty}^4 e^{4M_1C_1t} \\ &+ \frac{3}{2}\|\rho_0\|_{L^\infty}^2 e^{2M_1C_1t} + 24\kappa^{\frac{1}{2}} E^{\frac{3}{2}}(0)\big),\end{aligned}$$

and κ is defined in Lemma 2.1. If there are some $x_1,x_2\in\mathbb{S}$ such that

$$\rho_0(x_1)=0,\quad u_{0,x}(x_1)=\inf_{x\in\mathbb{S}}u_{0,x}(x),\tag{4.15}$$

and

$$\rho_0(x_2)=0,\quad u_{0,x}(x_2)=\sup_{x\in\mathbb{S}}u_{0,x}(x),\tag{4.16}$$

then the solution of (3.3) *blows up in finite time,*

$$T_2<-\frac{2(1+\delta)^2}{(m_1(0)+2C_5)\delta(\delta+2)},$$

where $C_5=\kappa E(0)\frac{\cosh(1/2)-1}{\sinh(1/2)}$ and $\delta\in\left(0,\frac{1}{2}\right]$.

Proof. Let $T>0$ be the maximal time of existence of the corresponding solution (u,ρ) to (1.6). By Theorem 2.1, we need only to prove this theorem for $s\geq 3$. According to Lemma 3.1, we can define $\xi(t)\in\mathbb{S}$ as

$$m_1(t)=u_x(t,\xi(t))=\inf_{x\in\mathbb{S}}u_x(t,x),\quad t\in[0,T).\tag{4.17}$$

Since $q(t,\cdot)$ defined by (2.5) is a diffeomorphism of the circle for any $t\in[0,T)$, we obtain that there exists a $x_1(t)\in\mathbb{S}$ such that

$$q(t,x_1(t))=\xi(t),\quad t\in[0,T).\tag{4.18}$$

Then (4.17) and (4.18) imply that

$$m_1(0)=u_x(0,\xi(0))=\inf_{x\in\mathbb{S}}u_{0,x}(x)=u_{0,x}(x_1).$$

Therefore we can choose $\xi(0)=x_1$ and

$$\rho_0(\xi(0))=\rho_0(x_1)=0.$$

Using Lemma 2.4, we have

$$\rho(t,q(t,x_1(t)))=\rho(t,\xi(t))=0,\quad\forall t\in[0,T).\tag{4.19}$$

On the other hand, since $\sup_{x\in\mathbb{S}}(v_x(t,x))=-\inf_{x\in\mathbb{S}}(-v_x(t,x))$, we similarly define

$$m_2(t)=u_x(t,\eta(t))=\sup_{x\in\mathbb{S}}u_x(t,x),\quad t\in[0,T),\tag{4.20}$$

then there exists a $x_2(t)\in\mathbb{S}$ such that $q(t,x_2(t))=\eta(t),t\in[0,T)$. Moreover, we have

$$\rho(t,q(t,x_2(t)))=\rho(t,\xi(t))=0,\quad\forall t\in[0,T).\tag{4.21}$$

Now, differentiating the first equation in (3.3) with respect to x, we have

$$\begin{aligned}u_{tx}+uu_{xx}=&u^2-\frac{1}{2}u_x^2+\frac{1}{2}\rho^2-3u^2\rho(\rho-1)-G*(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\\&-3\beta u^2\rho(\rho-1))-G*(24uu_x^2+12u^2u_{xx})\end{aligned}\tag{4.22}$$

