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GLOBAL CLASSICAL SOLUTIONS TO 1D FULL COMPRESSIBLE MICROPOLAR FLUIDS WITH THE NEUMANN/ROBIN BOUNDARY CONDITIONS AND VACUUM*

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Abstract. In this paper, we consider the initial boundary value problem for the one-dimensional micropolar fluids for viscous compressible and heat-conducting fluids in a bounded domain with the Neumann/Robin boundary conditions on temperature. There are few results until now about global existence of regular solutions to the equations of hydrodynamics with the Robin boundary conditions on temperature. By the analysis of the effect of boundary dissipation, we derive the global existence of classical solution to the corresponding initial boundary value problem with large initial data and vacuum.

Keywords. compressible micropolar fluids; heat-conducting fluids; vacuum; global classical solutions.

AMS subject classifications. 35Q30; 35K65; 76N10.

1. Introduction

In this paper we consider the one-dimensional flow of the non-isentropic compressible micropolar fluid flow being thermodynamically perfect and polytropic. In Eulerian coordinates, the motion of the fluid under consideration is given by the following system of four equations [17, 20]:

$$\begin{cases} \rho_t+(\rho u)_x=0, \\ \rho u_t+\rho u u_x+P_x=\mu_1 u_{xx}, \\ j_I(\rho w_t+\rho u w_x)+2\xi w=\mu_2 w_{xx}, \\ (\rho E)_t+(\rho u E)_x+(Pu)_x=(\mu_1 u u_x)_x+(\mu_2 w w_x)_x+(\kappa\theta_x)_x, \end{cases} \tag{1.1}$$

where $t\geq 0$ is time and $x\in[0,1]$ is the spatial coordinate. Here $\rho=\rho(x,t), u=u(x,t), w=w(x,t), P$ and θ denote the density, velocity, microrotation velocity, pressure and absolute temperature, respectively. Here $j_I>0$ denotes the microinertia density coefficient; $\kappa>0$ is the coefficient of heat conduction; $\mu_1=\lambda+2\mu$, where λ and μ are viscosity coefficients, they satisfy the conditions: $\mu>0, 3\lambda+2\mu\geq 0$; $\mu_2=c_0+2c_d$, where c_0 and c_d are coefficients of angular viscosity, they meet the conditions: $c_d>0, 3c_0+2c_d\geq 0$; $\xi>0$ is the dynamic microrotation viscosity (coupling coefficient). Then we deduce that $\mu_1>0$ and $\mu_2>0$. The total energy $E=e+\frac{1}{2}|u|^2+\frac{1}{2}|w|^2$, where e is the internal energy. The pressure P and the internal energy e have the following expressions:

$$P=A\rho\theta, \qquad e=c_v\theta,$$

where A and c_v are positive constants. For simplicity we let $j_I=A=c_v=1$.

There is much literature on the well-posedness of the micropolar system. For the case of one-dimensional compressible flow, Mujaković [20–22] studied the local-in-time

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existence and uniqueness, the global existence and regularity of the solutions to an initial-boundary value problem wherein the boundary conditions for u, w are the Dirichlet and for θ is the Neumann, and she also considered the non-homogeneous boundary conditions in [25–27], wherein the boundary conditions for u, w are the Dirichlet and for θ is nonzero. The large-time behavior of the solutions and the stabilization of solutions to the Cauchy problem on the microploar fluids was also analyzed by Mujaković in [18, 23, 24]. Recently, Duan [12] published global solutions for the one-dimensional compressible micropolar fluid model with zero heat conductivity. But all the above results are free from vacuum. For the case with initial vacuum, Chen [1] proved the global existence of strong solutions to the Cauchy problem. For three-dimensional model, Mujaković and Dražić studied the local existence, global existence, uniqueness, and large-time behavior of the spherical symmetry solutions [8–11, 19]. Chen [2] and Chen-Huang-Zhang [3] proved blowup criterions for strong solutions to the Cauchy problem. The global weak solutions with discontinuous initial data and vacuum was established by Chen-Xu-Zhang in [4]. Recently, Liu-Zhang [16] obtained the optimal time decay of the three-dimensional compressible flows.

If the microrotation velocity $w=0$, then the system (1.1) becomes the classical Navier-Stokes system. There is a lot of literature to study the well-posedness of this system. Wen-Zhu in [29, 30] obtained the global existence of classical solution to this system with large initial data and vacuum, where viscosity coefficients are constant and the coefficient of heat conduction is only a temperature-dependent function. Liang-Wu [15] also obtained the same result for the case where the coefficient of heat conduction is constant. Recently, Zhang-Zhu [31] proved the existence of this system with Nuemann conditions for u and Robin conditions for θ.

In this paper, we consider the global classical solutions of (1.1) with initial vacuum and the viscosity coefficients, microviscosity coefficients and the coefficient of heat conduction are constant or a function of temperature.

For simplicity, we rewrite the system (1.1) as follows:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ \rho u_t + \rho u u_x + P_x = \mu_1 u_{xx}, \\ \rho w_t + \rho u w_x + 2\xi w = \mu_2 w_{xx}, \\ \rho\theta_t + \rho u\theta_x + \rho\theta u_x = \mu_1 u_x^2 + \mu_2 w_x^2 + 2\xi w^2 + (\kappa\theta_x)_x. \end{cases} \tag{1.2}$$

The system satisfies the initial and boundary conditions:

$$(\rho, u, w, \theta)(x, 0) = (\rho_0, u_0, w_0, \theta_0)(x), \quad x \in [0, 1], \tag{1.3}$$

and

$$\begin{cases} u(0,t) = u(1,t) = 0, \\ w(0,t) = w(1,t) = 0, \\ (\theta_x - a\theta)(0,t) = (\theta_x + b\theta)(1,t) = 0, \end{cases} \tag{1.4}$$

where a, b are nonnegative constants.

To begin with, we briefly review some notation which will be used throughout the rest of the paper.

Notation:

(1) $I = [0,1]$, $\partial I = \{0,1\}$, $Q_T = I \times [0,T]$ for $T > 0$, and $\int f(x)dx = \int_I f(x)dx$.

(2) For $p\in[1,\infty]$, $L^p=L^p(I)$ denotes the L^p space with the norm $\|\cdot\|_{L^p}$. For $k\geq 1$ and $p\in[1,\infty]$, $W^{k,p}=W^{k,p}(I)$ denotes the Sobolev space, whose norm is denoted as $\|\cdot\|_{W^{k,p}}$, $H^k=W^{k,2}(I)$.

In this paper we assume that

$$\rho_0\geq 0,\ \int\rho_0 dx>0, \tag{1.5}$$

and if κ is a function of θ, then it satisfies $\kappa\in C^3(\mathbb{R}^+)$ and

$$\begin{cases} 0<\kappa_1(1+\theta^q)\leq\kappa(\theta)\leq\kappa_2(1+\theta^q), \\ 0\leq\kappa'(\theta)\leq C\left(1+\theta^{q-1}\right),\ 0\leq\kappa''(\theta)\leq C\left(1+\theta^{q-2}\right),\ q\geq 2, \end{cases} \tag{1.6}$$

where C, κ_1 and κ_2 are positive constants.

Now we are in a position to state our main results. The first result is on the global existence of classical solution of (1.2) with the $\kappa=k(\theta)$ for the Neumann boundary conditions.

THEOREM 1.1. *If $a=b=0$ in (1.4), in addition to (1.5)-(1.6), we assume that*

$$\rho_0\geq 0,\ \rho_0\in H^2,\ (\sqrt{\rho_0})_x\in L^\infty,\ u_0\in H_0^1\cap H^3,\ w_0\in H_0^1\cap H^3,\ 0\leq\theta_0\in H^3, \tag{1.7}$$

$$\theta_{0x}|_{x=0}=\theta_{0x}|_{x=1}=0, \tag{1.8}$$

and the following compatibility conditions

$$\begin{cases} \mu_1 u_{0xx}-[P(\rho_0,\theta_0)]_x=\sqrt{\rho_0}g_1, \\ \mu_2 w_{0xx}-2\xi w_0=\sqrt{\rho_0}g_2, \\ [\kappa(\theta_0)\theta_{0x}]_x+\mu_1 u_{0x}^2+\mu_2 w_{0x}^2+2\xi w_0^2=\sqrt{\rho_0}g_3 \end{cases} \tag{1.9}$$

hold for some $g_i\in L^2$ and $\sqrt{\rho_0}g_i\in H_0^1$, $i=1,2,3$. Then there exists a global classical solution (ρ,u,w,θ) to (1.2)-(1.4) such that

$$\begin{cases} \rho\in C([0,T];H^2), \quad \rho_t\in C([0,T];H^1), \quad \sqrt{\rho}\in W^{1,\infty}(Q_T), \\ (u,w)\in L^\infty(0,T;H_0^1\cap H^3), \quad (\sqrt{\rho}u_t,\sqrt{\rho}w_t)\in L^\infty(0,T;L^2), \\ (\rho u_t,\rho w_t)\in L^\infty(0,T;H_0^1), \quad (u_t,w_t)\in L^2(0,T;H_0^1), \\ \sqrt{\rho}\theta_t\in L^\infty(0,T;L^2), \quad \rho\theta_t\in L^\infty(0,T;H^1), \\ \theta\in L^\infty(0,T;H^3), \quad \theta_t\in L^2(0,T;H^1), \end{cases} \tag{1.10}$$

for any $T>0$.

The second result is on the global existence of classical solution for fixed positive constants a,b.

THEOREM 1.2. *In addition to (1.5)-(1.7), we assume that the initial data also satisfies*

$$(\theta_{0x}-a\theta_0)|_{x=0}=0,\ (\theta_{0x}+b\theta_0)|_{x=1}=0, \tag{1.11}$$

and the compatibility conditions (1.9). Then there exists a global classical solution (ρ,u,w,θ) to (1.2)-(1.4) satisfying the regularities (1.10).

The third result is on the limit behavior of the solution as $a,b\to 0^+$.

THEOREM 1.3. *For given $a,b>0$, we assume that $(\rho^{a,b},u^{a,b},w^{a,b},\theta^{a,b})$ is the solution as in Theorem 1.2 with the initial data replaced by $(\rho_0^{a,b},u_0^{a,b},w_0^{a,b},\theta_0^{a,b})$ satisfying*

$$\left|(\rho_0^{a,b}-\widetilde{\rho}_0,u_0^{a,b}-\widetilde{u}_0,w_0^{a,b}-\widetilde{w}_0,\theta_0^{a,b}-\widetilde{\theta}_0)\right|\le C\max\{a,b\}$$

for some positive constant C independent of a, b, and spatial variables. Then we have

$$(\rho^{a,b},u^{a,b},w^{a,b},\theta^{a,b})\to(\rho,u,w,\theta) \quad in \quad L^\infty(0,T;L^2),$$

as $a,b\to 0^+$, where (ρ,u,w,θ) is the solution as Theorem 1.1 with initial data $(\widetilde{\rho}_0,\widetilde{u}_0,\widetilde{w}_0,\widetilde{\theta}_0)$ for $\inf\widetilde{\rho}_0>0$.

The following results are similar to Theorems 1.1-1.3 for the system (1.2) with the coefficient of heat conduction κ being constant.

THEOREM 1.4. *If $a=b=0$ in (1.4), in addition to (1.5), we assume that the initial data satisfies (1.7)-(1.8) and the following compatibility conditions (1.9). Then there exists a global classical solution (ρ,u,w,θ) to (1.2)-(1.4) satisfying (1.10).*

THEOREM 1.5. *If a,b are positive constants in (1.4), in addition to (1.5), we assume that the initial data satisfy (1.7), (1.11) and the following compatibility conditions (1.9). Then there exists a global classical solution (ρ,u,w,θ) to (1.2)-(1.4) satisfying (1.10).*

The last result is on the limit behavior of the solution as $a,b\to 0^+$.

THEOREM 1.6. *For given $a,b>0$, we assume that $(\rho^{a,b},u^{a,b},w^{a,b},\theta^{a,b})$ is the solution as in Theorem 1.5 with the initial data replaced by $(\rho_0^{a,b},u_0^{a,b},w_0^{a,b},\theta_0^{a,b})$ satisfying*

$$\left|(\rho_0^{a,b}-\widetilde{\rho}_0,u_0^{a,b}-\widetilde{u}_0,w_0^{a,b}-\widetilde{w}_0,\theta_0^{a,b}-\widetilde{\theta}_0)\right|\le C\max\{a,b\}$$

for some positive constant C independent of a, b, and spatial variables. Then we have

$$(\rho^{a,b},u^{a,b},w^{a,b},\theta^{a,b})\to(\rho,u,w,\theta) \quad in \quad L^\infty(0,T;L^2),$$

as $a,b\to 0^+$, where (ρ,u,w,θ) is the solution as in Theorem 1.4 with initial data $(\widetilde{\rho}_0,\widetilde{u}_0,\widetilde{w}_0,\widetilde{\theta}_0)$.

REMARK 1.1. Theorem 1.6 suggests that the solution obtained in Theorem 1.5 converges to the one in Theorem 1.4 as $a,b\to 0^+$. Theorem 1.3 holds for the case with nonvacuum, because the term $\int\left(\left(\kappa-\kappa^{a,b}\right)\theta_x^{a,b}\right)_x\bar{\theta}dx$, which deduces a term $\|\bar{\theta}\|_{L^2}^2$, cannot be dealt with by the Grönwall inequality. But in Theorem 1.6, the term $\int\left(\left(\kappa-\kappa^{a,b}\right)\theta_x^{a,b}\right)_x\bar{\theta}dx$ is absent, because the coefficient of heat conduction is constant. Then, we can obtain Theorem 1.6 with initial vacuum.

2. Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used frequently in this paper.

LEMMA 2.1 ([29]). *Let $\Omega=[a,b]$ $(a<b)$ be a bounded domain in $\mathbb{R}$, and ρ be a nonnegative function such that*

$$0<m\le\int_\Omega\rho dx\le M,$$

for constants $m>0$ and $M>0$. Then, for any $v\in H^1(\Omega)$, we have

$$\|v\|_{L^\infty(\Omega)}\le\frac{M}{m}\|v_x\|_{L^1(\Omega)}+\frac{1}{m}\left|\int_\Omega\rho v dx\right|. \tag{2.1}$$

COROLLARY 2.1 ([29]). *Consider the same conditions as in Lemma 2.1, and in addition assume $\Omega=I$ and*

$$\|\rho v\|_{L^1}\le\bar{c}.$$

Then for any $k>0$, there exists a positive constant $C=C(m,M,k,\bar{c})$ such that

$$\|v^k\|_{L^\infty}\le C\left\|\left(v^k\right)_x\right\|_{L^2}+C, \tag{2.2}$$

for any $v^k\in H^1$.

LEMMA 2.2 ([29]). *For any $v\in H_0^1$, we have*

$$\|v\|_{L^\infty}\le C\|v_x\|_{L^1} \tag{2.3}$$

and for any $v\in H^1$, we have

$$\|v\|_{L^\infty}\le C(\|v_x\|_{L^1}+\|v\|_{L^1}) \tag{2.4}$$

LEMMA 2.3 (Calderón-Zygmund). *Let $\Omega=[a,b]$ be bounded. Suppose $0\le f\in L^1(\Omega)$ satisfies*

$$\frac{1}{|\Omega|}\int_\Omega f dx\le\alpha_0.$$

Then for any $\alpha>\alpha_0$, there exists a sequence (non-overlapping) Ω_j included in Ω such that

$$f(x)\le\alpha,\ \text{a.e.}\ \ x\in\Omega\setminus\Omega_j,\quad \text{and}\quad \alpha\le\frac{1}{|\Omega_j|}\int_{\Omega_j}f dx\le 2\alpha.$$

Moreover,

$$\left|\bigcup_j\Omega_j\right|\le\frac{\alpha_0|\Omega|}{\alpha},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

Proof. The proof is classical, see, e.g., [[14], Lemma 3.6, Chap. 3]. We omit the details here.

LEMMA 2.4 ([28]). *Assume $X\subset E\subset Y$ are Banach spaces and $X\hookrightarrow\hookrightarrow E$. Then the following imbeddings are compact:*

(i) $\{\varphi:\varphi\in L^q(0,T;X),\ \ \varphi_t\in L^1(0,T;Y)\}\hookrightarrow\hookrightarrow L^q(0,T;E),\quad \text{if}\quad 1\le q\le\infty,$
(ii) $\{\varphi:\varphi\in L^\infty(0,T;X),\ \ \varphi_t\in L^r(0,T;Y)\}\hookrightarrow\hookrightarrow C([0,T];E),\quad \text{if}\quad 1<r\le\infty.$

3. Proof of Theorem 1.1

In this section, we get a global solution to (1.2)-(1.4) with initial density and initial temperature having, respectively, a lower bound $\delta > 0$ by using some global *a priori* estimates of the solutions based on the local existence. Theorem 1.1 will be obtained after we do some global *a priori* estimates uniformly for δ and take $\delta \to 0^+$.

Denote $\rho_0^\delta = \rho_0 + \delta$ and $\theta_0^\delta = \theta_0 + \delta$ for $\delta \in (0,1)$. Throughout this section, we denote C to be a generic constant depending on $\rho_0, u_0, w_0, \theta_0, T$ and some other known constants but independent of δ for any $\delta \in (0,1)$.

For any given $\delta \in (0,1)$, let u_0^δ, w_0^δ be the solution to the following elliptic equations:

$$\begin{cases} \mu_1 u_{0xx}^\delta - P_{0x}^\delta = \sqrt{\rho_0} g_1, \\ \mu_2 w_{0xx}^\delta - 2\xi w_0^\delta = \sqrt{\rho_0} g_2, \\ u_0^\delta|_{x=0,1} = w_0^\delta|_{x=0,1} = 0, \\ \theta_x^\delta(0,t) = \theta_x^\delta(1,t) = 0. \end{cases} \tag{3.1}$$

Since $\rho_0^\delta = \rho_0 + \delta \in H^2$, $\theta_0^\delta = \theta_0 + \delta \in H^3$ and $\sqrt{\rho_0} g_1, \sqrt{\rho_0} g_2 \in H_0^1$, by the elliptic theory, $(1.9)_1$, $(1.9)_2$ and (3.1), we have $u_0^\delta, w_0^\delta \in H_0^1 \cap H^3$ and

$$\begin{cases} u_0^\delta \to u_0,\ w_0^\delta \to w_0, \quad \text{in} \quad H^3 \quad \text{as} \quad \delta \to 0^+, \\ \|u_0^\delta - u_0\|_{H^2} \le C\delta,\ \|w_0^\delta - w_0\|_{H^2} \le C\delta. \end{cases} \tag{3.2}$$

Before proving Theorem 1.1, we need the following auxiliary theorem to construct a sequence of approximate solutions.

THEOREM 3.1. *Consider the same assumptions as in Theorem 1.1. Then for any given $\delta \in (0,1)$, there exists a global solution (ρ, u, w, θ) to (1.2)-(1.4) with initial data replaced by $(\rho_0^\delta, u_0^\delta, w_0^\delta, \theta_0^\delta)$, such that for any $T > 0$,*

$$\begin{cases} \rho \in C([0,T]; H^2),\ \rho_t \in C([0,T]; H^1),\ \rho_{tt} \in L^2(0,T; L^2),\ \rho \ge \frac{\delta}{C} > 0, \\ (u,w) \in C([0,T]; H_0^1 \cap H^3),\ (u_t, w_t) \in C([0,T]; H_0^1) \cap L^2(0,T; H^2), \\ (u_{tt}, w_{tt}) \in L^2(0,T; L^2),\ \theta \in C([0,T]; H^3),\ \theta \ge C_\delta > 0, \\ \theta_t \in C([0,T]; H^1) \cap L^2(0,T; H^2),\ \theta_{tt} \in L^2(0,T; L^2), \end{cases}$$

where C_δ is a constant depending on δ.

Proof. The local existence of the solutions as in Theorem 3.1 can be obtained by the successive approximations as in [6]. We omit it here for brevity. Based on it, Theorem 3.1 can be proved by the following global-in-time *a priori* estimates as follows. ∎

In this and the next two sections, for the sake of simplicity, we denote by C the generic positive constants, which may depend on $\gamma, T, \mu_1, \mu_2, \xi, \kappa_1, \kappa_2$, the initial data, $\|g_i\|_{L^2}, \|(\sqrt{\rho_0} g_i)_x\|_{L^2}$ $(i = 1,2,3)$, and the constants of the Sobolev inequalities, but not depend on a, b, δ. We also sometimes use $C(\alpha)$ to emphasize the dependence on α.

For any given $T \in (0, +\infty)$, let (ρ, u, w, θ) be the solution to (1.2)-(1.4) as in Theorem 3.1. Then we have the following lemmas.

LEMMA 3.1. *Under the conditions of Theorem 3.1, it holds that for any $0 \le t \le T$,*

$$\int \rho dx = \int \rho_0 dx, \tag{3.3}$$

$$\int \rho\left(\theta+u^2+w^2\right)dx\leq C\int \rho_0 E_0 dx. \tag{3.4}$$

Here $E_0=\theta_0+\frac{1}{2}u_0^2+\frac{1}{2}w_0^2$.

