LOCAL WELL-POSEDNESS OF ISENTROPIC COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH VACUUM*

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Abstract. In this paper, the local well-posedness of strong solutions to the Cauchy problem of the isentropic compressible Navier-Stokes equations is proved with the initial data being allowed to have vacuum. The main contribution of this paper is that the well-posedness is established without assuming any compatibility condition on the initial data, which was widely used before in many works concerning the well-posedness of compressible Navier-Stokes equations in the presence of vacuum.

Keywords. Isentropic Navier-Stokes equations; local well-posedness; vacuum; without compatibility condition.

AMS subject classifications. 35D35; 35Q30; 35Q35; 76N10.

1. Introduction

The isentropic compressible Navier-Stokes equations read as

$$\rho(u_t + (u \cdot \nabla)u) - \mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u + \nabla P = 0, \tag{1.1}$$

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.2}$$

in $\mathbb{R}^3 \times (0, T)$, where the density $\rho \geq 0$ and the velocity field $u \in \mathbb{R}^3$ are the unknowns. Here P is the scalar pressure given as $P = a\rho^\gamma$, for two constants $a > 0$ and $\gamma > 1$. The viscosity constants λ, μ satisfy the physical requirements:

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0.$$

System (1.1)–(1.2) is complemented with the following initial-boundary conditions

$$\begin{cases} (\rho, \rho u)|_{t=0} = (\rho_0, \rho_0 u_0), \\ u(x, t) \to 0, \text{ as } |x| \to \infty. \end{cases} \tag{1.3}$$

There is extensive literature on the studies of the compressible Navier-Stokes equations. In the absence of vacuum, that is the density has positive lower bound, the system is locally well-posed for large initial data, see, e.g., [24, 44, 49, 51, 52, 54]; however, the global well-posedness is still unknown. It has been known that system in one dimension is globally well-posed for large initial data, see, e.g., [2, 27–30, 60, 61] and the references therein, see [38] for the large-time behavior of the solutions, and also [35, 36, 40, 41] for the global well-posedness for the case with nonnegative density. For the multi-dimensional case, the global well-posedness holds for small initial data, see, e.g., [3, 7, 11–14, 20, 31, 45–48, 50, 53]. In the presence of vacuum, that is the density may vanish on some set, global existence (but without uniqueness) of weak solutions

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has been known, see [1, 15–17, 26, 42, 43]. Local well-posedness of strong solutions was proved for suitably regular initial data under some extra compatibility conditions (being mentioned in some details below) in [8–10]. In general, when the vacuum is involved, one can only expect solutions in the homogeneous Sobolev spaces, that is, the L^2 integrability of u is not expectable, see [32]. Global well-posedness holds if the initial basic energy is sufficiently small, see [21, 22, 37, 39, 57]; however, due to the blow-up results in [58,59], the corresponding entropy of the global solutions in [21,37,57] must be infinite somewhere in the vacuum region, if the initial density is compactly supported.

In this paper, we focus on the well-posedness of the Cauchy problem to system (1.1)–(1.2) in the presence of vacuum. As mentioned in the previous paragraph, local well-posedness of strong solutions to the compressible Navier-Stokes in the presence of vacuum has already been studied in [8–10], where, among some other conditions, the regularity assumption

$$\rho_0 - \rho_\infty \in H^1 \cap W^{1,q}, \quad u_0 \in D_0^1 \cap D^2, \tag{1.4}$$

for some constant $\rho_\infty \in [0,\infty)$, and the compatibility condition

$$-\mu\Delta u_0 - (\mu+\lambda)\nabla\mathrm{div}\,u_0 + \nabla P(\rho_0) = \sqrt{\rho_0}g, \tag{1.5}$$

for some $g \in L^2$, were used. Similar assumptions as (1.4) and (1.5) were also widely used in studying many other fluid dynamical systems when the vacuum is involved, see, e.g., [4–6, 18, 19, 21–23, 33, 39, 55–57].