In view of the definitions of $m_i(t)(i=1,2)$ in (4.7) and (4.8), let $x=x_i(t),(t=1,2)$, we obtain that

$$\begin{aligned}\frac{dm_1}{dt}=&-\frac{1}{2}m_1^2+u^2-G*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2-3\beta u^2\rho(\rho-1)+24uu_x^2\right)\\&-G*(12u^2u_{xx})\\=&-\frac{1}{2}m_1^2+u^2-\int_0^1 G(\xi(t)-y)\big(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2-3\beta u^2\rho(\rho-1)\\&+24uu_x^2\big)dy-\int_0^1 G(\xi(t)-y)(12u^2u_{xx})dy.\end{aligned}\tag{4.23}$$

and

$$\begin{aligned}\frac{dm_2}{dt}=&-\frac{1}{2}m_2^2+u^2+4u^3-\int_0^1 G(\xi(t)-y)\big(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2+4u^3-3u^2\rho(\rho-1)\\&+24uu_x^2\big)dy-\int_0^1 G(\xi(t)-y)(12u^2u_{xx})dy.\end{aligned}\tag{4.24}$$

In view of $\frac{1}{2\sinh(1/2)}\le G(y)\le\frac{\cosh(1/2)}{2\sinh(1/2)}$ and thanks to Lemma 2.1, we have

$$\begin{aligned}&u^2-\int_0^1 G(x-y)\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2-3u^2\rho(\rho-1)+24uu_x^2\right)dy\\ \le&\kappa\int_{\mathbb{S}}(u^2+u_x^2)dx+\frac{\cosh(1/2)}{2\sinh(1/2)}\big(\int_{\mathbb{S}}(u^2+\frac{1}{2}u_x^2+(\rho-1)^2+1)dx\\&+3\kappa^2E^2(0)+\frac{3}{2}\|\rho_0\|_{L^\infty}^4e^{4M_1C_1t}+\frac{3}{2}\|\rho_0\|_{L^\infty}^2e^{2M_1C_1t}+24\|u\|_{L^\infty}E(0)\big)\\ \le&\kappa E(0)+\frac{\cosh(1/2)}{2\sinh(1/2)}\big(E(0)+1+3\kappa^2E^2(0)+\frac{3}{2}\|\rho_0\|_{L^\infty}^4e^{4M_1C_1t}\\&+\frac{3}{2}\|\rho_0\|_{L^\infty}^2e^{2M_1C_1t}+24\kappa^{\frac{1}{2}}E^{\frac{3}{2}}(0)\big)=C_4^2.\end{aligned}\tag{4.25}$$

The function $G(y)$ is continuous, decreasing on $[0,\frac{1}{2}]$ and increasing on $[\frac{1}{2},1]$, with $G\left(\frac{1}{2}\right)=\frac{1}{2\sinh(1/2)}$ and $G(0)=G(1)=\frac{\cosh(1/2)}{2\sinh(1/2)}$. So that we choose the function

$$g(y)=G(y)-\frac{1}{2\sinh(1/2)},\qquad y\in\mathbb{S},$$

which is continuous, decreasing on $[0,\frac{1}{2}]$ and increasing on $[\frac{1}{2},1]$, with $g(\frac{1}{2})=0$ and $g(0)=g(1)=\frac{\cosh(1/2)-1}{2\sinh(1/2)}$. Noting the periodicity of u_{xx}, we find for $i=1,2$, that

$$\begin{aligned}&\left|\int_0^1 G(y)u^2u_{xx}(t,x_i-y)dy\right|\\=&\|u\|_{L^\infty}^2\left|\int_0^1 g(y)u_{xx}(t,x_i-y)dy\right|\\ \le&\kappa E(0)\left(\left|\int_0^{\frac{1}{2}}g(y)u_{xx}(t,x_i-y)dy\right|+\left|\int_{\frac{1}{2}}^1 g(y)u_{xx}(t,x_i-y)dy\right|\right).\end{aligned}\tag{4.26}$$

Using Lemma 2.2, we have

$$\begin{aligned}
&\left|\int_0^{\frac{1}{2}} g(y)u_{xx}(t,x_i-y)dy\right| \\
&=\left|g(0)\int_0^{\varphi} u_{xx}(t,x_i-y)dy+g\left(\frac{1}{2}\right)\int_{\varphi}^{\frac{1}{2}} u_{xx}(t,x_i-y)dy\right| \\
&=\left|\frac{\cosh(1/2)-1}{2\sinh(1/2)}\left(u_x(t,x_i)-u_x(t,x_i-\varphi)\right)\right| \\
&\leq\frac{\cosh(1/2)-1}{2\sinh(1/2)}\left(m_2(t)-m_1(t)\right).
\end{aligned} \tag{4.27}$$