Proof. Integrating $(1.1)_1$ and $(1.1)_4$ over $I\times[0,t]$, using (1.5) and the boundary conditions (1.4), we have

$$\int \rho dx=\int \rho_0 dx,$$

and

$$\int \rho E dx=\int \rho_0 E_0 dx=\int \rho_0\left(\theta_0+\frac{1}{2}u_0^2+\frac{1}{2}w_0^2\right)dx.$$

Then we have (3.4). □

LEMMA 3.2. *Under the conditions of Theorem 3.1, it holds that for any* $(x,t)\in Q_T$,

$$0<\rho\leq C \quad \text{and} \quad \theta>0.$$

Proof. The proof of ρ is the same as Lemma 3.2 in [29]. We only need to prove $\theta>0$. By $(1.2)_4$, we have

$$\theta_t+u\theta_x-\frac{1}{\rho}(\kappa\theta_x)_x+\frac{\rho\theta^2}{4\mu_1}=\frac{1}{\rho}\left(\sqrt{\mu_1}u_x-\frac{\rho\theta}{2\sqrt{\mu_1}}\right)^2+\frac{\mu_2}{\rho}w_x^2+\frac{2\xi}{\rho}w^2\geq 0. \tag{3.5}$$

Let

$$\bar\theta=\frac{1}{Kt+1}\min_I\theta_0$$

with $K>0$. Denote $\tilde\theta=\theta-\bar\theta$. We find that

$$\tilde\theta_x|_{x=0,1}=\theta_x|_{x=0,1},\quad \tilde\theta|_{t=0}\geq 0,$$

and

$$\begin{aligned}&\tilde\theta_t+u\tilde\theta_x-\frac{1}{\rho}(\kappa\tilde\theta_x)_x-\frac{K\min_I\theta_0}{(Kt+1)^2}+\frac{\rho\tilde\theta^2}{4\mu_1}+\frac{\rho\tilde\theta}{2\mu_1}\cdot\frac{\min_I\theta_0}{Kt+1}-\frac{\rho}{4\mu_1}\left(\frac{\min_I\theta_0}{Kt+1}\right)^2\\
=&\theta_t+u\theta_x-\frac{1}{\rho}(\kappa\theta_x)_x+\frac{\rho\theta^2}{4\mu_1}\geq 0,\end{aligned}$$

where we have used (3.5). This gives

$$\begin{aligned}&\tilde\theta_t+u\tilde\theta_x-\frac{1}{\rho}(\kappa\tilde\theta_x)_x+\rho\tilde\theta\Big(\frac{\tilde\theta}{4\mu_1}+\frac{1}{2\mu_1}\cdot\frac{\min_I\theta_0}{Kt+1}\Big)\\
\geq&\frac{\min_I\theta_0}{(Kt+1)^2}\Big(K+\frac{\rho\min_I\theta_0}{4\mu_1}\Big)>0.\end{aligned}$$

Thus using the maximum principle for parabolic equations, we have $\tilde\theta\geq 0$ implying that $\theta\geq\bar\theta>0$. □

LEMMA 3.3. *Under the conditions of Theorem 3.1, there exists* $\alpha\in(0,1)$*, such that*

$$\int_{Q_T}\left(\frac{u_x^2}{\theta^\alpha}+\frac{w_x^2}{\theta^\alpha}+\frac{w^2}{\theta^\alpha}+\frac{(1+\theta^q)\theta_x^2}{\theta^{1+\alpha}}\right)dxdt\le C,$$

where C *may depend on* α.

Proof. Multiplying $(1.2)_4$ by $\theta^{-\alpha}$, integrating the resulting equation over Q_T, and using integration by parts, we have

$$\begin{aligned}&\int_0^T\int\left(\frac{\mu_1u_x^2}{\theta^\alpha}+\frac{\mu_2w_x^2}{\theta^\alpha}+\frac{2\xi w^2}{\theta^\alpha}+\frac{\alpha\kappa\theta_x^2}{\theta^{1+\alpha}}\right)dxdt\\&=\frac{1}{1-\alpha}\int\rho\theta^{1-\alpha}dx\Big|_0^T+\int_0^T\int\rho\theta^{1-\alpha}u_xdxdt\\&\le C+\frac{\mu_1}{2}\int_0^T\int\frac{u_x^2}{\theta^\alpha}dxdt+C\int_0^T\int\rho^2\theta^{2-\alpha}dxdt\\&\le C+\frac{\mu_1}{2}\int_0^T\int\frac{u_x^2}{\theta^\alpha}dxdt+C\int_0^T\|\theta\|_{L^\infty}^{1-\alpha}dt,\end{aligned}$$

where we have used the Cauchy inequality, Lemmas 3.1 and 3.2.

Then we have

$$\int_0^T\int\left(\frac{\mu_1u_x^2}{2\theta^\alpha}+\frac{\mu_2w_x^2}{\theta^\alpha}+\frac{2\xi w^2}{\theta^\alpha}+\frac{\alpha\kappa\theta_x^2}{\theta^{1+\alpha}}\right)dxdt\le C\int_0^T\|\theta\|_{L^\infty}^{1-\alpha}dt+C.\tag{3.6}$$

Now we estimate the right-hand side term of (3.6) as follows:

$$\begin{aligned}\int_0^T\|\theta\|_{L^\infty}^{1-\alpha}dt&\le C+C\int_0^T\|\theta^{-\alpha}\theta_x\|_{L^2}dt\\&\le C+C\int_0^T\left(\int\frac{\theta_x^2}{\theta^{2\alpha}}dx\right)^{\frac12}dt\\&\le C+C\int_0^T\left(\int\frac{\theta^{1-\alpha}\theta_x^2}{\theta^{1+\alpha}}dx\right)^{\frac12}dt\\&\le C+\frac{\alpha}{2}\int_0^T\int\frac{(1+\theta^q)\theta_x^2}{\theta^{1+\alpha}}dxdt,\quad(q\ge1-\alpha),\end{aligned}\tag{3.7}$$

where we have used Corollary 2.1, Lemma 3.1 and the Cauchy inequality. By (3.6) and (3.7), we complete the proof. ∎

LEMMA 3.4. *Under the conditions of Theorem 3.1, it holds that*

$$\int_0^T\|\theta\|_{L^\infty}^{q+1-\alpha}dt\le C,$$

$$\sup_{0\le t\le T}\int\left(\rho w^2+\rho u^2\right)dx+\int_{Q_T}\left(u_x^2+w^2+w_x^2\right)dxdt\le C.$$

Proof. By Corollary 2.1 and Lemma 3.1, we have

$$\int_0^T\|\theta\|_{L^\infty}^{q+1-\alpha}dt=\int_0^T\|\theta^{\frac{q+1-\alpha}{2}}\|_{L^\infty}^2dt$$

$$\begin{aligned}&\le C+C\int_0^T\int\theta^{q-1-\alpha}\theta_x^2dxdt\\&\le C+C\int_0^T\int\frac{(1+\theta^q)\theta_x^2}{\theta^{1+\alpha}}dxdt\\&\le C.\end{aligned}$$

Multiplying $(1.2)_2$ by u, integrating over I, and using integration by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\rho u^2dx+\mu_1\int u_x^2=&\int Pu_xdx\\&\le\frac{\mu_1}{2}\int u_x^2dx+C\int\rho^2\theta^2dx\\&\le C+\frac{\mu_1}{2}\int u_x^2dx+C\|\theta\|_{L^\infty}\\&\le C+C\|\theta\|_{L^\infty}^{q-\alpha+1}+\frac{\mu_1}{2}\int u_x^2dx,\end{aligned}$$

where we have used the Cauchy inequality and Lemma 3.2. Then, we have

$$\frac{d}{dt}\int\rho u^2dx+\mu_1\int u_x^2\le C+C\|\theta\|_{L^\infty}^{q-\alpha+1}.$$

Integrating it over $[0,T]$ and using the first inequality, we have the second inequality. Multiplying $(1.2)_3$ by w, integrating over I, using integration by parts and $(1.2)_1$, we have

$$\frac{1}{2}\frac{d}{dt}\int\rho w^2dx+\int\left(2\xi w^2+\mu_2w_x^2\right)dx=0.$$

Then we get the third inequality after integrating this inequality over $[0,T]$. ∎

LEMMA 3.5. *Under the conditions of Theorem 3.1, it holds that*

$$\sup_{0\le t\le T}\int\left(u_x^2+w_x^2+w^2+\rho\theta^2(1+\theta^q)\right)dx+\int_{Q_T}\left(\rho u_t^2+\rho w_t^2+(1+\theta^q)^2\theta_x^2\right)dxdt\le C.$$

Proof. Multiplying $(1.2)_2$ by u_t, integrating over I, and using integration by parts, Lemmas 2.2 and 3.2, and the Cauchy inequality, we have

$$\begin{aligned}&\int\rho u_t^2dx+\frac{\mu_1}{2}\frac{d}{dt}\int u_x^2dx\\&=\frac{d}{dt}\int Pu_xdx-\int\rho uu_xu_tdx-\int P_tu_xdx\\&\le\frac{1}{2}\int\rho u_t^2dx+\frac{1}{2}\int\rho u^2u_x^2dx+\frac{d}{dt}\int Pu_xdx-\int P_tu_xdx\\&\le\frac{1}{2}\int\rho u_t^2dx+C\left(\int u_x^2dx\right)^2+\frac{d}{dt}\int Pu_xdx-\int P_t(u_x-P)dx-\frac{1}{2}\frac{d}{dt}\int P^2dx,\end{aligned}$$

which implies

$$\int\rho u_t^2dx+\mu_1\frac{d}{dt}\int u_x^2\le C\left(\int u_x^2dx\right)^2+2\frac{d}{dt}\int Pu_xdx-\frac{d}{dt}\int P^2dx-2\int P_t(u_x-P)dx.\tag{3.8}$$

We are going to estimate the last term of the right side of (3.8). Using (1.2), and integration by parts, we have

$$\begin{aligned}
-2\int P_t(u_x-P)dx &= -2\int(\rho\theta)_t(u_x-P)dx\\
&= -2\int\left[(\kappa\theta_x)_x+\mu_1u_x^2+\mu_2w_x^2+2\xi w^2-(\rho u\theta)_x-\rho\theta u_x\right](u_x-P)dx\\
&= 2\int\kappa\theta_x(u_{xx}-P_x)dx-2\int\left(\mu_1u_x^2+\mu_2w_x^2+2\xi w^2\right)(u_x-P)dx\\
&\quad -2\int\rho u\theta(u_{xx}-P_x)dx+2\int\rho\theta u_x(u_x-P)dx\\
&\triangleq\sum_{i=1}^4 I_i.
\end{aligned}$$

By (1.2), (1.4), (2.2)-(2.4), the Young inequality and $W^{1,1}\hookrightarrow L^\infty$, we estimate I_i as follows:

$$\begin{aligned}
|I_1| &\le C\left|\int\kappa\theta_x(\rho u_t+\rho uu_x)dx\right| \le \frac18\int\rho u_t^2dx+C\int\kappa^2\theta_x^2dx+C\int\rho^2u^2u_x^2dx\\
&\le \frac18\int\rho u_t^2dx+C\int\kappa^2\theta_x^2dx+C\left(\int u_x^2dx\right)^2,
\end{aligned}$$

$$\begin{aligned}
|I_2| &\le C\|u_x-P\|_{L^\infty}\int\left(u_x^2+w_x^2+w^2\right)dx\\
&\le C\left(\|u_x-P\|_{L^1}+\|u_{xx}-P_x\|_{L^1}\right)\int\left(u_x^2+w_x^2+w^2\right)dx\\
&\le C\left(\|u_x-P\|_{L^1}+\|\rho u_t+\rho uu_x\|_{L^1}\right)\int\left(u_x^2+w_x^2+w^2\right)dx\\
&\le \frac18\int\rho u_t^2dx+C\left(\int u_x^2dx\right)^2+C\left(\int w_x^2dx\right)^2+C\left(\int w^2dx\right)^2+C,
\end{aligned}$$

$$\begin{aligned}
|I_3| &\le C\left|\int\rho^2uu_t\theta dx\right|+C\left|\int\rho^2u^2u_x\theta dx\right|\\
&\le \frac18\int\rho u_t^2dx+C\int u^2\theta^2dx+C\int\rho^4u^4dx+C\int\theta^2u_x^2dx\\
&\le \frac18\int\rho u_t^2dx+C\|\theta\|_{L^\infty}^2\int u^2dx+C\|u\|_{L^\infty}^2\int\rho u^2dx+C\|\theta\|_{L^\infty}^2\int u_x^2dx\\
&\le \frac18\int\rho u_t^2dx+C\left(1+\|\theta\|_{L^\infty}^2\right)\int u_x^2dx,
\end{aligned}$$

$$\begin{aligned}
|I_4| &\le C\|u_x-P\|_{L^\infty}\left|\int\rho\theta u_xdx\right|\\
&\le C\left(\|u_x-P\|_{L^1}+\|\rho u_t+\rho uu_x\|_{L^1}\right)\left|\int\rho\theta u_xdx\right|\\
&\le \frac18\int\rho u_t^2dx+C(1+\|\theta\|_{L^\infty})\int u_x^2dx+C.
\end{aligned}$$

Then, we have

$$-2\int P_t(u_x-P)dx\le\frac{1}{2}\int\rho u_t^2dx+C\int\kappa^2\theta_x^2dx+C\left(1+\|\theta\|_{L^\infty}^2\right)\int u_x^2dx\\+C\left(\int u_x^2dx\right)^2+C\left(\int w_x^2dx\right)^2+C\left(\int w^2dx\right)^2+C.\quad(3.9)$$

By (3.8) and (3.9), we have

$$\frac{1}{2}\int\rho u_t^2dx+\mu_1\frac{d}{dt}\int u_x^2dx\\\le2\frac{d}{dt}\int Pu_xdx-\frac{d}{dt}\int P^2dx+C\left(1+\|\theta\|_{L^\infty}^2\right)\int u_x^2dx+C\int\kappa^2\theta_x^2dx\\+C\left(\int u_x^2dx\right)^2+C\left(\int w_x^2dx\right)^2+C\left(\int w^2dx\right)^2+C.\quad(3.10)$$

Integrating (3.10) over $(0,t)$, using the Cauchy inequality and Young inequality, by Lemmas 3.2-3.4, we have

$$\int_0^T\int\rho u_t^2dxds+\mu_1\int u_x^2dx\le C\int\rho\theta u_xdx-\int\rho^2\theta^2dx+C\int_0^T\left(1+\|\theta\|_{L^\infty}^2\right)\int u_x^2dxds\\+C\int_0^T\left(\int u_x^2dx\right)^2ds+C\int_0^T\left(\int w_x^2dx\right)^2ds\\+C\int_0^T\left(\int w^2dx\right)^2ds+C\int_0^T\int\kappa^2\theta_x^2dxds+C\\\le C\int_0^T\left(1+\|\theta\|_{L^\infty}^2\right)\int u_x^2dxds+C\int_0^T\left(\int u_x^2dx\right)^2ds\\+C\int_0^T\left(\int w_x^2dx\right)^2ds+C\int_0^T\left(\int w^2dx\right)^2ds\\+C\int_0^T\int\kappa^2\theta_x^2dxds+C\|\theta\|_{L^\infty}+C.\quad(3.11)$$

Multiplying $(1.2)_3$ by w_t, integrating over I, using integration by parts, Lemmas 2.2 and 3.2, we have

$$\int\rho w_t^2dx+\frac{1}{2}\frac{d}{dt}\int\left(\mu_2w_x^2+2\xi w^2\right)dx=-\int\rho uw_xw_t\\\le\frac{1}{2}\int\rho w_t^2dx+C\|u\|_{L^\infty}^2\int w_x^2dx\\\le\frac{1}{2}\int\rho w_t^2dx+C\left(\int u_x^2dx\right)^2+C\left(\int w_x^2dx\right)^2,$$

after integrating it over $(0,t)$, we have

$$\int_0^T\int\rho w_t^2dxdt+\int\left(w_x^2+w^2\right)dx\le C+C\int_0^T\left(\int u_x^2dx\right)^2dt+C\int_0^T\left(\int w_x^2dx\right)^2dt.\quad(3.12)$$

Multiplying (1.2)$_4$ by $\int_0^\theta \kappa(\tau)d\tau$, integrating over I, and using integration by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\int\rho\int_0^\theta\int_0^y \kappa(\tau)dyd\tau dx+\int\kappa^2\theta_x^2dx\\
&=\mu_1\int u_x^2\int_0^\theta\kappa(\tau)d\tau dx+\mu_2\int w_x^2\int_0^\theta\kappa(\tau)d\tau dx+2\xi\int w^2\int_0^\theta\kappa(\tau)d\tau dx\\
&\quad-\int\rho\theta u_x\int_0^\theta\kappa(\tau)d\tau dx\\
&\le C\|\kappa\theta\|_{L^\infty}\left(\int u_x^2dx+\int w_x^2dx+\int w^2dx+\int\rho\theta u_xdx\right)\\
&\le C\left(\|\kappa\theta_x\|_{L^2}+1\right)\left(\int u_x^2dx+\int w_x^2dx+\int w^2dx+\int\rho\theta u_xdx\right).
\end{aligned}\tag{3.13}$$

Integrating (3.13) and using the Young inequality, we obtain that

$$\begin{aligned}
&\int\rho\theta^2(1+\theta^q)dx+\int_0^T\int\kappa^2\theta_x^2dxdt\\
&\le C\int_0^T\left(\int u_x^2dx\right)^2dt+C\int_0^T\left(\int w_x^2dx\right)^2dt+C\int_0^T\left(\int w^2dx\right)^2dt\\
&\quad+C\int_0^T\|\theta\|_{L^\infty}^2\int u_x^2dxdt+C.
\end{aligned}\tag{3.14}$$

Using (3.11), (3.12), (3.14), Lemma 3.4 and the Grönwall inequality, we complete the proof. □

LEMMA 3.6. *Under the conditions of Theorem 3.1, it holds that*

$$\sup_{0\le t\le T}\int\left(\rho_x^2+\rho_t^2\right)dx+\int_{Q_T}u_{xx}^2dxdt\le C,$$
$$\int_{Q_T}w_{xx}^2dxdt\le C.$$

Proof. The proof of the first inequality is the same as in the Lemma 3.6 in [29]. We prove the second. By (1.2)$_3$ and the L^2-estimates, we have

$$\int w_{xx}^2dx\le C\int\rho w_t^2dx+C\left(\int w_x^2dx\right)^2+C\int w^2dx\le C.$$

Integrating it over $[0,t]$ and using Lemmas 3.4-3.5, we get the second inequality. □

LEMMA 3.7. *Under the conditions of Theorem 3.1, it holds that*

$$\sup_{0\le t\le T}\int\left(\rho u_t^2+\rho w_t^2+(1+\theta^q)^2\theta_x^2\right)dx+\int_{Q_T}\left(u_{xt}^2+w_{xt}^2+w_t^2+\rho(1+\theta^q)\theta_t^2\right)dxdt\le C.$$

Proof. Differentiating (1.2)$_2$ with respect to t, we have

$$\rho u_{tt}+\rho_tu_t+\rho_tuu_x+\rho u_tu_x+\rho uu_{xt}+P_{xt}=\mu_1u_{xxt}.\tag{3.15}$$

Multiplying (3.15) by u_t and integrating the resulting equation over I, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho u_t^2dx+\mu_1\int u_{xt}^2dx\\
&=-2\int\rho uu_tu_{xt}dx-\int\rho_tuu_xu_tdx-\int\rho u_t^2u_xdx+\int P_tu_{xt}dx\\
&\le 2\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}u\|_{L^\infty}\|u_{xt}\|_{L^2}+\|u_t\|_{L^\infty}\|u\|_{L^\infty}\|\rho_t\|_{L^2}\|u_x\|_{L^2}+\|u_x\|_{L^\infty}\int\rho u_t^2dx\\
&\quad+\|\theta\|_{L^\infty}\|\rho_t\|_{L^2}\|u_{xt}\|_{L^2}+\|\rho\theta_t\|_{L^2}\|u_{xt}\|_{L^2}\\
&\le\frac{\mu_1}{2}\|u_{xt}\|_{L^2}^2+C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|u_{xx}\|_{L^2}^2\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\rho\theta_t\|_{L^2}^2+\|\theta\|_{L^\infty}^2+C.
\end{aligned}$$

Here we have used $(1.2)_1$ and Lemmas 2.2, 3.2, 3.5-3.6.