Assumptions (1.4) and (1.5) are so widely used when the initial vacuum is taken into consideration, one may ask if the regularities on the initial data stated in (1.4) can be relaxed and if the compatibility condition (1.5) is necessary for the local well-posedness of strong solutions to the corresponding system. In a previous work [34], the second author of this paper considered these questions for the inhomogeneous incompressible Navier-Stokes equations, and found that the compatibility condition is not necessary for the local well-posedness. The aim of the current paper is to give the same answer for the isentropic compressible Navier-Stokes equations. As will be shown in this paper that we can indeed reduce the regularities of the initial velocity in (1.4) and remove the compatibility condition (1.5), without losing the existence and uniqueness, but the prices that we need to pay are the following: (i) the corresponding strong solutions do not have as high regularities as those in [8–10] where both (1.4) and (1.5) were assumed; (ii) one can only ask for the continuity, at the initial time, of the momentum ρu, instead of the velocity u itself.

Before stating our main results, let us introduce some notations. Throughout this paper, we use $L^r = L^r(\mathbb{R}^3)$ and $W^{k,r} = W^{k,r}(\mathbb{R}^3)$ to denote, respectively, the standard Lebesgue and Sobolev spaces in $\mathbb{R}^3$, where k is a positive integer and $r \in [1,\infty]$. When $r = 2$, we use H^k instead of $W^{k,2}$. For simplicity, we use $\|\cdot\|_r = \|\cdot\|_{L^r}$. We denote

$$D^{k,r} = \left\{ u \in L^1_{loc}(\mathbb{R}^3) \,\middle|\, \|\nabla^k u\|_r < \infty \right\}, \quad D^k = D^{k,2},$$

$$D_0^1 = \left\{ u \in L^6 \,\middle|\, \|\nabla u\|_2 < \infty \right\}.$$

For simplicity of notations, we adopt the notation

$$\int f\,dx = \int_{\mathbb{R}^3} f\,dx.$$

Our main result is the following:

THEOREM 1.1.   *Suppose that the initial data (ρ_0, u_0) satisfies*

$$\rho_0 \geq 0, \quad \rho_0 - \rho_\infty \in H^1 \cap W^{1,q}, \quad u_0 \in D_0^1, \quad \rho_0 u_0 \in L^2,$$

for some $\rho_\infty \in [0, \infty)$ and some $q \in (3,6)$.

Then, there exists a positive time $\mathcal{T}$, depending only on μ, λ, a, γ, q, and the upper bound of $\psi_0 := \|\rho_0\|_\infty + \|\rho_0 - \rho_\infty\|_2 + \|\nabla \rho_0\|_{L^2 \cap L^q} + \|\nabla u_0\|_2$, such that system (1.1)–(1.2), subject to (1.3), admits a unique solution (ρ, u) on $\mathbb{R}^3 \times (0, \mathcal{T})$, satisfying

$$\rho - \rho_\infty \in C([0, \mathcal{T}]; L^2) \cap L^\infty(0, \mathcal{T}; H^1 \cap W^{1,q}), \quad \rho_t \in L^\infty(0, \mathcal{T}; L^2),$$
$$\rho u \in C([0, \mathcal{T}]; L^2), \quad u \in L^\infty(0, \mathcal{T}; D_0^1) \cap L^2(0, \mathcal{T}; D^2), \quad \sqrt{\rho} u_t \in L^2(0, \mathcal{T}; L^2),$$
$$\sqrt{t} u \in L^\infty(0, \mathcal{T}; D^2) \cap L^2(0, \mathcal{T}; D^{2,q}), \quad \sqrt{t} u_t \in L^2(0, \mathcal{T}; D_0^1).$$

REMARK 1.1.

(i) Compared with the local well-posedness results established in [8,9], in Theorem 1.1, u_0 is not required to be in D^2 and we do not need any compatibility conditions on the initial data.

(ii) The same result as in Theorem 1.1 also holds for the initial boundary value problem if imposing suitable boundary conditions on the velocity.

2. Lifespan estimate and some a priori estimates

As preparations of proving the main result being carried out in the next section, the aim of this section is to give the lifespan estimate and some a priori estimates, under the condition that the initial velocity $u_0 \in D_0^1 \cap D^2$ and some compatibility condition; however, it should be emphasized that all these estimates depend neither on $\|\nabla^2 u_0\|_2$ nor on the compatibility condition.

We start with the following local existence and uniqueness result, which has been essentially proved in [8–10].

