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# DECAY ESTIMATE TO A COMPRESSIBLE EULER SYSTEM WITH NON-LOCAL VELOCITY ALIGNMENT*

LINING TONG† AND LI CHEN‡

**Abstract.** In this paper, the asymptotic behavior of the solutions for compressible Euler system with a non-local interaction term is studied. Using velocity damping to restrain the singularity caused by the anisotropic interaction between individuals, the exponential decay estimate of the solutions is obtained.

**Keywords.** Compressible Euler system; Exponential decay; Velocity alignment.

**AMS subject classifications.** 35Q70; 35L65.

## 1. Introduction

This paper is concerned with the asymptotic behavior of the solutions to the following compressible Euler system with non-local velocity effect:

$$\partial_t \rho + \nabla\cdot(\rho u) = 0, \quad (x,t)\in\mathbb{T}^N\times(0,\infty), \tag{1.1}$$

$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla P(\rho) = -\frac{1}{\tau}\rho u - \rho\int_{\mathbb{T}^N}\mathbf{\Gamma}(x,y)(u(x,t)-u(y,t))\rho(y,t)\mathrm{d}y, \tag{1.2}$$

subject to the initial conditions

$$\rho|_{t=0} = \rho_0(x)\geq 0, \quad u|_{t=0} = u_0(x), \quad x\in\mathbb{T}^N, \tag{1.3}$$

where $\mathbb{T}^N$ is the N-dimensional torus with $N\geq 1$ and without loss of generality it is assumed that $|\mathbb{T}^N|=1$, $\rho$ and $u$ are the unknown density and velocity, respectively, and $P(\rho)=A\rho^\gamma$ is the pressure, with $A>0$, $\gamma\geq 1$ being constants. For simplicity, it is assumed that $A=1$. The damping constant $\tau>0$ is given and $\mathbf{\Gamma}(x,y)$ is a communication weight matrix.

The system (1.1)-(1.2) is the macroscopic description of the microscopic multi-particle model, which reflects the velocity alignment of individuals in complex systems. When $\mathbf{\Gamma}(x,y)$ is a zero matrix, the model is reduced to the compressible Euler system with damping. It has been proved in general that for symmetrizable hyperbolic system without damping, smooth solutions exist locally in time for smooth initial data and the formation of shock waves will breakdown the smoothness of the solutions in finite time even in the scalar case [15]. However, if an additional velocity damping is added, shock waves can be avoided for small perturbation of the diffusion waves [8, 11]. In the literature, the global existence and asymptotic behavior of solutions to initial-boundary value problems for compressible Euler equations with damping are well studied for example in [6, 12, 13, 17–19] and references therein.

For the Euler systems with non-local interaction term, if the communication weight function $\mathbf{\Gamma}(x,y)$ in velocity alignment is of scalar type, i.e. $\mathbf{\Gamma}(x,y)=\psi(x,y)\mathbf{1}$, where $\mathbf{1}$ is the identity matrix and $\psi(x,y)$ has a positive lower bound, then the non-local term will

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†Department of Mathematics, Shanghai University, 200444, China (tongln@shu.edu.cn); Supported by National Natural Science Foundation of China (No.11901379 and 11771274).

‡Department of Mathematics, University of Mannheim, 68131 Mannheim, Germany (chen@math.uni-mannheim.de).

act as damping and can restrain the formation of the shock waves, readers are referred to [3, 10, 14] and references therein for more details. In case that the communication kernel is smooth, the convergence to equilibrium by using Wasserstein distance has been shown rigorously for 1-D Euler system with damping in [2]. However, the interaction between individuals is neither positive definite nor smooth in some systems, for example it happens in the complex material flow model given in [9] and the pedestrian flow model in [5]. In these cases, it is obvious that the non-local velocity involved interaction can in general not prevent the formation of shocks, instead new singularity will appear. Thus one has to require that the damping effect from velocity be strong enough to compete with the non-local effect for obtaining the global existence of the solutions.