Using integration by parts, the Hölder inequality, the Cauchy inequality, the Sobolev inequality, Lemmas 2.2, 3.2, 3.5 and 3.6, we have

$$\begin{aligned}
&\frac{d}{dt}\int\rho u_t^2dx+\mu_1\int u_{xt}^2dx\\
&\le C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|u_{xx}\|_{L^2}^2\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\rho\theta_t\|_{L^2}^2+\|\theta\|_{L^\infty}^2+C.
\end{aligned}\tag{3.16}$$

Integrating (3.16) over $(0,t)$, and using Lemmas 3.4-3.5, we have

$$\begin{aligned}
&\int\rho u_t^2dx+\mu_1\int_0^T\int u_{xt}^2dxdt\\
&\le\int\rho u_t^2dx\Big|_{t=0}+C\int_0^T\int u_{xx}^2dx\int\rho u_t^2dxdt+C\int_0^T\int\rho\theta_t^2dxdt+\|\theta\|_{L^\infty}^2+C.
\end{aligned}\tag{3.17}$$

Multiplying $(1.2)_2$ by $(\sqrt{\rho})^{-1}$, taking $t\to0^+$, and using (3.1) and (3.2), we have

$$\begin{aligned}
|\rho u_t^2(x,0)|&\le\frac{|u_{0xx}^\delta-P(\rho_0^\delta,\theta_0^\delta)_x|}{\sqrt{\rho_0^\delta}}+\sqrt{\rho_0^\delta}\,|u_0^\delta u_{0x}^\delta|\\
&=\frac{|\sqrt{\rho_0}g_1|}{\sqrt{\rho_0^\delta}}+\sqrt{\rho_0^\delta}\,|u_0^\delta u_{0x}^\delta|\\
&\le|g_1|+C,
\end{aligned}$$

which implies

$$\int\rho u_t^2(x,0)dx\le C.\tag{3.18}$$

Substituting (3.18) into (3.17), we have

$$\begin{aligned}
&\int\rho u_t^2dx+\mu_1\int_0^T\int u_{xt}^2dxdt\\
&\le C\int_0^T\int u_{xx}^2dx\int\rho u_t^2dxdt+C\int_0^T\int\rho\theta_t^2dxdt+\|\theta\|_{L^\infty}^2+C.
\end{aligned}\tag{3.19}$$

Differentiating $(1.2)_3$ with respect to t, we have

$$\rho w_{tt}+\rho_tw_t+\rho_tuw_x+\rho u_tw_x+\rho uw_{xt}+2\xi w_t=\mu_2w_{xxt}.\tag{3.20}$$

Multiplying (3.20) by w_t and integrating the resulting equation over I, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho w_t^2 dx+\mu_2\int w_{xt}^2 dx+2\xi\int w_t^2 dx\\
=&-2\int \rho u w_t w_{xt} dx-\int \rho_t u w_x w_t dx-\int \rho u_t w_x w_t dx\\
\leq& 2\|\sqrt{\rho}w_t\|_{L^2}\|\sqrt{\rho}u\|_{L^\infty}\|w_{xt}\|_{L^2}+\|w_t\|_{L^\infty}\|u\|_{L^\infty}\|\rho_t\|_{L^2}\|w_x\|_{L^2}\\
&+\|w_x\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}w_t\|_{L^2}\\
\leq&\frac{\mu_2}{2}\|w_{xt}\|_{L^2}^2+C\left(\|w_{xx}\|_{L^2}^2+1\right)\left(\|\sqrt{\rho}w_t\|_{L^2}^2+\|\sqrt{\rho}w_t\|_{L^2}^2\right)+C.
\end{aligned}$$

Similar to (3.19), we have

$$\begin{aligned}
&\int \rho w_t^2 dx+\int_0^T\int w_{xt}^2 dxdt+\int_0^T\int w_t^2 dxdt\\
\leq& C\int_0^T\int w_{xx}^2 dx\int \rho u_t^2 dxdt+C\int_0^T\int w_{xx}^2 dx\int \rho w_t^2 dxdt+C.
\end{aligned}\tag{3.21}$$

Multiplying $(1.2)_4$ by $\kappa\theta_t$, integrating the resulting equation over I, and using integration by parts, Lemmas 2.2, 3.2 and 3.5, and the Cauchy inequality, we have, for suitably small $\varepsilon>0$,

$$\begin{aligned}
&\int \rho\kappa\theta_t^2 dx+\frac{1}{2}\frac{d}{dt}\int \kappa^2\theta_x^2 dx\\
=&\int \kappa\theta_t u_x^2 dx+\int \kappa\theta_t w_x^2 dx+\int \kappa\theta_t w^2 dx-\int \kappa\rho u\theta_x\theta_t dx-\int \kappa\rho\theta u_x\theta_t dx\\
\leq&\frac{1}{2}\int \rho\kappa\theta_t^2 dx+\frac{d}{dt}\left(\int u_x^2\int_0^\theta \kappa(\tau)d\tau dx+\int w_x^2\int_0^\theta \kappa(\tau)d\tau dx+\int w^2\int_0^\theta \kappa(\tau)d\tau dx\right)\\
&-2\int u_x u_{xt}\int_0^\theta \kappa(\tau)d\tau dx-2\int w_x w_{xt}\int_0^\theta \kappa(\tau)d\tau dx-2\int w w_t\int_0^\theta \kappa(\tau)d\tau dx\\
&+\int \rho\kappa(\theta^2+\theta_x^2)dx\\
\leq&\frac{1}{2}\int \rho\kappa\theta_t^2 dx+\frac{d}{dt}\left(\int u_x^2\int_0^\theta \kappa(\tau)d\tau dx+\int w_x^2\int_0^\theta \kappa(\tau)d\tau dx+\int w^2\int_0^\theta \kappa(\tau)d\tau dx\right)\\
&+\varepsilon\int u_{xt}^2 dx+\varepsilon\int w_{xt}^2 dx+C\int (1+\theta^q)^2\theta_x^2 dx+C.
\end{aligned}\tag{3.22}$$

Integrating (3.22) over $[0,T]$, and using Lemmas 2.2, 3.5-3.6 and the Cauchy inequality, we obtain

$$\begin{aligned}
&\int_0^T\int \rho(1+\theta^q)\theta_t^2 dxdt+\int (1+\theta^q)^2\theta_x^2 dx\\
\leq& C\left[\int u_x^2\int_0^\theta \kappa(\tau)d\tau dx+\int w_x^2\int_0^\theta \kappa(\tau)d\tau dx+\int w^2\int_0^\theta \kappa(\tau)d\tau dx\right]_0^T\\
&+\int (1+\theta^q)^2\theta_x^2 dx\Big|_{t=0}+C\varepsilon\int_0^T\int u_{xt}^2 dxdt+C\varepsilon\int_0^T\int w_{xt}^2 dxdt+C
\end{aligned}$$

$$
\begin{aligned}
&\le C\|(1+\theta^q)\theta\|_{L^\infty}+C\varepsilon\int_0^T\int u_{xt}^2dxdt+C\varepsilon\int_0^T\int w_{xt}^2dxdt+C\\
&\le C\|(1+\theta^q)\theta_x\|_{L^2}+C\varepsilon\int_0^T\int u_{xt}^2dxdt+C\varepsilon\int_0^T\int w_{xt}^2dxdt+C\\
&\le \frac{1}{2}\int(1+\theta^q)^2\theta_x^2dx+C\varepsilon\int_0^T\int u_{xt}^2dxdt+C\varepsilon\int_0^T\int w_{xt}^2dxdt+C.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\int_0^T\int\rho(1+\theta^q)\theta_t^2dxdt+\int(1+\theta^q)^2\theta_x^2dx\\
&\le C\varepsilon\int_0^T\int u_{xt}^2dxdt+C\varepsilon\int_0^T\int w_{xt}^2dxdt+C.
\end{aligned}
\tag{3.23}
$$

By (3.19), (3.21) and (3.23), choosing suitably small $\varepsilon>0$, using the Grönwall inequality and Lemma 3.6, we complete the proof of Lemma 3.7. □

With Corollary 2.1, Lemmas 3.1-3.7, one can easily derive the following estimates of the solution (ρ,u,w,θ) in a similar manner as those obtained in [29]. More precisely, we get the following proposition.

PROPOSITION 3.1. *Under the conditions of Theorem 3.1, it holds that*

$$
\begin{aligned}
&\|\theta\|_{L^\infty(Q_T)}\le C,\\
&\|u\|_{W^{1,\infty}(Q_T)}+\|w\|_{W^{1,\infty}(Q_T)}+\sup_{0\le t\le T}\int\left(u_{xx}^2+w_{xx}^2\right)dx+\int_{Q_T}\theta_{xx}dxdt\le C,\\
&\|\rho\|_{W^{1,\infty}(Q_T)}+\|\rho_t\|_{L^\infty(Q_T)}+\sup_{0\le t\le T}\int\left(\rho_{xx}^2+\rho_{xt}^2\right)dx+\int_{Q_T}\left(\rho_{tt}^2+u_{xxx}^2\right)dxdt\le C.
\end{aligned}
$$

LEMMA 3.8. *Under the conditions of Theorem 3.1, it holds that*

$$
\sup_{0\le t\le T}\int\rho\theta_t^2dx+\int_{Q_T}|[(1+\theta^q)\theta_x]_t|^2dxdt\le C.
$$

Proof. Differentiating $(1.2)_4$ with respect to t, we have

$$
\rho\theta_{tt}+\rho_t\theta_t+(\rho u\theta_x)_t+(\rho\theta u_x)_t=2\mu_1u_xu_{xt}+2\mu_2w_xw_{xt}+4\xi ww_t+(\kappa\theta_x)_{xt}. \tag{3.24}
$$

Similar to [29], multiplying (3.24) by $\kappa\theta_t$, integrating over I, and using integration by parts, $(1.2)_1$, Proposition 3.1, Lemmas 2.1, 3.2, and the Hölder inequality, we have

$$
\begin{aligned}
&\frac{d}{dt}\int\rho\kappa\theta_t^2dx+\int|[(1+\theta^q)\theta_x]_t|^2dx\\
&\le C\int\theta_{xx}^2dx+C\left(\int\rho\theta_t^2dx\right)^2+C\left(\int u_{xt}^2dx+\int w_{xt}^2dx+\int w_t^2dx\right)+C.
\end{aligned}
\tag{3.25}
$$

Integrating (3.25) over $[0,T]$, using Proposition 3.1, we obtain

$$
\int\rho\theta_t^2dx+\int_0^T\int|[(1+\theta^q)\theta_x]_t|^2dxdt\le\int\rho\kappa\theta_t^2dx\Big|_{t=0}+C\int_0^T\left(\int\rho\theta_t^2dx\right)^2dt+C. \tag{3.26}
$$

From [29], we have

$$\int \rho\kappa\theta_t^2 dx\Big|_{t=0} \le C\int g_2^2 dx + C\int \theta_{0xx}^2 dx + C \le C. \tag{3.27}$$

Substituting (3.27) into (3.26), using the Grönwall inequality and Proposition 3.1, we complete this proof. □

With the help of Lemmas 3.1-3.8 and Proposition 3.1, one can easily derive the following estimates of the solution (ρ,u,w,θ) in a similar manner as those obtained in [29].

PROPOSITION 3.2. *Under the conditions of Theorem 3.1, it holds that*

$$\begin{gathered}
\int_0^T \|\theta_t\|_{L^\infty} dt \le C,\\
\int_{Q_T} \theta_{xt}^2 dxdt \le C,\\
\|\theta\|_{W^{1,\infty}(Q_T)} + \sup_{0\le t\le T}\int \theta_{xx}^2 dx + \int_{Q_T} \theta_{xxx}^2 dxdt \le C,\\
\|(\sqrt{\rho})_x\|_{L^\infty(Q_T)} + \|(\sqrt{\rho})_t\|_{L^\infty(Q_T)} \le C.
\end{gathered}$$

LEMMA 3.9. *Under the conditions of Theorem 3.1, it holds that*

$$\sup_{0\le t\le T}\int \rho^2|(\kappa\theta_x)_t|^2 dx + \int_{Q_T} \rho^3\theta_{tt}^2 dxdt \le C.$$

Proof. Multiplying (3.24) by $\rho^2(\kappa\theta_t)_t$, using integration by parts, Propositions 3.1-3.2 and the Cauchy inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\int \rho^2|(\kappa\theta_x)_t|^2 dx + \int \rho^3\theta_{tt}^2 dx\\
&\le C\int |(\kappa\theta_x)_t|^2 dx + C\int \theta_{xt}^2 dx + C\left(\int u_{xt}^2 dx + \int w_{xt}^2 dx + \int w_t^2 dx\right)\\
&\quad + C\|\theta_t\|_{L^\infty} + C.
\end{aligned} \tag{3.28}$$

Integrating (3.28) on $[0,T]$, by (1.6), (1.8), Lemmas 3.7, 3.8, Propositions 3.1-3.2 and the Young inequality, we have

$$\int \rho^2|(\kappa\theta_x)_t|^2 dx + \int_0^T\int \rho^3\theta_{tt}^2 dxdt \le \int \rho^2|(\kappa\theta_x)_t|^2 dx\Big|_{t=0} + C.$$

By [29], $\int \rho^2|(\kappa\theta_x)_t|^2 dx\big|_{t=0} \le C$. Then we complete the proof. □

With the help of Lemmas 3.1-3.9, and Propositions 3.1-3.2, one can easily derive the estimates of the solution (ρ,u,w,θ) in a similar manner as those obtained in [29].

PROPOSITION 3.3. *Under the conditions of Theorem 3.1, it holds that*

$$\sup_{0\le t\le T}\int \left(\theta_{xxx}^2 + \rho^2\theta_{xt}^2\right) dx \le C,$$

$$\sup_{0\le t\le T}\int \rho^2 u_{xt}^2 dx+\int_{Q_T}\rho^3 u_{tt}^2 dxdt\le C,$$
$$\sup_{0\le t\le T}\int u_{xxx}^2 dx\le C.$$

From all the above estimates, we get

$$\begin{aligned}&\|(\sqrt{\rho})_x\|_{L^\infty}+\|(\sqrt{\rho})_t\|_{L^\infty}+\|\rho\|_{H^2}+\|\rho_t\|_{H^1}+\|u\|_{H^3}+\|\rho u_t\|_{H^1}+\|\sqrt{\rho}u_t\|_{L^2}+\|\theta\|_{H^3}\\&+\|\sqrt{\rho}\theta_t\|_{L^2}+\|\rho\theta_t\|_{H^1}+\int_{Q_T}\left(u_{xt}^2+\rho_{tt}^2+\theta_t^2+\theta_{xt}^2+\rho^3u_{tt}^2+\rho^3\theta_{tt}^2\right)dxdt\le C.\end{aligned}\tag{3.29}$$

Corollary 3.1. *Under the conditions of Theorem 3.1, there exists a positive constant C_δ depending on δ such that for any $(x,t)\in Q_T$, it holds that,*

$$\begin{cases}\rho\ge\frac{\delta}{C}>0,\\ \theta\ge C_\delta>0.\end{cases}$$

Proof. The proof is the same as in the Corollary 3.9 in [29]. □

Then from (3.29), the above lemmas and corollary, we have

$$\begin{aligned}&\sup_{0\le t\le T}(\|\rho\|_{H^2}+\|\rho_t\|_{H^1}+\|u\|_{H^3}+\|u_t\|_{H^1}+\|w\|_{H^3}+\|w_t\|_{H^1}+\|\theta\|_{H^3}+\|\rho\theta_t\|_{H^1})\\&\quad+\int_{Q_T}\left(u_{xt}^2+u_{xxt}^2+w_{xt}^2+w_{xxt}^2+\rho_{tt}^2+\theta_t^2+\theta_{xt}^2+\theta_{xxt}^2+u_{tt}^2+w_{tt}^2+\theta_{tt}^2\right)dxdt\\&\le C(\delta).\end{aligned}\tag{3.30}$$

With (3.30), we complete the proof of Theorem 3.1.

Proof. **(Proof of Theorem 1.1.)** Consider (1.2)-(1.4) with initial data replaced by $(\rho_0^\delta,u_0^\delta,w_0^\delta,\theta_0^\delta)$. We obtain from Theorem 3.1 that there exists the solution $(\rho^\delta,u^\delta,w^\delta,\theta^\delta)$, such that (3.29) and (3.30) are valid when we replace (ρ,u,w,θ) by $(\rho^\delta,u^\delta,w^\delta,\theta^\delta)$. With the estimates uniform for δ, we take $\delta\to0^+$ (take the subsequence if necessary) to get a solution to (1.2)-(1.4) still denoted by (ρ,u,w,θ) which satisfies (3.29) by the lower semicontinuity of the norms. This proves Theorem 1.1. □

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by the same method as in Section 3.

Denote $\rho_0^\delta=\rho_0+\delta$ and $\theta_0^\delta=\theta_0+\delta$ for $\delta\in(0,1)$. Throughout this section, we denote C to be a generic constant depending on $\rho_0,u_0,w_0,\theta_0,T$ and some other known constants but independent of δ for any $\delta\in(0,1)$.

For any given $\delta\in(0,1)$, let u_0^δ, w_0^δ be the solution to the following elliptic equations:

$$\begin{cases}\mu_1u_{0xx}^\delta-P_{0x}^\delta=\sqrt{\rho_0}g_1,\\ \mu_2w_{0xx}^\delta-2\xi w_0^\delta=\sqrt{\rho_0}g_2,\\ u_0^\delta|_{x=0,1}=0,\ w_0^\delta|_{x=0,1}=0.\end{cases}\tag{4.1}$$

Since $\rho_0^\delta=\rho_0+\delta\in H^2$, $\theta_0^\delta=\theta_0+\delta\in H^3$ and $\sqrt{\rho_0}g_1,\sqrt{\rho_0}g_2\in H_0^1$, by the elliptic theory, (1.11) and (4.1), we have $u_0^\delta,w_0^\delta\in H_0^1\cap H^3$ and

$$\begin{cases}u_0^\delta\to u_0,\ w_0^\delta\to u_0,\quad \text{in}\quad H^3\quad \text{as}\quad \delta\to0^+,\\ \|u_0^\delta-u_0\|_{H^2}\le C\delta,\ \|w_0^\delta-w_0\|_{H^2}\le C\delta.\end{cases}\tag{4.2}$$

The boundary conditions of θ^δ is

$$(\theta^\delta_x - a\theta^\delta)(0,t) = -a\delta, \quad (\theta^\delta_x + b\theta^\delta)(1,t) = b\delta, \tag{4.3}$$

Before proving Theorem 1.2, we need the following auxiliary theorem to construct a sequence of approximate solutions, which is the same as in Theorem 3.1.

THEOREM 4.1. *Consider the same assumptions as in Theorem 1.2. Then for any given $\delta \in (0,1)$, there exists a global solution (ρ,u,w,θ) to (1.2)-(1.4) with initial data replaced by $(\rho_0^\delta, u_0^\delta, w_0^\delta, \theta_0^\delta)$, such that for any $T>0$,*

$$\begin{cases} \rho \in C([0,T];H^2),\ \rho_t \in C([0,T];H^1),\ \rho_{tt} \in L^2(0,T;L^2),\ \rho \geq \frac{\delta}{C} > 0, \\ (u,w) \in C([0,T];H_0^1 \cap H^3),\ (u_t,w_t) \in C([0,T];H_0^1) \cap L^2(0,T;H^2), \\ (u_{tt},w_{tt}) \in L^2(0,T;L^2),\ \theta \in C([0,T];H^3),\ \theta \geq C_\delta > 0, \\ \theta_t \in C([0,T];H^1) \cap L^2(0,T;H^2),\ \theta_{tt} \in L^2(0,T;L^2), \end{cases}$$

where C_δ is a constant depending on δ.

For any given $T \in (0,+\infty)$, let (ρ,u,w,θ) be the solution to (1.2)-(1.4) as in Theorem 4.1. Then we have the following lemmas.

LEMMA 4.1. *Under the conditions of Theorem 4.1, it holds that for any $0 \leq t \leq T$*

$$\int \rho dx = \int \rho_0 dx, \tag{4.4}$$

$$\int \rho\left(\theta + u^2 + w^2\right) dx + \int_0^T [a\kappa_1(1+\theta^q)\theta(0,t) + b\kappa_1(1+\theta^q)\theta(1,t)]\, dt \leq C. \tag{4.5}$$

Proof. Integrating $(1.1)_1$ and $(1.1)_4$ over $I \times [0,t]$, using (1.6) and the boundary conditions (1.4), we have

$$\int \rho dx = \int \rho_0 dx,$$

and

$$\int \rho E dx = \int \rho_0 E_0 dx - \int_0^t [a\kappa(\theta-\delta)(0,s) + b\kappa(\theta-\delta)(1,s)]\, ds,$$

then we have

$$\begin{aligned} &\int \rho E dx + \int_0^T [a\kappa_1(1+\theta^q)\theta(0,t) + b\kappa_1(1+\theta^q)\theta(1,t)]\, dt \\ \leq &\int \rho_0 E_0 dx + \delta \int_0^T [a\kappa_2(1+\theta^q)(0,t) + b\kappa_2(1+\theta^q)(1,t)]\, dt. \end{aligned}$$

Using the Young inequality to the last term in the above inequality, then we have (4.5). □

Similar to Lemma 3.2, we give the following lemma.

LEMMA 4.2. *Under the conditions of Theorem 4.1, it holds that for any $(x,t) \in Q_T$,*

$$0 < \rho \leq C \quad \textit{and} \quad \theta > 0.$$

Lemma 4.3. *Under the conditions of Theorem 4.1, there exists* $\alpha\in(0,1)$,

$$\int_{Q_T}\left(\frac{u_x^2}{\theta^\alpha}+\frac{w_x^2}{\theta^\alpha}+\frac{w^2}{\theta^\alpha}+\frac{(1+\theta^q)\theta_x^2}{\theta^{1+\alpha}}\right)dxdt\le C,$$

where C may depend on α.

