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NON-DEGENERATE STATIONARY SOLUTION FOR OUTFLOW PROBLEM ON THE 1-D VISCOUS HEAT-CONDUCTING GAS WITH RADIATION*

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Abstract. This paper studies the asymptotic behavior of the solution to the initial boundary value problem of a one-dimensional compressible viscous heat-conducting gas with radiation. We consider an outflow problem, where the gas blows out the region through the boundary, of the general gases including ideal polytropic gas. First, we give the necessary and sufficient conditions for an existence of the non-degenerate stationary solution. In addition, using the energy method, it proves the asymptotic stability of the solutions under the assumption that the initial perturbation and the boundary data in the Sobolev space is small. We also demonstrate the convergence rate for the exponential and logarithmic decay of the solver. Note that it is the result of the outflow problem of the viscous heat-conducting gas with radiation in the half line.

Keywords. compressible radiation hydrodynamics; outflow problem; stationary solution; stability; convergence rate.

AMS subject classifications. 35Q30; 35B35; 35L65; 76D33; 74J40.

1. Introduction and main result

The equations describing the one-dimensional motion of a compressible viscous heat-conducting gas with radiation in Eulerian coordinates, can be written in the following form (see [25])

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = \mu u_{xx}, \\ [\rho(e + \frac{u^2}{2})]_t + [\rho u(e + \frac{u^2}{2}) + pu]_x + q_x = \kappa\theta_{xx} + \mu(uu_x)_x, \\ -q_{xx} + q + (\theta^4)_x = 0, \end{cases} \tag{1.1}$$

where the unknown functions are the densities $\rho(x,t) > 0$, the velocities $u(x,t)$, the temperatures $\theta(x,t) > 0$, and the radiation heat flux $q(x,t)$. Also, $p = p(\rho,\theta)$ and $e = e(\rho,\theta)$ are the pressure and internal energy respectively, while $\mu > 0$ denotes the viscosity and $\kappa > 0$ denotes the heat-conductivity.

We consider the system (1.1) on $[0,\infty)$ replenished with the initial data, the far field conditions and the boundary condition.

$$\begin{cases} (\rho,u,\theta)\,|_{t=0} = (\rho_0,u_0,\theta_0)(x), \quad x \in [0,\infty), \\ \lim_{x\to+\infty}(\rho,u,\theta,q)(x,t) = (\rho_+,u_+,\theta_+,0), \end{cases} \tag{1.2}$$

$$u\,|_{x=0} = u_-, \quad \theta\,|_{x=0} = \theta_-, \quad q\,|_{x=0} = 0, \tag{1.3}$$

where $\rho_+ > 0$, $u_\pm$, $\theta_\pm > 0$ are constants.

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We are interested in the large-time behavior of the solutions for the initial-boundary value problem (1.1)-(1.3) in the case of $u_-<0$, that is, outflow problem.

Throughout this paper, we assume that

$$p_\rho(\rho,\theta)>0, \quad e_\theta(\rho,\theta)>0. \tag{1.4}$$

Notation: Throughout this paper, $O(1), c$ or C represents a generic constant and $C_i(\cdot,\cdot)$ or $c_i(\cdot,\cdot)(i\in Z_+)$ denotes general constants relating only to quantities indicated in parentheses. General Sobolev space with norm $\|\cdot\|_k$ denoted by $H^k:=H^k(0,\infty)$ and $\|\cdot\|_0=\|\cdot\|$ denote the usual L_2-norm.

Now, we state the main results of this paper. The stationary solution $(\hat\rho,\hat u,\hat\theta,\hat q)(x)$ of the system (1.1)-(1.3) must satisfy the following equations:

$$\begin{aligned}
&(\hat\rho\hat u)_x=0, \qquad x>0,\\
&(\hat\rho\hat u^2+\hat p)_x=\mu\hat u_{xx},\\
&[\hat\rho\hat u(\hat e+\frac{\hat u^2}{2})+\hat p\hat u]_x+q_x=\kappa\hat\theta_{xx}+\mu(\hat u\hat u_x)_x,\\
&-\hat q_{xx}+\hat q+(\hat\theta^4)_x=0,\\
&(\hat u,\hat\theta,\hat q)(0)=(u_-,\theta_-,0), \quad \lim_{x\to\infty}(\hat\rho,\hat u,\hat\theta,\hat q)(x)=(\rho_+,u_+,\theta_+,0),
\end{aligned} \tag{1.5}$$

where $\hat p=p(\hat\rho,\hat\theta)$, $\hat e=e(\hat\rho,\hat\theta)$.

The sound speed and the Mach number are defined, respectively, by

$$c(v,\theta)=\sqrt{\frac{\partial p(\rho,s)}{\partial\rho}}=\sqrt{-v^2\widetilde{p}_v(v,s)}, \quad M(v,u,\theta)=\frac{|u|}{c(v,\theta)}, \tag{1.6}$$

where s is the entropy.

Then, we first state the result for the following existence and the properties of solutions $(\hat\rho,\hat u,\hat\theta,\hat q)(x)$ to the system (1.5):

THEOREM 1.1 (Existence of non-degenerate stationary solution). *Let $u_-<0,\rho_+>0,\theta_\pm>0$. Following equation is a necessary condition for existence of the solution to the system (1.5).*

$$\hat\rho\hat u=\rho_+u_+=\hat\rho(0)u_-, \forall x>0. \tag{1.7}$$

If $u_+\geq 0$, then solution of system (1.5) does not exist.

For the case $M_+\equiv M(v_+,u_+,\theta_+)\neq 1$, if $u_+<0$ and (1.4) hold, then there exists a positive constant δ_0 and a local manifold $\mathcal{M}\subset\mathcal{M}_{\delta_0}:=\{(u,\theta)\in R_+^2\,|\,0<|(u-u_+,\theta-\theta_+)|\leq\delta_0\}$ such that if $(u_-,\theta_-)\in\mathcal{M}$, then the system (1.5) has a unique smooth solution $(\hat\rho,\hat u,\hat\theta,\hat q)(x)$ satisfying

$$|\partial_x^k(\hat\rho-\rho_+,\hat u-u_+,\hat\theta-\theta_+,\hat q)|\leq C\delta\exp(-\hat c x), \quad k=0,1,2, \tag{1.8}$$

where $\delta=|(u_--u_+,\theta_--\theta_+)|$ and $C,\hat c$ are positive constants independent of x,δ.

REMARK 1.1. Equation (1.8) denotes that for $M_+\neq 1$, the solution of (1.5) converges to the spatial asymptotic state with an exponential decay rate, this is called non-degenerate stationary solution. For the case $M_+=1$, the solution of the system

(1.5) may converge with an algebraic decay rate, this is called degenerate stationary solution. The case will be pursued by the authors in the future.

Next, we state the result for the stability of the non-degenerate stationary solutions for the outflow problem (1.1)-(1.4).

THEOREM 1.2 (Asymptotic stability of non-degenerate stationary solution). *Let $u_\pm<0,\rho_\pm>0,\theta_\pm>0$. Suppose that there exists the solution $(\hat\rho,\hat u,\hat\theta,\hat q)(x)$ to the system (1.5) satisfying (1.8). In addition, suppose that the initial data (ρ_0,u_0,θ_0) satisfies*

$$(\rho_0-\hat\rho,u_0-\hat u,\theta_0-\hat\theta)\in H^1(0,\infty),\ u_0(0)=u_-,\ \theta_0(0)=\theta_-. \tag{1.9}$$

If there exists a proper positive constant ε_0, such that

$$\|(\rho_0-\hat\rho,u_0-\hat u,\theta_0-\hat\theta)\|_1+\delta\le\varepsilon_0, \tag{1.10}$$

where $\delta=|(u_--u_+,\theta_--\theta_+)|$, then the system (1.1)-(1.4) has a unique solution $(\rho,u,\theta,q)(x,t)$ that satisfies the following conditions:

$$\begin{aligned}&(\rho-\hat\rho,u-\hat u,\theta-\hat\theta,q,q_x)\in C([0,\infty);H^1(0,\infty)),\\&\rho_x\in L_2(0,\infty;L_2(0,\infty)),\ (u_x,\theta_x,q,q_x)\in L_2(0,\infty;H^1(0,\infty)).\end{aligned}$$

The solution $(\rho,u,\theta,q)(x,t)$ tends time-asymptotically from the stationary solution $(\hat\rho,\hat u,\hat\theta,\hat q)(x)$ in the sense that

$$\lim_{t\to\infty}\sup_{x\in(0,\infty)}|(\rho,u,\theta,q)(x,t)-(\hat\rho,\hat u,\hat\theta,\hat q)(x)|=0.$$

The next theorem shows the convergence rate of the non-degenerate stationary solutions to the system (1.1)-(1.4) for $M_+>1$.

THEOREM 1.3 (Convergence rate of non-degenerate stationary solution). *Let $\rho_\pm>0,u_\pm<0,\theta_\pm>0$. In the case of $M_+>1$ suppose that there exists the solution $(\hat\rho,\hat u,\hat\theta)(x)$ to the system (1.5) satisfying (1.8). Assume (1.9) and (1.10). Then, we have the following property:*

(1) (exponential decay) *If $(\rho_0-\hat\rho,u_0-\hat u,\theta_0-\hat\theta)\in L^2_{\varsigma,\exp}(0,h\infty)$, there is a proper constant $\beta>0$ depending on ς such that the solution $(\rho,u,\theta,q)(x,t)$ to the system (1.1)-(1.4) satisfies the following condition.*

$$\sup_{x\in(0,\infty)}|(\rho,u,\theta,q)(x,t)-(\hat\rho,\hat u,\hat\theta,\hat q)(x)|\le Ce^{-\beta t}.$$

(2) (algebraic decay) *If $(\rho_0-\hat\rho,u_0-\hat u,\theta_0-\hat\theta)\in L^2_\varsigma(0,\infty)$, then the solution $(\rho,u,\theta,q)(x,t)$ to the system (1.1)-(1.4) satisfies the following condition*

$$\sup_{x\in(0,\infty)}|(\rho,u,\theta,q)(x,t)-(\hat\rho,\hat u,\hat\theta,\hat q)(x)|\le C(1+t)^{-\frac{\varsigma}{2}},$$

where $\varsigma>0$ and

$$\begin{aligned}L^2_{\varsigma,\exp}(0,\infty)&:=\{f\in L_{2,loc}(0,\infty);\int_0^\infty e^{\varsigma x}f^2(x)dx<\infty\},\\L^2_\varsigma(0,\infty)&:=\{f\in L_{2,loc}(0,\infty);\int_0^\infty(1+x)^\varsigma f^2(x)dx<\infty\}.\end{aligned}$$

Related results: When the radiation effect is involved, the mathematical study of this field starts from Hamer's work [5]. The model considered in [5] can be understood as Burgers equation coupled with an elliptic equation:

$$\begin{cases} w_t + f(w)_x + q_x = 0, \\ -q_{xx} + q + w_x = 0, \end{cases} \tag{1.11}$$

where w is a scalar unknown function. It is the simplest possible model and the third-order approximation of the compressible Euler system with radiation (see Appendix A in [3]):

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ [\rho(e + \frac{u^2}{2})]_t + [\rho u(e + \frac{u^2}{2}) + pu]_x + q_x = 0, \\ -q_{xx} + q + (\theta^4)_x = 0. \end{cases} \tag{1.12}$$

For the Hamer's model (1.11), Kawashima-Nishibata [8] proved asymptotic stability of shock profiles. Kawashima-Tanaka [11] showed the stability of rarefaction waves. Then this result was extended to multi-D cases by Gao-Ruan-Zhu in [3, 4, 22]. Recently, Ohnawa [19] has extended the result in [8] to continuous shock cases. On the other hand, there are also some results on the nonlinear stability of elementary waves for the Euler system with radiation (1.12). In [12] the authors proved the global existence of shock profiles for the Euler-Poison system, and Lattanzio-Mascia-Serre [14] extended the proof to a general hyperbolic-elliptic system. Lin-Coulombel-Goudon studied the stability of shock profiles under the zero mass perturbation assumption in [13]. Then Nguyen-Plaza-Zumbrun removed the zero mass perturbation assumption by using a Green function method in [18]. The stability of a single "viscous contact wave" is studied in [21, 26] and the stability of a rarefaction wave is considered in [15]. Xie [27] proved the stability for the combination of viscous contact wave with rarefaction waves. Also, for the system (1.1) of compressible viscous heat-conducting gas with radiation, there are a few mathematical results for the stability toward elementary waves. Wang-Xie [25] proved the stability of a single viscous contact wave and Hong [6] showed the stability of the combination of contact discontinuity with rarefaction waves. However, to the best of our knowledge, there is little known about the stability of nonlinear wave patterns for the initial boundary value problem in half line on the system (1.1) of compressible viscous heat-conducting gas with radiation, which is of interest in this paper.