The main result of this paper is as follows:

THEOREM 1.1. *Assume that* $\mathbf{\Gamma}\in L^2(\mathbb{T}^N\times\mathbb{T}^N)$, $\mathbf{\Gamma}(x,y)=\mathbf{\Gamma}(y,x)$, *and* $(\rho,u)$ *be a solution to the system* (1.1)–(1.3) *satisfying* $(\rho,u)\in L^\infty(\mathbb{T}^N\times\mathbb{R}^+)$ *and*

$$\frac{1}{\tau}-2\sup_{(x,t)\in\mathbb{T}^N\times\mathbb{R}^+}\int_{\mathbb{T}^N}|\mathbf{\Gamma}(x,y)\rho(y,t)|\mathrm{d}y\geq\mu>0. \tag{1.4}$$

*Then the following estimate holds,*

$$\int_{\mathbb{T}^N}\Big(\rho|u|^2+(\rho-m_0)^2\Big)\mathrm{d}x\leq C\cdot E_0e^{-Ct}. \tag{1.5}$$

*where* $m_0=\int_{\mathbb{T}^N}\rho\mathrm{d}x$ *and* $E_0=\int_{\mathbb{T}^N}\Big(\rho_0u_0^2+(\rho_0-m_0)^2\Big)\mathrm{d}x$ *is the initial total energy, and where* $C$ *is a constant depending on* $\mu,\tau,\|\mathbf{\Gamma}\|^2_{L^2(\mathbb{T}^N\times\mathbb{T}^N)}$, $\|u\|_{L^\infty}$, *and* $\|\rho\|_{L^\infty}$.

REMARK 1.1. The results are also valid for those $\mathbf{\Gamma}$ which are not symmetric, in this case, instead of using condition (1.4), one can assume that

$$\frac{1}{\tau}-\frac{3}{2}\sup_{(x,t)\in\mathbb{T}^N\times\mathbb{R}^+}\int_{\mathbb{T}^N}|\mathbf{\Gamma}(x,y)\rho(y,t)|\mathrm{d}y-\frac{1}{2}\sup_{(y,t)\in\mathbb{T}^N\times\mathbb{R}^+}\int_{\mathbb{T}^N}|\mathbf{\Gamma}(x,y)\rho(x,t)|\mathrm{d}x\geq\mu>0.$$

REMARK 1.2. In case $\mathbf{\Gamma}(x,y)=\mathbf{\Gamma}(x-y)$, the global existence of classical solution, as a perturbation of a constant background solution, to the system (1.1)–(1.3) is established when the domain is $\mathbb{R}^N$ in [20], where an assumption on strong damping is required. More precisely, if $\frac{1}{\tau}-2\bar{\kappa}^\gamma\|\mathbf{\Gamma}\|_{L^1}>0$, where $(\rho_c,u_c)=(\bar{\kappa}^\gamma,0)$ is the constant background solution, then the stability of this constant solution is shown in [20]. However, no decay estimate or asymptotic stability has been given.

When the domain is periodic, because of the conservation of mass, the only possible constant density solution is the mean value $m_0$. One can follow the same method as in [20] and obtain the global bounded solution as has been mentioned in the assumption in the above theorem, at the same time the assumption (1.4) holds because of the stability and the special form of the integral kernel $\mathbf{\Gamma}(x,y)=\mathbf{\Gamma}(x-y)$.

The periodic domain is chosen because of technical difficulties. On the one hand, the complicated boundary condition can be avoided, on the other hand the boundedness of the domain is needed in many of the estimates in the proof.

REMARK 1.3. The idea in this paper mainly follows the study of the long-time behavior to the solutions for the isentropic compressible Navier-Stokes equation in [7]. For the case of $\gamma>1$, it is noticed that the time derivatives of the pressure term $\frac{1}{\gamma-1}\int_{\mathbb{T}^N}\rho^\gamma\mathrm{d}x$ in the energy and $\int_{\mathbb{T}^N}\rho\int_{m_0}^{\rho}\frac{h^\gamma-m_0^\gamma}{h^2}\mathrm{d}h\mathrm{d}x$ are the same. The equivalence of the term