Proof. Multiplying $(1.2)_4$ by $\theta^{-\alpha}$, integrating the resulting equation over Q_T, and using integration by parts, we have

$$\begin{aligned}
&\int_0^T\int\left(\frac{\mu_1u_x^2}{\theta^\alpha}+\frac{\mu_2w_x^2}{\theta^\alpha}+\frac{2\xi w^2}{\theta^\alpha}+\frac{\alpha\kappa\theta_x^2}{\theta^{1+\alpha}}\right)dxdt+\int_0^T\frac{\kappa\theta_x}{\theta^\alpha}\Big|_0^1dt\\
&=\frac{1}{1-\alpha}\int\rho\theta^{1-\alpha}dx\Big|_0^T+\int_0^T\int\rho\theta^{1-\alpha}u_xdxdt\\
&\le C+\frac{\mu_1}{2}\int_0^T\int\frac{u_x^2}{\theta^\alpha}dxdt+C\int_0^T\int\rho^2\theta^{2-\alpha}dxdt\\
&\le C+\frac{\mu_1}{2}\int_0^T\int\frac{u_x^2}{\theta^\alpha}dxdt+C\int_0^T\|\theta\|_{L^\infty}^{1-\alpha}dt,
\end{aligned}\tag{4.6}$$

where we have used the Cauchy inequality, Lemmas 4.1 and 4.2. From (1.4), we have

$$\int_0^T\frac{\kappa\theta_x}{\theta^\alpha}\Big|_0^1dt=-\int_0^T\left[\frac{b\kappa(\theta-\delta)}{\theta^\alpha}(1,t)+\frac{a\kappa(\theta-\delta)}{\theta^\alpha}(0,t)\right]dt.\tag{4.7}$$

From (4.6), (4.7) and the Young inequality, we have

$$\begin{aligned}
&\int_0^T\int\left(\frac{\mu_1u_x^2}{\theta^\alpha}+\frac{\mu_2w_x^2}{\theta^\alpha}+\frac{2\xi w^2}{\theta^\alpha}+\frac{\alpha\kappa\theta_x^2}{\theta^{1+\alpha}}\right)dxdt+\delta\int_0^T\left[\frac{b\kappa}{\theta^\alpha}(1,t)+\frac{a\kappa}{\theta^\alpha}(0,t)\right]dt\\
&\le\int_0^T\left[b\kappa\theta^{1-\alpha}(1,t)+a\kappa\theta^{1-\alpha}(0,t)\right]dt+\frac{\mu_1}{2}\int_0^T\int\frac{u_x^2}{\theta^\alpha}dxdt+C\int_0^T\|\theta\|_{L^\infty}^{1-\alpha}dt+C\\
&\le C\int_0^T[(1+\theta^q)\theta(0,t)+(1+\theta^q)\theta(1,t)]dt+\frac{\mu_1}{2}\int_0^T\int\frac{u_x^2}{\theta^\alpha}dxdt+C\int_0^T\|\theta\|_{L^\infty}^{1-\alpha}dt+C.
\end{aligned}\tag{4.8}$$

Equations (4.8) and (4.5) yield

$$\int_0^T\int\left(\frac{\mu_1u_x^2}{2\theta^\alpha}+\frac{\mu_2w_x^2}{\theta^\alpha}+\frac{2\xi w^2}{\theta^\alpha}+\frac{\alpha\kappa\theta_x^2}{\theta^{1+\alpha}}\right)dxdt\le C\int_0^T\|\theta\|_{L^\infty}^{1-\alpha}dt+C.$$

The rest is the same as in Lemma 3.3. Then we complete the proof of this lemma. □

Lemma 4.4. *Under the conditions of Theorem 4.1, it holds that*

$$\int_0^T\|\theta\|_{L^\infty}^{q+1-\alpha}dt\le C,$$

$$\sup_{0\le t\le T}\int\rho u^2dx+\int_{Q_T}u_x^2dxdt\le C,$$

$$\sup_{0\le t\le T}\int\rho w^2dx+\int_{Q_T}(w^2+w_x^2)dxdt\le C.$$

Proof. The proof of this is same as that of Lemma 3.4. □

LEMMA 4.5. *Under the conditions of Theorem 4.1, it holds that*

$$\sup_{0\le t\le T}\int\left(u_x^2+w_x^2+w^2+\rho\theta^2(1+\theta^q)\right)dx+\int_{Q_T}\left(\rho u_t^2+\rho w_t^2+(1+\theta^q)^2\theta_x^2\right)dxdt\le C.$$

Proof. Similar to (3.8), we get

$$\int\rho u_t^2dx+\mu_1\frac{d}{dt}\int u_x^2\le C\left(\int u_x^2dx\right)^2+2\frac{d}{dt}\int Pu_xdx-\frac{d}{dt}\int P^2dx-2\int P_t(u_x-P)dx.$$

Using (1.2) and integration by parts, we have

$$\begin{aligned}-2\int P_t(u_x-P)dx&=-2\int(\rho\theta)_t(u_x-P)dx\\&=-2\int\left[(\kappa\theta_x)_x+u_x^2-(\rho u\theta)_x-\rho\theta u_x\right](u_x-P)dx\\&=-2\kappa\theta_x(u_x-P)\big|_0^1+2\int\kappa\theta_x(u_{xx}-P_x)dx-2\int u_x^2(u_x-P)dx\\&\quad-2\int\rho u\theta(u_{xx}-P_x)dx+2\int\rho\theta u_x(u_x-P)dx\\&\triangleq\sum_{i=1}^{5}J_i.\end{aligned}$$

By (1.2), (1.4), (2.2)-(2.4) and the Young inequality, we estimate the right-hand side of the first term of above equality as follows:

$$\begin{aligned}|J_1|&=2\left|\{\kappa\theta_x[u_x-P](1,t)-\kappa\theta_x[u_x-P](0,t)\}\right|\\&=2\left|\{b\kappa(\theta-\delta)[u_x-P](1,t)+a\kappa(\theta-\delta)[u_x-P](0,t)\}\right|\\&\le C(a+b)\|u_x-P\|_{L^\infty}\|(1+\theta^q)(\theta-\delta)\|_{L^\infty}\\&\le C(a+b)\|u_x-P\|_{L^\infty}(\|\kappa\theta_x\|_{L^2}+1)\\&\le C(a+b)(\|u_x-P\|_{L^1}+\|\rho u_t+\rho uu_x\|_{L^1})(\|\kappa\theta_x\|_{L^2}+1)\\&\le C(a+b)(\|\kappa\theta_x\|_{L^2}+1)\left(\int u_xdx+\int Pdx+\int\rho u_tdx+\int\rho uu_xdx\right)\\&\le C(a+b)(\|\kappa\theta_x\|_{L^2}+1)\left[\left(\int u_x^2dx\right)^2+\int\rho\theta^2dx+\int\rho u_t^2dx+\|u\|_{L^\infty}^2\int u_x^2dx+1\right]\\&\le\frac{1}{10}\int\rho u_t^2dx+C\left(\int u_x^2dx\right)^2+\int\rho\theta^2dx+C(a+b)^2\left(\|\kappa\theta_x\|_{L^2}^2+1\right)+C,\end{aligned}$$

Then, combining Lemma 3.5, we get

$$\begin{aligned}\int_0^T\int\rho u_t^2dxdt+\mu_1\int u_x^2dx\le C\int_0^T(1+\|\theta\|_{L^\infty}^2)\int u_x^2dxdt+C\int_0^T\left(\int u_x^2dx\right)^2dt\\+C\int_0^T\int\kappa^2\theta_x^2dxdt+C\|\theta\|_{L^\infty}+C,\end{aligned}\tag{4.9}$$

and

$$\int_0^T\int\rho w_t^2dxdt+\int\left(\mu_2w_x^2+2\xi w^2\right)dx$$
$$\leq C\int_0^T\left(\int u_x^2dx\right)^2dt+C\int_0^T\left(\int w_x^2dx\right)^2dt. \tag{4.10}$$

Multiplying $(1.2)_4$ by $\int_0^\theta\kappa(\tau)d\tau$, integrating over I, and using integration by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\int\rho\int_0^\theta\int_0^y\kappa(\tau)dyd\tau dx+\int\kappa^2\theta_x^2dx-\left[\kappa\theta_x\int_0^\theta\kappa(\tau)d\tau\right]_0^1\\
=&\mu_1\int u_x^2\int_0^\theta\kappa(\tau)d\tau dx+\mu_2\int w_x^2\int_0^\theta\kappa(\tau)d\tau dx+2\xi\int w^2\int_0^\theta\kappa(\tau)d\tau dx\\
&-\int\rho\theta u_x\int_0^\theta\kappa(\tau)d\tau dx\\
\leq& C\|\kappa\theta\|_{L^\infty}\left(\int u_x^2dx+\int w_x^2dx+\int w^2dx+\int\rho\theta u_xdx\right)\\
\leq& C\left(\|\kappa\theta_x\|_{L^2}+1\right)\left(\int u_x^2dx+\int w_x^2dx+\int w^2dx+\int\rho\theta u_xdx\right).
\end{aligned} \tag{4.11}$$

From (1.4), we have

$$\left[\kappa\theta_x\int_0^\theta\kappa(\tau)d\tau\right]_0^1=-b\left(\kappa(\theta-\delta)\int_0^\theta\kappa(\tau)d\tau\right)(1,t)-a\left(\kappa(\theta-\delta)\int_0^\theta\kappa(\tau)d\tau\right)(0,t).$$

Then, from (4.11), we get

$$\begin{aligned}
&\frac{d}{dt}\int\rho\theta\int_0^\theta\kappa(\tau)d\tau dx+\int\kappa^2\theta_x^2dx+b\left(\kappa\theta\int_0^\theta\kappa(\tau)d\tau\right)(1,t)+a\left(\kappa\theta\int_0^\theta\kappa(\tau)d\tau\right)(0,t)\\
\leq& C\left(\|\kappa\theta_x\|_{L^2}+1\right)\left(\int u_x^2dx+\int w_x^2dx+\int w^2dx+\int\rho\theta u_xdx\right)\\
&+\delta b\left(\kappa\int_0^\theta\kappa(\tau)d\tau\right)(1,t)+\delta a\left(\kappa\int_0^\theta\kappa(\tau)d\tau\right)(0,t).
\end{aligned}$$

By (1.6), we can deduce that

$$\begin{aligned}
&\frac{d}{dt}\int\rho\theta\int_0^\theta\kappa(\tau)d\tau dx+\int\kappa^2\theta_x^2dx+\frac{b}{q+1}\kappa_1^2(1+\theta^q)^2\theta^2(1,t)+\frac{a}{q+1}\kappa_1^2(1+\theta^q)^2\theta^2(0,t)\\
\leq&\int u_x^2\int_0^\theta\kappa(\tau)d\tau dx-\int\rho\theta u_x\int_0^\theta\kappa(\tau)d\tau dx+\delta b\kappa_2^2(1+\theta^q)^2\theta(1,t)+\delta a\kappa_2^2(1+\theta^q)^2\theta(0,t)\\
\leq& C\left(\|\kappa\theta_x\|_{L^2}+1\right)\left(\int u_x^2dx+\int w_x^2dx+\int w^2dx+\int\rho\theta u_xdx\right)\\
&+\delta b\kappa_2^2(1+\theta^q)^2\theta(1,t)+\delta a\kappa_2^2(1+\theta^q)^2\theta(0,t)\\
\leq&\frac{1}{2}\|\kappa\theta_x\|_{L^2}^2+C\left(\int u_x^2dx\right)^2+C\left(\int w_x^2dx\right)^2+C\left(\int w^2dx\right)^2+C\int\rho^2\theta^2u_x^2dx
\end{aligned}$$

$$
\begin{aligned}
&+\delta b\kappa_2^2(1+\theta^q)^2\theta(1,t)+\delta a\kappa_2^2(1+\theta^q)^2\theta(0,t)+C\\
\le{}&\frac{1}{2}\|\kappa\theta_x\|_{L^2}^2+C\left(\int u_x^2dx\right)^2+C\left(\int w_x^2dx\right)^2+C\left(\int w^2dx\right)^2+C\|\theta\|_{L^\infty}^2\int u_x^2dx\\
&+\delta b\kappa_2^2(1+\theta^q)^2\theta(1,t)+\delta a\kappa_2^2(1+\theta^q)^2\theta(0,t)+C. \qquad (4.12)
\end{aligned}
$$

Using the Young inequality, we obtain from (4.12) that

$$
\begin{aligned}
&\int\rho\theta^2(1+\theta^q)dx+\int_0^T\int\kappa^2\theta_x^2dxdt\\
\le{}&C\int_0^T\left(\int u_x^2dx\right)^2dt+C\int_0^T\left(\int w_x^2dx\right)^2dt+C\int_0^T\left(\int w^2dx\right)^2dt\\
&+C\int_0^T\|\theta\|_{L^\infty}^2\int u_x^2dxdt+C. \qquad (4.13)
\end{aligned}
$$

Using (4.9), (4.10), (4.13), Lemma 4.4 and the Grönwall inequality, we complete the proof. ∎

LEMMA 4.6. *Under the conditions of Theorem 4.1, it holds that*

$$
\sup_{0\le t\le T}\int\left(\rho_x^2+\rho_t^2\right)dx+\int_{Q_T}u_{xx}^2dxdt\le C,
$$
$$
\int_{Q_T}w_{xx}^2dxdt\le C.
$$

Proof. The proof of this lemma is the same as in Lemma 3.6. ∎

LEMMA 4.7. *Under the conditions of Theorem 4.1, it holds that*

$$
\sup_{0\le t\le T}\int\left(\rho u_t^2+\rho w_t^2+(1+\theta^q)^2\theta_x^2\right)dx+\int_{Q_T}\left(u_{xt}^2+w_{xt}^2+w_t^2+\rho(1+\theta^q)\theta_t^2\right)dxdt\le C.
$$

Proof. From Lemma 3.7, we have

$$
\begin{aligned}
&\int\rho u_t^2dx+\int_0^T\int u_{xt}^2dxdt\\
\le{}&C\int_0^T\int u_{xx}^2dx\int\rho u_t^2dxdt+C\int_0^T\int\rho\theta_t^2dxdt+\|\theta\|_{L^\infty}^2+C, \qquad (4.14)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int\rho w_t^2dx+\int_0^T\int w_{xt}^2dxdt+\int_0^T\int w_t^2dxdt\\
\le{}&C\int_0^T\int w_{xx}^2dx\int\rho u_t^2dxdt+C\int_0^T\int w_{xx}^2dx\int\rho w_t^2dxdt+C. \qquad (4.15)
\end{aligned}
$$

Multiplying $(1.2)_4$ by $\kappa\theta_t$, integrating the resulting equation over I, and using integration by parts, Lemmas 2.2, 4.2 and 4.5, and the Cauchy inequality, we have, for suitably small $\varepsilon>0$,

$$
\int\rho\kappa\theta_t^2dx+\frac{1}{2}\frac{d}{dt}\int\kappa^2\theta_x^2dx
$$

$$
\begin{aligned}
&=\kappa^2\theta_x\theta_t\big|_0^1+\int\kappa\theta_t u_x^2dx+\int\kappa\theta_t w_x^2dx+\int\kappa\theta_t w^2dx-\int\kappa\rho u\theta_x\theta_tdx-\int\kappa\rho\theta u_x\theta_tdx\\
&\le\frac{d}{dt}\left(\int u_x^2\int_0^\theta\kappa(\tau)d\tau dx+\int w_x^2\int_0^\theta\kappa(\tau)d\tau dx+\int w^2\int_0^\theta\kappa(\tau)d\tau dx\right)\\
&\quad+\frac{1}{2}\int\rho\kappa\theta_t^2dx-2\int u_xu_{xt}\int_0^\theta\kappa(\tau)d\tau dx-2\int w_xw_{xt}\int_0^\theta\kappa(\tau)d\tau dx\\
&\quad-2\int ww_t\int_0^\theta\kappa(\tau)d\tau dx+\int\rho\kappa(\theta^2+\theta_x^2)dx+\kappa^2\theta_x\theta_t\big|_0^1\\
&\le\frac{d}{dt}\left(\int u_x^2\int_0^\theta\kappa(\tau)d\tau dx+\int w_x^2\int_0^\theta\kappa(\tau)d\tau dx+\int w^2\int_0^\theta\kappa(\tau)d\tau dx\right)\\
&\quad+\frac{1}{2}\int\rho\kappa\theta_t^2dx+\varepsilon\int u_{xt}^2dx+\varepsilon\int w_{xt}^2dx+C\int(1+\theta^q)^2\theta_x^2dx+\kappa^2\theta_x\theta_t\big|_0^1+C. \qquad (4.16)
\end{aligned}
$$

From (1.4), we have

$$
\begin{aligned}
\kappa^2\theta_x\theta_t\big|_0^1&=\kappa^2\theta_x\theta_t(1,t)-\kappa^2\theta_x\theta_t(0,t)\\
&=-b\kappa^2(\theta-\delta)\theta_t(1,t)-a\kappa^2(\theta-\delta)\theta_t(0,t)\\
&=-b\kappa^2(\theta-\delta)\theta_t(1,t)-a\kappa^2(\theta-\delta)\theta_t(0,t)+b\delta\kappa^2\theta_t(1,t)-a\delta\kappa^2\theta_t(0,t).
\end{aligned}
$$

Integrating the above equality over $(0,t)$, using the Young inequality, choosing $\delta>0$ small suitably, we have

$$
\begin{aligned}
&-\int_0^t\kappa^2\theta_x\theta_t(x,s)\big|_0^1ds\\
&\ge Cb\left[\theta^2+\theta^{q+2}+\theta^{2q+2}\right](1,t)+Ca\left[\theta^2+\theta^{q+2}+\theta^{2q+2}\right](0,t)\\
&\quad-Cb\left[\theta^2+\theta^{q+2}+\theta^{2q+2}\right](1,0)-Ca\left[\theta^2+\theta^{q+2}+\theta^{2q+2}\right](0,0)-C\\
&\ge Cb\left[\theta^2+\theta^{q+2}+\theta^{2q+2}\right](1,t)+Ca\left[\theta^2+\theta^{q+2}+\theta^{2q+2}\right](0,t)-C. \qquad (4.17)
\end{aligned}
$$

Integrating (4.16) over $(0,t)$, and using Lemmas 2.2, 3.5-3.6, (4.17) and the Cauchy inequality, we obtain

$$
\begin{aligned}
&\int_0^T\int\rho(1+\theta^q)\theta_t^2dxdt+\int(1+\theta^q)^2\theta_x^2dx\\
&\quad+Cb\left[\theta^2+\theta^{q+2}+\theta^{2q+2}\right](1,T)+Ca\left[\theta^2+\theta^{q+2}+\theta^{2q+2}\right](0,T)\\
&\le C\left[\int u_x^2\int_0^\theta\kappa(\tau)d\tau dx+\int w_x^2\int_0^\theta\kappa(\tau)d\tau dx+\int w^2\int_0^\theta\kappa(\tau)d\tau dx\right]_0^T\\
&\quad+\int(1+\theta^q)^2\theta_x^2dx\Big|_{t=0}+C\varepsilon\int_0^T\int u_{xt}^2dxdt+C\varepsilon\int_0^T\int w_{xt}^2dxdt+C\\
&\le C\|(1+\theta^q)\theta\|_{L^\infty}+C\varepsilon\int_0^T\int u_{xt}^2dxdt+C\varepsilon\int_0^T\int w_{xt}^2dxdt+C\\
&\le C\|(1+\theta^q)\theta_x\|_{L^2}+C\varepsilon\int_0^T\int u_{xt}^2dxdt+C\varepsilon\int_0^T\int w_{xt}^2dxdt+C\\
&\le\frac{1}{2}\int(1+\theta^q)^2\theta_x^2dx+C\varepsilon\int_0^T\int u_{xt}^2dxdt+C\varepsilon\int_0^T\int w_{xt}^2dxdt+C.
\end{aligned}
$$

Because the third and fourth terms on the left-hand side are positive, then we have

$$\begin{aligned}&\int_0^T\int\rho(1+\theta^q)\theta_t^2dxdt+\int(1+\theta^q)^2\theta_x^2dx\\&\le C\varepsilon\int_0^T\int u_{xt}^2dxdt+C\varepsilon\int_0^T\int w_{xt}^2dxdt+C.\end{aligned}\tag{4.18}$$

By (4.14), (4.15) and (4.18), choosing suitably small $\varepsilon>0$, using Grönwall inequality and Lemma 3.6, we complete the proof of Lemma 4.7. □

Like Proposition 3.1, we get the following proposition.

PROPOSITION 4.1. *Under the conditions of Theorem 4.1, it holds that*

$$\begin{gathered}\|\theta\|_{L^\infty(Q_T)}\le C,\\ \|u\|_{W^{1,\infty}(Q_T)}+\|w\|_{W^{1,\infty}(Q_T)}+\sup_{0\le t\le T}\int\left(u_{xx}^2+w_{xx}^2\right)dx+\int_{Q_T}\theta_{xx}dxdt\le C,\\ \|\rho\|_{W^{1,\infty}(Q_T)}+\|\rho_t\|_{L^\infty(Q_T)}+\sup_{0\le t\le T}\int\left(\rho_{xx}^2+\rho_{xt}^2\right)dx+\int_{Q_T}\left(\rho_{tt}^2+u_{xxx}^2\right)dxdt\le C.\end{gathered}$$

LEMMA 4.8. *Under the conditions of Theorem 4.1, it holds that*

$$\sup_{0\le t\le T}\int\rho\theta_t^2dx+\int_{Q_T}|[(1+\theta^q)\theta_x]_t|^2dxdt\le C.$$

Proof. Multiplying (3.24) by $\kappa\theta_t$, integrating over I, and using integration by parts, $(1.2)_1$, Proposition 4.1, Lemmas 2.1, 4.2, and the Hölder inequality, we have

$$\begin{aligned}&\frac{d}{dt}\int\rho\kappa\theta_t^2dx+\int|[(1+\theta^q)\theta_x]_t|^2dx\\&\le[(\kappa\theta_x)_t\kappa\theta_t]_0^1+C\int\theta_{xx}^2dx+C\left(\int\rho\theta_t^2dx\right)^2\\&\quad+C\left(\int u_{xt}^2dx+\int w_{xt}^2dx+\int w_t^2dx\right)+C.\end{aligned}\tag{4.19}$$

The first term in the right-hand side can be estimated as follows:

$$\begin{aligned}[(\kappa\theta_x)_t\kappa\theta_t]_0^1&=\kappa[(\kappa\theta_x)_t\theta_t](1,t)-\kappa[(\kappa\theta_x)_t\theta_t](0,t)\\&=-b\kappa[(\kappa(\theta-\delta))_t\theta_t](1,t)-a\kappa[(\kappa(\theta-\delta))_t\theta_t](0,t)\\&=-b(\kappa'\theta+\kappa)\theta_t^2(1,t)-a(\kappa'\theta+\kappa)\theta_t^2(0,t)+b\delta\kappa'\theta_t^2(1,t)+a\delta\kappa'\theta_t^2(0,t).\end{aligned}\tag{4.20}$$

Then, from (4.19) and (4.20), we have

$$\begin{aligned}&\frac{d}{dt}\int\rho\kappa\theta_t^2dx+\int|[(1+\theta^q)\theta_x]_t|^2dx+b(\kappa'\theta+\kappa)\theta_t^2(1,t)+a(\kappa'\theta+\kappa)\theta_t^2(0,t)\\&\le b\delta\kappa'\theta_t^2(1,t)+a\delta\kappa'\theta_t^2(0,t)+C\int\theta_{xx}^2dx+C\left(\int\rho\theta_t^2dx\right)^2\\&\quad+C\left(\int u_{xt}^2dx+\int w_{xt}^2dx+\int w_t^2dx\right)+C.\end{aligned}$$

Noticing that the third and the fourth term on the left-hand side of the above inequality is nonnegative, by the Young inequality and Lemma 4.7, we have

$$\begin{aligned}&\frac{d}{dt}\int\rho\kappa\theta_t^2dx+\int|[(1+\theta^q)\theta_x]_t|^2dx\\&\le C\int\theta_{xx}^2dx+C\left(\int\rho\theta_t^2dx\right)^2+C\left(\int u_{xt}^2dx+\int w_{xt}^2dx+\int w_t^2dx\right)+C.\end{aligned}\tag{4.21}$$

Integrating (4.21) over $[0,T]$, using (4.20) and Proposition 4.1, we obtain

$$\begin{aligned}\int\rho\theta_t^2dx+\int_0^T\int|[(1+\theta^q)\theta_x]_t|^2dxdt&\le\int\rho\kappa\theta_t^2dx\Big|_{t=0}+C\int_0^T\left(\int\rho\theta_t^2dx\right)^2dt+C\\&\le C\int_0^T\left(\int\rho\theta_t^2dx\right)^2dt+C.\end{aligned}\tag{4.22}$$

Using the Grönwall inequality and Proposition 4.1, we complete the proof. □

Similar to Proposition 3.2, we have the following proposition.