Here, we briefly review some main difficulties of our problem, compared to the Cauchy problem of the system (1.1) or the outflow problem to the compressible Navier-Stokes equations. As we know, the Cauchy problem of the system (1.1) can be reduced into a more simple system in Lagrangian coordinates, which is not applicable for the outflow problem of the system (1.1). This brings some difficulties in our analysis because the system in Eulerian coordinates is more complicated than one in Lagrangian coordinates. On the other hand, when omitting the radiation effect, the system (1.1) reduces to the classical compressible Navier-Stokes equations. For the outflow problem of compressible Navier-Stokes equations, there have been many mathematical studies about the existence, stability and convergence rate of the stationary solutions, please refer to [2, 7, 9, 10, 16, 17, 20, 23, 24] and the references therein. Compared to the Navier-Stokes equations, system (1.5) is more general and more complex for the radiation effect is taken into account. For instance, in order to obtain the existence of stationary solutions, they in [10] considered 2×2 system of autonomous ordinary differential equations, but we have to introduce the new variable to deduce the stationary equations to a 4×4 system of autonomous ordinary differential equations, and examine dynamics around an equilibrium by applying the manifold theory (Section 2). Next, to deduce our results desired for the stability of the stationary solutions by the elementary energy method, it is sufficient to deduce certain uniform (with respect to the time t) a priori estimates on the perturbations $(\varphi,\psi,\zeta,\omega)$ around stationary solutions $(\hat\rho,\hat u,\hat\theta,\hat q)$. In the first step of a priori estimates, comparing with the Navier-Stokes equations, the main difficulty is to control the energy form (3.5) so that we get the uniform estimate for L_2-norm of the perturbations, which is not trivial due to control of the new term $-\zeta\left(\frac{q_x}{\theta}-\frac{\hat q_x}{\hat\theta}\right)$ (see (3.13) in Section 3). Last, the main point in proof of the convergence rate for the stationary solutions is how to get the lower estimate on the term $-w_xG^1$ in weighted energy form (4.3). For this, in the case of the ideal polytropic gas, they in [10] essentially utilize the expression $p=R\rho\theta$, $e=R(\gamma-1)^{-1}\theta$ on the pressure p and the inertial energy e as the function for independent variables (v,θ), where $\gamma>1$ denotes the adiabatic exponent and R is gas constant, which is not applicable for the general gas case (see Section 4).

This paper consists of the following. In Section 2, we prove the existence of the non-degenerate stationary solutions. Section 3 is devoted to showing the stability result (Theorem 1.2) of the non-degenerate stationary solutions. In Section 4, for the supersonic case, the convergence rate mentioned in Theorem 1.3 is obtained by a time- and space-weighted energy method.

2. The existence of non-degenerate stationary solutions

2.1. Reformulation of stationary problem. Integrating the first, second and third equations of (1.5) over $[x,\infty)$ yields

$$\begin{aligned}&\hat\rho\hat u=\rho_+u_+,\ x>0,\\&\hat\rho\hat u^2+\hat p=\mu\hat u_x+\rho_+u_+^2+p_+,\\&\hat\rho\hat u(\hat e+\frac{\hat u^2}{2})+\hat p\hat u+\hat q=\kappa\hat\theta_x+\mu\hat u\hat u_x+\rho_+u_+(e_++\frac{u_+^2}{2})+p_+u_+,\\&\hat q_x-\hat E-(\hat\theta^4-\theta_+^4)=0,\end{aligned}\tag{2.1}$$

where $p_+=p(v_+,\theta_+)$, $e_+=e(v_+,\theta_+)$, $\hat E(x)=-\int_x^\infty\hat q(y)dy$.

Integration of the first equation of (1.5) over $[0,x)$ is as following.

$$\hat\rho\hat u=\hat\rho(0)u_-,\ x>0.\tag{2.2}$$

By (2.1) and (2.2), (1.7) holds.

We set $\hat u=\frac{u_+}{v_+}\hat v\,(\hat v=\hat\rho^{-1},v_+=\rho_+^{-1})$, $\hat u_1=\hat u_x$. Then, we have from (2.1)

$$\begin{cases}\hat v_x=\frac{u_+}{\mu v_+}(\hat v-v_+)+\frac{v_+}{\mu u_+}(p(\hat v,\hat\theta)-p_+),\\\hat\theta_x=\frac{u_+}{\kappa v_+}(e(\hat v,\hat\theta)-e_+)-\frac{u_+^3}{2\kappa v_+^3}(\hat v-v_+)^2+\frac{u_+}{\kappa v_+}p_+(\hat v-v_+)+\frac{1}{\kappa}\hat q,\\\hat q_x=\hat E+(\hat\theta^4-\theta_+^4),\\\hat E_x=\hat q.\end{cases}\tag{2.3}$$

Also, we have from (1.5)

$$(\hat v,\hat\theta,\hat q)(0)=(v_-,\theta_-,0)\ \text{with}\ v_-=\frac{u_-}{u_+}v_+,\quad(\hat v,\hat\theta,\hat q,\hat E)(\infty)=(v_+,\theta_+,0,0).\tag{2.4}$$

To discuss the solvability of the system (2.3), (2.4) near the infinity asymptotic state $(v_+,\theta_+,0,0)$, we need to introduce the stationary perturbation variables given by

$$(\widetilde{v},\widetilde{\theta},\widetilde{q},\widetilde{E}) := (\hat{v},\hat{\theta},\hat{q},\hat{E}) - (v_+,\theta_+,0,0).$$

Then, the system (2.3), (2.4) is transformed into the vector equations for $(\widetilde{v},\widetilde{\theta},\widetilde{q},\widetilde{E})$

$$\begin{aligned} &\frac{d}{dx}\begin{pmatrix}\widetilde{v}\\ \widetilde{\theta}\\ \widetilde{q}\\ \widetilde{E}\end{pmatrix} = J_+\begin{pmatrix}\widetilde{v}\\ \widetilde{\theta}\\ \widetilde{q}\\ \widetilde{E}\end{pmatrix} + \begin{pmatrix}g_1(\widetilde{v},\widetilde{\theta})\\ g_2(\widetilde{v},\widetilde{\theta})\\ g_3(\widetilde{\theta})\\ 0\end{pmatrix}, \quad x>0 \\ &(\widetilde{v},\widetilde{\theta},\widetilde{q})(0) = (v_- - v_+,\theta_- - \theta_+,0), \quad (\widetilde{v},\widetilde{\theta},\widetilde{q},\widetilde{E})(\infty) = (0,0,0,0), \end{aligned} \tag{2.5}$$

where J_+ is the Jacobian matrix at an equilibrium point $(0,0,0,0)$ defined by

$$J_+ = \begin{pmatrix} \frac{v_+}{\mu u_+}\left(\frac{u_+^2}{v_+^2}+p_v^+\right) & \frac{v_+}{\mu u_+}p_\theta^+ & 0 & 0\\ \frac{u_+}{\kappa v_+}(e_v^+ + p_+) & \frac{u_+}{\kappa v_+}e_\theta^+ & \frac{1}{\kappa} & 0\\ 0 & 4\theta_+^3 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} \equiv \begin{pmatrix} a_{11} & a_{12} & 0 & 0\\ a_{21} & a_{22} & a_{23} & 0\\ 0 & a_{32} & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{2.6}$$

and $g_i (i=1,\cdots,4)$ are nonlinear terms that

$$\begin{aligned} g_1(\widetilde{v},\widetilde{\theta}) &= \frac{v_+}{\mu u_+}(\hat{p} - p_+ - p_v^+\widetilde{v} - p_\theta^+\widetilde{\theta}) = O(\widetilde{v}^2 + \widetilde{\theta}^2),\\ g_2(\widetilde{v},\widetilde{\theta}) &= \frac{u_+}{\kappa v_+}(\hat{e} - e_+ - e_v^+\widetilde{v} - e_\theta^+\widetilde{\theta}) - \frac{u_+^3}{2\kappa v_+^3}\widetilde{v}^2 = O(\widetilde{v}^2 + \widetilde{\theta}^2),\\ g_3(\widetilde{\theta}) &= (\widetilde{\theta}+\theta_+)^4 - \theta_+^4 - 4\theta_+^3\widetilde{\theta} = O(\widetilde{\theta}^2), \end{aligned}$$

where $p_v^+ = p_v(v_+,\theta_+)$, $e_v^+ = e_v(v_+,\theta_+)$ and so on.

2.2. Proof of Theorem 1.1. By (2.6), we have

$$J_+ - \lambda I = \begin{pmatrix} a_{11}-\lambda & a_{12} & 0 & 0\\ a_{21} & a_{22}-\lambda & a_{23} & 0\\ 0 & a_{32} & -\lambda & 1\\ 0 & 0 & 1 & -\lambda\end{pmatrix}$$

and the characteristic determinant of J_+ is

$$\begin{aligned} |J_+ - \lambda I| &= (-\lambda)\begin{vmatrix} a_{11}-\lambda & a_{12} & 0\\ a_{21} & a_{22}-\lambda & a_{23}\\ 0 & a_{32} & -\lambda\end{vmatrix} - \begin{vmatrix} a_{11}-\lambda & a_{12} & 0\\ a_{21} & a_{22}-\lambda & a_{23}\\ 0 & 0 & 1\end{vmatrix}\\ &= (\lambda^2-1)\begin{vmatrix} a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{vmatrix} + a_{23}\lambda\begin{vmatrix} a_{11}-\lambda & a_{12}\\ 0 & a_{32}\end{vmatrix}. \end{aligned}$$

Assume that $u_+ < 0$ and (1.4). Then, the eigenvalues $\lambda_i (i=1,\cdots,4)$ of J_+ must be satisfied

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0, \tag{2.7}$$

where

$$\begin{aligned} b_1 &= -(a_{11}+a_{22}) = -\frac{v_+}{\mu u_+}\left(\frac{u_+^2}{v_+^2}+p_v^+\right) - \frac{u_+}{\kappa v_+}e_\theta^+, \\ b_2 &= a_{11}a_{22}-a_{12}a_{21}-1-a_{23}a_{32} \\ &= \frac{1}{\mu\kappa}\left(\frac{u_+^2}{v_+^2}+p_v^+\right)e_\theta^+ - \frac{1}{\mu\kappa}\left(e_v^+ + p_+\right)p_\theta^+ - 1 - \frac{4\theta_+^3}{\kappa}, \\ b_3 &= a_{11}+a_{22}+a_{23}a_{32}a_{11} = -b_1 + \frac{4\theta_+^3}{\kappa}\frac{v_+}{\mu u_+}\left(\frac{u_+^2}{v_+^2}+p_v^+\right), \\ b_4 &= -(a_{11}a_{22}-a_{21}a_{12}) = -\frac{1}{\mu\kappa}\left(\frac{u_+^2}{v_+^2}+p_v^+\right)e_\theta^+ + \frac{1}{\mu\kappa}\left(e_v^+ + p_+\right)p_\theta^+. \end{aligned} \tag{2.8}$$

Using (1.6), we have

$$M_+ > 1(<1) \leftrightarrows \left(\frac{u_+^2}{v_+^2}+\widetilde{p}_v(v_+,s_+)\right) > 0(<0). \tag{2.9}$$

Noticing that

$$\widetilde{p}_v(v_+,s_+) = p_v^+ - \frac{\theta_+(p_\theta^+)^2}{e_\theta^+}, \quad e_v^+ = \theta_+ p_\theta^+ - p^+, \tag{2.10}$$

we have

$$\begin{aligned} a_{11}a_{22}-a_{12}a_{21} &\equiv \frac{v_+^2}{\mu\kappa}\left(\frac{u_+^2}{v_+^2}+p_v^+\right)e_\theta^+ - \frac{v_+^2}{\mu\kappa}\left(e_v^+ + p_+\right)p_\theta^+ \\ &= \frac{v_+^2}{\mu\kappa}\left(\frac{u_+^2}{v_+^2}+\widetilde{p}_v(v_+,s_+)\right)e_\theta^+, \\ a_{11} = \frac{v_+}{\mu u_+}&\left(\frac{u_+^2}{v_+^2}+\widetilde{p}_v(v_+,s_+)+\frac{\theta_+(p_\theta^+)^2}{e_\theta^+}\right). \end{aligned} \tag{2.11}$$