PROPOSITION 2.1.   *Let $\rho_\infty \in [0, \infty)$ and $q \in (3,6)$ be fixed constants. Assume that the data ρ_0 and u_0 satisfy the regularity condition*

$$\rho_0 \geq 0, \quad \rho_0 - \rho_\infty \in H^1 \cap W^{1,q}, \quad u_0 \in D_0^1 \cap D^2,$$

and the compatibility condition

$$-\mu \Delta u_0 - (\mu + \lambda) \nabla \, div u_0 + \nabla P_0 = \sqrt{\rho_0} g,$$

for some $g \in L^2$, where $P_0 = a \rho_0^\gamma$.

Then, there exists a small time $T_* > 0$ and a unique strong solution (ρ, u) to (1.1)–(1.2), subject to (1.3), such that

$$\rho - \rho_\infty \in C([0, T_*]; H^1 \cap W^{1,q}), \quad u \in C([0, T_*]; D_0^1 \cap D^2) \cap L^2(0, T_*; D^{2,q}),$$
$$\rho_t \in C([0, T_*]; L^2 \cap L^q), \quad u_t \in L^2(0, T_*; D_0^1), \quad \sqrt{\rho} u_t \in L^\infty(0, T_*; L^2).$$

As will be shown in this section, the existence time T_* in the above proposition can be chosen depending only on μ, λ, a, γ, q, and the upper bound of

$$\Psi_0 := \|\rho_0\|_\infty + \|P_0 - P_\infty\|_2 + \|P_0\|_\infty + \|\nabla P_0\|_2 + \|\nabla P_0\|_q + \|\nabla u_0\|_2,$$

with $P_\infty = a\rho_\infty^\gamma$, and, in particular, independent of $\|\nabla^2 u_0\|_2$ and $\|g\|_2$. The following quantity plays the key role in this section

$$\Psi(t) := (\|\rho\|_\infty + \|P - P_\infty\|_2 + \|P\|_\infty + \|\nabla P\|_2 + \|\nabla P\|_q + \|\nabla u\|_2 + \|\sqrt{t}\sqrt{\rho}u_t\|_2)(t) + 1.$$

In the rest of this section, until the last proposition, we always assume (ρ, u) is a solution to system (1.1)–(1.2), subject to (1.3), on $\mathbb{R}^3 \times (0, T)$, satisfying the regularities stated in Proposition 2.1, with T_* there replaced with T.

Throughout this section, except otherwise explicitly mentioned, we denote by C a generic constant depending only on μ, λ, a, γ, and the upper bound of Ψ_0.

PROPOSITION 2.2.   *The following estimates hold*

$$\|\nabla^2 u\|_2^2 \leq C(\Psi^{10} + \Psi\|\sqrt{\rho}u_t\|_2^2),$$
$$\|\sqrt{\rho}(u \cdot \nabla)u\|_2^2 \leq C(\Psi^9 + \Psi^5\|\sqrt{\rho}u_t\|_2),$$
$$\|\nabla^2 u\|_q \leq C(\|\sqrt{t}\nabla u_t\|_2^2 + \|\sqrt{\rho}u_t\|_2^2 + t^{-\frac{5q-6}{4q}} + \Psi^{\alpha_1}), \quad q \in (3, 6),$$

with $\alpha_1 := \max\left\{12, \frac{(5q-6)^2}{2q(6-q)}\right\}.$

Proof.   Applying the elliptic estimates to (1.1) yields

$$\|\nabla^2 u\|_2^2 \leq C\|\rho\|_\infty(\|\sqrt{\rho}u_t\|_2^2 + \|\sqrt{\rho}(u \cdot \nabla)u\|_2^2) + C\|\nabla P\|_2^2.$$

By the Hölder and Sobolev inequality, one has

$$\|\sqrt{\rho}(u \cdot \nabla)u\|_2^2 \leq \|\rho\|_\infty\|u\|_6^2\|\nabla u\|_2\|\nabla u\|_6 \leq C\Psi^4\|\nabla^2 u\|_2.$$

Substituting the above inequality into the previous one and using the Cauchy inequality, one gets

$$\|\nabla^2 u\|_2^2 \leq C(\Psi\|\sqrt{\rho}u_t\|_2^2 + \Psi^5\|\nabla^2 u\|_2 + \Psi^2) \leq \frac{1}{2}\|\nabla^2 u\|_2^2 + C(\Psi\|\sqrt{\rho}u_t\|_2^2 + \Psi^{10}),$$

that is

$$\|\nabla^2 u\|_2^2 \leq C(\Psi^{10} + \Psi\|\sqrt{\rho}u_t\|_2^2), \tag{2.1}$$

and, consequently,

$$\|\sqrt{\rho}(u \cdot \nabla)u\|_2^2 \leq C(\Psi^9 + \Psi^5\|\sqrt{\rho}u_t\|_2),$$

proving the first two conclusions.