In the same way, we obtain

$$\left|\int_{\frac{1}{2}}^1 g(y)u_{xx}(t,x_i-y)dy\right|\leq\frac{\cosh(1/2)-1}{2\sinh(1/2)}\left(m_2(t)-m_1(t)\right). \tag{4.28}$$

Substituting (4.27) and (4.28) into (4.26), we deduce that

$$\left|\int_0^1 G(y)u^2u_{xx}(t,x_i-y)dy\right|\leq\kappa E(0)\frac{\cosh(1/2)-1}{\sinh(1/2)}\left(m_2(t)-m_1(t)\right). \tag{4.29}$$

In view of (4.26), (4.27), (4.28), (4.29), we obtain for a.e. $t\in(0,T)$ that

$$\frac{dm_1}{dt}\leq-\frac{1}{2}m_1^2+\kappa E(0)\frac{\cosh(1/2)-1}{\sinh(1/2)}(m_2-m_1)+C_4^2, \tag{4.30}$$

$$\frac{dm_2}{dt}\leq-\frac{1}{2}m_2^2+\kappa E(0)\frac{\cosh(1/2)-1}{\sinh(1/2)}(m_2-m_1)+C_4^2. \tag{4.31}$$

Summing up the above two equations gives

$$\begin{aligned}
\frac{d(m_1+m_2)}{dt}&\leq-\frac{1}{2}(m_1^2+m_2^2)+2\kappa E(0)\frac{\cosh(1/2)-1}{\sinh(1/2)}(m_2-m_1)+2C_4^2 \\
&=-\frac{1}{2}(m_1^2+m_2^2)+2\kappa E(0)\frac{\cosh(1/2)-1}{\sinh(1/2)}(m_2+m_1) \\
&\quad-4\kappa E(0)\frac{\cosh(1/2)-1}{\sinh(1/2)}m_1+2C_4^2.
\end{aligned} \tag{4.32}$$

Let

$$C_5=\kappa E(0)\frac{\cosh(1/2)-1}{\sinh(1/2)}. \tag{4.33}$$

Then (4.30) and (4.31) become the following equations

$$\frac{dm_1}{dt}\leq-\frac{1}{2}m_1^2+C_5(m_2-m_1)+C_4^2, \tag{4.34}$$

and

$$\frac{d(m_1+m_2)}{dt}\leq-\frac{1}{2}(m_1^2+m_2^2)+2C_5(m_2+m_1)-4C_5m_1+2C_4^2. \tag{4.35}$$

Since $(m_1+m_2)(0) < -4\kappa E(0)\dfrac{\cosh(1/2)-1}{\sinh(1/2)} - 2\sqrt{2}C_4 = -4C_5 - 2\sqrt{2}C_4$, there is $\delta \in \left(0,\frac{1}{2}\right]$ such that $(m_1+m_2)(0) \le -\alpha - 2\sqrt{2}(1+\delta)C_4$ with $\alpha = 4C_5+\delta$, $\alpha > 4C_5$.