PROPOSITION 4.2. *Under the conditions of Theorem 4.1, it holds that*

$$\int_0^T\|\theta_t\|_{L^\infty}dt\le C,\ \int_{Q_T}\theta_{xt}^2dxdt\le C,$$
$$\|\theta\|_{W^{1,\infty}(Q_T)}+\sup_{0\le t\le T}\int\theta_{xx}^2dx+\int_{Q_T}\theta_{xxx}^2dxdt\le C,$$
$$\|(\sqrt{\rho})_x\|_{L^\infty(Q_T)}+\|(\sqrt{\rho})_t\|_{L^\infty(Q_T)}\le C.$$

LEMMA 4.9. *Under the conditions of Theorem 4.1, it holds that*

$$\sup_{0\le t\le T}\int\rho^2|(\kappa\theta_x)_t|^2dx+\int_{Q_T}\rho^3\theta_{tt}^2dxdt\le C.$$

Proof. Multiplying (3.24) by $\rho^2(\kappa\theta_t)_t$, using integration by parts, Propositions 4.1-4.2 and the Cauchy inequality, we have

$$\begin{aligned}&\frac{d}{dt}\int\rho^2|(\kappa\theta_x)_t|^2dx+\int\rho^3\theta_{tt}^2dx\\&\le\left[\rho^2(\kappa\theta_x)_t(\kappa\theta_t)_t\right]_0^1+C\int|(\kappa\theta_x)_t|^2dx+C\int\theta_{xt}^2dx\\&\quad+C\left(\int u_{xt}^2dx+\int w_{xt}^2dx+\int w_t^2dx\right)+C\|\theta_t\|_{L^\infty}+C.\end{aligned}\tag{4.23}$$

By (1.4), we have

$$\begin{aligned}\left[\rho^2(\kappa\theta_x)_t(\kappa\theta_t)_t\right]_0^1&=-b\rho^2\left[\kappa(\theta-\delta)\right]_t(\kappa\theta_t)_t(1,t)-a\rho^2\left[\kappa(\theta-\delta)\right]_t(\kappa\theta_t)_t(0,t)\\&\triangleq-bM(1,t)-aM(0,t),\end{aligned}$$

where the function $M(x,t)$ is defined as follows:

$$\begin{aligned}M(x,t)&=\rho^2\left[\kappa\kappa'+(\kappa')^2\theta\right]\theta_t^3+\rho^2(\kappa^2+\kappa\kappa'\theta)\theta_t\theta_{tt}-\delta\rho^2(\kappa')^2\theta_t^3-\delta\rho^2\kappa\kappa'\theta_t\theta_{tt}\\&\triangleq\frac{d}{dt}\left[H(\theta)\theta_t^2\right]+N(\theta)\theta_t^3\end{aligned}$$

with $H(\theta)=\frac{1}{2}\rho^2\left[\kappa^2+\kappa\kappa'(\theta-\delta)\right]$ and

$$N(\theta)=-\frac{1}{2}\rho^2\left[\kappa\kappa'\theta_t-(\kappa')^2(\theta-\delta)-\kappa\kappa''(\theta-\delta)\right]-2\rho\rho_t\left[\kappa^2+\kappa\kappa'(\theta-\delta)\right].$$

Then, from (4.23) and the definition of M, we have

$$\begin{aligned}&\frac{d}{dt}\int\rho^2|(\kappa\theta_x)_t|^2dx+\int\rho^3\theta_{tt}^2dx+\frac{d}{dt}\left\{b\left[H(\theta)\theta_t^2\right](1,t)+a\left[H(\theta)\theta_t^2\right](0,t)\right\}\\&\leq bN(\theta)\theta_t^3(1,t)+aN(\theta)\theta_t^3(0,t)+C\int|(\kappa\theta_x)_t|^2dx\\&\quad+C\int\theta_{xt}^2dx+C\left(\int u_{xt}^2dx+\int w_{xt}^2dx+\int w_t^2dx\right)+C\|\theta_t\|_{L^\infty}+C.\end{aligned}\tag{4.24}$$

Integrating (4.24) on $[0,t]$, by (1.6), (1.11), Lemmas 4.7, 4.9, Propositions 4.1-4.2 and the Young inequality, we have

$$\begin{aligned}&\int\rho^2|(\kappa\theta_x)_t|^2dx+\int_0^T\int\rho^3\theta_{tt}^2dxdt+\frac{1}{2}b\rho^2\left[\kappa^2+\kappa\kappa'\theta\right]\theta_t^2(1,t)+\frac{1}{2}a\rho^2\left[\kappa^2+\kappa\kappa'\theta\right]\theta_t^2(0,t)\\&\leq C\int_0^T\left[bN(\theta)\theta_t^3(1,t)+aN(\theta)\theta_t^3(0,t)\right]dt+\int\rho^2|(\kappa\theta_x)_t|^2dx\Big|_{t=0}\\&\quad+b\left[H(\theta)\theta_t^2\right](1,0)+a\left[H(\theta)\theta_t^2\right](0,0)+\frac{1}{2}b\delta\rho^2\kappa\kappa'\theta_t^2(1,t)+\frac{1}{2}a\delta\rho^2\kappa\kappa'\theta_t^2(0,t)+C\\&\leq C\int_0^T\|\theta_t\|_{L^\infty}^3dt+\int\rho^2|(\kappa\theta_x)_t|^2dx\Big|_{t=0}+\frac{1}{2}b\delta\rho^2\kappa\kappa'\theta_t^2(1,t)+\frac{1}{2}a\delta\rho^2\kappa\kappa'\theta_t^2(0,T)+C\\&\leq\int\rho^2|(\kappa\theta_x)_t|^2dx\Big|_{t=0}+\frac{1}{2}b\delta\rho^2\kappa\kappa'\theta_t^2(1,t)+\frac{1}{2}a\delta\rho^2\kappa\kappa'\theta_t^2(0,t)+C.\end{aligned}\tag{4.25}$$

Using the Young inequality, by (1.6), the Proposition 4.2, we deduce from (4.25) that

$$\int\rho^2|(\kappa\theta_x)_t|^2dx+\int_0^T\int\rho^3\theta_{tt}^2dxdt\leq\int\rho^2|(\kappa\theta_x)_t|^2dx\Big|_{t=0}+C\leq C.$$

Then we complete the proof. □

Similar to Proposition 3.3, we get the following proposition.

PROPOSITION 4.3. *Under the conditions of Theorem 4.1, it holds that*

$$\sup_{0\leq t\leq T}\int\left(\theta_{xxx}^2+\rho^2\theta_{xt}^2\right)dx\leq C,$$

$$\sup_{0\leq t\leq T}\int\rho^2u_{xt}^2dx+\int_{Q_T}\rho^3u_{tt}^2dxdt\leq C,$$

$$\sup_{0\leq t\leq T}\int u_{xxx}^2dx\leq C.$$

From all the above estimates, we get

$$\begin{aligned}&\sup_{0\leq t\leq T}(\|(\sqrt{\rho})_x\|_{L^\infty}+\|(\sqrt{\rho})_t\|_{L^\infty}+\|\rho\|_{H^2}+\|\rho_t\|_{H^1}+\|u\|_{H^3}+\|\rho u_t\|_{H^1}+\|\sqrt{\rho}u_t\|_{L^2}\\&+\|\theta\|_{H^3}+\|\sqrt{\rho}\theta_t\|_{L^2}+\|\rho\theta_t\|_{H^1})+\int_{Q_T}\left(u_{xt}^2+\rho_{tt}^2+\theta_t^2+\theta_{xt}^2+\rho^3u_{tt}^2+\rho^3\theta_{tt}^2\right)dxdt\leq C.\end{aligned}\tag{4.26}$$

COROLLARY 4.1. *Under the conditions of Theorem 4.1, there exists a positive constant C_δ depending on δ such that for any $(x,t)\in Q_T$, it holds that,*

$$\begin{cases}\rho\geq\frac{\delta}{C}>0,\\ \theta\geq C_\delta>0.\end{cases}$$

Then from (4.26), the above lemmas and corollaries, we have

$$\begin{aligned}&\|\rho\|_{H^2}+\|\rho_t\|_{H^1}+\|u\|_{H^3}+\|u_t\|_{H^1}+\|w\|_{H^3}+\|w_t\|_{H^1}+\|\theta\|_{H^3}+\|\rho\theta_t\|_{H^1}\\&\quad+\int_{Q_T}\left(u_{xt}^2+u_{xxt}^2+w_{xt}^2+w_{xxt}^2+\rho_{tt}^2+\theta_t^2+\theta_{xt}^2+\theta_{xxt}^2+u_{tt}^2+w_{tt}^2+\theta_{tt}^2\right)dxdt\\&\leq C(\delta).\end{aligned}\tag{4.27}$$

With (4.27), we complete the proof of Theorem 4.1.

Proof. **(Proof of Theorem 1.2.)** The proof is similar to that of Theorem 1.1. ∎

5. Proof of Theorem 1.3

Assume that $(\rho^{a,b},u^{a,b},w^{a,b},\theta^{a,b})$ is the solution as in Theorem 1.2, and that (ρ,u,w,θ) is the solution as in Theorem 1.1 and $\inf\widetilde{\rho}_0>0$.

Denote $\bar\rho=\rho-\rho^{a,b}$, $\bar u=u-u^{a,b}$, $\bar w=w-w^{a,b}$, $\bar\theta=\theta-\theta^{a,b}$. Then we can obtain that $(\bar\rho,\bar u,\bar w,\bar\theta)$ satisfies the following system:

$$\begin{cases}\bar\rho_t+(\bar\rho u)_x+(\rho^{a,b}\bar u)_x=0,\\ \rho\bar u_t+\bar\rho u_t^{a,b}+\rho u\bar u_x+\rho\bar u u_x^{a,b}+\bar\rho u^{a,b}u_x^{a,b}+(\rho\bar\theta+\bar\rho\theta^{a,b})_x=\mu_1\bar u_{xx},\\ \rho\bar w_t+\bar\rho w_t^{a,b}+\rho u\bar w_x+\rho\bar u w_x^{a,b}+\bar\rho u^{a,b}w_x^{a,b}+2\xi\bar w=\mu_2\bar w_{xx},\\ \rho\bar\theta_t+\bar\rho\theta_t^{a,b}+\rho u\bar\theta_x+\rho\bar u\theta_x^{a,b}+\bar\rho u^{a,b}\theta_x^{a,b}+\rho\bar\theta u_x+\rho\theta^{a,b}\bar u_x+\bar\rho\theta^{a,b}u_x^{a,b}\\ =\mu_1\bar u_x(u_x+u_x^{a,b})+\mu_2\bar w_x(w_x+w_x^{a,b})+2\xi\bar w(w+w^{a,b})+(\kappa\bar\theta_x)_x+\left(\left(\kappa-\kappa^{a,b}\right)\theta_x^{a,b}\right)_x.\end{cases}\tag{5.1}$$

Multiplying $(5.1)_1$ by $\bar\rho$, integrating over I, by (3.30) and (4.27), we have

$$\begin{aligned}\frac12\frac{d}{dt}\int|\bar\rho|^2dx&=-\frac12\int\bar\rho^2u_xdx-\int\rho_x^{a,b}\bar u\bar\rho dx-\int\rho^{a,b}\bar u_x\bar\rho dx\\&\leq C\|u_x\|_{L^\infty}\|\bar\rho\|_{L^2}^2+\|\rho_x^{a,b}\|_{L^2}\|\bar u\|_{L^\infty}\|\bar\rho\|_{L^2}+\|\rho^{a,b}\|_{L^\infty}\|\bar u_x\|_{L^2}\|\bar\rho\|_{L^2}\\&\leq\varepsilon_1\|\bar u_x\|_{L^2}^2+C(\varepsilon_1)\|\bar\rho\|_{L^2}^2,\end{aligned}\tag{5.2}$$

where we use the inequality $\|\bar u\|_{L^\infty}\leq C\|\bar u_x\|_{L^1}\leq C\|\bar u_x\|_{L^2}$, which can be obtained by (2.3) and the Hölder inequality.

Multiplying $(5.1)_2$ by $\bar u$, integrating over I, by (3.30) and (4.27), we have

$$\begin{aligned}&\frac12\frac{d}{dt}\int\rho\bar u^2dx+\mu_1\int\bar u_x^2dx\\&=-\int\bar\rho\bar u u_t^{a,b}dx-\int\rho\bar u^2u_x^{a,b}dx-\int\bar\rho\bar u u^{a,b}u_x^{a,b}dx+\int\rho\bar\theta\bar u_xdx+\int\bar\rho\theta^{a,b}\bar u_xdx\\&\leq\|\bar\rho\|_{L^2}\|\bar u\|_{L^\infty}\|u_t^{a,b}\|_{L^2}+\|\sqrt{\rho}\bar u\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|\bar u\|_{L^\infty}\|u_x^{a,b}\|_{L^2}\\&\qquad+\|\bar\rho\|_{L^2}\|\bar u\|_{L^\infty}\|u^{a,b}\|_{L^\infty}\|u_x^{a,b}\|_{L^2}+\|\sqrt{\rho}\bar\theta\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|\bar u_x\|_{L^2}+\|\bar\rho\|_{L^2}\|\theta^{a,b}\|_{L^\infty}\|\bar u_x\|_{L^2}\\&\leq\varepsilon_2\|\bar u_x\|_{L^2}^2+C(\varepsilon_2)\left(\|\bar\rho\|_{L^2}^2+\|\sqrt{\rho}\bar u\|_{L^2}^2+\|\sqrt{\rho}\bar\theta\|_{L^2}^2\right).\end{aligned}\tag{5.3}$$

Multiplying $(5.1)_3$ by $\bar{w}$, integrating over I, by (3.30) and (4.27), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho\bar{w}^2dx+\mu_2\int\bar{w}_x^2dx+2\xi\int\bar{w}^2dx\\
&=-\int\bar{\rho}\bar{w}w_t^{a,b}dx-\int\rho\bar{u}\bar{w}w_x^{a,b}dx-\int\bar{\rho}\bar{w}u^{a,b}w_x^{a,b}dx\\
&\le\|\bar{\rho}\|_{L^2}\|\bar{w}\|_{L^\infty}\|w_t^{a,b}\|_{L^2}+\|\sqrt{\rho}\bar{u}\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|\bar{w}\|_{L^\infty}\|w_x^{a,b}\|_{L^2}\\
&\qquad+\|\bar{\rho}\|_{L^2}\|\bar{w}\|_{L^\infty}\|u^{a,b}\|_{L^\infty}\|w_x^{a,b}\|_{L^2}\\
&\le\varepsilon_2\|\bar{w}_x\|_{L^2}^2+C(\varepsilon_2)\left(\|\bar{\rho}\|_{L^2}^2+\|\sqrt{\rho}\bar{u}\|_{L^2}^2\right).
\end{aligned}\tag{5.4}$$

Multiplying $(5.1)_4$ by $\bar{\theta}$, integrating over I, by (3.30) and (4.27), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho\bar{\theta}^2dx+\int\kappa\bar{\theta}_x^2dx\\
&=\kappa\bar{\theta}_x\bar{\theta}|_0^1-\int\bar{\rho}\bar{\theta}\theta_t^{a,b}dx-\int\rho\bar{u}\bar{\theta}\theta_x^{a,b}dx+\int\bar{\rho}u^{a,b}\bar{\theta}\theta_x^{a,b}dx-\int\rho\bar{\theta}^2u_xdx\\
&\quad-\int\rho\theta^{a,b}\bar{\theta}\bar{u}_xdx-\int\bar{\rho}\theta^{a,b}\bar{\theta}u_x^{a,b}dx+\mu_1\int\bar{u}_x(u_x+u_x^{a,b})\bar{\theta}dx\\
&\quad+\mu_2\int\bar{w}_x(w_x+w_x^{a,b})\bar{\theta}dx+2\xi\int\bar{w}(w+w^{a,b})\bar{\theta}dx+\int\left(\left(\kappa-\kappa^{a,b}\right)\theta_x^{a,b}\right)_x\bar{\theta}dx\\
&\le\kappa\bar{\theta}\theta_x^{a,b}|_0^1+\|\bar{\rho}\|_{L^2}\|\bar{\theta}\|_{L^\infty}\|\theta_t^{a,b}\|_{L^2}+\|\sqrt{\rho}\bar{u}\|_{L^2}\|\sqrt{\rho}\bar{\theta}\|_{L^2}\|\theta_x^{a,b}\|_{L^\infty}+\|\sqrt{\rho}\bar{\theta}\|_{L^2}^2\|u_x\|_{L^\infty}\\
&\quad+\|\bar{\rho}\|_{L^2}\|\bar{\theta}\|_{L^\infty}\|u^{a,b}\|_{L^2}\|\theta_x^{a,b}\|_{L^\infty}+\|\sqrt{\rho}\bar{\theta}\|_{L^2}\|\bar{u}_x\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|\theta^{a,b}\|_{L^\infty}\\
&\quad+\|\bar{\rho}\|_{L^2}\|\bar{\theta}\|_{L^\infty}\|u_x^{a,b}\|_{L^2}\|\theta^{a,b}\|_{L^\infty}+C\|\bar{\theta}\|_{L^\infty}\|\bar{u}_x\|_{L^2}\left(\|u_x\|_{L^2}+\|u_x^{a,b}\|_{L^2}\right)\\
&\quad+C\|\bar{\theta}\|_{L^\infty}\|\bar{w}_x\|_{L^2}\left(\|w_x\|_{L^2}+\|w_x^{a,b}\|_{L^2}\right)+C\|\bar{\theta}\|_{L^\infty}\|\bar{w}\|_{L^2}\left(\|w\|_{L^2}+\|w^{a,b}\|_{L^2}\right)\\
&\quad+C\max\{|\kappa'|,|(\kappa^{a,b})'|\}\|\bar{\theta}_x\|_{L^2}\|\theta_x^{a,b}\|_{L^\infty}\|\bar{\theta}\|_{L^2}+C\left(\|\kappa\|_{L^\infty}+\|\kappa^{a,b}\|_{L^\infty}\right)\|\theta_{xx}^{a,b}\|_{L^2}\|\bar{\theta}\|_{L^2}\\
&\le\varepsilon_3\|\bar{\theta}_x\|_{L^2}^2+C_{\varepsilon_3}\left(\|\bar{u}_x\|_{L^2}^2+\|\bar{w}_x\|_{L^2}^2+\|\bar{w}\|_{L^2}^2\right)+C\|\bar{\theta}\|_{L^2}^2\\
&\quad+C_{\varepsilon_3}\left(\|\theta_t^{a,b}\|_{L^2}^2+\|\theta_{xt}^{a,b}\|_{L^2}+1\right)\left(\|\bar{\rho}\|_{L^2}^2+\|\sqrt{\rho}\bar{u}\|_{L^2}^2+\|\sqrt{\rho}\bar{\theta}\|_{L^2}^2\right).
\end{aligned}\tag{5.5}$$

Then, choosing suitably small $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$, we obtain from (5.2)-(5.5) that

$$\begin{aligned}
&\frac{d}{dt}\left(\|\bar{\rho}\|_{L^2}^2+\|\sqrt{\rho}\bar{u}\|_{L^2}^2+\|\sqrt{\rho}\bar{\theta}\|_{L^2}^2\right)+\int\left(\bar{u}_x^2+\bar{w}_x^2+\bar{w}^2+\kappa\bar{\theta}_x^2\right)dx\\
&\le C\left(\|\theta_t^{a,b}\|_{L^2}^2+\|\theta_{xt}^{a,b}\|_{L^2}+1\right)\left(\|\bar{\rho}\|_{L^2}^2+\|\sqrt{\rho}\bar{u}\|_{L^2}^2+\|\sqrt{\rho}\bar{\theta}\|_{L^2}^2\right)+C\|\sqrt{\rho}\bar{\theta}\|_{L^2}^2,
\end{aligned}\tag{5.6}$$

where we have used that ρ is bounded away from zero since its initial data $\widetilde{\rho}_0$ is assumed to be positive.

Using Grönwall inequality over (5.6), we complete the proof of Theorem 1.3.

6. Proof of Theorem 1.4

In terms of the local existence (Lemma 6.1), we can complete Theorem 1.4 by combining the global *a priori* estimates with continuity arguments (cf. [6]). Therefore, in this section we only need to achieve the global *a priori* estimates, and for this purpose we assume that (ρ,u,w,θ) is the classical solution over $[0,T]$ for any $T\in(0,\infty)$. Additionally, let C be a generic constant in this section and the next two sections which depends only on $T,\mu_1,\mu_2,\xi,\kappa,\gamma$, the initial data, and $\|g_i\|_{L^2},\|(\sqrt{\rho_0}g_i)_x\|_{L^2}$ $(i=1,2,3)$, but does not depend on a,b; we also use $C(\alpha)$ to emphasize that C relies upon α.

The first lemma is for the local well posedness of classical solutions.