From Vieta's formula, the roots of the system (2.7) have the following properties:

$$\begin{aligned} &\lambda_1+\lambda_2+\lambda_3+\lambda_4 = -b_1, \\ &\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4 = b_2, \\ &\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4 = -b_3, \\ &\lambda_1\lambda_2\lambda_3\lambda_4 = b_4. \end{aligned} \tag{2.12}$$

For the case $M_+>1$: Using (1.4), (2.4)-(2.11) and $u_+<0$, we obtain from (2.8)

$$b_1>0, \quad b_3<0, \quad b_4<0,$$

which implies together with (2.12)

$$\begin{aligned} &\lambda_1\lambda_2\lambda_3\lambda_4 < 0, \\ &\lambda_1\lambda_2(\lambda_3+\lambda_4)+(\lambda_1+\lambda_2)\lambda_3\lambda_4 > 0, \\ &\lambda_1+\lambda_2+\lambda_3+\lambda_4 < 0. \end{aligned} \tag{2.13}$$

The first inequality of (2.13) implies (2.7) doesn't have any zero real root and we can assume $\lambda_1\lambda_2>0, \lambda_3\lambda_4<0$ without the loss of generality. Also, using the second and third inequalities of (2.13), we have $\lambda_1+\lambda_2<0$. So, without the loss of generality, we can assume

$$Re\lambda_1<0,\ Re\lambda_2<0,\ \lambda_3<0 \text{ and } \lambda_4>0.$$

For the case $M_+<1$: Using (1.4), (2.4)-(2.11) and $u_+<0$, we obtain from (2.8)

$$b_2<0, \quad b_4>0,$$

which implies together with (2.12)

$$\begin{aligned}&\lambda_1\lambda_2\lambda_3\lambda_4>0,\\ &\lambda_1\lambda_2+\lambda_3\lambda_4+(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)<0.\end{aligned} \tag{2.14}$$

Using (2.14), we deduce that (2.7) doesn't have any zero real root and the following possible cases:

$$\begin{aligned}&(1)\quad \lambda_1\lambda_2>0,\ \lambda_3\lambda_4>0,\ (\lambda_1+\lambda_2)(\lambda_3+\lambda_4)<0,\\ &(2)\quad \lambda_1\lambda_2<0,\ \lambda_3\lambda_4<0.\end{aligned} \tag{2.15}$$

Therefore, we can assume from (2.15), without the loss of generality,

$$Re\lambda_1<0,\ Re\lambda_2<0,\ \lambda_3>0 \text{ and } \lambda_4>0.$$

Now, we stand in position for the proof of Theorem 1.1. We will only discuss the case of $M_+<1$ because the case of $M_+>1$ is similar and more easy.

In order to make the manifold theory directly applicable, we need to reduce the system (2.5) to block diagonal form. By Jordan theorem in linear algebra, there is a real nonsingular matrix $Q=(q_{ij})_{4\times 4}$ such that

$$Q^{-1}J_+Q=\mathrm{diag(B,A)}, \tag{2.16}$$

where A is a 2×2 matrix having eigenvalues with positive real part, and B is a 2×2 matrix having eigenvalues with negative real part. Therefore, the linear transformation

$$\begin{pmatrix}\widetilde{V}\\ \widetilde{\Theta}\\ \widetilde{Q}\\ \widetilde{\Xi}\end{pmatrix}=Q^{-1}\begin{pmatrix}\widetilde{v}\\ \widetilde{\theta}\\ \widetilde{q}\\ \widetilde{E}\end{pmatrix}$$

applied to the system (2.5) yields the equivalent boundary value problem

$$\frac{d}{dx}\begin{pmatrix}\widetilde{V}\\ \widetilde{\Theta}\\ \widetilde{Q}\\ \widetilde{\Xi}\end{pmatrix}=\mathrm{diag(B,A)}\begin{pmatrix}\widetilde{V}\\ \widetilde{\Theta}\\ \widetilde{Q}\\ \widetilde{\Xi}\end{pmatrix}+\begin{pmatrix}H_1(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\\ H_2(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\\ H_3(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\\ H_4(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\end{pmatrix},\quad \mathrm{x}>0, \tag{2.17}$$

$$\begin{aligned}&q_{11}\widetilde{V}(0)+q_{12}\widetilde{\Theta}(0)+q_{13}\widetilde{Q}(0)+q_{14}\widetilde{\Xi}(0)=v_--v_+,\\ &q_{21}\widetilde{V}(0)+q_{22}\widetilde{\Theta}(0)+q_{23}\widetilde{Q}(0)+q_{24}\widetilde{\Xi}(0)=\theta_--\theta_+,\\ &q_{31}\widetilde{Q}(0)+q_{32}\widetilde{\Theta}(0)+q_{33}\widetilde{Q}(0)+q_{34}\widetilde{\Xi}(0)=0,\end{aligned} \tag{2.18}$$

$$(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})(\infty)=(0,0,0,0), \tag{2.19}$$

where $H_i(i=1,\cdots,4)$ are defined by

$$\begin{pmatrix} H_1(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi}) \\ H_2(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi}) \\ H_3(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi}) \\ H_4(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi}) \end{pmatrix} = Q^{-1} \begin{pmatrix} g_1(\widetilde{v},\widetilde{\theta}) \\ g_2(\widetilde{v},\widetilde{\theta}) \\ g_1(\widetilde{v},\widetilde{\theta}) \\ g_3(\widetilde{\theta}) \\ 0 \end{pmatrix}.$$

For the sake of technique only, it is convenient to introduce an undetermined parameter $\widetilde{E}_0 := \widetilde{E}(0)$, simultaneously add the auxiliary boundary condition $q_{41}\widetilde{V}(0)+q_{42}\widetilde{\Theta}(0)+q_{43}\widetilde{Q}(0)+q_{44}\widetilde{\Xi}(0)=\widetilde{E}_0$ which is combined with (2.18) and described very succinctly as

$$\begin{pmatrix} \widetilde{V}(0) \\ \widetilde{\Theta}(0) \\ \widetilde{Q}(0) \\ \widetilde{\Xi}(0) \end{pmatrix} = Q^{-1} \begin{pmatrix} v_- - v_+ \\ \theta_- - \theta_+ \\ 0 \\ \widetilde{E}_0 \end{pmatrix}. \tag{2.20}$$

Since the previous argument proceeds inductively to yield the fact that J_+ has two negative eigenvalues $\lambda_i(i=1,2)$ as well as two eigenvalues with positive real part. By virtue of the manifold theory in [1], there exist a C^∞ local stable manifold $W^s_{loc}(0,0,0,0)$ corresponding to $\lambda_i(i=1,2)$ and a C^∞ local unstable manifold $W^u_{loc}(0,0,0,0)$ corresponding to $\lambda_i(i=3,4)$. More specifically, $W^s_{loc}(0,0,0,0)$ can locally be represented by a graph over the $(\widetilde{V},\widetilde{\Theta})$ variables, i.e.,

$$\begin{aligned} W^s_{loc}(0,0,0,0) = \{&(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\in R^4 \,|\, \exists C^\infty \text{ functions } h^s_{\widetilde{Q}} \text{ and } h^s_{\widetilde{\Xi}} \\ &\text{s.t. } \widetilde{Q}=h^s_{\widetilde{Q}}(\widetilde{V},\widetilde{\Theta}),\ \widetilde{\Xi}=h^s_{\widetilde{\Xi}}(\widetilde{V},\widetilde{\Theta}) \text{ with } h^s_{\widetilde{Q}}(0,0)=Dh^s_{\widetilde{Q}}(0,0)=0, \\ &h^s_{\widetilde{\Xi}}(0,0)=Dh^s_{\widetilde{\Xi}}(0,0)=0, \text{ for } |(\widetilde{V},\widetilde{\Theta})| \text{ sufficiently small}\}. \end{aligned}$$

Furthermore, if $(\widetilde{V}(0),\widetilde{\Theta}(0),\widetilde{Q}(0),\widetilde{\Xi}(0))$ is located on the stable manifold $W^s_{loc}(0,0,0,0)$, then the problem (2.17), (2.19) and (2.20) has a unique smooth solution $(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})$ which approaches the origin $(0,0,0,0)$ at an exponential rate asymptotically as $x\to\infty$, i.e.,

$$|\partial^k_x(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})(\xi)| \le C(|\widetilde{V}(0)|+|\widetilde{\Theta}(0)|)e^{-cx}, \text{ for } k=0,1,2,\cdots. \tag{2.21}$$

Next we assert that if

$$(\widetilde{V}(0),\widetilde{\Theta}(0),\widetilde{Q}(0),\widetilde{\Xi}(0)) \in \{(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\in R^4 \,|\, \widetilde{Q}=h^s_{\widetilde{Q}}(\widetilde{V},\widetilde{\Theta}),\ \widetilde{\Xi}=h^s_{\widetilde{\Xi}}(\widetilde{V},\widetilde{\Theta})\},$$

the original stationary problem (1.5) with $|v_- - v_+|+|\theta_- - \theta_+| \ll 1$ is equivalent to the boundary value problem (2.17), (2.19) and (2.20) with $|\widetilde{V}(0)|+|\widetilde{\Theta}(0)| \ll 1$. It suffices to show that $(\widetilde{V}(0),\widetilde{\Theta}(0))$ depends locally and only on the original data $(v_- - v_+, \theta_- - \theta_+)$ in a continuous differentiable way. In fact, by premultiplying both sides of the equality (2.16) by Q and using (2.6), we immediately deduce that $J_+Q = Q\text{diag}(\text{B},\text{A})$ including the following algebraic equations:

$$\begin{cases} a_{11}q_{11}+a_{12}q_{21}+a_{14}q_{41}=b_{11}q_{11}+b_{21}q_{12}, \\ a_{21}q_{11}+a_{22}q_{21}=b_{11}q_{21}+b_{21}q_{22}, \\ a_{11}q_{12}+a_{12}q_{22}+a_{14}q_{42}=b_{12}q_{11}+b_{22}q_{12}, \\ a_{21}q_{12}+a_{22}q_{22}=b_{12}q_{21}+b_{22}q_{22}, \end{cases} \tag{2.22}$$

and

$$q_{31}=b_{11}q_{41}+b_{21}q_{42}, \quad q_{32}=b_{12}q_{41}+b_{22}q_{42} \tag{2.23}$$

according to the definition of matrix multiplication, where $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}$.

Using (2.22) and (2.23), we show that the matrix $\hat{Q}=\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22}\end{pmatrix}$ is nonsingular.

If the matrix $\hat{Q}$ is singular, then we know that there exists a real number β such that

$$q_{11}=\beta q_{12} \quad \text{and} \quad q_{21}=\beta q_{22}. \tag{2.24}$$

Substituting (2.24) into (2.22) yields

$$\begin{cases} a_{11}q_{11}+a_{12}q_{21}+a_{14}q_{41}=\beta_1 q_{12}, \\ a_{21}q_{11}+a_{22}q_{21}=\beta_1 q_{22}, \\ a_{11}q_{12}+a_{12}q_{22}+a_{14}q_{42}=\beta_2 q_{12}, \\ a_{21}q_{12}+a_{22}q_{22}=\beta_2 q_{22} \end{cases} \tag{2.25}$$

with $\beta_1=b_{11}\beta+b_{21}$ and $\beta_2=b_{12}\beta+b_{22}$.

By (2.24), $(2.25)_2$ and $(2.25)_4$, we obtain

$$\beta_1=\beta\beta_2. \tag{2.26}$$

Using (2.24), (2.26), $(2.25)_1$ and $(2.25)_3$ yields

$$q_{41}=\beta q_{42}. \tag{2.27}$$

Also, by (2.27), (2.26) and (2.23), we get

$$q_{31}=\beta q_{32}. \tag{2.28}$$

By (2.24), (2.27) and (2.28), we obtain the fact that the vector $(q_{11},q_{21},q_{31},q_{41})$ is parallel to the vector $(q_{12},q_{22},q_{32},q_{42})$, which is impossible since the matrix Q is nonsingular. Therefore, the matrix $\hat{Q}$ is nonsingular.