We first claim that there holds for all $t\in(0,T]$

$$(m_1+m_2)(t) \le -\alpha - 2\sqrt{2}(1+\delta)C_4. \tag{4.36}$$

Let $\overline{m}(t) =: (m_1+m_2)(t)+\alpha+2\sqrt{2}(1+\delta)C_4$. Then we claim that $\overline{m}(t)\le 0$. It is observed that $\overline{m}$ is continuous on $[0,T)$. If (4.36) does not hold, we can find a $t_0\in(0,T)$ such that $\overline{m}(t)>0$. Denote

$$t_1 = \max(t<t_0 : \overline{m}(t_0)=0).$$

Then

$$\overline{m}(t_1)=0, \qquad \overline{m}'(t_1)\ge 0. \tag{4.37}$$

Thanks to

$$m_1(t_1) \le \frac{1}{2}(m_1+m_2)(t_1) = -\frac{1}{2}\alpha - \sqrt{2}(1+\delta)C_4$$

and

$$m_2(t_1) = -\alpha - 2\sqrt{2}(1+\delta)C_4 - m_1(t_1),$$

using (4.35), we get

$$\begin{aligned}
\overline{m}'(t_1) =& (m_1+m_2)'(t_1)\\
\le& -\frac{1}{2}m_1^2(t_1) - \frac{1}{2}m_2^2(t_1) + 2C_5(m_2+m_1)(t_1) - 4C_5 m_1(t_1) + 2C_4^2\\
=& -\frac{1}{2}m_1^2(t_1) - \frac{1}{2}\left(-\alpha - 2\sqrt{2}(1+\delta)C_4 - m_1(t_1)\right)^2\\
&+2C_5\left(-\alpha-2\sqrt{2}(1+\delta)C_4\right) - 4C_5 m_1(t_1) + 2C_4^2\\
=& -m_1^2(t_1) - m_1(t_1)\left(\alpha + 2\sqrt{2}(1+\delta)C_4 + 4C_5\right)\\
&-\frac{1}{2}\left(\alpha+2\sqrt{2}(1+\delta)C_4\right)^2 - 2C_5\left(\alpha+2\sqrt{2}(1+\delta)C_4\right) + 2C_4^2\\
=& -\left(m_1(t_1) + \frac{1}{2}\left(\alpha+2\sqrt{2}(1+\delta)C_4+4C_5\right)\right)^2\\
&-\frac{1}{4}\left(\alpha+2\sqrt{2}(1+\delta)C_4\right)^2 + 4C_5^2 + 2C_4^2,
\end{aligned} \tag{4.38}$$

which, together with the fact that $\alpha > 4C_5$, implies

$$\overline{m}'(t_1) \le -\frac{1}{4}\left(\alpha+2\sqrt{2}(1+\delta)C_4\right)^2 + 4C_5^2 + 2C_4^2 < 0.$$

This yields a contradiction with (4.37).

Putting (4.36) back to (4.34), we have

$$\frac{d(m_1(t)+2C_5)}{dt} = \frac{dm_1}{dt} \le -\frac{1}{2}m_1^2(t) + C_5(m_2-m_1)(t) + C_4^2$$

$$
\begin{aligned}
&= -\frac{1}{2}m_1^2(t)+C_5(m_2+m_1)(t)-2C_5m_1(t)+C_4^2(t)\\
&\leq -\frac{1}{2}m_1^2(t)+C_5\left(-\alpha-2\sqrt{2}(1+\delta)C_4\right)-2C_5m_1(t)+C_4^2\\
&= -\frac{1}{2}(m_1(t)+2C_5)^2-C_5\alpha-2\sqrt{2}(1+\delta)C_4C_5+2C_5^2+C_4^2\\
&= -\frac{1}{2}(m_1(t)+2C_5)^2+C_4^2-C_5(\alpha-2C_5+2\sqrt{2}(1+\delta)C_4)\\
&< -\frac{\delta(\delta+2)}{2(1+\delta)^2}(m_1(t)+2C_5)^2.
\end{aligned}\tag{4.39}
$$

Since $m_1(t)$ is locally Lipshitz on $(0,T)$, we have that $\frac{1}{m_1(t)+2C_5}$ is also locally Lipshitz on $(0,T)$. Being locally Lipshitz, $\frac{1}{m_1(t)+2C_5}$ is absolutely continuous on $(0,T)$, it is then inferred from (4.39) that

$$
\frac{d}{dt}\left(\frac{1}{m_1(t)+2C_5}\right)>\frac{\delta(\delta+2)}{2(1+\delta)^2},\qquad t\in(0,T).
$$