Lemma 6.1 (See [5–7]). *Under the same assumptions given in Theorem 1.4, the problem (1.2)-(1.4) has a unique classical solution* (ρ,u,w,θ) *over* $[0,T_*]$ *for some (small) time* $T_*>0$ *which satisfies the regularities (1.10).*

For any given $T\in(0,+\infty)$, let (ρ,u,w,θ) be the solution to (1.2)-(1.4) as in Theorem 1.4. Then we have the following lemmas.

Lemma 6.2. *Under the conditions of Theorem 1.4, it holds that for any* $0\le t\le T$,

$$\int\rho dx=\int\rho_0dx, \tag{6.1}$$

$$\int\rho\left(\theta+\frac{1}{2}u^2+\frac{1}{2}w^2\right)dx\le\int\rho_0\left(\theta_0+\frac{1}{2}u_0^2+\frac{1}{2}w_0^2\right)dx\triangleq\mathcal{E}_0. \tag{6.2}$$

Proof. The proof of this lemma is similar to that of Lemma 3.1. □

Lemma 6.3. *Under the conditions of Theorem 1.4, it holds that for any* $(x,t)\in Q_T$,

$$0\le\rho\le C\quad\text{and}\quad\theta>0. \tag{6.3}$$

Proof. The proof of this lemma is similar to that of Lemma 3.2. □

Lemma 6.4. *Given* $\alpha>(\gamma-1)\mathcal{E}_0$, *there exists a sequence of (non-overlapping) intervals* Ω_j *in* I *such that for every* $t\in[0,T]$,

$$P(x,t)\le\alpha,\ x\in I\setminus\bigcup_j\Omega_j\quad\text{and}\quad\alpha\le\frac{1}{|\Omega_j|}\int_{\Omega_j}P(x,t)dx\le2\alpha. \tag{6.4}$$

Moreover,

$$\left|\bigcup_j\Omega_j\right|\le\frac{(\gamma-1)\mathcal{E}_0}{\alpha}. \tag{6.5}$$

Proof. The proof comes from $P=\rho\theta$ and Lemmas 2.3, 6.2. □

Lemma 6.5. *Under the conditions of Theorem 1.4, it holds that,*

$$\sup_{0\le t\le T}\int\rho\left(u^4+w^4+\theta^2\right)dx+\int_0^T\int\left(\mu_1u^2u_x^2+\mu_2w^2w_x^2+\kappa\theta_x^2+\xi w^4\right)dxdt\le C. \tag{6.6}$$

Proof. Rewrite $(1.1)_4$ as follows:

$$(\rho E)_t+(\rho uE)_x+(Pu)_x=\kappa E_{xx}-(\mu_1-\kappa)(uu_x)_x-(\mu_2-\kappa)(ww_x)_x. \tag{6.7}$$

Multiplying (6.7) by E, integrating by parts over $I\times(0,t)$, by (1.4), (6.1)-(6.3), (2.1) and the Young inequality, it yields that

$$\int\rho E^2dx+\kappa\int_0^T\int|E_x|^2dxdt\le C+C\int_0^T\int\left(\rho^2\theta^2u^2+u^2u_x^2+w^2w_x^2\right)dxdt.$$

On the other hand, we multiply $(1.2)_2$ and $(1.2)_3$ by u^3 and w^3, respectively, integrate by parts over $I\times(0,t)$, to deduce that

$$\int\rho u^4dx+\mu_1\int_0^T\int u^2u_x^2dxdt\le C+C\int_0^T\int\rho^2\theta^2u^2dxdt,$$

$$\int \rho w^4 dx + \mu_2 \int_0^T \int w^2 w_x^2 dxdt + \xi \int_0^T \int w^4 dxdt \leq C.$$

From the last three inequalities, we have

$$\begin{aligned}
&\int \rho \left(u^4 + w^4 + \theta^2\right) dx + \int_0^T \int \left(\mu_1 u^2 u_x^2 + \mu_2 w^2 w_x^2 + \kappa \theta_x^2 + \xi w^4\right) dxdt \\
\leq & C + C \int_0^T \int \rho^2 \theta^2 u^2 dxdt \\
\leq & C + C \int_0^T \left(\int_{I \setminus \bigcup_j \Omega_j} + \int_{\bigcup_j \Omega_j} \right) \rho^2 \theta^2 u^2 dxdt \triangleq C + A_1 + A_2,
\end{aligned} \tag{6.8}$$

where the intervals Ω_j are as in Lemma 6.4.

From (2.1), (6.1) and (6.2) that

$$\max_{x \in I} \theta \leq C \int |\theta_x| dx + C \int \rho \theta dx \leq C + C \int |\theta_x| dx. \tag{6.9}$$

Together with (6.1), (6.2), (6.4), (6.9) and the Cauchy inequality, deduces

$$\begin{aligned}
A_1 &\leq C\alpha \int_0^T \int_{I \setminus \bigcup_j \Omega_j} \rho \theta u^2 dxdt \leq C \alpha \mathcal{E}_0 \int_0^T \|\theta\|_{L^\infty} dt \\
&\leq \frac{C}{\alpha} \int_0^T \int |\theta_x|^2 dxdt + C(\alpha).
\end{aligned} \tag{6.10}$$

Next, we estimate A_2.

If $\rho \leq \varepsilon \leq 1$, $\forall\, x \in \Omega_j$. Then, we have

$$\int_{\Omega_j} \rho^2 \theta^2 u^2 dx \leq \varepsilon \|\theta\|_{L^\infty}^2 \int_{\Omega_j} \rho u^2 dx \leq C \varepsilon \left(1 + \int |\theta_x|^2 dx\right) \int_{\Omega_j} \rho u^2 dx. \tag{6.11}$$

If there exists at least a $x_j^* \in \Omega_j$ such that $\rho(x_j^*, t) > \varepsilon$, since ρ is uniformly continuous in $I \times [0, T]$, one has

$$|\rho(x,t) - \rho(x_j^*, t)| \leq \frac{\varepsilon}{2},$$

for all $x \in U(x_j^*, \delta)$ with δ independent of x_j^* or t. On the other hand, by Lemma 6.4 we may choose α so large as to

$$|\Omega_j| \leq \left|\bigcup_j \Omega_j\right| \leq \frac{(\gamma - 1)\mathcal{E}_0}{\alpha} \leq \varepsilon.$$

Then, for all $x \in \Omega_j$, $\rho(x,t) \geq \rho(x_j^*, t) - \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2}$, from which we obtain

$$\begin{aligned}
\|\theta\|_{L^\infty(\Omega_j)} &\leq C \|\theta_x\|_{L^1(\Omega_j)} + C \|\theta\|_{L^1(\Omega_j)} \leq C \|\theta_x\|_{L^1(\Omega_j)} + \frac{C}{\varepsilon} \int_{\Omega_j} P dx \\
&\leq C \varepsilon^{1/2} \|\theta_x\|_{L^2(\Omega_j)} + C(\alpha, \varepsilon).
\end{aligned}$$

So, we have

$$\int_{\Omega_j} \rho^2 \theta^2 u^2 dx \leq C \|\theta\|_{L^\infty(\Omega_j)}^2 \int_{\Omega_j} \rho u^2 dx \leq \left(C \varepsilon \|\theta_x\|_{L^2(\Omega_j)}^2 + C(\alpha, \varepsilon)\right) \int_{\Omega_j} \rho u^2 dx. \tag{6.12}$$

Combining (6.11) and (6.12), we have

$$\int_{\Omega_j}\rho^2\theta^2u^2dx\le\left(C\varepsilon\int|\theta_x|^2dx+C(\alpha,\varepsilon)\right)\int_{\Omega_j}\rho u^2dx. \tag{6.13}$$

Since $\Omega_i\cap\Omega_j=\emptyset$ for $i\neq j$, it satisfies from (6.13) that

$$\begin{aligned}A_2&\le\int_0^T\left(C\varepsilon\int|\theta_x|^2dx+C(\alpha,\varepsilon)\right)\sum_j\int_{\Omega_j}\rho u^2dxdt\\&=\int_0^T\left(C\varepsilon\int|\theta_x|^2dx+C(\alpha,\varepsilon)\right)\int_{\bigcup\limits_j\Omega_j}\rho u^2dxdt\\&\le C\mathcal{E}_0\varepsilon\int_0^T\int|\theta_x|^2dxdt+C(\alpha,\varepsilon,\mathcal{E}_0,T).\end{aligned} \tag{6.14}$$

By (6.11) and (6.14), choosing $\varepsilon>0$ small first and then α large, we have

$$\int_0^T\int\rho^2\theta^2u^2dxdt\le C+\frac{\kappa}{2}\int_0^T\int|\theta_x|^2dxdt. \tag{6.15}$$

Substituting (6.15) into (6.8), choosing $\varepsilon>0$ small enough, we get the desired (6.6). □

Corollary 6.1. *Under the conditions of Theorem 1.4, it holds that*

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}w\|_{L^2}^2\right)+\int_0^T\left(\|\theta\|_{L^\infty}+\|u_x\|_{L^2}^2+\|w\|_{H^1}^2\right)dt\le C. \tag{6.16}$$

Proof. It follows from (2.1), Lemma 6.2 and (6.6) that

$$\int_0^T\|\theta\|_{L^\infty}dt\le C\int_0^T\int|\theta_x|dxdt+C\int_0^T\int\rho\theta dxdt\le C\int_0^T\int|\theta_x|^2dxdt+C\le C.$$

Multiplying $(1.2)_2$ and $(1.2)_3$ by u and w, respectively, integrating by parts, we have

$$\sup_{0\le t\le T}\|\sqrt{\rho}u\|_{L^2}^2+\mu_1\int_0^T\|u_x\|_{L^2}^2\le C\int_0^T\|\rho\theta\|_{L^2}^2dt\le C\int_0^T\|\theta\|_{L^\infty}^2dt+C\le C$$

and

$$\sup_{0\le t\le T}\|\sqrt{\rho}w\|_{L^2}^2+\int_0^T\left(\mu_2\|w_x\|_{L^2}^2+2\xi\|w\|_{L^2}^2\right)dt\le C.$$

Then we complete the proof of Corollary 6.1. □

Lemma 6.6. *Under the conditions of Theorem 1.4, it holds that*

$$\sup_{0\le t\le T}\left(\|u_x\|_{L^2}^2+\|w\|_{H^1}^2\right)+\int_0^T\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\sqrt{\rho}\dot{w}\|_{L^2}^2+\|u_x\|_{L^\infty}^2\right)dt\le C. \tag{6.17}$$

Proof. Multiplying $(1.2)_2$ by $\dot{u}$, where $\dot{f}=\partial_tf+u\partial_xf$, integrating over $I\times(0,t)$, we have

$$\int\left(\frac{\mu_1}{2}|u_x|^2-Pu_x\right)(x,t)dx+\int_0^T\int\rho|\dot{u}|^2dxdt$$

$$
\begin{aligned}
&= \int \left(\frac{\mu_1}{2}|u_x|^2 - Pu_x\right)(x,0)dx - \int_0^T \int (P_t + (Pu)_x)(\mu_1 u_x - P)dxdt \\
&\quad + \int_0^T \int (P_t + (Pu)_x)Pdxdt + \int_0^T \int \left(P|u_x|^2 - \frac{\mu_1}{2}(u_x)^3\right)dxdt. \qquad (6.18)
\end{aligned}
$$

Using (6.3), (6.6) and the Cauchy inequality, we get

$$
\int \left(\frac{\mu_1}{2}|u_x|^2 - Pu_x\right)dx \geq \frac{\mu_1}{4}\int |u_x|^2 dx - C\int \rho^2\theta^2 dx \geq \frac{\mu_1}{4}\int |u_x|^2 dx - C. \qquad (6.19)
$$

By $(1.2)_2$, Lemma 6.2 and the Sobolev inequality $W^{1,1} \hookrightarrow L^\infty$, it yields that

$$
\|\mu_1 u_x - P\|_{L^\infty} \leq C(\|\mu_1 u_x - P\|_{L^1} + \|\mu_1 u_{xx} - P_x\|_{L^1}) \leq C + C\int (|u_x| + \rho|\dot{u}|)dx. \qquad (6.20)
$$

By (6.20), we have

$$
\begin{aligned}
&\int_0^T \int \left(P|u_x|^2 - \frac{\mu_1}{2}(u_x)^3\right)dxdt \\
&= \frac{1}{2}\int_0^T \int P|u_x|^2 dxdt + \frac{1}{2}\int_0^T \int (P - \mu_1 u_x)|u_x|^2 dxdt \\
&\leq C\int_0^T \left(\|\theta\|_{L^\infty}\|u_x\|_{L^2}^2\right)dt + C\int_0^T \left(\|\mu_1 u_x - P\|_{L^\infty}\|u_x\|_{L^2}^2\right)dt \\
&\leq \frac{1}{6}\int_0^T \int \rho|\dot{u}|^2 dxdt + C\int_0^T \left(1 + \|\theta\|_{L^\infty} + \|u_x\|_{L^2}^2\right)\|u_x\|_{L^2}^2 dt. \qquad (6.21)
\end{aligned}
$$

Next, it follows from (1.4), (1.2), (6.3) and (6.6) and the Cauchy inequality, choosing suitably small $\varepsilon > 0$, we get

$$
\begin{aligned}
&\int_0^T \int (P_t + (Pu)_x)(\mu_1 u_x - P)dxdt \\
&= \int_0^T \int \left(\kappa\theta_{xx} + \mu_1 u_x^2 + \mu_2 w_x^2 + 2\xi w^2 - Pu_x\right)(\mu_1 u_x - P)dxdt \\
&= \kappa\int_0^T \int \theta_{xx}(\mu_1 u_x - P)dxdt + \int_0^T \int \left(\mu_1 u_x^2 + \mu_2 w_x^2 + 2\xi w^2 - Pu_x\right)(\mu_1 u_x - P)dxdt \\
&\leq C(\varepsilon)\int_0^T \int |\theta_x|^2 dxdt + \varepsilon\int_0^T \int |(\mu_1 u_x - P)_x|^2 dxdt \\
&\quad + C\int_0^T \|\mu_1 u_x - P\|_{L^\infty}\int \left(P^2 + u_x^2 + w_x^2 + w^2\right)dxdt \\
&\leq \frac{1}{6}\int_0^T \int \rho|\dot{u}|^2 dxdt + C\int_0^T \|\mu_1 u_x - P\|_{L^\infty}(1 + \|u_x\|_{L^2}^2 + \|w\|_{H^1}^2)dt + C. \qquad (6.22)
\end{aligned}
$$

From (6.3) and (6.6), it yields that

$$
\begin{aligned}
\int_0^T \int (P_t + (Pu)_x)Pdxdt &= \frac{1}{2}\int_0^T \frac{d}{dt}\int P^2 dxdt + \frac{1}{2}\int_0^T \int P^2 u_x dxdt \\
&\leq \frac{1}{2}\int P^2 dx + C\int_0^T \|\theta\|_{L^\infty}^2\left(\|u_x\|_{L^2}^2 + 1\right)dt + C
\end{aligned}
$$

$$\leq C\int_0^T \|\theta\|_{L^\infty}^2 \left(\|u_x\|_{L^2}^2+1\right)dt+C. \tag{6.23}$$

Taking (6.19) and (6.21)-(6.23) into (6.18), we deduce

$$\begin{aligned}&\sup_{0\leq t\leq T}\left(\mu_1\|u_x\|_{L^2}^2\right)+\int_0^T \|\sqrt{\rho}\dot{u}\|_{L^2}^2dt\\ \leq& C\int_0^T \left(\|u_x\|_{L^2}^2+\|\theta\|_{L^\infty}^2+1\right)\left(\|u_x\|_{L^2}^2+\|w\|_{H^1}^2+1\right)dt+C.\end{aligned} \tag{6.24}$$

Using the Grönwall inequality and (6.16), from (6.24) we can deduce that

$$\sup_{0\leq t\leq T}\left(\mu_1\|u_x\|_{L^2}^2\right)+\int_0^T \|\sqrt{\rho}\dot{u}\|_{L^2}^2dt\leq C. \tag{6.25}$$

Consequently, it yields from (6.16), (6.21) and (6.25) that

$$\int_0^T \|u_x\|_{L^\infty}^2dt\leq C\int_0^T \left(\|\mu_1 u_x-P\|_{L^\infty}^2+\|\theta\|_{L^\infty}^2\right)dt\leq C. \tag{6.26}$$

Multiplying $(1.2)_3$ by $\dot{w}$, integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\mu_2\|w_x\|_{L^2}^2+2\xi\|w\|_{L^2}^2\right)+\|\sqrt{\rho}\dot{w}\|_{L^2}^2\\ =&-\mu_2\int u_x w_x^2dx-2\xi\int wuw_xdx\\ \leq& C\left(\|u_x\|_{L^\infty}^2+\|w_x\|_{L^2}^4+\|u\|_{L^2}^2+\|ww_x\|_{L^2}^2\right)\\ \leq& C\left(\|u_x\|_{L^\infty}^2+\|w_x\|_{L^2}^4+\|u_x\|_{L^2}^2+\|ww_x\|_{L^2}^2\right).\end{aligned} \tag{6.27}$$

By (6.6), (6.16), (6.26) and the Grönwall inequality, we have from (6.27)

$$\sup_{0\leq t\leq T}\left(\mu_2\|w_x\|_{L^2}^2+2\xi\|w\|_{L^2}^2\right)+\int_0^T \|\sqrt{\rho}\dot{w}\|_{L^2}^2dt\leq C. \tag{6.28}$$

Combining (6.25), (6.26) and (6.28), we complete the proof of Lemma 6.6. □

LEMMA 6.7. *Under the conditions of Theorem 1.4, it holds that*

$$\begin{aligned}&\sup_{0\leq t\leq T}\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|u\|_{W^{1,\infty}}+\|\sqrt{\rho}\dot{w}\|_{L^2}^2+\|w\|_{W^{1,\infty}}+\|\theta_x\|_{L^2}^2+\|\theta\|_{L^\infty}\right)\\ &+\int_0^T \left(\|\dot{u}_x\|_{L^2}^2+\|\dot{u}\|_{L^\infty}^2+\|\dot{w}_x\|_{L^2}^2+\|\dot{w}\|_{L^\infty}^2+\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2\right)dt\leq C.\end{aligned} \tag{6.29}$$

Proof. Operating $\partial_t+\partial_x(u\cdot)$ to $(1.2)_2$, utilizing $P_t+(Pu)_x=\rho\dot{\theta}$, we calculate

$$\begin{aligned}&(\rho\dot{u})_t+(\rho u\dot{u})_x-\mu_1\dot{u}_{xx}\\ =&-\mu_1(|u_x|^2)_x+(Pu_x)_x-(P_t+(Pu)_x)_x\\ =&-\mu_1(|u_x|^2)_x+(Pu_x)_x-(\rho\dot{\theta})_x.\end{aligned} \tag{6.30}$$

Multiplying (6.30) by $\dot{u}$, using (1.9), (6.6), (6.16), (6.17) and integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\mu_1\int_0^T\|\dot{u}_x\|_{L^2}^2dt\\
&=\frac{1}{2}\|g_1\|_{L^2}^2+\int_0^T\int\left(\mu_1|u_x|^2+\rho\dot{\theta}-Pu_x\right)\dot{u}_xdxdt\\
&\le\frac{\mu_1}{2}\int_0^T\|\dot{u}_x\|_{L^2}^2dt+C\int_0^T\int\left(|u_x|^4+\rho|\dot{\theta}|^2+\rho^4\theta^4\right)dxdt+C\\
&\le\frac{\mu_1}{2}\int_0^T\|\dot{u}_x\|_{L^2}^2dt+C\int_0^T\int\rho|\dot{\theta}|^2dxdt+C,
\end{aligned}$$

which implies

$$\sup_{0\le t\le T}\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\mu_1\int_0^T\|\dot{u}_x\|_{L^2}^2dt\le C\int_0^T\int\rho|\dot{\theta}|^2dxdt+C. \tag{6.31}$$

Operating $\partial_t+\partial_x(u\cdot)$ to $(1.2)_3$, we calculate

$$(\rho\dot{w})_t+(\rho u\dot{w})_x-\mu_2\dot{w}_{xx}+2\xi\dot{w}=-\mu_2(u_xw_x)_x-2\xi u_xw. \tag{6.32}$$

Multiplying (6.32) by $\dot{w}$, using (1.9), (6.6), (6.16), (6.17) and the Poincaré inequality, integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\|\sqrt{\rho}\dot{w}\|_{L^2}^2+\int_0^T\left(\mu_2\|\dot{w}_x\|_{L^2}^2+2\xi\|\dot{w}\|_{L^2}^2\right)dt\\
&=\frac{1}{2}\|g_2\|_{L^2}^2+\int_0^T\int\left(\mu_2u_xw_x\dot{w}_x-2\xi u_xw\dot{w}\right)dxdt\\
&\le\frac{\mu_2}{2}\int_0^T\|\dot{w}_x\|_{L^2}^2dt+C\int_0^T\int\left(u_x^4+w_x^4+w^4\right)dxdt+C\\
&\le\frac{\mu_2}{2}\int_0^T\|\dot{w}_x\|_{L^2}^2dt+C,
\end{aligned}$$

which implies

$$\sup_{0\le t\le T}\|\sqrt{\rho}\dot{w}\|_{L^2}^2+\int_0^T\left(\mu_2\|\dot{w}_x\|_{L^2}^2+2\xi\|\dot{w}\|_{L^2}^2\right)dt\le C. \tag{6.33}$$

Multiplying $(1.2)_4$ by $\dot{\theta}$, and integrating the resulting equation over $I\times(0,t)$, it gives

$$\begin{aligned}
&\frac{\kappa}{2}\|\theta_x\|_{L^2}^2+\int_0^T\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2dt\\
&\le\frac{1}{2}\|g_3\|_{L^2}^2+C\int\theta\left(|u_x|^2+|w_x|^2+w^2\right)dx+C\int_0^T\int\Big(|u_x||\theta_x|^2+\rho\theta|u_x||\dot{\theta}|\\
&\quad+\theta|u_x|^3+\theta|u_x||\dot{u}_x|+\theta|u_x||w_x|^2+\theta|w_x||\dot{w}_x|+\theta|w||\dot{w}|+\theta|u_x||w|^2\Big)dxdt+C.
\end{aligned} \tag{6.34}$$