It follows from the Hölder and Gagliardo-Nirenberg inequalities that

$$\|\rho(u \cdot \nabla)u\|_q \leq \|\rho\|_\infty\|u\|_{\frac{6q}{6-q}}\|\nabla u\|_6$$
$$\leq C\|\rho\|_\infty\|u\|_6^{\frac{3}{q}}\|\nabla u\|_6^{2-\frac{3}{q}} \leq C\|\rho\|_\infty\|\nabla u\|_2^{\frac{3}{q}}\|\nabla^2 u\|_2^{\frac{2q-3}{q}}, \tag{2.2}$$

from which, by the Young inequality and using (2.1), one has

$$\|\rho(u \cdot \nabla)u\|_q \leq C\Psi^{\frac{q+3}{q}}(\Psi^{10} + \Psi\|\sqrt{\rho}u_t\|_2^2)^{\frac{2q-3}{2q}}$$
$$\leq C\Psi^3(\Psi^9 + \|\sqrt{\rho}u_t\|_2^2)^{\frac{2q-3}{2q}}$$
$$\leq C(\Psi^{2q} + \Psi^9 + \|\sqrt{\rho}u_t\|_2^2) \leq C(\Psi^{12} + \|\sqrt{\rho}u_t\|_2^2).$$

It follows from the Hölder and Sobolev inequalities that

$$\|\rho u_t\|_q \le \|\rho\|_\infty^{\frac{5q-6}{4q}} \|\sqrt{\rho}u_t\|_2^{\frac{6-q}{2q}} \|u_t\|_6^{\frac{3q-6}{2q}} \le C\|\rho\|_\infty^{\frac{5q-6}{4q}} \|\sqrt{\rho}u_t\|_2^{\frac{6-q}{2q}} \|\nabla u_t\|_2^{\frac{3q-6}{2q}}, \qquad (2.3)$$

and further by the Young inequality that

$$\|\rho u_t\|_q \le C\Psi^{\frac{5q-6}{4q}} t^{-\frac{3q-6}{4q}} \|\sqrt{t}\nabla u_t\|_2^{\frac{3q-6}{2q}} \|\sqrt{\rho}u_t\|_2^{\frac{6-q}{2q}}$$

$$\le C(\|\sqrt{t}\nabla u_t\|_2^2 + \|\sqrt{\rho}u_t\|_2^2 + t^{-\frac{5q-6}{4q}} + \Psi^{\frac{(5q-6)^2}{2q(6-q)}}).$$

Thanks to the above two, and applying the elliptic estimates to (1.1), one obtains

$$\|\nabla^2 u\|_q \le C(\|\rho u_t\|_q + \|\rho(u\cdot\nabla)u\|_q + \|\nabla P\|_q)$$

$$\le C(\|\sqrt{t}\nabla u_t\|_2^2 + \|\sqrt{\rho}u_t\|_2^2 + t^{-\frac{5q-6}{4q}} + \Psi^{\alpha_1}),$$

proving the conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

PROPOSITION 2.3.   *The following estimate holds*

$$\sup_{0\le t\le T}(\|\nabla u\|_2^2 + \|P - P_\infty\|_2^2) + \int_0^T \|\sqrt{\rho}u_t\|_2^2 dt \le C + C\int_0^T \Psi^{10} dt.$$

Proof.   Multiplying (1.1) with u_t, it follows from integration by parts that

$$\frac{1}{2}\frac{d}{dt}(\mu\|\nabla u\|_2^2 + (\mu+\lambda)\|\operatorname{div} u\|_2^2) + \|\sqrt{\rho}u_t\|_2^2 = -\int(\rho(u\cdot\nabla)u + \nabla P)\cdot u_t dx.$$