Consequently,

$$
m_1(t)<\frac{2(1+\delta)^2(m_1(0)+2C_5)}{(m_1(0)+2C_5)\delta(\delta+2)t+2(1+\delta)^2}-2C_5,\qquad t\in(0,T).
$$

Using Lemma 3.1, the above equation implies that $T_2<-\dfrac{2(1+\delta)^2}{(m_1(0)+2C_5)\delta(\delta+2)}$. Therefore, the proof of the theorem is complete. □

5. Global existence

Now a sufficient condition of existence of global solution about the periodic RCH2 system can be obtained in the following theorem.

THEOREM 5.1. *Let $(u_0,\rho_0-1)\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$ with $s>\frac{3}{2}$ and $T>0$ be the maximal time of existence of the solution (u,ρ) of the system (3.3) with initial data (u_0,ρ_0-1). If*

$$
\rho_0(x)\neq 0,\forall x\in\mathbb{R},E(0)\leq\frac{1}{6\kappa},\tag{5.1}
$$

then $T=+\infty$, i.e. the solution (u,ρ) is global in time t.

Proof. It is noted that the nonzero assumption in (3.2) implied that

$$
\rho_0(x)\neq 0,\forall x\in\mathbb{R},
$$

since $\rho_0(x)\to 1$ as $|x|\to\infty$. Arguing by condition (5.1), suppose wave breaking happens at the point (x_{T^*},T^*), for some time T^* when the wave breaking first occurs at a time T^* from Lemma 3.1. Then

$$
M(t)\to-\infty\qquad t\to T^*.\tag{5.2}
$$

where $M(t)=u_x(t,q(t,x_T))$. Let

$$
P(t)=\rho(t,q(t,x_{T^*}))\quad\textit{and}\quad K(t)=u(t,q(t,x_{T^*}))
$$

for $t\in[0,T^*)$ where $q(t,x_{T^*})$ is the characteristic at the point x_{T^*} defined as in (2.5).

Using these notations, (3.5) can be rewritten as

$$\begin{cases} M'(t) = -\frac{1}{2}M^2(t) + \frac{1}{2}P^2(t) + K^2 + 4K^3 - 3K^2P(P-1) - F(t), \\ P'(t) = -PM(1-3K^2). \end{cases} \tag{5.3}$$

for $t \in [0,T^*)$, where

$$F = G*(K^2 + \frac{1}{2}M^2 + \frac{1}{2}P^2 + 4K^3 - 3K^2P(P-1)),$$

Claim: $\frac{M^2(t)}{P^2(t)}$ is decreasing for t near T^*. In fact, a simple computation reveals that

$$\begin{aligned} \frac{d}{dt}(\frac{M^2}{P^2}) &= \frac{M}{P^2}(P^2 - M^2 + 2K^2 + 8K^3 - 6K^2P(P-1) - 2F + 2M^2(1-3K^2)) \\ &= \frac{M}{P^2}(P^2 - 6K^2P(P-1) + M^2(1-6K^2) + 2K^2 + 8K^3 - 2F) \\ &= \frac{M}{P^2}((1-6K^2)P^2 + 6K^2P + M^2(1-6K^2) + 2K^2 + 8K^3 - 2F) \\ &= \frac{M}{P^2}((1-6K^2)(P(t) + \frac{6K^2}{1-6K^2})^2 + M^2(1-6K^2) + 2K^2 + 8K^3 \\ &\qquad - 2F - \frac{72K^4}{(1-6K^2)^2}). \end{aligned} \tag{5.4}$$