From (6.17) and (2.1), we get

$$\int\theta\left(|u_x|^2+|w_x|^2+w^2\right)dx\le C\|\theta\|_{L^\infty}\le\frac{\kappa}{4}\|\theta_x\|_{L^2}^2+C, \tag{6.35}$$

and then

$$
\begin{aligned}
&C\int_0^T\int\Big(|u_x||\theta_x|^2+\rho\theta|u_x||\dot\theta|+\theta|u_x|^3+\theta|u_x||\dot u_x|+\theta|u_x||w_x|^2\\
&\quad+\theta|w_x||\dot w_x|+\theta|w||\dot w|+\theta|u_x||w|^2\Big)dxdt\\
&\le\frac12\int_0^T\|\sqrt{\rho}\dot\theta\|_{L^2}^2dt+\varepsilon\int_0^T\|\dot w_x\|_{L^2}^2dt+\varepsilon\int_0^T\|\dot u_x\|_{L^2}^2dt\\
&\quad+C(\varepsilon)\int_0^T(\|u_x\|_{L^\infty}+\|\theta\|_{L^\infty})\left(\|\theta_x\|_{L^2}^2+\|\theta|u_x|^2\|_{L^1}\right)dt\\
&\le\frac12\int_0^T\|\sqrt{\rho}\dot\theta\|_{L^2}^2dt+\varepsilon\int_0^T\|\dot w_x\|_{L^2}^2dt+\varepsilon\int_0^T\|\dot u_x\|_{L^2}^2dt\\
&\quad+C(\varepsilon)\int_0^T(\|u_x\|_{L^\infty}+\|\theta\|_{L^\infty})\left(\|\theta_x\|_{L^2}^2+1\right)dt.
\end{aligned}\tag{6.36}
$$

Combining (6.34)-(6.36), we have

$$
\begin{aligned}
&\|\theta_x\|_{L^2}^2+\int_0^T\|\sqrt{\rho}\dot\theta\|_{L^2}^2dt\\
&\le\varepsilon\int_0^T\|\dot u_x\|_{L^2}^2dt+\varepsilon\int_0^T\|\dot w_x\|_{L^2}^2dt+C\int_0^T(\|u_x\|_{L^\infty}+\|\theta\|_{L^\infty})\left(\|\theta_x\|_{L^2}^2+1\right)dt+C.
\end{aligned}\tag{6.37}
$$

From (6.31), (6.33) and (6.37), choosing $\varepsilon>0$ suitably small, we have

$$
\begin{aligned}
&\sup_{0\le t\le T}\left(\|\theta_x\|_{L^2}^2+\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\sqrt{\rho}\dot w\|_{L^2}^2\right)+\int_0^T\int\Big(\rho|\dot\theta|^2+|\dot u_x|^2+|\dot w_x|^2\Big)dxdt\\
&\le C\int_0^T(\|u_x\|_{L^\infty}+\|\theta\|_{L^\infty})\left(\|\theta_x\|_{L^2}^2+1\right)dt+C.
\end{aligned}
$$

In terms of (6.16), (6.17) and the Grönwall inequality, we have

$$
\sup_{0\le t\le T}\left(\|\theta_x\|_{L^2}^2+\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\sqrt{\rho}\dot w\|_{L^2}^2\right)+\int_0^T\int\Big(\rho|\dot\theta|^2+|\dot u_x|^2+|\dot w_x|^2\Big)dxdt\le C.\tag{6.38}
$$

From (2.1), (6.4), (6.17), (6.20) and (6.38), we deduce

$$
\begin{aligned}
&\|u\|_{W^{1,\infty}}+\|w\|_{W^{1,\infty}}+\|\theta\|_{L^\infty}\\
&\le C\int(P+|u_x|+\rho|\dot u|+\rho|\dot w|+|w_x|+|w|+|\theta_x|)\,dx\le C.
\end{aligned}\tag{6.39}
$$

By (2.1) and (6.38), we have

$$
\begin{aligned}
&\int_0^T\left(\|\dot u\|_{L^\infty}^2+\|\dot w\|_{L^\infty}^2\right)dt\\
&\le C\int_0^T\left(\|\dot u_x\|_{L^2}^2+\|\dot w_x\|_{L^2}^2\right)dt+C\int_0^T\left(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\sqrt{\rho}\dot u\|_{L^2}^2\right)dt\le C.
\end{aligned}\tag{6.40}
$$

Combining (6.31), (6.33) and (6.38)-(6.40), we complete the proof of Lemma 6.7. □

LEMMA 6.8. *Under the conditions of Theorem 1.4, it holds that*

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}\dot\theta\|_{L^2}^2+\|\theta_{xx}\|_{L^2}^2+\|\theta_x\|_{L^\infty}\right)+\int_0^T\left(\|\theta_{tx}\|_{L^2}^2+\|\dot\theta_x\|_{L^2}^2+\|\dot\theta\|_{L^\infty}^2\right)dt\le C. \tag{6.41}$$

Proof. Operating $\partial_t+\partial_x(u\cdot)$ to $(1.2)_4$ and using direct calculation, we have

$$\begin{aligned}&(\rho\dot\theta)_t+(\rho u\dot\theta)_x-\kappa\dot\theta_{xx}\\ =&-\kappa(u_x\theta_x)_x-(P_t+(Pu)_x)u_x+P|u_x|^2-P\dot u_x+2\mu_1u_x\dot u_x-\mu_1(u_x)^3\\ &+2\mu_2w_x\dot w_x-\mu_2u_xw_x^2+4\xi w\dot w-2\xi u_xw^2.\end{aligned} \tag{6.42}$$

Multiplying (6.42) by $\dot\theta$, using the initial compatibility conditions (1.9) and $P_t+(Pu)_x=\rho\dot\theta$, we have

$$\begin{aligned}&\int\rho|\dot\theta|^2dx+\kappa\int_0^T\int|\dot\theta_x|^2dxdt\\ \le&\, C\int_0^T\int|u_x||\theta_x||\dot\theta|dxdt+C\int_0^T\int\left(\rho|\dot\theta||u_x|+P|u_x|^2+P|\dot u|+|u_x||\dot u_x|+|u_x|^3\right)|\dot\theta|dxdt\\ &+C\int_0^T\int\left(|w_x||\dot w_x|+|u_x|w_x^2+|w||\dot w|+|u_x|w^2\right)|\dot\theta|dxdt+C\|g_3\|_{L^2}^2+C.\end{aligned} \tag{6.43}$$

From (6.29), we have

$$\begin{aligned}&\int_0^T\int|u_x||\theta_x||\dot\theta_x|dxdt+\int_0^T\int\rho|u_x||\dot\theta|dxdt\\ \le&\,\frac{\kappa}{8}\int_0^T\int|\dot\theta_x|^2dxdt+C\int_0^T\left(\|u_x\|_{L^\infty}^2+1\right)\left(\|\theta_x\|_{L^2}^2+\|\sqrt{\rho}\dot\theta\|_{L^2}^2\right)dt.\end{aligned}$$

By (2.1), (6.6), (6.17) and (6.29), it yields

$$\int_0^T\int(P|u_x|^2+|u_x|^3)|\dot\theta|dxdt\le C\int_0^T\|\dot\theta\|_{L^\infty}dt\le\frac{\kappa}{8}\int_0^T\int|\dot\theta_x|^2dxdt+C,$$

and

$$\begin{aligned}&\int_0^T\int(P|\dot u_x|+|u_x||\dot u_x|)|\dot\theta|dxdt\le\frac{\kappa}{8}\int_0^T\|\dot\theta\|_{L^\infty}^2dt+C\int_0^T\|\dot u_x\|_{L^2}^2dt\\ \le&\,\frac{\kappa}{8}\int_0^T\int|\dot\theta_x|^2dxdt+C.\end{aligned}$$

Similarly, we have

$$\int_0^T\int\left(|w_x||\dot w_x|+|u_x|w_x^2+|w||\dot w|+|u_x|w^2\right)|\dot\theta|dxdt\le\frac{\kappa}{8}\int_0^T\int|\dot\theta_x|^2dxdt+C.$$

Taking the last four inequalities into account, from (6.43) we infer

$$\sup_{0\le t\le T}\|\sqrt{\rho}\dot\theta\|_{L^2}^2+\int_0^T\|\dot\theta_x\|_{L^2}^2dt\le C. \tag{6.44}$$

From $(1.2)_4$, (6.6), (6.29) and (6.44), we have

$$\int|\theta_{xx}|^2dx\le C\int\rho|\dot\theta|^2dx+C\le C$$

and

$$\int_0^T\int|\theta_{tx}|^2dxdt\le C\int_0^T\int|\dot\theta_x|^2dxdt+C\int_0^T\int\left(|u|^2|\theta_{xx}|^2+|u_x|^2|\theta_x|^2\right)dxdt\le C.$$

Due to (2.1), we use (6.19) and the above inequalities to conclude

$$\sup_{0\le t\le T}\|\theta_x\|_{L^\infty}+\int_0^T\|\dot\theta\|_{L^\infty}^2dt\le C.$$

The last three inequalities and (6.44) complete the proof of Lemma 6.8. □

LEMMA 6.9. *Under the conditions of Theorem 1.4, it holds that*

$$\sup_{0\le t\le T}\left(\|\rho_x\|_{L^2}^2+\|u_{xx}\|_{L^2}^2+\|w_{xx}\|_{L^2}^2\right)+\int_0^T\left(\|u_{tx}\|_{L^2}^2+\|w_{tx}\|_{L^2}^2\right)dt\le C.$$

Proof. Differentiating $(1.2)_1$ with respect to x yields

$$\rho_{tx}+\rho_{xx}u+\rho_xu_x+\rho u_{xx}=0. \tag{6.45}$$

Multiplying (6.45) by ρ_x, and using integration by parts, we infer

$$\int|\rho_x|^2dx\le C+C\int_0^T(1+\|u_x\|_{L^\infty})\|\rho_x\|_{L^2}^2dt+\int_0^T\int|u_{xx}|^2dxdt. \tag{6.46}$$

By (1.2), (6.2) and (6.39), we have

$$\int\left(|u_{xx}|^2+|w_{xx}|^2\right)dx\le C\int\left(\rho|\dot u|^2+P_x^2+\rho|\dot w|^2+|w|^2\right)dx\le C\|\rho_x\|_{L^2}^2+C. \tag{6.47}$$

Combining (6.46) with (6.47) together, and using the Grönwall inequality, we have

$$\sup_{0\le t\le T}\left(\|\rho_x\|_{L^2}^2+\|u_{xx}\|_{L^2}^2+\|w_{xx}\|_{L^2}^2\right)\le C.$$

Then, together (6.29) and (6.39), it yields

$$\int_0^T\int|u_{tx}|^2dxdt\le C\int_0^T\int|\dot u_x|^2dxdt+C\int_0^T\int\left(|u_x|^4+|u|^2|u_{xx}|^2\right)dxdt\le C,$$

and

$$\int_0^T\int|w_{tx}|^2dxdt\le C\int_0^T\int|\dot w_x|^2dxdt+C\int_0^T\int\left(|u_x|^2|w_x|^2+|u|^2|w_{xx}|^2\right)dxdt\le C.$$

Then we complete the proof of Lemma 6.9. □

LEMMA 6.10. *Under the conditions of Theorem 1.4, it holds that*

$$\sup_{0\le t\le T}\left(\|\rho_x\|_{L^\infty}^2+\|\rho_{xx}\|_{L^2}^2\right)+\int_0^T\left(\|\rho_{tt}\|_{L^2}^2+\|u_{xxx}\|_{L^2}^2+\|w_{xxx}\|_{L^2}^2\right)dt\le C. \tag{6.48}$$

Proof. Differentiating (6.45) by x and multiplying it by $2\rho_{xx}$ gives rise to

$$\int |\rho_{xx}|^2 dx \le C\int_0^T \int \left(|u_x||\rho_{xx}|^2 + |\rho_x||\rho_{xx}||u_{xx}| + \rho|\rho_{xx}||u_{xxx}|\right) dxdt. \tag{6.49}$$

By (6.9), we have

$$\|\rho_x\|_{L^\infty} \le C(\|\rho_x\|_{L^2} + \|\rho_{xx}\|_{L^2}) \le C(\|\rho_{xx}\|_{L^2} + 1). \tag{6.50}$$

By (6.3), (6.29), (6.39)-(6.41) and (6.50), we deduce from $(1.2)_2$ that

$$\begin{aligned}
&\int_0^T \int |u_{xxx}|^2 dxdt \\
&\le C\int_0^T \int \left(|P_{xx}|^2 + (\rho\dot{u})_x^2\right) dxdt \\
&\le C\int_0^T \int \left(|\rho_{xx}|^2|\theta|^2 + |\rho_x|^2|\theta_x|^2 + |\rho|^2|\theta_{xx}|^2 + |\rho_x|^2|\dot{u}|^2 + |\rho|^2|\dot{u}_x|^2\right) dxdt \\
&\le C\int_0^T \int \left(|\rho_{xx}|^2 + |\theta_{xx}|^2 + |\dot{u}_x|^2\right) dxdt + C\int_0^T \left(\|\rho_x\|_{L^\infty}^2 + \|\dot{u}\|_{L^\infty}^2\right) dt \\
&\le C\int_0^T \int |\rho_{xx}|^2 dxdt + C,
\end{aligned} \tag{6.51}$$

and

$$\int_0^T \int \rho|\rho_{xx}||u_{xxx}| dxdt \le C\int_0^T \|\rho_{xx}\|_{L^2}\|u_{xxx}\|_{L^2} dt \le C\int_0^T \int |\rho_{xx}|^2 dxdt + C.$$

Then, by (6.49), we have

$$\begin{aligned}
\int |\rho_{xx}|^2 dx &\le C\int_0^T (1 + \|u_x\|_{L^\infty})\|\rho_{xx}\|_{L^2}^2 dt + C\int_0^T \|\rho_x\|_{L^\infty}\|\rho_{xx}\|_{L^2}\|u_{xx}\|_{L^2} dt + C \\
&\le C\int_0^T (1 + \|u_x\|_{L^\infty})\|\rho_{xx}\|_{L^2}^2 dt + C.
\end{aligned} \tag{6.52}$$

Combining (6.50)-(6.52) and (6.16), we have

$$\|\rho_x\|_{L^\infty}^2 + \|\rho_{xx}\|_{L^2}^2 + C\int_0^T \|u_{xxx}\|_{L^2}^2 dt \le C.$$

By (6.5), (6.6), (6.16), (6.17) and $(1.2)_3$, we have

$$\int_0^T \|w_{xxx}\|_{L^2}^2 dt \le C\int_0^T \left(\|(\rho\dot{w})_x\|_{L^2} + \|w_x\|_{L^2}\right)^2 dt \le C.$$

By (1.2), (6.17), (6.41) and (6.9), one has

$$\int_0^T \int |\rho_{tt}|^2 dxdt \le C\int_0^T \int \left(|\rho_x| + |\rho_{xx}| + |u_{xx}| + |\theta_{xx}| + |u_{xxx}|\right)^2 dt \le C.$$

Then the proof of this lemma is completed. ∎

The proof of the following lemma about ρ is the same as that of Lemma 3.10 in [29].

LEMMA 6.11. *Under the conditions of Theorem 1.4, it holds that*

$$\sup_{0\le t\le T}(\|\partial_x\sqrt{\rho}\|_{L^\infty}+\|\partial_t\sqrt{\rho}\|_{L^\infty})\le C.$$

LEMMA 6.12. *Under the conditions of Theorem 1.4, it holds that*

$$\sup_{0\le t\le T}(\|\rho\theta_{xt}\|_{L^2}^2+\|\theta_{xxx}\|_{L^2}^2)+\int_0^T\|\sqrt{\rho^3}\theta_{tt}\|_{L^2}^2dt\le C.$$

Proof. Differentiating $(1.2)_4$ with respect to t and multiplying it by $\rho^2\theta_{tt}$, using Lemmas 6.2-6.11, Corollary 6.1 and the Cauchy inequality, integrating by parts, we have

$$\begin{aligned}&\frac{\kappa}{2}\frac{d}{dt}\int\rho^2|\theta_{xt}|^2dx+\int\rho^3\theta_{tt}^2dx\\&\le C\int\theta_{xt}^2dx+C\int u_{xt}^2dx+C\int w_{xt}^2dx+C\|\theta_t\|_{L^\infty}^2+C.\end{aligned}\tag{6.53}$$

Integrating (6.53) on $[0,t]$, by Lemmas 6.2-6.11, Corollary 6.1, the compatibility conditions (1.9) and the Young inequality, we have

$$\int\rho^2|\theta_{xt}|^2dx+\int_0^T\int\rho^3\theta_{tt}^2dxdt\le\int\rho^2|\theta_{xt}|^2dx\bigg|_{t=0}+C\le C.\tag{6.54}$$

Differentiating $(1.2)_4$ with respect to x, we have

$$(\rho\theta_t)_x+(\rho u\theta_x)_x+(\rho\theta u_x)_x=2\mu_1u_xu_{xx}+2\mu_2w_xw_{xx}+4\xi ww_x+\kappa\theta_{xxx}.$$

From the above equation, by Lemmas 6.2-6.11, Corollary 6.1, (6.54) and the Young inequality, we have

$$\int|\theta_{xxx}|^2dx\le C\int\rho^2|\theta_{xt}|^2dx+C\le C.\tag{6.55}$$

Combining (6.54) and (6.55), we complete the proof. □

The proof of the last lemma is similar to the one in [29].

LEMMA 6.13. *Under the conditions of Theorem 1.4, it holds that*

$$\begin{aligned}&\sup_{0\le t\le T}(\|\rho u_{xt}\|_{L^2}^2+\|\rho w_{xt}\|_{L^2}^2+\|u_{xxx}\|_{L^2}^2+\|w_{xxx}\|_{L^2}^2)\\&\quad+\int_0^T\left(\|\sqrt{\rho^3}u_{tt}\|_{L^2}^2+\|\sqrt{\rho^3}w_{tt}\|_{L^2}^2\right)dt\le C.\end{aligned}$$

Proof. **(Proof of Theorem 1.4.)** Collecting Lemmas 6.2-6.13 and Corollary 6.1, we have

$$\begin{aligned}&\sup_{0\le t\le T}(\|\rho\|_{H^2}+\|\rho_t\|_{H^1}+\|u\|_{H^3}+\|u_t\|_{H^1}+\|w\|_{H^3}+\|w_t\|_{H^1}+\|\theta\|_{H^3}+\|\rho\theta_t\|_{H^1})\\&\quad+\int_0^T\int\left(u_{xt}^2+u_{xxt}^2+w_{xt}^2+w_{xxt}^2+\rho_{tt}^2+\theta_t^2+\theta_{xt}^2+\theta_{xxt}^2+u_{tt}^2+\theta_{tt}^2\right)dxdt\le C.\end{aligned}\tag{6.56}$$

Then we complete the proof of Theorem 1.4. □

7. Proof of Theorem 1.5

Similar to the proof of Theorem 1.4, we prove the following Lemmas. The first lemma is for the local well posedness of classical solutions.