$\int_{\mathbb{T}^N}\rho\int_{m_0}^{\rho}\frac{h^\gamma-m_0^\gamma}{h^2}\mathrm{d}h\mathrm{d}x$ and the $L^2$-norm of $\rho-m_0$ is proved when $\rho$ is upper bounded in [7]. When $\gamma=1$, the same result is obtained for the pressure term $\int_{\mathbb{T}^N}\rho\ln\rho\mathrm{d}x$ in the energy [16]. Based on these facts, we will show that the relative free energy decays with a positive density dissipation to zero exponentially in time, where, due to the assumption of strong damping in the theorem, the anisotropic interaction force between individuals in the system is controlled.

The remaining part of this paper, Section 2, is to give the proof of the main theorem.

## 2. The proof of Theorem 1.1

We first provide a list of useful lemmata which will be used for the proof of Theorem 1.1.

LEMMA 2.1 (Lemma 3.1 in [7] the quantitative estimate for the pressure $P(\rho)=\rho^\gamma$.). *(1) Let $r_0,\bar{r}>0$, and $\gamma>1$ be arbitrary fixed constants,*

$$f(r,r_0)=r\int_{r_0}^{r}\frac{h^\gamma-r_0^\gamma}{h^2}\mathrm{d}h,\quad for\ r\in[0,\bar{r}].$$

*Then there exist positive constants $c_1$ and $c_2$ depending on $r_0$ and $\bar{r}$, such that*

$$c_1(r-r_0)^2\le f(r,r_0)\le c_2(r-r_0)^2. \tag{2.1}$$

*(2) It holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\gamma-1}\int_{\mathbb{T}^N}\rho^\gamma\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^N}f(\rho,m_0)\mathrm{d}x.$$

LEMMA 2.2. *Let $(\rho,u)$ be a global classical solution to (1.1)−(1.2). Then it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^N}\rho\mathrm{d}x=0, \tag{2.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^N}\rho^\gamma\mathrm{d}x=-(\gamma-1)\int_{\mathbb{T}^N}\rho^\gamma\nabla\cdot u\mathrm{d}x, \tag{2.3}$$

*Proof.* Using the divergence theorem, we can directly get (2.2), the conservation of mass, from (1.1). By multiplying $\gamma\rho^{\gamma-1}$ on both sides of (1.1), we have

$$(\rho^\gamma)_t=-\gamma\rho^\gamma\nabla\cdot u-\nabla\rho^\gamma\cdot u=-(\gamma-1)\rho^\gamma\nabla\cdot u-\nabla\cdot(\rho^\gamma u). \tag{2.4}$$

Then (2.3) can be derived after integrating (2.4) over $\mathbb{T}^N$. ◻

REMARK 2.1. Equation (2.2) implies directly $\int_{\mathbb{T}^N}\rho(x,t)\mathrm{d}x=\int_{\mathbb{T}^N}\rho_0(x)\mathrm{d}x=m_0$, which is a positive constant independent of time.

In the following we will use the operator $\mathcal{B}$ which was introduced in [1], defined in the following, $\forall f\in L^2(\mathbb{T}^N)$ with $\int_{\mathbb{T}^N}f\mathrm{d}x=0$, the unique solution of

$$\nabla\cdot v=f,\quad \nabla\times v=0,\quad \text{and}\ \int_{\mathbb{T}^N}v\mathrm{d}x=0, \tag{2.5}$$

is defined to be the image of $\mathcal{B}(f)$. This operator has been used in [7] for the bounded domain with no-slip boundary condition. The version for torus has been listed in [4]. A list of properties can be found for example in [4,7][1]

*Proof.* **(Proof of Theorem 1.1.)** Without loss of generality we prove it for the case $\gamma>1$. As has been mentioned in Remark 1.3, the same result holds also for $\gamma=1$. The proof is split into 3 steps:

**Step 1. (The energy inequality.)** We first derive the following energy inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x+\frac{1}{\gamma-1}\int_{\mathbb{T}^N}\rho^\gamma\mathrm{d}x\Big)+\mu\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x\le 0, \tag{2.6}$$

where $\mu>0$ is defined in (1.4).