Integration by parts and noticing that

$$P_t + u\cdot\nabla P + \gamma\operatorname{div} u P = 0, \qquad\qquad (2.4)$$

one deduces

$$-\int \nabla P\cdot u_t dx = \int(P - P_\infty)\operatorname{div} u_t dx$$

$$= \frac{d}{dt}\int(P - P_\infty)\operatorname{div} u dx - \int P_t\operatorname{div} u dx$$

$$= \frac{d}{dt}\int(P - P_\infty)\operatorname{div} u dx + \int(u\cdot\nabla P + \gamma\operatorname{div} u P)\operatorname{div} u dx.$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\left(\mu\|\nabla u\|_2^2 + (\mu+\lambda)\|\operatorname{div} u\|_2^2 - 2\int(P - P_\infty)\operatorname{div} u dx\right) + \|\sqrt{\rho}u_t\|_2^2$$

$$= \int(u\cdot\nabla P + \gamma\operatorname{div} u P)\operatorname{div} u dx - \int\rho(u\cdot\nabla)u\cdot u_t dx =: I_1 + I_2.$$

By the Hölder, Sobolev, and Young inequalities, and applying Proposition 2.2, one has

$$I_1 \le \|u\|_6\|\nabla P\|_2\|\operatorname{div} u\|_3 + \gamma\|\operatorname{div} u\|_2^2\|P\|_\infty$$

$$\le C\|\nabla u\|_2\|\nabla P\|_2\|\nabla u\|_2^{\frac{1}{2}}\|\nabla^2 u\|_2^{\frac{1}{2}} + C\Psi^3$$

$$\le C\Psi^{\frac{5}{2}}(\Psi^{\frac{5}{2}} + \Psi^{\frac{1}{4}}\|\sqrt{\rho}u_t\|_2^{\frac{1}{2}}) + C\Psi^3 \le \frac{1}{4}\|\sqrt{\rho}u_t\|_2^2 + C\Psi^5,$$

and

$$
\begin{aligned}
I_2 &\leq \|\sqrt{\rho}u_t\|_2 \|\sqrt{\rho}(u\cdot\nabla)u\|_2 \\
&\leq C(\Psi^{\frac{9}{2}} + \Psi^{\frac{5}{2}}\|\sqrt{\rho}u_t\|_2^{\frac{1}{2}})\|\sqrt{\rho}u_t\|_2 \\
&\leq \frac{1}{4}\|\sqrt{\rho}u_t\|_2^2 + C\Psi^{10}.
\end{aligned}
$$

Therefore

$$
\frac{1}{2}\frac{d}{dt}\left(\mu\|\nabla u\|_2^2 + (\mu+\lambda)\|\operatorname{div}u\|_2^2 - 2\int(P-P_\infty)\operatorname{div}u\,dx\right) + \|\sqrt{\rho}u_t\|_2^2 \leq C\Psi^{10},
$$

from which, one obtains by the Cauchy inequality that

$$
\sup_{0\leq t\leq T}\|\nabla u\|_2^2 + \int_0^T\|\sqrt{\rho}u_t\|_2^2 dt \leq C\left(1 + \sup_{0\leq t\leq T}\|P-P_\infty\|_2^2 + \int_0^T\Psi^{10}dt\right). \tag{2.5}
$$

Multiplying (2.4) with $P-P_\infty$, it follows from integration by parts and the Sobolev inequality that

$$
\begin{aligned}
\frac{d}{dt}\|P-P_\infty\|_2^2 &= -(\gamma-\frac{1}{2})\int\operatorname{div}u(P-P_\infty)^2 dx - \gamma P_\infty\int\operatorname{div}u(P-P_\infty)dx \\
&\leq C\|\nabla u\|_2\|P-P_\infty\|_2^{\frac{1}{2}}\|\nabla P\|_2^{\frac{3}{2}} + C\|\nabla u\|_2\|P-P_\infty\|_2 \leq C\Psi^3,
\end{aligned}
$$

which gives

$$
\sup_{0\leq t\leq T}\|P-P_\infty\|_2^2 \leq C + C\int_0^T\Psi^3 dt.
$$

This, combined with (2.5), leads to the conclusion.   □

The t-weighted estimate in the next proposition is the key to remove the compatibility condition on the initial data.