Notice that

$$(\widetilde{V}(0),\widetilde{\Theta}(0),\widetilde{Q}(0),\widetilde{\Xi}(0))\in\{(\widetilde{V},\widetilde{\Theta},\widetilde{Q},\widetilde{\Xi})\in R^4 \mid \widetilde{Q}=h^s_{\widetilde{Q}}(\widetilde{V},\widetilde{\Theta}),\ \widetilde{\Xi}=h^s_{\widetilde{\Xi}}(\widetilde{V},\widetilde{\Theta})\},$$

therefore the first and second equations in (2.18) can be rewritten as

$$\begin{aligned} &q_{11}\widetilde{V}(0)+q_{12}\widetilde{\Theta}(0)+q_{13}h^s_{\widetilde{Q}}(\widetilde{V}(0),\widetilde{\Theta}(0))+q_{14}h^s_{\widetilde{\Xi}}(\widetilde{V}(0),\widetilde{\Theta}(0))=v_- - v_+, \\ &q_{21}\widetilde{V}(0)+q_{22}\widetilde{\Theta}(0)+q_{23}h^s_{\widetilde{Q}}(\widetilde{V}(0),\widetilde{\Theta}(0))+q_{24}h^s_{\widetilde{\Xi}}(\widetilde{V}(0),\widetilde{\Theta}(0))=\theta_- - \theta_+. \end{aligned} \tag{2.29}$$

Because the matrix $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ is nonsingular, by using the implicit function theorem, one easily solves the Equation (2.29) for $(\widetilde{V}(0),\widetilde{\Theta}(0))$ to obtain a unique C^1 function of $(v_- - v_+, \theta_- - \theta_+)$ in a neighborhood of the origin $(0,0)$. Thus, by using the differential mean value theorem, we have

$$c(|v_- - v_+| + |\theta_- - \theta_+|) \leq |\widetilde{V}(0)| + |\widetilde{\Theta}(0)| \leq C(|v_- - v_+| + |\theta_- - \theta_+|), \tag{2.30}$$

if $|v_- - v_+| + |\theta_- - \theta_+| \ll 1$. This implies the assertion mentioned at the beginning of this paragraph holds. In addition, from (2.30), it follows that the condition (2.21) is also equivalent to (1.8). By combining the information as above, we complete the proof of Theorem 1.1.

3. Asymptotic stability of stationary solutions

We rewrite (1.1) and (1.5) as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \qquad t>0,\ x>0, \\ \rho(u_t + uu_x) + p_x = \mu u_{xx}, \\ \rho(e_t + ue_x) + pu_x + q_x = \kappa\theta_{xx} + \mu u_x^2, \\ \rho\theta(s_t + us_x) + q_x = \kappa\theta_{xx} + \mu u_x^2,\ s = s(\rho,\theta), \\ -q_{xx} + q + (\theta^4)_x = 0 \end{cases} \tag{3.1}$$

and

$$\begin{cases} (\hat\rho\hat u)_x = 0, \qquad x>0,\ t>0, \\ \hat\rho\hat u\hat u_x + \hat p_x = +\mu\hat u_{xx},\ \hat p = p(\hat\rho,\hat\theta), \\ \hat\rho\hat u\hat e_x + \hat p\hat u_x + \hat q_x = \kappa\hat\theta_{xx} + \mu\hat u_x^2,\ \hat e = e(\hat\rho,\hat\theta), \\ \hat\rho\hat\theta\hat u\hat s_x + \hat q_x = \kappa\hat\theta_{xx} + \mu\hat u_x^2,\ \hat s = s(\hat\rho,\hat\theta), \\ -\hat q_{xx} + \hat q + (\hat\theta^4)_x = 0. \end{cases} \tag{3.2}$$

Perturbation $(\varphi,\psi,\zeta,\omega)$ and the solution space $X(I)$ is as following, respectively.

$$\begin{aligned} &(\varphi,\psi,\zeta,\omega)(x,t) = (\rho,u,\theta,q)(x,t) - (\hat\rho,\hat u,\hat\theta,\hat q)(x) \\ &\qquad X(I) = \{(\varphi,\psi,\zeta,\omega) \,|\, (\varphi,\psi,\zeta,\omega,\omega_x) \in C(I;H^1), \\ &\qquad\qquad \varphi_x \in L_2(I;L_2),\ (\psi_x,\zeta_x,\omega,\omega_x) \in L_2(I;H^1)\} \end{aligned}$$

for any interval $I \subset [0,\infty)$.

Local existence of the stationary solution to the system (1.1)-(1.4) can be established by the standard iteration argument and hence will be skipped in the paper. To prove Theorem 1.2, a crucial step is to show the following a priori estimate.

PROPOSITION 3.1 (A priori estimate). *Suppose that (ρ,u,θ,q) is the solution of the system (1.1)-(1.4) satisfying $(\phi,\psi,\zeta,\chi) \in X([0,T])$. Then, there is a suitable positive constant ε_1 that satisfies*

$$\sup_{0\leq t\leq T} \|(\varphi,\psi,\zeta,\omega,\omega_x)(t)\|_1 \leq \varepsilon_1 \quad \textit{and} \quad \delta = |(u_- - u_+, \theta_- - \theta_+)| \leq \varepsilon_1, \tag{3.3}$$

for any $t \in [0,T]$, it that

$$\|(\varphi,\psi,\zeta,\omega,\omega_x)(t)\|_1^2 + \int_0^t \left(\|\varphi_x(\tau)\|^2 + \|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)\|_1^2\right) d\tau$$

$$+\int_0^t |(\varphi,\varphi_x)(0,\tau)|^2 d\tau \le C\|(\varphi,\psi,\zeta)(0)\|_1^2, \tag{3.4}$$

In the remainder of this section, we will prove Proposition 3.1.

3.1. Energy form. Let

$$\begin{aligned}\mathcal{E} :=& (e-\hat{e}) - \hat{\theta}(s-\hat{s}) + \frac{\psi^2}{2} + \hat{p}\left(\frac{1}{\rho} - \frac{1}{\hat{\rho}}\right)\\ =& (e-\hat{\theta}s) + \frac{\psi^2}{2} + \hat{p}(v-\hat{v}) - (\hat{e}-\hat{\theta}\hat{s}),\end{aligned} \tag{3.5}$$

from (3.1) and (3.2),

$$\begin{aligned}&(\rho\mathcal{E})_t + (\rho u\mathcal{E})_x = \rho\mathcal{E}_t + (\rho u)\mathcal{E}_x\\ =&\rho\Big((e_t+ue_x) - \hat{\theta}(s_t+us_x)\Big) - \rho us\hat{\theta}_x + \rho\psi(u_t+uu_x-\psi\hat{u}_x-\hat{u}\hat{u}_x)\\ &+\rho\hat{p}v_t + \rho u(\hat{p}(v-\hat{v}))_x - \rho u\left(\hat{e}-\hat{\theta}\hat{s}\right)_x\\ =&\left(1-\frac{\hat{\theta}}{\theta}\right)(\kappa\theta_{xx}+\mu u_x^2) - pu_x - \left(1-\frac{\hat{\theta}}{\theta}\right)q_x - \rho us\hat{\theta}_x\\ &+\mu\psi\psi_{xx} + \mu\rho\psi\hat{u}_{xx}(v-\hat{v}) - \rho\psi^2\hat{u}_x - \rho\psi\left(\frac{p_x}{\rho} - \frac{\hat{p}_x}{\hat{\rho}}\right) + \rho\psi\chi\\ &+\hat{p}u_x + \rho u(\hat{p}_x(v-\hat{v}) - \hat{p}\hat{v}_x) - \rho u\left(\hat{e}_x - \hat{\theta}_x\hat{s} - \hat{\theta}\hat{s}_x\right).\end{aligned} \tag{3.6}$$

Using the calculation results in the literature [2], the Equation (3.6) can be written as

$$(\rho\mathcal{E})_t + (\rho u\mathcal{E})_x + \mu\frac{\hat{\theta}}{\theta}\psi_x^2 + \kappa\frac{\hat{\theta}}{\theta^2}\zeta_x^2 = \Delta_{1x} + \Delta_2 + \Delta_3 - \zeta\left(\frac{q_x}{\theta} - \frac{\hat{q}_x}{\hat{\theta}}\right), \tag{3.7}$$

where

$$\begin{aligned}&\Delta_1 = \mu\psi\psi_x + \kappa\frac{\zeta\zeta_x}{\theta} - (p-\hat{p})\psi,\\ &\Delta_2 = \kappa\frac{\hat{\theta}_x\zeta\zeta_x}{\theta^2} - (\kappa\hat{\theta}_{xx}+\mu\hat{u}_x^2)\frac{\zeta^2}{\theta\hat{\theta}} + 2\mu\frac{\zeta}{\theta}\psi_x\hat{u}_x + \mu\rho\psi\hat{u}_{xx}(v-\hat{v}) - \rho\psi^2\hat{u}_x,\\ &\Delta_3 = -(p-\hat{p})\hat{u}_x - \rho\psi\hat{p}_x(v-\hat{v}) + \rho u\left(\hat{p}_x(v-\hat{v}) - \hat{\theta}_x(s-\hat{s})\right) + \hat{\rho}\hat{u}\hat{s}_x\zeta.\end{aligned}$$

3.2. The proof of Proposition 3.1. We prove Proposition 3.1 by the following five steps.

Step 1: Energy estimate.

For notational simplicity, we introduce $A \lesssim B$ if $A \le C_0 B$ holds uniformly on the constant C_0 independently of t, x, T, ε_1.

Due to the assumptions of Proposition 3.1, it is easy to check that

$$(\varphi^2+\psi^2+\zeta^2) \lesssim \mathcal{E}(x,t) \lesssim (\varphi^2+\psi^2+\zeta^2). \tag{3.8}$$

from (3.8), $u_- < 0$ and $(\psi,\zeta)|_{x=0} = 0$, we have

$$\Delta_1|_{x=0} = 0, \quad -(\rho u\mathcal{E})|_{x=0} \gtrsim \varphi^2(0,t). \tag{3.9}$$

Using the last equations in (3.1) yields

$$4\theta^3\zeta_x = \omega_{xx} - \omega - 4\hat{\theta}_x\left(\theta^3 - \hat{\theta}^3\right). \tag{3.10}$$

Noticing that

$$-\zeta\left(\frac{q_x}{\theta} - \frac{\hat{q}_x}{\hat{\theta}}\right) = -\frac{\zeta}{\theta}\omega_x + \hat{q}_x\frac{\zeta^2}{\theta\hat{\theta}}, \tag{3.11}$$

and using (3.10), (1.8), (3.3) and the inequality

$$|f(x)| \le |f(0)| + \sqrt{x}\|f_x\|, \forall f \in H^1(0,\infty), \tag{3.12}$$

we have

$$\begin{aligned}
&-\int_0^\infty \zeta\left(\frac{q_x}{\theta} - \frac{\hat{q}_x}{\hat{\theta}}\right)dx = \int_0^\infty \zeta_x\frac{\omega}{\theta}dx + \int_0^\infty \zeta\omega\left(\frac{1}{\theta}\right)_x dx + \int_0^\infty \hat{q}_x\frac{\zeta^2}{\theta\hat{\theta}}dx\\
&= \int_0^\infty \frac{\omega\omega_{xx}}{4\theta^4}dx - \int_0^\infty \frac{\omega^2}{4\theta^4}dx - \int_0^\infty \hat{\theta}_x\omega\left(1 - \frac{\theta^3}{\hat{\theta}^3}\right)dx\\
&\quad - \int_0^\infty \zeta\frac{\omega\zeta_x}{\theta^2}dx - \int_0^\infty \hat{\theta}_x\frac{\omega\zeta}{\theta^2}dx + \int_0^\infty \hat{q}_x\frac{\zeta^2}{\theta\hat{\theta}}dx\\
&\le -\int_0^\infty \frac{\omega_x^2}{4\theta^4}dx - \int_0^\infty \frac{\omega^2}{4\theta^4}dx + C_0\int_0^\infty |\omega|\left(|\omega_x||\zeta_x| + |\omega_x||\hat{\theta}_x| + |\hat{\theta}_x||\zeta|\right)dx\\
&\quad + C_0\varepsilon_1\int_0^\infty |\omega||\zeta_x|dx + C_0\int_0^\infty \left(|\hat{\theta}_x||\omega||\zeta| + |\hat{q}_x||\zeta|^2\right)dx\\
&\le -\int_0^\infty \frac{\omega_x^2}{4\theta^4}dx - \int_0^\infty \frac{\omega^2}{4\theta^4}dx + C_0\varepsilon_1\int_0^\infty (|\omega_x| + |\omega|)|\zeta_x|dx + C_0\delta\int_0^\infty |\omega_x||\omega|dx\\
&\quad + C_0\delta\int_0^\infty \omega_x^2 dx + C_0\delta\int_0^\infty x\exp(-\hat{c}x)(\|\omega_x\|\|\zeta_x\| + \|\zeta_x\|^2)dx\\
&\le -\int_0^\infty \frac{\omega_x^2}{4\theta^4}dx - \int_0^\infty \frac{\omega^2}{4\theta^4}dx + C_0(\varepsilon_1 + \delta)\left(\|\omega_x\|^2 + \|\omega\|^2 + \|\zeta_x\|^2\right).
\end{aligned} \tag{3.13}$$