LEMMA 7.1 (See [5–7]). *Under the same assumptions given in Theorem 1.5, the problem (1.2)-(1.4) has a unique classical solution (ρ,u,w,θ) over $[0,T_*]$ for some (small) time $T_*>0$ which satisfies the regularities (1.10).*

LEMMA 7.2. *Under the conditions of Theorem 1.5, it holds that for any $0\le t\le T$,*

$$\int \rho dx = \int \rho_0 dx, \tag{7.1}$$

$$\int \rho\left(\theta+\frac{1}{2}u^2+\frac{1}{2}w^2\right)dx+\kappa\int_0^T [a\theta(0,t)+b\theta(1,t)]\,dt \le \int \rho_0\left(\theta_0+\frac{1}{2}u_0^2+\frac{1}{2}w_0^2\right)dx \triangleq \mathcal{E}_0. \tag{7.2}$$

Proof. Integrating $(1.1)_1$ and $(1.1)_4$ directly, and integrating by parts over $I\times[0,t]$, we have

$$\int \rho dx = \int \rho_0 dx,$$

and

$$\int \rho\left(\theta+\frac{1}{2}u^2+\frac{1}{2}w^2\right)dx = \int \rho_0\left(\theta_0+\frac{1}{2}u_0^2+\frac{1}{2}w_0^2\right)dx-\kappa\int_0^T [a\theta(0,s)+b\theta(1,s)]\,ds,$$

which implies (7.1) and (7.2). □

LEMMA 7.3 ([13, 29]). *Under the conditions of Theorem 1.5, it holds that for any $(x,t)\in Q_T$,*

$$0\le\rho\le C \quad \text{and} \quad \theta>0. \tag{7.3}$$

LEMMA 7.4. *Given $\alpha>(\gamma-1)\mathcal{E}_0$, there exists a sequence of (non-overlapping) intervals Ω_j in I such that for every $t\in[0,T]$,*

$$P(x,t)\le\alpha,\ x\in I\setminus\cup_j\Omega_j \quad \text{and} \quad \alpha\le\frac{1}{|\Omega_j|}\int_{\Omega_j}P(x,t)dx\le 2\alpha.$$

Moreover,

$$\left|\cup_j\Omega_j\right|\le\frac{(\gamma-1)\mathcal{E}_0}{\alpha}.$$

LEMMA 7.5. *Under the conditions of Theorem 1.5, it holds that,*

$$\sup_{0\le t\le T}\int\rho\left(u^4+w^4+\theta^2\right)dx+\int_0^T\int\left(\mu_1u^2u_x^2+\mu_2w^2w_x^2+\kappa\theta_x^2+\xi w^4\right)dxdt\le C. \tag{7.4}$$

Proof. Multiplying (6.7) by E, integrating by parts over $I\times(0,t)$, by (1.4), (7.1)-(7.3), (2.1) and the Young inequality, it yields that

$$\begin{aligned}
&\int \rho E^2 dx+\kappa\int_0^T\int |E_x|^2 dxdt\\
&\le C+\kappa\int_0^T E_xE|_0^1 dt+C\int_0^T\int \left(\rho^2\theta^2u^2+u^2u_x^2+w^2w_x^2\right)dxdt\\
&\le C-a\kappa\int_0^T\theta^2(0,t)dt-b\kappa\int_0^T\theta^2(1,t)dt+C\int_0^T\int \left(\rho^2\theta^2u^2+u^2u_x^2+w^2w_x^2\right)dxdt\\
&\le C+C\int_0^T\int \left(\rho^2\theta^2u^2+u^2u_x^2+w^2w_x^2\right)dxdt,
\end{aligned}$$

where we note the second and third terms in the second step are negative, so we omit it in the next step. And the remaining proof of this lemma is the same as that in Lemma 6.4. Then, we complete the proof of this lemma. □

COROLLARY 7.1. *Under the conditions of Theorem 1.5, it holds that*

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}w\|_{L^2}^2\right)+\int_0^T\left(\|\theta\|_{L^\infty}+\|u_x\|_{L^2}^2+\|w\|_{H^1}^2\right)dt\le C.$$

Proof. The proof is the same as that in Corollary 6.1. □

LEMMA 7.6. *Under the conditions of Theorem 1.5, it holds that*

$$\sup_{0\le t\le T}\left(\|u_x\|_{L^2}^2+\|w\|_{H^1}^2\right)+\int_0^T\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\|\sqrt{\rho}\dot{w}\|_{L^2}^2+\|u_x\|_{L^\infty}^2\right)dt\le C.$$

Proof. Multiplying $(1.2)_2$ by $\dot{u}$, where $\dot{f}=\partial_t f+u\partial_x f$, integrating over $I\times(0,t)$, we have

$$\begin{aligned}
&\int\left(\frac{\mu_1}{2}|u_x|^2-Pu_x\right)(x,t)dx+\int_0^T\int\rho|\dot{u}|^2dxdt\\
&=\int\left(\frac{\mu_1}{2}|u_x|^2-Pu_x\right)(x,0)dx-\int_0^T\int(P_t+(Pu)_x)(\mu_1u_x-P)dxdt\\
&\quad+\int_0^T\int(P_t+(Pu)_x)Pdxdt+\int_0^T\int\left(P|u_x|^2-\frac{\mu_1}{2}(u_x)^3\right)dxdt.
\end{aligned}\tag{7.5}$$

By (1.1), (1.4), (7.3) and (7.4) and the Cauchy inequality, we get

$$\begin{aligned}
&\int_0^T\int(P_t+(Pu)_x)(\mu_1u_x-P)dxdt\\
&=\int_0^T\int\left(\kappa\theta_{xx}+\mu_1u_x^2+\mu_2w_x^2+2\xi w^2-Pu_x\right)(\mu_1u_x-P)dxdt\\
&=\kappa\int_0^T\int\theta_{xx}(\mu_1u_x-P)dxdt+\int_0^T\int\left(\mu_1u_x^2+\mu_2w_x^2+2\xi w^2-Pu_x\right)(\mu_1u_x-P)dxdt\\
&\le\kappa\int_0^T\theta_x(\mu_1u_x-P)|_0^1dt+C(\varepsilon)\int_0^T\int|\theta_x|^2dxdt+\varepsilon\int_0^T\int|(\mu_1u_x-P)_x|^2dxdt\\
&\quad+C\int_0^T\|\mu_1u_x-P\|_{L^\infty}\int\left(P^2+u_x^2+w_x^2+w^2\right)dxdt,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\kappa\int_0^T \theta_x(\mu_1 u_x - P)|_0^1 dt\right| \\
&\le C\int_0^T |\{\theta_x[\mu_1 u_x - P](1,t) - \theta_x[\mu_1 u_x - P](0,t)\}| dt \\
&\le C\int_0^T |\{b\theta[\mu_1 u_x - P](1,t) + a\theta[\mu_1 u_x - P](0,t)\}| dt \\
&\le C\int_0^T \|\mu_1 u_x - P\|_{L^\infty}\|\theta\|_{L^\infty} dt \\
&\le \frac{1}{6}\int_0^T\int \rho|\dot{u}|^2 dxdt + C\int_0^T \|\theta\|_{L^\infty}^2 dt + C\int_0^T \|u_x\|_{L^2}^2 dt + C \\
&\le \frac{1}{6}\int_0^T\int \rho|\dot{u}|^2 dxdt + C,
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_0^T\int (P_t + (Pu)_x)(\mu_1 u_x - P) dxdt \\
&\le \frac{1}{6}\int_0^T\int \rho|\dot{u}|^2 dxdt + C\int_0^T \|\mu_1 u_x - P\|_{L^\infty}(1 + \|u_x\|_{L^2}^2 + \|w\|_{H^1}^2) dt + C.
\end{aligned}$$

The rest of the proof of this lemma is the same as that in Lemma 6.6. □

LEMMA 7.7. *Under the conditions of Theorem 1.5, it holds that*

$$\begin{aligned}
&\sup_{0\le t\le T}\left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|u\|_{W^{1,\infty}} + \|\sqrt{\rho}\dot{w}\|_{L^2}^2 + \|w\|_{W^{1,\infty}} + \|\theta_x\|_{L^2}^2 + \|\theta\|_{L^\infty}\right) \\
&\quad + \int_0^T \left(\|\dot{u}_x\|_{L^2}^2 + \|\dot{u}\|_{L^\infty}^2 + \|\dot{w}_x\|_{L^2}^2 + \|\dot{w}\|_{L^\infty}^2 + \|\sqrt{\rho}\dot{\theta}\|_{L^2}^2\right) dt \le C.
\end{aligned}$$

Proof. Multiplying $(1.2)_4$ by $\dot{\theta}$, and integrating the resulting equation over $I\times(0,t)$, it gives

$$\begin{aligned}
&\frac{\kappa}{2}\|\theta_x\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 dt - \kappa\int_0^T \theta_x\dot{\theta}|_0^1 dt \\
&\le \frac{1}{2}\|g_3\|_{L^2}^2 + C\int \theta\left(|u_x|^2 + |w_x|^2 + w^2\right) dx + C\int_0^T\int \Big(|u_x||\theta_x|^2 + \rho\theta|u_x||\dot{\theta}| \\
&\qquad + \theta|u_x|^3 + \theta|u_x||\dot{u}_x| + \theta|u_x||w_x|^2 + \theta|w_x||\dot{w}_x| + \theta|w||\dot{w}| + \theta|u_x||w|^2\Big) dxdt + C.
\end{aligned}$$

By (1.4), we have

$$\begin{aligned}
\kappa\int_0^T \theta_x\dot{\theta}|_0^1 dt &= -\frac{\kappa}{2}\int_0^T \frac{d}{dt}\left(a\theta^2(0,t) + b\theta^2(1,t)\right) dt \\
&= -\frac{\kappa}{2}\left(a\theta^2(0,T) + b\theta^2(1,T)\right) + \frac{\kappa}{2}\left(a\theta^2(0,0) + b\theta^2(1,0)\right).
\end{aligned}$$

By Lemma 6.7, we have

$$\|\theta_x\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 dt$$

$$\leq \varepsilon \int_0^T \|\dot{u}_x\|_{L^2}^2 dt + \varepsilon \int_0^T \|\dot{w}_x\|_{L^2}^2 dt + C\int_0^T (\|u_x\|_{L^\infty} + \|\theta\|_{L^\infty})\left(\|\theta_x\|_{L^2}^2 + 1\right) dt + C.$$

The rest of the proof is the same as that in Lemma 6.7. □

LEMMA 7.8. *Under the conditions of Theorem 1.5, it holds that*

$$\sup_{0\leq t\leq T}\left(\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + \|\theta_{xx}\|_{L^2}^2 + \|\theta_x\|_{L^\infty}\right) + \int_0^T \left(\|\theta_{tx}\|_{L^2}^2 + \|\dot{\theta}_x\|_{L^2}^2 + \|\dot{\theta}\|_{L^\infty}^2\right) dt \leq C.$$

Proof. Multiplying (6.42) by $\dot{\theta}$, using the initial compatibility conditions (1.9) and $P_t + (Pu)_x = \rho\dot{\theta}$, we have

$$\begin{aligned}
&\int \rho|\dot{\theta}|^2 dx + \kappa\int_0^T \int |\dot{\theta}_x|^2 dxdt \\
&\leq \kappa\int_0^T \dot{\theta}_x\dot{\theta}|_0^1 dt - \kappa\int_0^T u_x\theta_x\dot{\theta}|_0^1 dt + C\int_0^T \int |u_x||\theta_x||\dot{\theta}| dxdt \\
&\quad + C\int_0^T \int \left(\rho|\dot{\theta}||u_x| + P|u_x|^2 + P|\dot{u}| + |u_x||\dot{u}_x| + |u_x|^3\right)|\dot{\theta}| dxdt \\
&\quad + C\int_0^T \int (|w_x||\dot{w}_x| + |u_x|w_x^2 + |w||\dot{w}| + |u_x|w^2)|\dot{\theta}| dxdt + C\|g_3\|_{L^2}^2 + C.
\end{aligned}$$

Reminding that $\dot{f} = f_t + uf_x$, by (1.4), we obtain,

$$\begin{aligned}
&\kappa\int_0^T \dot{\theta}_x\dot{\theta}|_0^1 dt - \kappa\int_0^T u_x\theta_x\dot{\theta}|_0^1 dt = \kappa\int_0^T \theta_{tx}\theta_t|_0^1 dt = \frac{\kappa}{2}\int_0^T (|\theta_t|^2)_x dt \\
&= -\frac{\kappa}{2}\int_0^T \left(a\theta_t^2(0,t) + b\theta_t^2(1,t)\right) dt \leq 0.
\end{aligned}$$

Thus we can omit this term. The rest of the proof is the same as that in Lemma 6.8. □

The proof of the following lemma is the same as those in Lemmas 6.9-6.10.

LEMMA 7.9. *Under the conditions of Theorem 1.5, it holds that*

$$\sup_{0\leq t\leq T}(\|\rho_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + \|w_{xx}\|_{L^2}^2) + \int_0^T (\|u_{tx}\|_{L^2}^2 + \|w_{tx}\|_{L^2}^2) dt \leq C.$$

$$\sup_{0\leq t\leq T}(\|\rho_x\|_{L^\infty}^2 + \|\rho_{xx}\|_{L^2}^2) + \int_0^T (\|\rho_{tt}\|_{L^2}^2 + \|u_{xxx}\|_{L^2}^2 + \|w_{xxx}\|_{L^2}^2) dt \leq C.$$

$$\sup_{0\leq t\leq T}(\|\partial_x\sqrt{\rho}\|_{L^\infty} + \|\partial_t\sqrt{\rho}\|_{L^\infty}) \leq C.$$

LEMMA 7.10. *Under the conditions of Theorem 1.5, it holds that*

$$\sup_{0\leq t\leq T}(\|\rho\theta_{xt}\|_{L^2}^2 + \|\theta_{xxx}\|_{L^2}^2) + \int_0^T \|\sqrt{\rho^3}\theta_{tt}\|_{L^2}^2 dt \leq C.$$

Proof. Differentiating (1.2)$_4$ with respect to t and multiplying it by $\rho^2\theta_{tt}$, using Lemmas 7.2-7.9, Corollary 7.1 and the Cauchy inequality, integrating by parts, we have

$$\begin{aligned}&\frac{\kappa}{2}\frac{d}{dt}\int \rho^2|\theta_{xt}|^2dx+\int \rho^3\theta_{tt}^2dx\\ &\le \kappa\left[\rho^2\theta_{xt}\theta_{tt}\right]_0^1+C\int \theta_{xt}^2dx+C\int u_{xt}^2dx+C\int w_{xt}^2dx+C\|\theta_t\|_{L^\infty}+C.\end{aligned}\tag{7.6}$$

By (1.4), we have

$$\begin{aligned}\kappa\left[\rho^2\theta_{xt}\theta_{tt}\right]_0^1&=-b\kappa\rho^2\theta_t\theta_{tt}(1,t)-a\kappa\rho^2\theta_t\theta_{tt}(1,t)(0,t)\\ &=-\frac{\kappa}{2}\frac{d}{dt}\left[b\rho^2\theta_t^2(1,t)+a\rho^2\theta_t^2(0,t)\right]+b\kappa\rho\rho_t\theta_t^2(1,t)+a\kappa\rho\rho_t\theta_t^2(0,t).\end{aligned}$$

Then, from (7.6) and Lemma 7.9, we have

$$\begin{aligned}&\frac{d}{dt}\int \rho^2|\theta_{xt}|^2dx+\int \rho^3\theta_{tt}^2dx+\frac{\kappa}{2}\frac{d}{dt}\left[b\rho^2\theta_t^2(1,t)+a\rho^2\theta_t^2(0,t)\right]\\ &\le C\int \theta_{xt}^2dx+C\int u_{xt}^2dx+C\int w_{xt}^2dx+C\|\theta_t\|_{L^\infty}^2+C.\end{aligned}\tag{7.7}$$

Integrating (7.7) on $[0,T]$, by Lemmas 7.2-7.9, Corollary 7.1 and the Young inequality, we have

$$\begin{aligned}&\int \rho^2|\theta_{xt}|^2dx+\int_0^T\int \rho^3\theta_{tt}^2dxdt+\frac{\kappa}{2}\left[b\rho^2\theta_t^2(1,T)+a\rho^2\theta_t^2(0,T)\right]\\ &\le \int \rho^2|\theta_{xt}|^2dx\Big|_{t=0}+\frac{\kappa}{2}\left[b\rho^2\theta_t^2(1,0)+a\rho^2\theta_t^2(0,0)\right]+C\le C.\end{aligned}$$

The rest of the proof is the same as that in Lemma 6.7. □

LEMMA 7.11. *Under the conditions of Theorem 1.5, it holds that*

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\rho u_{xt}\|_{L^2}^2+\|\rho w_{xt}\|_{L^2}^2+\|u_{xxx}\|_{L^2}^2+\|w_{xxx}\|_{L^2}^2\right)\\ &\quad+\int_0^T\left(\|\sqrt{\rho^3}u_{tt}\|_{L^2}^2+\|\sqrt{\rho^3}w_{tt}\|_{L^2}^2\right)dt\le C.\end{aligned}$$

Proof. **(Proof of Theorem 1.5.)** Collecting Lemmas 7.2-7.11 and Corollary 7.1, we have

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|\rho\|_{H^2}+\|\rho_t\|_{H^1}+\|u\|_{H^3}+\|u_t\|_{H^1}+\|w\|_{H^3}+\|w_t\|_{H^1}+\|\theta\|_{H^3}+\|\rho\theta_t\|_{H^1}\right)\\ &\quad+\int_0^T\int\left(u_{xt}^2+u_{xxt}^2+w_{xt}^2+w_{xxt}^2+\rho_{tt}^2+\theta_t^2+\theta_{xt}^2+\theta_{xxt}^2+u_{tt}^2+\theta_{tt}^2\right)dxdt\le C.\end{aligned}\tag{7.8}$$

Then we complete the proof of Theorem 1.5. □

8. The proof of Theorem 1.6

Assume that (ρ,u,w,θ) is the solution as in Theorem 1.4, and that $(\rho^{a,b},u^{a,b},w^{a,b},\theta^{a,b})$ is the solution as in Theorem 1.5.

Denote $\bar{\rho}=\rho-\rho^{a,b}$, $\bar{u}=u-u^{a,b}$, $\bar{w}=w-w^{a,b}$, $\bar{\theta}=\theta-\theta^{a,b}$. Then we can obtain that $(\bar{\rho},\bar{u},\bar{w},\bar{\theta})$ satisfies the following system:

$$\begin{cases}\bar{\rho}_t+(\bar{\rho}u)_x+(\rho^{a,b}\bar{u})_x=0,\\ \rho\bar{u}_t+\bar{\rho}u_t^{a,b}+\rho u\bar{u}_x+\rho\bar{u}u_x^{a,b}+\bar{\rho}u^{a,b}u_x^{a,b}+(\rho\bar{\theta}+\bar{\rho}\theta^{a,b})_x=\mu_1\bar{u}_{xx},\\ \rho\bar{w}_t+\bar{\rho}w_t^{a,b}+\rho u\bar{w}_x+\rho\bar{u}w_x^{a,b}+\bar{\rho}u^{a,b}w_x^{a,b}+2\xi\bar{w}=\mu_2\bar{w}_{xx},\\ \rho\bar{\theta}_t+\bar{\rho}\theta_t^{a,b}+\rho u\bar{\theta}_x+\rho\bar{u}\theta_x^{a,b}+\bar{\rho}u^{a,b}\theta_x^{a,b}+\rho\bar{\theta}u_x+\rho\theta^{a,b}\bar{u}_x+\bar{\rho}\theta^{a,b}u_x^{a,b}\\ \quad=\mu_1\bar{u}_x(u_x+u_x^{a,b})+\mu_2\bar{w}_x(w_x+w_x^{a,b})+2\xi\bar{w}(w+w^{a,b})+\kappa\bar{\theta}_{xx}.\end{cases}\tag{8.1}$$

Multiplying the first three equations of (8.1) by $\bar{\rho},\bar{u},\bar{w}$, respectively, integrating over I, by (6.56) and (7.8), we have the same results as (5.2)-(5.4).

Multiplying $(8.1)_4$ by $\bar{\theta}$, integrating over I, by (6.56) and (7.8), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho\bar{\theta}^2dx+\int\kappa\bar{\theta}_x^2dx\\ =&\kappa\bar{\theta}_x\bar{\theta}|_0^1-\int\bar{\rho}\bar{\theta}\theta_t^{a,b}dx-\int\rho\bar{u}\bar{\theta}\theta_x^{a,b}dx+\int\bar{\rho}u^{a,b}\bar{\theta}\theta_x^{a,b}dx-\int\rho\bar{\theta}^2u_xdx\\ &-\int\rho\theta^{a,b}\bar{\theta}\bar{u}_xdx-\int\bar{\rho}\theta^{a,b}\bar{\theta}u_x^{a,b}dx+\mu_1\int\bar{u}_x(u_x+u_x^{a,b})\bar{\theta}dx\\ &+\mu_2\int\bar{w}_x(w_x+w_x^{a,b})\bar{\theta}dx+2\xi\int\bar{w}(w+w^{a,b})\bar{\theta}dx\\ \le&\kappa\bar{\theta}\theta_x^{a,b}|_0^1+\|\bar{\rho}\|_{L^2}\|\bar{\theta}\|_{L^\infty}\|\theta_t^{a,b}\|_{L^2}+\|\sqrt{\rho}\bar{u}\|_{L^2}\|\sqrt{\rho}\bar{\theta}\|_{L^2}\|\theta_x^{a,b}\|_{L^\infty}+\|\sqrt{\rho}\bar{\theta}\|_{L^2}^2\|u_x\|_{L^\infty}\\ &+\|\bar{\rho}\|_{L^2}\|\bar{\theta}\|_{L^\infty}\|u^{a,b}\|_{L^2}\|\theta_x^{a,b}\|_{L^\infty}+\|\sqrt{\rho}\bar{\theta}\|_{L^2}\|\bar{u}_x\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|\theta^{a,b}\|_{L^\infty}\\ &+\|\bar{\rho}\|_{L^2}\|\bar{\theta}\|_{L^\infty}\|u_x^{a,b}\|_{L^2}\|\theta^{a,b}\|_{L^\infty}+C\|\bar{\theta}\|_{L^\infty}\|\bar{u}_x\|_{L^2}\left(\|u_x\|_{L^2}+\|u_x^{a,b}\|_{L^2}\right)\\ &+C\|\bar{\theta}\|_{L^\infty}\|\bar{w}_x\|_{L^2}\left(\|w_x\|_{L^2}+\|w_x^{a,b}\|_{L^2}\right)+C\|\bar{\theta}\|_{L^\infty}\|\bar{w}\|_{L^2}\left(\|w\|_{L^2}+\|w^{a,b}\|_{L^2}\right)\\ \le&\varepsilon_3\|\bar{\theta}_x\|_{L^2}^2+C_{\varepsilon_3}\left(\|\bar{u}_x\|_{L^2}^2+\|\bar{w}_x\|_{L^2}^2+\|\bar{w}\|_{L^2}^2\right)\\ &+C_{\varepsilon_3}\left(\|\theta_t^{a,b}\|_{L^2}^2+\|\theta_{xt}^{a,b}\|_{L^2}+1\right)\left(\|\bar{\rho}\|_{L^2}^2+\|\sqrt{\rho}\bar{u}\|_{L^2}^2+\|\sqrt{\rho}\bar{\theta}\|_{L^2}^2\right).\end{aligned}\tag{8.2}$$

Then, choosing suitably small $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$, we obtain from (5.2)-(5.4), (8.2) that

$$\begin{aligned}&\frac{d}{dt}\left(\|\bar{\rho}\|_{L^2}^2+\|\sqrt{\rho}\bar{u}\|_{L^2}^2+\|\sqrt{\rho}\bar{\theta}\|_{L^2}^2\right)+\int\left(\bar{u}_x^2+\bar{w}_x^2+\bar{w}^2+\kappa\bar{\theta}_x^2\right)dx\\ \le&C\left(\|\theta_t^{a,b}\|_{L^2}^2+\|\theta_{xt}^{a,b}\|_{L^2}+1\right)\left(\|\bar{\rho}\|_{L^2}^2+\|\sqrt{\rho}\bar{u}\|_{L^2}^2+\|\sqrt{\rho}\bar{\theta}\|_{L^2}^2\right).\end{aligned}\tag{8.3}$$

Using Grönwall inequality over (8.3), we complete the proof of Theorem 1.6.

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