We start from computing the time derivative of $\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x=&\frac{1}{2}\int_{\mathbb{T}^N}\rho_t|u|^2\mathrm{d}x+\int_{\mathbb{T}^N}\rho u_t\cdot u\mathrm{d}x\\ =&-\frac{1}{2}\int_{\mathbb{T}^N}\nabla\cdot(\rho u)|u|^2\mathrm{d}x+\int_{\mathbb{T}^N}\rho u_t\cdot u\mathrm{d}x\\ =&-\int_{\mathbb{T}^N}\nabla\rho^\gamma\cdot u\mathrm{d}x-\frac{1}{\tau}\int_{\mathbb{T}^N}\rho u\cdot u\mathrm{d}x\\ &-\int_{\mathbb{T}^N}\int_{\mathbb{T}^N}\mathbf{\Gamma}(x,y)(u(x,t)-u(y,t))\cdot u(x,t)\rho(y,t)\rho(x,t)\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Thus we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{T}^N}\rho|u|^2(t)\mathrm{d}x+\frac{1}{\gamma-1}\int_{\mathbb{T}^N}\rho^\gamma(t)\mathrm{d}x\Big)+\frac{1}{\tau}\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x\\ =&-\int_{\mathbb{T}^N}\int_{\mathbb{T}^N}\mathbf{\Gamma}(x,y)(u(x,t)-u(y,t))\cdot u(x,t)\rho(y,t)\rho(x,t)\mathrm{d}y\mathrm{d}x\\ \le&2\sup_{(x,t)\in\mathbb{T}^N\times\mathbb{R}^+}\int_{\mathbb{T}^N}|\mathbf{\Gamma}(x,y)\rho(y,t)|\mathrm{d}y\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x,\end{aligned}$$

where we have used (2.3). Then we obtain the energy inequality (2.6).

**Step 2. (A proper dissipation of $\|\rho-m_0\|_{L^2}$.)** Using the linear operator $\mathcal{B}$ introduced before, we define the following functionals:

$$V_\delta=\int_{\mathbb{T}^N}\Big(\frac{1}{2}\rho|u|^2+\rho\int_{m_0}^{\rho}\frac{h^\gamma-m_0^\gamma}{h^2}dh-\delta\rho u\cdot\mathcal{B}[\rho-m_0]\Big)\mathrm{d}x,$$

and

$$\begin{aligned}W_\delta=&\mu\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x+\delta\int_{\mathbb{T}^N}(\rho^\gamma-m_0^\gamma)\nabla\cdot\mathcal{B}[\rho-m_0]\mathrm{d}x\\ &+\delta\int_{\mathbb{T}^N}\rho u\otimes u:\nabla\mathcal{B}[\rho-m_0]\mathrm{d}x-\frac{\delta}{\tau}\int_{\mathbb{T}^N}\rho u\cdot\mathcal{B}[\rho-m_0]\mathrm{d}x\end{aligned}$$

[1] Since the field $v$ is curl-free, the operator $\mathcal{B}$ is equivalent to $\nabla\Delta^{-1}$ where $\Delta^{-1}$ is the inverse Laplacian operator with periodic boundary condition. The unique solvability of Laplacian equation with periodic boundary condition is understood in the sense that two solutions are the same if there is only a constant difference between them. Then the following estimates hold due to the regularity estimate of elliptic equations. (i) $\|\mathcal{B}[f]\|_{H^1(\mathbb{T}^N)}\le C\|f\|_{L^2(\mathbb{T}^N)}$. (ii) If a function $f\in L^2(\mathbb{T}^N)$ can be written in the form $f=\nabla\cdot g$ with $g\in[H^1(\mathbb{T}^N)]^N$, then $\|\mathcal{B}[f]\|_{L^2(\mathbb{T}^N)}\le C\|g\|_{L^2(\mathbb{T}^N)}$.

$$-\delta\int_{\mathbb{T}^N}\int_{\mathbb{T}^N}\mathbf{\Gamma}(x,y)(u(x,t)-u(y,t))\cdot\mathcal{B}[\rho(x,t)-m_0]\rho(y,t)\rho(x,t)\mathrm{d}x\mathrm{d}y$$
$$+\delta\int_{\mathbb{T}^N}\rho u\cdot\mathcal{B}[\rho_t]\mathrm{d}x=\sum_{i=1}^{6}J_i,$$

where $0<\delta<1$. The notation means $A:B=\sum_{i,j=1}^N a_{ij}b_{ij}$ for two $N\times N$ square matrixes.