After integrating (3.7) for (x,t), using (3.8), (3.9) and (3.13) yields

$$\begin{aligned}
&\|(\varphi,\psi,\zeta)(t)\|^2 + \int_0^t \|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)\|^2 d\tau + \int_0^t |\varphi(0,\tau)|^2 d\tau\\
&\lesssim \|(\varphi,\psi,\zeta)(0)\|^2 + \sum_{k=2}^{3}\int_0^t\int_0^\infty |\Delta_k|dxd\tau.
\end{aligned} \tag{3.14}$$

Using (1.8) and (3.12) yields

$$\begin{aligned}
|\Delta_2| &\lesssim \delta|(\psi_x,\zeta_x)|^2 + \delta|(\varphi,\psi,\zeta)|^2\exp(-\hat{c}x)\\
&\lesssim \delta|(\psi_x,\zeta_x)|^2 + \delta|\varphi(0,\tau)|^2 + \delta\|(\varphi_x,\psi_x,\zeta_x)\|^2 x\exp(-\hat{c}x).
\end{aligned} \tag{3.15}$$

By (3.15), we have

$$\int_0^t\int_0^\infty |\Delta_2|dxd\tau \lesssim \delta\int_0^t \|(\varphi_x,\psi_x,\zeta_x)(\tau)\|^2 d\tau + \delta\int_0^t |\varphi(0,\tau)|^2 d\tau. \tag{3.16}$$

Let us estimate Δ_3.
From $\rho^2\theta_\rho(\rho,s)=p_s(\rho,s)$ and $\hat{p}_x=p_\rho(\hat{\rho},\hat{s})\hat{\rho}_x+p_s(\hat{\rho},\hat{s})\hat{s}_x$, $\quad \hat{u}\hat{\rho}_x=-\hat{u}_x\hat{\rho}$,

$$\begin{aligned}&-\rho\psi\hat{p}_x(v-\hat{v})+\rho u\hat{p}_x(v-\hat{v})=-\frac{\hat{u}}{\hat{\rho}}\hat{p}_x(\rho-\hat{\rho})\\ =&p_\rho(\hat{\rho},\hat{s})(\rho-\hat{\rho})\hat{u}_x-\theta_\rho(\hat{\rho},\hat{s})\hat{\rho}\hat{u}\hat{s}_x(\rho-\hat{\rho}).\end{aligned} \tag{3.17}$$

By using (3.17) and $\hat{\theta}_v=-\hat{\rho}^2\hat{\theta}_\rho=-\hat{p}_s$,

$$\begin{aligned}\Delta_3=&-(p-\hat{p})\hat{u}_x+\hat{p}_\rho(\rho-\hat{\rho})\hat{u}_x+\hat{\rho}\hat{u}\hat{s}_x\hat{\theta}_\rho(\hat{\rho}-\rho)\\ &-\hat{\rho}\hat{u}\left(\hat{\theta}_v\hat{v}_x+\hat{\theta}_s\hat{s}_x\right)(s-\hat{s})+\hat{\rho}\hat{u}\hat{s}_x(\theta-\hat{\theta})+(\hat{\rho}\hat{u}-\rho u)\hat{\theta}_x(s-\hat{s})\\ =&-\hat{u}_x\left(p-\hat{p}-\hat{p}_\rho(\rho-\hat{\rho})-\hat{p}_s(s-\hat{s})\right)\\ &+\hat{\rho}\hat{u}\hat{s}_x\left(\theta-\hat{\theta}-\hat{\theta}_\rho(\rho-\hat{\rho})-\hat{\theta}_s(s-\hat{s})\right)+(\hat{\rho}\hat{u}-\rho u)\hat{\theta}_x(s-\hat{s}),\end{aligned} \tag{3.18}$$

where $\hat{p}_s=p_s(\hat{\rho},\hat{s})$, $\hat{\theta}_s=\theta_s(\hat{\rho},\hat{s})$, $\hat{p}_\rho=p_\rho(\hat{\rho},\hat{s})$ and $\hat{\theta}_\rho=\theta_\rho(\hat{\rho},\hat{s})$.
Using (3.18), by the same methods as in (3.16), we have

$$\begin{aligned}\int_0^t\int_0^\infty|\Delta_3|dxd\tau\lesssim&\,\delta\int_0^t\int_0^\infty\left(|\varphi|^2+|\psi|^2+|\zeta|^2\right)\exp(-\hat{c}x)dxd\tau\\ \lesssim&\,\delta\int_0^t\|(\varphi_x,\psi_x,\zeta_x)(\tau)\|^2d\tau+\delta\int_0^t|\varphi(0,\tau)|^2d\tau.\end{aligned}$$

By the estimations for $\Delta_k(k=2,3)$ and (3.14), we have

$$\begin{aligned}&\|(\varphi,\psi,\zeta)(t)\|^2+\int_0^t\|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)\|^2d\tau+\int_0^t|\varphi(0,\tau)|^2d\tau\\ \lesssim&\|(\varphi,\psi,\zeta)(0)\|^2+\delta\int_0^t\|\varphi_x(\tau)\|^2d\tau.\end{aligned} \tag{3.19}$$

Step 2: Estimation of $\|\varphi_x(t)\|$.
Using the calculation results in the literature [2], we have

$$\begin{aligned}&\left(\frac{\mu\varphi_x^2}{2\rho^3}+\frac{\varphi_x\psi}{\rho}\right)_t+\left(\frac{\mu u\varphi_x^2}{2\rho^3}-\frac{\varphi_t}{\rho}\psi\right)_x+\frac{p_\rho}{\rho^2}\varphi_x^2=\sum_{k=1}^3 f_k,\\ &f_3:=-\frac{\varphi_x}{\rho^2}\left(p_\theta\zeta_x+\hat{\rho}_x(p_\rho-\hat{p}_\rho)+\hat{\theta}_x(p_\theta-\hat{p}_\theta)\right).\end{aligned} \tag{3.20}$$

By using $\psi_i(0,t)=0$, $u(0,t)=u_-<0$ and (1.8), we have

$$\begin{aligned}&\int_0^\infty\left(\frac{\mu u\varphi_x^2}{2\rho^3}-\frac{\varphi_t}{\rho}\psi\right)_x dx=-\frac{\mu u_-}{2\rho^3(0,t)}\varphi_x^2(0,t)\gtrsim\varphi_x^2(0,t),\\ &|f_1|\lesssim\delta\exp(-\hat{c}x)\left(\varphi_x^2+\psi_x^2+\varphi^2+\psi^2\right),\\ &|f_2|\lesssim\delta\exp(-\hat{c}x)\left(\varphi_x^2+\psi_x^2+\varphi^2+\psi^2\right)+\epsilon\varphi_x^2+\epsilon^{-1}\psi_x^2,\\ &|f_3|\lesssim\delta\exp(-\hat{c}x)\left(\varphi^2+\zeta^2\right)+\epsilon\varphi_x^2+\epsilon^{-1}\zeta_x^2,\ \forall\epsilon>0.\end{aligned} \tag{3.21}$$

After integrating (3.20) for (x,t) and using $p_\rho(\rho,\theta)>0$ and (3.21), by the same arguments as in step 1, we have

$$\|\varphi_x(t)\|^2+\int_0^t\|\varphi_x(\tau)\|^2d\tau+\int_0^t|\varphi_x(0,\tau)|^2d\tau$$

$$\lesssim \left(\|\varphi_x(0)\|^2+\|\psi(t)\|^2\right)+\int_0^t \|(\psi_x,\zeta_x)(\tau)\|^2 d\tau. \tag{3.22}$$

By (3.22) and (3.19), we get

$$\|\varphi_x(t)\|^2+\int_0^t \|\varphi_x\|^2 d\tau+\int_0^t |\varphi_x(0,\tau)|^2 d\tau \lesssim \|(\varphi,\psi,\zeta,\varphi_x)(0)\|^2. \tag{3.23}$$

Step 3: Estimation for $\|\psi_x(t)\|$. Subtracting the second equation in (3.2) from the second equation in (3.1) and multiplying it by $-\psi_{xx}\rho^{-1}$ yields

$$\begin{aligned}&\left(\frac{\psi_x^2}{2}\right)_t-(\psi_t\psi_x)_x+\rho^{-1}\mu\psi_{xx}^2=f_4,\\ &f_4=u\psi_x\psi_{xx}+\rho^{-1}(p-\hat{p})_x\psi_{xx}+\rho^{-1}(\hat{\rho}\hat{u}-\rho u)\hat{u}_x\psi_{xx}.\end{aligned} \tag{3.24}$$

By using (1.8) and $\psi|_{x=0}=0$, we have

$$\begin{aligned}&\int_0^\infty (\psi_t\psi_x)_x dx=0,\\ &|f_4|\lesssim(\epsilon+\delta)\psi_{xx}^2+\epsilon^{-1}|(\varphi_x,\psi_x,\zeta_x)|^2+\delta\exp(-\hat{c}x)|(\varphi,\psi,\zeta)|^2,\ \forall\epsilon>0.\end{aligned} \tag{3.25}$$

After integrating (3.24) for (x,t), using (3.25), we have

$$\|\psi_x(t)\|^2+\int_0^t \|\psi_{xx}\|^2 d\tau \lesssim \|\psi_x(0)\|^2+\int_0^t \|(\varphi_x,\psi_x,\zeta_x)\|^2 d\tau+\int_0^t \varphi^2(0,\tau)d\tau.$$

Step 4: Estimation for $\|\zeta_x(t)\|$.

Also, subtracting $(3.2)_3$ from $(3.1)_3$ and using $e_t=e_\theta(\rho,\theta)\theta_t-e_\rho(\rho,\theta)(\rho u)_x$, we have

$$\begin{aligned}\rho e_\theta(\rho,\theta)\zeta_t+\rho u\zeta_x-\kappa\zeta_{xx}+\omega_x&=\rho e_\rho(\rho,\theta)(\rho u-\hat{\rho}\hat{u})_x\\ &\quad+(\hat{u}\hat{\rho}-\rho u)\hat{e}_x-(pu_x-\hat{p}\hat{u}_x)+\mu(u_x^2-\hat{u}_x^2).\end{aligned}$$

Multiplying it by $-\frac{\zeta_{xx}}{\rho e_\theta(\rho,\theta)}$ yields

$$\left(\frac{\zeta_x^2}{2}\right)_t-(\zeta_t\zeta_x)_x+\frac{\kappa\zeta_{xx}^2}{\rho e_\theta(\rho,\theta)}=\frac{\omega_x\zeta_{xx}}{\rho e_\theta(\rho,\theta)}+f_5, \tag{3.26}$$

where

$$\begin{aligned}f_5=&\frac{\zeta_{xx}}{\rho e_\theta(\rho,\theta)}\left(\rho u\zeta_x+(pu_x-\hat{p}\hat{u}_x)-\rho e_\rho(\rho,\theta)(\rho u-\hat{\rho}\hat{u})_x\right)\\ &-\frac{\zeta_{xx}}{\rho e_\theta(\rho,\theta)}\left((\hat{u}\hat{\rho}-\rho u)\hat{e}_x+\mu(u_x^2-\hat{u}_x^2)\right).\end{aligned}$$

By using (1.8) and $\zeta|_{x=0}=0$, we have

$$\begin{aligned}&\int_0^\infty (\zeta_t\zeta_x)_x dx=0,\\ &|f_5|\lesssim(\epsilon+\delta)\zeta_{xx}^2+\epsilon^{-1}|(\varphi_x,\psi_x,\zeta_x)|^2+\delta\exp(-\hat{c}x)|(\varphi,\psi,\zeta)|^2,\ \forall\epsilon>0.\end{aligned} \tag{3.27}$$