PROPOSITION 2.4.   *The following estimate holds*

$$
\sup_{0\leq t\leq T}\|\sqrt{t}\sqrt{\rho}u_t\|_2^2 + \int_0^T\|\sqrt{t}\nabla u_t\|_2^2 dt \leq C + C\int_0^T\Psi^{16} dt.
$$

Proof.   Differentiating (1.1) in t and using (1.2) yield

$$
\begin{aligned}
&\rho(\partial_t u_t + (u\cdot\nabla)u_t) - \mu\Delta u_t - (\lambda+\mu)\nabla\operatorname{div}u_t \\
&= -\nabla P_t + \operatorname{div}(\rho u)(u_t + (u\cdot\nabla)u) - \rho(u_t\cdot\nabla)u.
\end{aligned}
$$

Multiplying it by u_t, integrating by parts over $\mathbb{R}^3$ and then using the continuity equation (1.2), one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u_t\|_2^2 + \mu\|\nabla u_t\|_2^2 + (\lambda+\mu)\|\operatorname{div}u_t\|_2^2 \\
&= \int P_t\operatorname{div}u_t\,dx + \int\operatorname{div}(\rho u)|u_t|^2 dx + \int\operatorname{div}(\rho u)(u\cdot\nabla)u\cdot u_t\,dx \\
&\quad -\int\rho(u_t\cdot\nabla u)\cdot u_t\,dx =: II_1 + II_2 + II_3 + II_4.
\end{aligned}
$$

Recalling (2.4) and using the Sobolev and Young inequalities, one deduces

$$
\begin{aligned}
II_1 &= -\int (\gamma \operatorname{div} u P + u \cdot \nabla P) \operatorname{div} u_t \, dx \\
&\leq C(\|P\|_\infty \|\nabla u\|_2 \|\nabla u_t\|_2 + \|u\|_6 \|\nabla P\|_3 \|\nabla u_t\|_2) \\
&\leq C\left(\Psi^4 + \|\nabla u\|_2^2 \|\nabla P\|_{L^2 \cap L^q}^2\right) + \frac{\mu}{8}\|\nabla u_t\|_2^2 \\
&\leq C\Psi^4 + \frac{\mu}{8}\|\nabla u_t\|_2^2 .
\end{aligned}
$$

Integrating by parts, using the Hölder, Sobolev and Young inequalities, and applying Proposition 2.2, we have

$$
\begin{aligned}
II_2 &= -\int_{\mathbb{R}^3} \rho u \cdot \nabla |u_t|^2 \, dx \\
&\leq C\|\rho\|_\infty^{\frac{1}{2}} \|u\|_6 \|\sqrt{\rho} u_t\|_2^{\frac{1}{2}} \|\sqrt{\rho} u_t\|_6^{\frac{1}{2}} \|\nabla u_t\|_2 \\
&\leq C\|\rho\|_\infty^{\frac{3}{4}} \|\nabla u\|_2 \|\sqrt{\rho} u_t\|_2^{\frac{1}{2}} \|\nabla u_t\|_2^{\frac{3}{2}} \\
&\leq C\Psi^7 \|\sqrt{\rho} u_t\|_2^2 + \frac{\mu}{8}\|\nabla u_t\|_2^2 , \\
II_3 &\leq \int_{\mathbb{R}^3} \rho|u|(|\nabla u|^2 |u_t| + |u||\nabla^2 u||u_t| + |u||\nabla u||\nabla u_t|) dx \\
&\leq C\|\rho\|_\infty (\|u\|_6 \|\nabla u\|_3^2 \|u_t\|_6 + \|u\|_6^2 \|\nabla^2 u\|_2 \|u_t\|_6 + \|u\|_6^2 \|\nabla u\|_6 \|\nabla u_t\|_2) \\
&\leq C\|\rho\|_\infty \|\nabla u\|_2^2 \|\nabla^2 u\|_2 \|\nabla u_t\|_2 \\
&\leq C\Psi^6 \|\nabla^2 u\|_2^2 + \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 \\
&\leq C(\Psi^{16} + \Psi^7 \|\sqrt{\rho} u_t\|_2^2) + \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 ,
\end{aligned}
$$