Using the assumption in (5.1), we have

$$K^2 \leq \kappa E(0) < \frac{1}{6}. \tag{5.5}$$

It follows that

$$\begin{aligned} |F(t)| &< |G*(K^2 + \frac{1}{2}M^2 + \frac{1}{2}P^2)| + |4G*K^3| + 3|G*K^2P(P-1)| \\ &< C(\|u\|_{H^1}^2 + \|\rho-1\|_{L^2}^2 + 1 + \|u\|_{L^\infty}^3 + \|u\|_{L^\infty}^2\|\rho-1\|_{L^2}^2 + \|\rho-1\|_{L^2})) \\ &= C_6. \end{aligned}$$

where C_6 is a constant depending only on $E(0)$. In view of (5.4) and the boundedness of F and K, there exists some time $t_1 \in [0,T^*)$ such that

$$M^2(t)(1-6K^2) > M^2(t)(1-6\kappa E(0)) > 8K^3 - 2F(t) - \frac{72K^4}{(1-6K^2)^2}. \tag{5.6}$$

and $M(t) < 0$, for $t \in [t_1, T^*)$. It then follows from (5.4) that

$$\frac{d}{dt}(\frac{M^2(t)}{P^2(t)}) < 0, \quad \forall t \in [t_1, T^*).$$

Then, the claim is established, and the limit of $\frac{M^2(t)}{P^2(t)}$ as $t \to T^*$ exists. A simple computation also gives that

$$-\int_{t_1}^{t} M(1-3K^2)ds = -\int_{t_1}^{t} \frac{(1-3K^2)}{1-6K^2} M(1-6K^2)ds \leq -C_6 \int_{t_1}^{t} M(1-6K^2)ds \tag{5.7}$$

with $C_6=\frac{1-\frac{3}{2}E(0)}{1-3E(0)}$, this implies that

$$P(t)=P(t_1)e^{-\int_{t_1}^{t}M(1-3K^2)ds}\le P(t_1)e^{-C_6\int_{t_1}^{T^*}M(1-6K^2)ds}. \tag{5.8}$$

To show that $\lim_{t\to T^*}P(t)$ exists, it suffices to verify that the increasing function $P(t)$ is bounded above for $t\in[t_1,T^*)$. Then we have

$$\begin{aligned}
0&<-\int_{t_1}^{T^*}M(s)(1-6K^2(s))ds\\
&\le-\int_{t_1}^{T^*}M(s)(1-6K^2(s))+\frac{M(s)}{(P(s)+\frac{6K^2(s)}{1-6K^2(s)})^2}\big(M^2(s)(1-6K^2(s))+2K^2(s)\\
&\quad+8K^3(s)-2F(s)-\frac{72K^4(s)}{(1-6K^2(s))^2}\big)ds\\
&=-\int_{t_1}^{T^*}\frac{M}{(P+\frac{6K^2}{1-6K^2})^2}\big((1-6K^2)(P+\frac{6K^2}{1-6K^2})^2+M^2(1-6K^2)\\
&\quad+2K^2+8K^3-2F-\frac{72K^4}{(1-6K^2)^2}\big)ds\\
&\le-\int_{t_1}^{T^*}\frac{M}{P^2}\big((1-6K^2)(P+\frac{6K^2}{1-6K^2})^2+M^2(1-6K^2)+2K^2+8K^3-2F\\
&\quad-\frac{72K^4}{(1-6K^2)^2}\big)ds\\
&=\frac{M^2(t_1)}{P^2(t_1)}-\lim_{t\to T^*}\frac{M^2(T^*)}{P^2(T^*)}<\frac{M^2(t_1)}{P^2(t_1)}<\infty.
\end{aligned} \tag{5.9}$$

It follows from above that $P(t)$ is bounded for $t\in[t_1,T^*)$ and $\lim_{t\to T^*}P(t)$ exists. As a consequence,

$$\lim_{t\to T^*}M^2(t)=(\lim_{t\to T^*}P^2(t))(\lim_{t\to T^*}\frac{M^2(t)}{P^2(t)})<\infty. \tag{5.10}$$

Which contradicts with (5.2). Then the proof of the Theorem 5.1 is complete. ∎

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