Noticing that

$$-\omega_{xx}+\omega+4\theta^3\zeta_x+4(\theta^2+\theta\hat{\theta}+\hat{\theta}^2)\zeta\hat{\theta}_x=0$$

due to $(3.1)_5$ and $(3.2)_5$, we have

$$\zeta_x=\frac{\omega_{xx}}{4\theta^3}-\frac{\omega}{4\theta^3}-\zeta\hat{\theta}_x\frac{(\theta^2+\theta\hat{\theta}+\hat{\theta}^2)}{\theta^3}. \tag{3.28}$$

Therefore, using (3.28) yields

$$\begin{aligned}
&\int_0^\infty \frac{\omega_x\zeta_{xx}}{\rho e_\theta(\rho,\theta)}dx=-\int_0^\infty \zeta_x\left(\frac{\omega_x}{\rho e_\theta(\rho,\theta)}\right)_x dx\\
=&-\int_0^\infty \frac{\omega_{xx}^2}{4\theta^3\rho e_\theta(\rho,\theta)}dx-\int_0^\infty \frac{\omega_{xx}\omega_x}{4\theta^3}\left(\frac{1}{\rho e_\theta(\rho,\theta)}\right)_x dx\\
&-\int_0^\infty \left(\frac{\omega}{4\theta^3}\right)_x\frac{\omega_x}{\rho e_\theta(\rho,\theta)}dx-\int_0^\infty\left(\zeta\hat{\theta}_x\frac{(\theta^2+\theta\hat{\theta}+\hat{\theta}^2)}{\theta^3}\right)_x\frac{\omega_x}{\rho e_\theta(\rho,\theta)}dx\\
=&-\int_0^\infty \frac{\omega_{xx}^2}{4\theta^3\rho e_\theta(\rho,\theta)}dx+I_1+I_2+I_3.
\end{aligned}\tag{3.29}$$

After integrating (3.26) for (x,t), using (3.29), (3.27) and $e_\theta(\rho,\theta)>0$, we have

$$\begin{aligned}
&\|\zeta_x(t)\|^2+\int_0^t\|(\zeta_{xx},\omega_{xx})\|^2d\tau\\
\lesssim&\|\zeta_x(0)\|^2+\int_0^t\|(\varphi_x,\psi_x,\zeta_x)\|^2d\tau+\int_0^t\varphi^2(0,\tau)d\tau+\sum_{j=1}^3\int_0^t|I_j|d\tau.
\end{aligned}\tag{3.30}$$

We estimate $I_j(j=1,2,3)$. Using (3.3), (1.8) and (3.12), we have

$$\begin{aligned}
|I_1|\lesssim&\int_0^\infty|\omega_{xx}||\omega_x|(|\varphi_x|+|\zeta_x|+|\hat{\rho}_x|+|\hat{\theta}_x|)dx\\
\lesssim&\varepsilon_1\int_0^\infty|\omega_{xx}|(|\varphi_x|+|\zeta_x|)dx+\delta\int_0^\infty|\omega_{xx}||\omega_x|dx\\
\lesssim&(\varepsilon_1+\delta)\|(\omega_{xx},\varphi_x,\zeta_x,\omega_x)\|^2.
\end{aligned}\tag{3.31}$$

Also, by the same lines as in (3.31), we have

$$\begin{aligned}
|I_2|\lesssim&\int_0^\infty[|\omega_x|^2+|\omega|(|\zeta_x|+|\hat{\theta}_x|)|\omega_x|]dx\\
\lesssim&\|\omega_x\|^2+\varepsilon_1\|(\zeta_x,\omega_x)\|^2+\delta\|(\omega_x,\omega)\|^2
\end{aligned}\tag{3.32}$$

and

$$|I_3|\lesssim\int_0^\infty[|\zeta_x||\hat{\theta}_x|+|\zeta||\hat{\theta}_{xx}|+|\zeta||\hat{\theta}_x|(|\zeta_x|+|\hat{\theta}_x|)]|\omega_x|dx\lesssim\delta\|(\zeta_x,\omega_x)\|^2. \tag{3.33}$$

Substituting (3.31)-(3.33) into (3.30) yields

$$\begin{aligned}
&\|\zeta_x(t)\|^2+\int_0^t\|(\zeta_{xx},\omega_{xx})\|^2d\tau\\
\lesssim&\|\zeta_x(0)\|^2+\int_0^t\|(\varphi_x,\psi_x,\zeta_x,\omega,\omega_x)\|^2d\tau+\int_0^t\varphi^2(0,\tau)d\tau.
\end{aligned}\tag{3.34}$$

The proof of Proposition 3.1: By (3.19), (3.23), (3.30) and (3.34), we get

$$\begin{aligned}&\|(\varphi,\psi,\zeta)(t)\|_1^2+\int_0^t\left(\|\varphi_x\|^2+\|(\psi_x,\zeta_x,\omega,\omega_x)\|_1^2\right)d\tau+\int_0^t|(\varphi,\varphi_x)(0,\tau)|^2d\tau\\ \lesssim&\|(\varphi,\psi,\zeta)(0)\|_1^2.\end{aligned}\tag{3.35}$$

Noticing that

$$-\omega_{xx}+\omega=-4\theta^3\zeta_x-4\hat{\theta}_x\left(\theta^3-\hat{\theta}^3\right),$$

it is easy to check that

$$\begin{aligned}&\int_0^\infty\left(\omega_x^2+\omega^2\right)dx\lesssim\int_0^\infty\left(|\zeta_x|+|\zeta||\hat{\theta}_x|\right)|\omega|dx\lesssim\|\zeta_x\|\|\omega\|,\\ &\int_0^\infty\left(\omega_{xx}^2+\omega_x^2\right)dx\lesssim\int_0^\infty\left(|\zeta_x|+|\zeta||\hat{\theta}_x|\right)|\omega_{xx}|dx\lesssim\|\zeta_x\|\|\omega_{xx}\|.\end{aligned}\tag{3.36}$$

By (3.35) and (3.36), we get (3.4) which completes the proof of Proposition 3.1.

4. Convergence rate of the stationary solutions

In this section, we show the convergence rate stated in Theorem 1.3 by using a time- and space-weighted energy method.

The a priori estimate obtained in the weighted Sobolev space $X_\nu(0,T)$ defined by

$$X_\nu(0,T):=\{(\varphi,\psi,\zeta,\omega)\in X(0,T)\,|\,\sqrt{\nu}(\varphi,\psi,\zeta,\omega)\in C([0,T];L_2(0,\infty))\}.$$

For the weight function $\nu(x):=(1+x)^\alpha$ or $\nu(x)=e^{\alpha x}$, we use the following notation

$$|f|_{2,\nu}:=\left(\int_0^\infty\nu(x)f^2(x)dx\right)^{\frac{1}{2}},\ \|f\|_{a,\alpha}:=|f|_{2,(1+x)^\alpha},\ \|f\|_{e,\alpha}:=|f|_{2,e^{\alpha x}}.$$

For proof of Theorem 1.3, we show the following weighted norm estimates.

PROPOSITION 4.1. *Suppose that the same assumptions as in Theorem 1.3 hold.*
(1) (Exponential Decay) *Suppose that (ρ,u,θ,q) is the solution to the outflow problem (1.1)-(1.4) satisfying $(\varphi,\psi,\zeta,\omega)\in X_{e^{\varsigma x}}(0,T)$ for certain positive constants $\varsigma>0$ and $T>0$. Then there are frown positive constants $\varepsilon_1,\alpha(<\varsigma),\beta(\ll\alpha)$ such that*

$$\text{if}\quad\sup_{t\in[0,T]}\|(\varphi,\psi,\zeta,\omega,\omega_x)(t)\|_1+\delta\le\varepsilon_1,$$

then the following weighted estimates are satisfied:

$$\begin{aligned}&e^{\beta t}\left(\|(\varphi,\psi,\zeta,\omega,\omega_x)(t)\|_1^2+\|(\varphi,\psi,\zeta)(t)\|_{e,\alpha}^2\right)\\ \le&C\left(\|(\varphi,\psi,\zeta)(0)\|_1^2+\|(\varphi,\psi,\zeta)(0)\|_{e,\alpha}^2\right),\end{aligned}\tag{4.1}$$

where C is a positive constant independent of t,x,T,ε_1.
(2) (Algebraic Decay) *Suppose that (ρ,u,θ,q) is the solution to the outflow problem (1.1)-(1.4) satisfying $(\varphi,\psi,\zeta,\omega)\in X_{(1+x)^\varsigma}(0,T)$ for certain positive constants $\varsigma>0$ and $T>0$. Then there exist positive constants ε_1 such that*

$$\text{if}\quad\sup_{t\in[0,T]}\|(\varphi,\psi,\zeta,\omega,\omega_x)(t)\|_1+\delta\le\varepsilon_1,$$

then the following weighted estimates are satisfied:

$$(1+t)^{\varsigma}\|(\varphi,\psi,\zeta,\omega,\omega_x)(t)\|_1^2\le C\left(\|(\varphi,\psi,\zeta)(0)\|_1^2+\|(\varphi,\psi,\zeta)(0)\|_{a,\varsigma}^2\right), \tag{4.2}$$

where C is a positive constant independent of t,x,T,ε_1.

In the remainder of this section, we will prove Proposition 4.1. As in Section 3, we denote $A\lesssim B$ if $A\le C_0B$ holds uniformly on the constant C_0 independent of t,x,T,ε_1.

Step 1: Weighted energy estimates.

Suppose that $\eta(t)$ and $\nu(x)$ is the weight function like $(1+t)^{\beta}$ (or $e^{\beta t}$) and $(1+x)^{\alpha}$ (or $e^{\alpha x}, \alpha\le\frac{\hat{c}}{2}$, where $\hat{c}$ is the positive number in (1.8)) respectively.

Setting $w(x,t)=\eta(t)\nu(x)$, from (3.7), we have

$$\begin{aligned}&(w\rho\mathcal{E})_t+\{w(\rho u\mathcal{E}-\Delta_1)\}_x-w_xG^1+w\left(\mu\theta^{-1}\hat{\theta}\psi_x^2+\kappa\theta^{-2}\hat{\theta}\zeta_x^2\right)\\ =&w_t\rho\mathcal{E}-w_xG^2+w(\Delta_2+\Delta_3)-w\zeta\left(\theta q_x-\hat{\theta}\hat{q}_x\right),\end{aligned} \tag{4.3}$$

where

$$G^1=\rho u\mathcal{E}+(p-\hat{p})\psi,\quad G^2=\mu\psi\psi_x+\kappa\zeta\zeta_x\theta^{-1}.$$

Using (3.9) yields

$$\int_0^{\infty}\{w(\rho u\mathcal{E}-\Delta_1)\}_xdx\gtrsim\eta(t)\varphi^2(0,t). \tag{4.4}$$

Using the calculation results from [2], we obtain

$$-\int_0^{\infty}w_xG^1dx\gtrsim\eta(t)|(\phi,\psi,\zeta)|_{2,\nu_x}^2 \tag{4.5}$$

It is easy to check that

$$\begin{aligned}&\int_0^{\infty}|w_xG^2|dx\lesssim\eta(t)\left(\epsilon|(\psi,\zeta)|_{2,\nu_x}^2+\epsilon^{-1}|(\psi_x,\zeta_x)|_{2,\nu_x}^2\right),\forall\epsilon>0,\\ &\int_0^{\infty}|w_t\rho\mathcal{E}|dx\lesssim\eta'(t)|(\varphi,\psi,\zeta)|_{2,\nu}^2.\end{aligned} \tag{4.6}$$

By (3.11), we get

$$-w\zeta\left(\frac{q_x}{\theta}-\frac{\hat{q}_x}{\hat{\theta}}\right)=-\eta(t)\frac{\nu\zeta}{\theta}\omega_x+\hat{q}_x\frac{\zeta^2}{\theta\hat{\theta}}\eta(t)\nu. \tag{4.7}$$

Using (4.7), we have

$$\begin{aligned}&-\int_0^{\infty}w\zeta\left(\frac{q_x}{\theta}-\frac{\hat{q}_x}{\hat{\theta}}\right)dx\\ =&\eta(t)\int_0^{\infty}\zeta_x\frac{\omega}{\theta}\nu dx+\eta(t)\int_0^{\infty}\zeta\omega\left(\frac{1}{\theta}\right)_x\nu dx\\ &+\eta(t)\int_0^{\infty}\frac{\zeta\omega}{\theta}\nu_xdx+\eta(t)\int_0^{\infty}\hat{q}_x\frac{\zeta^2}{\theta\hat{\theta}}\nu dx.\end{aligned} \tag{4.8}$$