It is easy to check with the help of (2.6) that

$$\frac{d}{dt}V_\delta+W_\delta\le 0. \tag{2.7}$$

Using the properties of the operator $\mathcal{B}$, we have

$$\begin{aligned}\Big|-\int_{\mathbb{T}^N}\delta\rho u\cdot\mathcal{B}[\rho-m_0]\mathrm{d}x\Big|&\le\delta\|\rho\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho}u\|_{L^2}\|\mathcal{B}[\rho-m_0]\|_{L^2}\\&\le\frac{\delta}{2}\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x+\frac{\|\rho\|_{L^\infty}\delta}{2}\int_{\mathbb{T}^N}|\rho-m_0|^2\mathrm{d}x.\end{aligned} \tag{2.8}$$

with the help of Hölder's and Young's inequalities.

Then it follows from (2.8), Lemma 2.1 and for sufficiently small $\delta$ that there exists a constant $C_0>0$ which depends on $\delta$ and $\|\rho\|_{L^\infty}$, such that

$$\frac{1}{C_0}\Big(\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x+\int_{\mathbb{T}^N}|\rho-m_0|^2\mathrm{d}x\Big)\le V_\delta\le C_0\Big(\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x+\int_{\mathbb{T}^N}|\rho-m_0|^2\mathrm{d}x\Big). \tag{2.9}$$

We investigate further for the terms in $W_\delta$. Notice that $J_1$ has a positive coefficient because of (1.4), and applying the properties of operator $\mathcal{B}$, the positivity of $J_2$ is obtained in the following:

$$J_2=\delta\int_{\mathbb{T}^N}(\rho^\gamma-m_0^\gamma)\nabla\cdot\mathcal{B}[\rho-m_0]\mathrm{d}x=\delta\int_{\mathbb{T}^N}(\rho^\gamma-m_0^\gamma)(\rho-m_0)\mathrm{d}x\ge C\delta\int_{\mathbb{T}^N}(\rho-m_0)^2\mathrm{d}x,$$

where the standard inequality $(r^\gamma-r_0^\gamma)(r-r_0)\ge C(r_0,\gamma)|r-r_0|^2$ for $r_0>0,\forall r\ge 0$ has been used. A short explanation of this inequality is the following. In case $r\ge\frac{r_0}{2}$, it can be obtained directly by mean value theorem, while for $0\le r<\frac{r_0}{2}$, it follows from the estimate

$$\frac{r^\gamma-r_0^\gamma}{r-r_0}=\frac{r_0^\gamma-r^\gamma}{r_0-r}\ge\frac{r_0^\gamma-(\frac{r_0}{2})^\gamma}{r_0}.$$

The estimates for $J_3-J_6$ in $W_\delta$ are dedicated to obtain the relation between $W_\delta$ and $\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x+\int_{\mathbb{T}^N}|\rho-m_0|^2\mathrm{d}x$. Using the properties of the operator $\mathcal{B}$ and Young's inequality, we obtain

$$|J_3|=\Big|\delta\int_{\mathbb{T}^N}\rho u\otimes u:\nabla\mathcal{B}[\rho-m_0]\mathrm{d}x\Big|\le\frac{\mu}{8}\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x+C\delta^2\|\rho\|_{L^\infty}\|u\|_{L^\infty}^2\int_{\mathbb{T}^N}|\rho-m_0|^2\mathrm{d}x,$$