and

$$
\begin{aligned}
II_4 &\leq \int_{\mathbb{R}^3} \rho|u_t|^2 |\nabla u| dx \\
&\leq C\|\rho\|_\infty^{\frac{1}{2}} \|\nabla u\|_2 \|\sqrt{\rho} u_t\|_2^{\frac{1}{2}} \|\sqrt{\rho} u_t\|_6^{\frac{1}{6}} \|u_t\|_6 \\
&\leq C\|\rho\|_\infty^{\frac{3}{4}} \|\nabla u\|_2 \|\sqrt{\rho} u_t\|_2^{\frac{1}{2}} \|\nabla u_t\|_2^{\frac{3}{2}} \\
&\leq C\Psi^7 \|\sqrt{\rho} u_t\|_2^2 + \frac{\mu}{8}\|\nabla u_t\|_2^2 .
\end{aligned}
$$

Therefore, we have

$$
\frac{d}{dt}\|\sqrt{\rho} u_t\|_2^2 + \mu\|\nabla u_t\|_2^2 \leq C(\Psi^{16} + \Psi^7 \|\sqrt{\rho} u_t\|_2^2),
$$

which, multiplied by t, gives

$$
\frac{d}{dt}\|\sqrt{t}\sqrt{\rho} u_t\|_2^2 + \mu\|\sqrt{t}\nabla u_t\|_2^2 \leq C(\Psi^{16} + \Psi^7 \|\sqrt{t}\sqrt{\rho} u_t\|_2^2 + \|\sqrt{\rho} u_t\|_2^2) \leq C(\Psi^{16} + \|\sqrt{\rho} u_t\|_2^2).
$$

Integrating this in t and applying Proposition 2.3, the conclusion follows.   □

PROPOSITION 2.5.   *The following estimate holds*

$$
\int_0^T (\|\nabla u\|_\infty + \|\nabla^2 u\|_q) dt \leq C + C\int_0^T \Psi^{\alpha_2} dt,
$$

with $\alpha_2 := \max\{16, \alpha_1\} = \max\left\{16, \frac{(5q-6)^2}{2q(6-q)}\right\}$.

    Proof.   Noticing that $t^{-\frac{5q-6}{4q}} \in L^1(0,1)$, for $q \in (3,6)$, and recalling the following estimate by Proposition 2.2

$$\|\nabla^2 u\|_q \le C(\|\sqrt{t}\nabla u_t\|_2^2 + \|\sqrt{\rho}u_t\|_2^2 + t^{-\frac{5q-6}{4q}} + \Psi^{\alpha_1}),$$

it follows from the Gagliardo-Nirenberg and Young inequalities and Propositions 2.3 and 2.4 that

$$
\begin{aligned}
\int_0^T \|\nabla u\|_\infty dt &\le C \int_0^T \|\nabla u\|_2^{1-\theta}\|\nabla^2 u\|_q^\theta dt \\
&\le C \int_0^T (\|\nabla u\|_2 + \|\nabla^2 u\|_q)dt \\
&\le C \int_0^T (\|\sqrt{t}\nabla u_t\|_2^2 + \|\sqrt{\rho}u_t\|_2^2 + t^{-\frac{5q-6}{4q}} + \Psi^{\alpha_1})dt \\
&\le C + C \int_0^T \Psi^{\alpha_2} dt,
\end{aligned}
$$

where $\theta = \frac{3q}{5q-6} \in (0,1)$, proving the conclusion.   □

PROPOSITION 2.6.   *The following estimate holds*

$$\sup_{0 \le t \le T}(\|\rho\|_\infty + \|P\|_\infty) \le C\exp\left(C\int_0^T \Psi^{\alpha_2} dt\right),$$

where α_2 is the number in Proposition 2.5.

    Proof.   Define $X(t;x)$ as

$$
\begin{cases}
\frac{d}{dt}X(t;x) = u(X(t;x),t), \\
X(0;x) = x.
\end{cases}
$$

One can show that for any $t \in (0,T)$, and for any $y \in \mathbb{R}^3$, there is a unique $x \in \mathbb{R}^3$, such that $X(t;x) = y$, and, in particular, $X(t;\mathbb{R}^3) = \mathbb{R}^3$; in fact, to show this, it suffices to consider the backward problem $\frac{d}{dt}Z(t) = u(Z(t),t), X(T;x) = y$. Then, by (1.2), it has

$$
\begin{aligned}
\frac{d}{dt}\rho(X(t;x),t) &= \partial_t\rho(X(t;x),t) + u(X(t;x),t)\cdot\nabla\rho(X(t;x),t) \\
&= -\operatorname{div}u(X(t;x),t)\rho(X(t;x),t),
\end{aligned}
$$

and, thus,

$$\rho(X(t;x),t) = \rho_0(x)\exp\left(-\int_0^t \operatorname{div}u(X(\tau;x),\tau)d\tau\right). \tag{2.6}$$