We estimate the right-hand side in (4.8). Using (3.10), (1.8), (3.3) and (3.12), we have

$$\begin{aligned}
&\int_0^\infty \zeta_x \frac{\omega}{\theta}\nu dx = \int_0^\infty \frac{\omega\omega_{xx}}{4\theta^4}\nu dx - \int_0^\infty \frac{\omega^2}{4\theta^4}\nu dx - \int_0^\infty \hat{\theta}_x\omega\left(1-\frac{\theta^3}{\hat{\theta}^3}\right)\nu dx \\
\leq & -\int_0^\infty \frac{\omega_x^2}{4\theta^4}\nu dx - \int_0^\infty \frac{\omega^2}{4\theta^4}\nu dx + C_0\int_0^\infty |\omega_x||\omega|\nu_x dx \\
&+C_0\int_0^\infty |\omega||\omega_x|\left(|\zeta_x|+|\hat{\theta}_x|\right)\nu dx + C_0\int_0^\infty |\hat{\theta}_x||\omega||\zeta|\nu dx \\
\leq & -\int_0^\infty \frac{\omega_x^2}{4\theta^4}\nu dx - \int_0^\infty \frac{\omega^2}{4\theta^4}\nu dx + C_0|(\omega,\omega_x)|_{2,\nu_x}^2 \\
&+C_0(\varepsilon_1+\delta)|(\omega,\omega_x,\zeta_x)|_{2,\nu}^2 + C_0\delta\int_0^\infty x\exp(-\hat{c}x)\nu(x)\|\omega_x\|\|\zeta_x\|dx \\
\leq & -\int_0^\infty \frac{\omega_x^2}{4\theta^4}\nu dx - \int_0^\infty \frac{\omega^2}{4\theta^4}\nu dx + C_0|(\omega,\omega_x)|_{2,\nu_x}^2 + C_0(\varepsilon_1+\delta)|(\omega,\omega_x,\zeta_x)|_{2,\nu}^2, \qquad (4.9)
\end{aligned}$$

where we used the fact that if $\nu(x) = (1+x)^\alpha$, then

$$\int_0^\infty x\exp(-\hat{c}x)\nu(x)dx \leq \int_0^\infty x\exp(-\hat{c}x)(1+x)^\alpha dx < \infty$$

and if $\nu(x) = e^{\alpha x}$, $\alpha \leq \frac{\hat{c}}{2}$, then

$$\int_0^\infty x\exp(-\hat{c}x)\nu(x)dx \leq \int_0^\infty x\exp(\alpha x - \hat{c}x)dx < \infty.$$

Also, we have

$$\begin{aligned}
&\int_0^\infty \zeta\omega\left(\frac{1}{\theta}\right)_x \nu dx + \int_0^\infty \frac{\zeta\omega}{\theta}\nu_x dx + \int_0^\infty \hat{q}_x\frac{\zeta^2}{\theta\hat{\theta}}\nu dx \\
\lesssim & \int_0^\infty |\zeta||\omega|(|\zeta_x|+|\hat{\theta}_x|)\nu dx + \int_0^\infty |\zeta||\omega|\nu_x dx + \int_0^\infty |\hat{q}_x|\zeta^2\nu dx \\
\lesssim & (\varepsilon_1+\delta)|(\omega,\omega_x,\zeta_x)|_{2,\nu}^2 + |(\omega,\zeta_x)|_{2,\nu_x}^2. \qquad (4.10)
\end{aligned}$$

Integrating (4.3) for x,t and using (4.4), (4.5), (4.6), (4.8)-(4.10), we have

$$\begin{aligned}
&\eta(t)|\Phi(t)|_{2,\nu}^2 + \int_0^t \eta(\tau)\left(|\Phi(\tau)|_{2,\nu_x}^2 + |(\psi_x,\zeta_x,\omega,\omega_x)(\tau)|_{2,\nu}^2 + |\varphi(0,\tau)|^2\right)d\tau \\
\lesssim & |\Phi(0)|_{2,\nu}^2 + \int_0^t \left(\eta'(\tau)|\Phi(\tau)|_{2,\nu}^2 + \eta(\tau)|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)|_{2,\nu_x}^2\right)d\tau \\
&+\int_0^t \eta(\tau)\int_0^\infty \nu(|\Delta_2|+|\Delta_3|)dxd\tau, \qquad (4.11)
\end{aligned}$$

where $\Phi = (\varphi,\psi,\zeta)$.

Using (3.15) and (3.18), it is easy to check that

$$\int_0^\infty \nu(|\Delta_2|+|\Delta_3|)dx \lesssim \delta\|(\varphi_x,\psi_x,\zeta_x)(t)\|^2 + \delta|\varphi(0,t)|^2. \qquad (4.12)$$

By (4.11), (4.12) and $\|\cdot\|\lesssim|\cdot|_{2,\nu}$, we have

$$\begin{aligned}&\eta(t)|\Phi(t)|_{2,\nu}^2+\int_0^t\eta(\tau)\left(|\Phi(\tau)|_{2,\nu_x}^2+|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)|_{2,\nu}^2+|\varphi(0,\tau)|^2\right)d\tau\\ \lesssim&|\Phi(0)|_{2,\nu}^2+\int_0^t\left(\eta'|\Phi|_{2,\nu}^2+\eta|(\psi_x,\zeta_x,\omega,\omega_x)|_{2,\nu_x}^2\right)d\tau+\delta\int_0^t\eta(\tau)\|\varphi_x\|^2d\tau.\end{aligned}\tag{4.13}$$

Setting $\nu=1$ in (4.13), we get

$$\begin{aligned}&\eta(t)\|\Phi(t)\|^2+\int_0^t\eta(\tau)\left(\|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)\|^2+|\varphi(0,\tau)|^2\right)d\tau\\ \lesssim&\|\Phi(0)\|^2+\delta\int_0^t\eta(\tau)\|\varphi_x(\tau)\|^2d\tau+\int_0^t\eta'(\tau)\|\Phi(\tau)\|^2d\tau.\end{aligned}\tag{4.14}$$

Step 2: Weighted estimation of $\|\varphi_x(t)\|$.
By (3.20), we have

$$\left(\eta(\frac{\mu\varphi_x^2}{2\rho^3}+\frac{\varphi_x\psi}{\rho})\right)_t+\left(\eta(\frac{\mu u\varphi_x^2}{2\rho^3}-\frac{\varphi_t}{\rho}\psi)\right)_x+\frac{\eta p_\rho(\rho,\theta)}{\rho^2}\varphi_x^2=G^3,\tag{4.15}$$

where

$$G^3=\eta(t)(f_1+f_2+f_3)+\eta'(t)\left(\frac{\mu\varphi_x^2}{2\rho^3}+\frac{\varphi_x\psi}{\rho}\right)=:G_1^3+G_2^3.$$

Using (3.21) and the assumptions of Proposition 4.1 yields

$$\begin{aligned}&\int_0^\infty|G_1^3|dx\lesssim(\delta+\epsilon)\eta(t)\|\varphi_x\|^2+\delta\eta(t)\varphi^2(0,t)+C_\epsilon\eta(t)\|(\psi_x,\zeta_x)\|^2,\\ &\int_0^\infty|G_2^3|dx\lesssim\eta'(t)\|(\varphi_x,\psi)\|^2,\ \forall\epsilon>0.\end{aligned}\tag{4.16}$$

Integrating (4.15) for (x,t), and using (3.21) and (4.16), we obtain

$$\begin{aligned}&\eta(t)\|\varphi_x(t)\|^2+\int_0^t\eta(\tau)\|\varphi_x(\tau)\|^2d\tau+\int_0^t\eta(\tau)\varphi_x^2(0,\tau)d\tau\\ \lesssim&\|\varphi_x(0)\|^2+\eta(t)\|\psi(t)\|^2+\int_0^t\eta(\tau)\|(\psi_x,\zeta_x)(\tau)\|^2d\tau\\ &+\delta\int_0^t\eta(\tau)|\varphi(0,\tau)|^2d\tau+\int_0^t\eta'(\tau)\|(\varphi_x,\psi)(\tau)\|^2d\tau.\end{aligned}\tag{4.17}$$

By (4.14) and (4.17), we obtain

$$\begin{aligned}&\eta(t)\|\varphi_x(t)\|^2+\int_0^t\eta(\tau)\left(\|\varphi_x(\tau)\|^2+\varphi_x^2(0,\tau)\right)d\tau\\ \lesssim&\|(\Phi,\varphi_x)(0)\|^2+\int_0^t(\eta'(\tau)\left(\|\Phi(\tau)\|^2+\|\varphi_x(\tau)\|^2\right)d\tau,\end{aligned}\tag{4.18}$$

where $\Phi=(\varphi,\psi,\zeta)$.

Step 3: Weighted estimation for $\|\psi_x(t)\|$.

By (3.24), we have

$$\frac{1}{2}\left(\eta\psi_x^2\right)_t - \eta(\psi_t\psi_x)_x + \eta\frac{\mu\psi_{xx}^2}{\rho} = G^4, \tag{4.19}$$

where $G^4 = \eta(t)f_4 + \frac{1}{2}\eta'(t)\psi_x^2$.

By (3.25), we have

$$\begin{aligned}\int_0^\infty |G^4|dx \lesssim & \eta(t)(\epsilon+\delta)\|\psi_{xx}\|^2 + \eta(t)\delta|\varphi(0,t)|^2 \\ & + \eta(t)\epsilon^{-1}\|(\varphi_x,\psi_x,\zeta_x)\|^2 + \eta'(t)\|\psi_x\|^2,\ \forall \epsilon > 0.\end{aligned} \tag{4.20}$$

Integrating (4.19) for (x,t), and using (4.20) and (3.25), we obtain

$$\begin{aligned}\eta(t)\|\psi_x(t)\|^2 + \int_0^t \eta(\tau)\|\psi_{xx}(\tau)\|^2 d\tau \lesssim & \|\psi_x(0)\|^2 + \int_0^t \eta'(t)\|\psi_x(\tau)\|^2 d\tau \\ + \int_0^t \eta(\tau)\|(\varphi_x,\psi_x,\zeta_x)(\tau)\|^2 d\tau + \delta\int_0^t \eta(\tau)|\varphi(0,\tau)|^2 d\tau.\end{aligned} \tag{4.21}$$

Step 4: Weighted estimation for $\|\zeta_x(t)\|$.

By (3.26), we have

$$\left(\eta\frac{\zeta_x^2}{2}\right)_t - \eta(\zeta_t\zeta_x)_x + \eta\frac{\kappa\zeta_{xx}^2}{\rho e_\theta(\rho,\theta)} = \eta\frac{\omega_x\zeta_{xx}}{\rho e_\theta(\rho,\theta)} + G^4, \tag{4.22}$$

where $G^4 = \eta(t)f_5 + \frac{1}{2}\eta'(t)\zeta_x^2$.