$$|J_4|=|-\frac{\delta}{\tau}\int_{\mathbb{T}^N}\rho u\cdot\mathcal{B}[\rho-m_0]\mathrm{d}x|\le\frac{\mu}{8}\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x+C\frac{\delta^2}{\tau^2}\|\rho\|_{L^\infty}\int_{\mathbb{T}^N}|\rho-m_0|^2\mathrm{d}x,$$

and

$$\begin{aligned}|J_5| &= \Big| -\delta \int_{\mathbb{T}^N}\int_{\mathbb{T}^N} \mathbf{\Gamma}(x,y)(u(x,t)-u(y,t))\cdot \mathcal{B}[\rho(x,t)-m_0]\rho(y,t)\rho(x,t)\mathrm{d}x\mathrm{d}y\Big| \\ &\le \frac{\mu}{8}\int_{\mathbb{T}^N} \rho|u|^2\mathrm{d}x + C\delta^2\|\mathbf{\Gamma}\|^2_{L^2(\mathbb{T}^N\times\mathbb{T}^N)}\|\rho\|^3_{L^\infty}\int_{\mathbb{T}^N}|\rho-m_0|^2\mathrm{d}x.\end{aligned}$$

Finally, we estimate $J_6$. Using the properties of operator $\mathcal{B}$, we have

$$|J_6| = \Big|\delta\int_{\mathbb{T}^N}\rho u\cdot\mathcal{B}[\rho_t]\mathrm{d}x\Big| = \Big|\delta\int_{\mathbb{T}^N}\rho u\cdot\mathcal{B}[\nabla\cdot\rho u]\mathrm{d}x\Big| \le \delta\|\rho\|_{L^\infty}\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x \le \frac{\mu}{8}\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x.$$

where $\delta$ is chosen to be smaller than $\frac{\mu}{8}\|\rho\|_{L^\infty}$.

Collecting all the estimates of $J_i$ , we have

$$W_\delta \ge \frac{\mu}{2}\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x + C_1\int_{\mathbb{T}^N}|\rho-m_0|^2\mathrm{d}x, \tag{2.10}$$

where $C_1 = C\delta\Big(1-\delta\|\rho\|_{L^\infty}(\tau^{-2}+\|u\|^2_{L^\infty}+\|\mathbf{\Gamma}\|^2_{L^2(\mathbb{T}^N\times\mathbb{T}^N)}\|\rho\|^2_{L^\infty})\Big)$. We can choose

$$\delta = \min\{\frac{\mu}{8}\|\rho\|_{L^\infty}, \frac{1}{2}\big(\|\rho\|_{L^\infty}(\tau^{-2}+\|u\|^2_{L^\infty}+\|\mathbf{\Gamma}\|^2_{L^2(\mathbb{T}^N\times\mathbb{T}^N)}\|\rho\|^2_{L^\infty})\big)^{-1}\}$$

so that $C_1 > 0$.

**Step 3. (The decay estimate.)** Combining (2.7), (2.9), and (2.10), we deduce that there exists a constant $C_2 > 0$ such that

$$\frac{d}{dt}V_\delta + C_2 V_\delta \le 0, \tag{2.11}$$

where $C_2$ is a constant depending on $\mu, \tau, \|\mathbf{\Gamma}\|^2_{L^2(\mathbb{T}^N\times\mathbb{T}^N)}$, $\|u\|_{L^\infty}$, and $\|\rho\|_{L^\infty}$. Applying Grönwall's inequality and (2.9), we have

$$\begin{aligned}&\Big(\int_{\mathbb{T}^N}\rho|u|^2\mathrm{d}x + \int_{\mathbb{T}^N}|\rho-m_0|^2\mathrm{d}x\Big) \le C_0V_\delta(t) \le C_0V_\delta(0)e^{-C_2t} \\ \le& C_0^2\Big[\int_{\mathbb{T}^N}\rho|u|^2(0)\mathrm{d}x + \int_{\mathbb{T}^N}|\rho-m_0|^2(0)\mathrm{d}x\Big]e^{-C_2t} \le C_0^2E_0e^{-C_2t}. \end{aligned}\tag{2.12}$$

In summary, the proof of the Theorem 1.1 is completed. □

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