By (3.27), we have

$$\begin{aligned}\int_0^\infty |G^5|dx \lesssim & \eta(t)(\epsilon+\delta)\|\zeta_{xx}\|^2 + \eta(t)\delta|\varphi(0,t)|^2 \\ & + \eta(t)\epsilon^{-1}\|(\varphi_x,\psi_x,\zeta_x)\|^2 + \eta'(t)\|\zeta_x\|^2,\ \forall \epsilon > 0.\end{aligned} \tag{4.23}$$

By (3.29), (3.31)-(3.33), we have

$$\begin{aligned}&\int_0^\infty \frac{\omega_x\zeta_{xx}}{\rho e_\theta(\rho,\theta)}dx \\ &\leq -\int_0^\infty \frac{\omega_{xx}^2}{4\theta^3\rho e_\theta(\rho,\theta)}dx + C_0(\varepsilon_1+\delta)\|\omega_{xx}\|^2 + C_0\|(\varphi_x,\zeta_x,\omega,\omega_x)\|^2.\end{aligned} \tag{4.24}$$

Integrating (4.22) for (x,t), and using (4.23), (3.27), $e_\theta(\rho,\theta) > 0$ and (4.24), we obtain

$$\begin{aligned}\eta(t)\|\zeta_x(t)\|^2 + \int_0^t \eta(\tau)\|(\zeta_{xx},\omega_{xx})(\tau)\|^2 d\tau \lesssim & \|\zeta_x(0)\|^2 + \int_0^t \eta'(t)\|\zeta_x(\tau)\|^2 d\tau \\ + \int_0^t \eta(\tau)\|(\varphi_x,\psi_x,\zeta_x,\omega,\omega_x)(\tau)\|^2 d\tau + \delta\int_0^t \eta(\tau)|\varphi(0,\tau)|^2 d\tau.\end{aligned} \tag{4.25}$$

The proof of Proposition 4.1:

Using (4.14), (4.18), (4.21), (4.25) and (3.36), we have

$$\eta(t)\left(\|\Phi(t)\|_1^2 + \|\omega(t)\|_2^2\right) + \int_0^t \eta(\tau)\left(\|\Phi_x\|^2 + \|(\psi_{xx},\zeta_{xx})\|^2 + \|\omega\|_2^2\right)d\tau$$

$$+\int_0^t \eta(\tau)|(\varphi,\varphi_x)(0,\tau)|^2 d\tau \lesssim \|\Phi(0)\|_1^2+\int_0^t \eta'(\tau)\|\Phi_x(\tau)\|^2 d\tau, \tag{4.26}$$

where $\Phi=(\varphi,\psi,\zeta)$. Also, by using (4.13), (4.26) and $\|\cdot\|\lesssim|\cdot|_{2,\nu}$, we get

$$\begin{aligned}
&\eta(t)\left(|\Phi(t)|_{2,\nu}^2+\|\Phi_x(t)\|^2+\|\omega(t)\|_2^2\right)\\
&\quad+\int_0^t \eta(\tau)\left(|\Phi(\tau)|_{2,\nu_x}^2+|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)|_{2,\nu}^2+|(\varphi,\varphi_x)(0,\tau)|^2\right)d\tau\\
&\quad+\int_0^t \eta(\tau)\left(\|\Phi_x(\tau)\|^2+\|(\psi_{xx},\zeta_{xx})(\tau)\|^2+\|\omega\|_2^2\right)d\tau\\
&\le C_1\left(|\Phi(0)|_{2,\nu}^2+\|\Phi_x(0)\|^2\right)+C_2\int_0^t \eta'(\tau)|\Phi(\tau)|_{2,\nu}^2 d\tau\\
&\quad+C_3\int_0^t \eta(\tau)|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)|_{2,\nu_x}^2 d\tau+C_4\int_0^t \eta'(\tau)\|\Phi_x(\tau)\|^2 d\tau,
\end{aligned} \tag{4.27}$$

where $C_i(i=1,\cdots,4)$ are positive constants independent of t,x,T,ε_1.

We first prove (4.1). Setting $\nu(x)=e^{\alpha x}$ and $\eta(t)=e^{\beta t}$, we obtain from (4.27)

$$\begin{aligned}
&e^{\beta t}\left(\|\Phi(t)\|_{e,\alpha}^2+\|\Phi_x(t)\|^2+\|\omega(t)\|_2^2\right)+(\alpha-C_2\beta)\int_0^t e^{\beta\tau}\|\Phi(\tau)\|_{e,\alpha}^2 d\tau\\
&\quad+(1-C_3\alpha)\int_0^t e^{\beta\tau}\|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)\|_{e,\alpha}^2 d\tau\\
&\quad+(1-C_4\beta)\int_0^t e^{\beta\tau}\|\Phi_x(\tau)\|^2 d\tau\le C_1\left(\|\Phi(0)\|_{e,\alpha}^2+\|\Phi_x(0)\|^2\right).
\end{aligned} \tag{4.28}$$

If we choose α and $\beta\,(0<\beta<\alpha<\varsigma)$ satisfying

$$\alpha-C_2\beta\ge 0,\quad 1-C_3\alpha\ge\frac{1}{2},\quad 1-C_4\beta\ge\frac{1}{2},$$

then (4.28) yields (4.1).

Next, we prove (4.2). Setting $\nu(x)=(1+x)^\alpha$ and $\eta(t)=(1+t)^\beta$, we obtain from (4.27)

$$\begin{aligned}
&(1+t)^\beta\left(\|\Phi(t)\|_{a,\alpha}^2+\|\Phi_x(t)\|^2+\|\omega(t)\|_2^2\right)\\
&\quad+\int_0^t (1+\tau)^\beta\left(\alpha\|\Phi(\tau)\|_{a,\alpha-1}^2+\|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)\|_{a,\alpha}^2+|(\varphi,\varphi_x)(0,\tau)|^2\right)d\tau\\
&\quad+\int_0^t (1+\tau)^\beta\left(\|\Phi_x(\tau)\|^2+\|(\psi_{xx},\zeta_{xx})(\tau)\|^2+\|\omega(\tau)\|_2^2\right)d\tau\\
&\lesssim\left(\|\Phi(0)\|_{a,\alpha}^2+\|\Phi_x(0)\|^2\right)+\alpha\int_0^t (1+\tau)^\beta\|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)\|_{a,\alpha-1}^2 d\tau\\
&\quad+\beta\int_0^t (1+\tau)^{\beta-1}\left(\|\Phi(\tau)\|_{a,\alpha}^2+\|\Phi_x(\tau)\|^2\right)d\tau.
\end{aligned} \tag{4.29}$$

Setting

$$\begin{aligned}
&E_\alpha(t)^2:=\|\Phi(t)\|_{a,\alpha}^2+\|\Phi_x(t)\|^2,\\
&D(t)^2:=\|\Phi_x(t)\|^2+\|(\psi_{xx},\zeta_{xx})(t)\|^2+|(\varphi,\varphi_x)(0,t)|^2,\\
&D_\alpha(t)^2:=D(t)^2+\alpha\|\Phi(t)\|_{a,\alpha-1}^2+\|(\psi_x,\zeta_x,\omega,\omega_x)(t)\|_{a,\alpha}^2,
\end{aligned}$$

we rewrite (4.29) as

$$\begin{aligned}&(1+t)^{\beta}\left(E_{\alpha}(t)^2+\|\omega(t)\|_2^2\right)+\int_0^t(1+\tau)^{\beta}\left(D_{\alpha}(\tau)^2+\|\omega(\tau)\|_2^2\right)d\tau\\ \lesssim& E_{\alpha}(0)^2+\alpha\int_0^t(1+\tau)^{\beta}\|(\psi_x,\zeta_x,\omega,\omega_x)(\tau)\|_{a,\alpha-1}^2d\tau+\beta\int_0^t(1+\tau)^{\beta-1}E_{\alpha}(\tau)^2d\tau.\end{aligned}\tag{4.30}$$

Using (3.4) and (4.30), we have

$$(1+t)^k\left(E_{\varsigma-k}(t)^2+\|\omega(t)\|_2^2\right)+\int_0^t(1+\tau)^k\left(D_{\varsigma-k}(\tau)^2+\|\omega(\tau)\|_2^2\right)d\tau\lesssim E_{\varsigma}(0)^2 \tag{4.31}$$

and

$$(1+t)^k\left(E_0(t)^2+\|\omega(t)\|_2^2\right)+\int_0^t(1+\tau)^k\left(D_0(\tau)^2+\|\omega(\tau)\|_2^2\right)d\tau\lesssim E_{\varsigma}(0)^2 \tag{4.32}$$

for any $\varsigma>0$ and integer $k=0,1,2,\cdots,[\varsigma]$. (Refer to [2]).

We will obtain (4.2) from (4.32). The proof of Proposition 4.1 is completed.

REFERENCES

[1] J. Caar, *Applications of Center Manifold Theory*, Springer-Verlag, 1981. 2.2

[2] Y. Chen, H.H. Hong, and X. Shi, *Convergence rate of stationary solutions to outflow problem for full Navier-Stokes equations*, Anal. Appl., 98(7):1267–1288, 2019. 1, 3.1, 3.2, 4, 4

[3] W.L. Gao and C.J. Zhu, *Asymptotic decay toward the planar rarefaction waves for a model system of the radiating gas in two dimensions*, Math. Model. Meth. Appl. Sci., 18:511–541, 2008. 1, 1

[4] W.L. Gao, L. Ruan, and C.J. Zhu, *Decay rates to the planar rarefaction waves for a model system of the radiating gas in n dimensions*, J. Diff. Eqs., 244:2614–2640, 2008. 1

[5] K. Hamer, *Nonlinear effects on the propagation of sound waves in a radiating gas*, Quart. J. Mech. Appl. Math., 24:155–168, 1971. 1

[6] H.H. Hong, *Asymptotic behavior toward the combination of contact discontinuity with rarefaction waves for 1-D compressible viscous gas with radiation*, Nonlinear Anal. Real World Appl., 35:175–199, 2017. 1

[7] F.M. Huang and X. Qin, *Stability of boundary layer and rarefaction wave to an outflow problem for compressible Navier-Stokes equations under large perturbation*, J. Diff. Eqs., 246:4077–4096, 2009. 1

[8] S. Kawashima and S. Nishibata, *Shock waves for a model system of the radiating gas*, SIAM J. Math. Anal., 30:95–117, 1999. 1

[9] S. Kawashima, S. Nishibata, and P. Zhu, *Asymptotic stability of the stationary solution to the compressible Navier-Stokes equations in the half space*, Comm. Math. Phys., 240:483–500, 2003. 1

[10] S. Kawashima, T. Nakamura, S. Nishibata, and P. Zhu, *Stationary waves to viscous heat-conductive gases in the half space: existence, stability and convergence rate*, Math. Model. Mech. Appl. Sci., 20(12):2201–2235, 2010. 1

[11] S. Kawashima and Y. Tanaka, *Stability of rarefaction waves for a model system of a radiating gas*, Kyushu J. Math., 58:211–250, 2004. 1

[12] C.J. Lin, J.-F. Coulombel, and T. Goudon, *Shock profiles for non-equilibrium radiating gases*, Phys. D, 218:83–94, 2006. 1

[13] C.J. Lin, J.-F. Coulombel, and T. Goudon, *Asymptotic stability of shock profiles in radiative hydrodynamics*, C.R. Math. Acad. Sci. Paris, 345:625–628, 2007. 1

[14] C. Lattanzio, C. Mascia, and D. Serre, *Shock waves for radiative hyperbolic-elliptic systems*, Indiana Univ. Math. J., 56:2601–2640, 2007. 1

[15] C.J. Lin, *Asymptotic stability of rarefaction waves in radiative hydrodynamics*, Commun. Math. Sci., 9:207–223, 2011. 1

[16] T. Nakamura and S. Nishibata, *Convergence rate toward planar stationary waves for compressible viscous fluid in multididmensional half line*, SIAM J. Math. Anal., 41(5):1757–1791, 2009. 1

[17] T. Nakamura, S. Nishibata, and T. Yuge, *Convergence rate of solutions toward stationary solutions to the compressible Navier-Stokes equation in a half line*, J. Diff. Eqs., 241:94–111, 2007. 1

[18] T. Nguyen, R.G. Plaza, and K. Zumbrun, *Stability of radiative shock profiles for hyperbolicelliptic coupled systems*, Phys. D., 239:428–453, 2010. 1

[19] M. Ohnawa, *L_∞-stability of continuous shock waves in a radiating gas model*, SIAM J. Math. Anal., 46:2136–2159, 2014. 1

[20] X. Qin, *Large-time behaviour of solutions to the outfow problem of full compressible Navier-Stokes equations*, Nonlinearity, 24:1369–1394, 2011. 1

[21] C. Rohde and F. Xie, *Decay rates to viscous contact wave for a 1D compressible radiation hydrodynamics model*, Math. Model. Meth. Appl. Sci., 23:441–469, 2013. 1

[22] L.Z. Ruan and C.J. Zhu, *Asymptotic behavior of solutions to a hyperbolic-elliptic coupled system in multi-dimensional radiating gas*, J. Diff. Eqs., 249:2076–2110, 2010. 1

[23] L. Wan, T. Wang, and H.J. Zhao, *Asymprotic stability of wave patterns to compressible viscous and heat-conducting gases in the half space*, J. Diff. Eqs., 261(11):5949–5991, 2016. 1

[24] L. Wan, T. Wang, and Q. Zou, *Stability of stationary solutions to the outflow problem for full compressible Navier-Stokes equations with large initial perturbation*, Nonlinearity, 29:1329–1354, 2016. 1

[25] J. Wang and F. Xie, *Asymptotic stability of viscous contact wave for the one-dimensional compressible viscous gas with radiation*, Nonlinear Anal., 74:4138–4151, 2011. 1, 1

[26] J. Wang and F. Xie, *Asymptotic stability of viscous contact wave for the 1D radiation hydrodynamics system*, J. Diff. Eqs., 251:1030–1055, 2011. 1

[27] F. Xie, *Nonlinear stability of combination of viscous contact wave with rarefaction waves for a 1D radiation hydrodynamics model*, Discrete Contin. Dyn. Syst. Ser. B, 17:1075–1100, 2012. 1