Therefore,

$$\|\rho\|_\infty(t) = \|\rho(X(t;x),t)\|_\infty(t) \le \|\rho_0\|_\infty\exp\left(\int_0^T \|\nabla u\|_\infty dt\right),$$

and the conclusion follows by applying Proposition 2.5.   □

PROPOSITION 2.7.   *The following estimate holds*

$$\sup_{0 \le t \le T} (\|\nabla P\|_2 + \|\nabla P\|_q) \le C \exp \left(C \int_0^T \Psi^{\alpha_2} dt \right), \quad q \in (3,6),$$

where α_2 is the number in Proposition 2.5.

   Proof.   From (2.4), one has

$$\partial_t \nabla P + \gamma \operatorname{div} u \nabla P + \gamma P \nabla \operatorname{div} u + (u \cdot \nabla) \nabla P + \nabla P \nabla u = 0.$$

Multiplying the above by $|\nabla P|^{p-2} \nabla P$, integrating over $\mathbb{R}^3$, one has

$$\frac{d}{dt} \|\nabla P\|_p^p \le C (\|\nabla u\|_\infty \|\nabla P\|_p^p + \|P\|_\infty \|\nabla^2 u\|_p \|\nabla P\|_p^{p-1}),$$

which gives

$$\frac{d}{dt} \|\nabla P\|_p \le C (\|\nabla u\|_\infty \|\nabla P\|_p + \|P\|_\infty \|\nabla^2 u\|_p).$$

By the Gronwall inequality, one has

$$\sup_{0 \le t \le T} \|\nabla P\|_p \le C \left(\|\nabla P_0\|_p + \int_0^T \|P\|_\infty \|\nabla^2 u\|_p dt \right) \exp \left(C \int_0^T \|\nabla u\|_\infty dt \right)$$

$$\le C \left(1 + \int_0^T \|\nabla^2 u\|_p dt \right) \exp \left(C \int_0^T \Psi^{\alpha_2} dt \right). \tag{2.7}$$

Thanks to the above, it follows from Proposition 2.5 and Proposition 2.6 that

$$\sup_{0 \le t \le T} \|\nabla P\|_q \le C \left(1 + \int_0^T \Psi^{\alpha_2} dt \right) \exp \left(C \int_0^T \Psi^{\alpha_2} dt \right) \le C \exp \left(C \int_0^T \Psi^{\alpha_2} dt \right),$$

where we have used the fact that $e^z \ge 1 + z$ for $z \ge 0$. By Proposition 2.2 and Proposition 2.3, it follows from (2.7) and the Cauchy inequality that

$$\sup_{0 \le t \le T} \|\nabla P\|_2 \le C \left[1 + \int_0^T (\Psi^5 + \Psi^{\frac{1}{2}} \|\sqrt{\rho} u_t\|_2) dt \right] \exp \left(C \int_0^T \Psi^{\alpha_2} dt \right)$$

$$\le C \left[1 + \int_0^T (\Psi^5 + \|\sqrt{\rho} u_t\|_2^2) dt \right] \exp \left(C \int_0^T \Psi^{\alpha_2} dt \right)$$

$$\le C \left(1 + \int_0^T \Psi^{10} dt \right) \exp \left(C \int_0^T \Psi^{\alpha_2} dt \right)$$

$$\le C \exp \left(C \int_0^T \Psi^{\alpha_2} dt \right).$$

This proves the conclusion.   □

PROPOSITION 2.8.   *The following estimates hold*

$$\sup_{0 \le t \le T} (\|\rho - \rho_\infty\|_2 + \|\nabla \rho\|_2 + \|\nabla \rho\|_q) \le C \exp \left(C \int_0^T \Psi^{\alpha_2} dt \right), \quad q \in (3,6),$$

with constant C depending also on $\|\rho_0 - \rho_\infty\|_2 + \|\nabla\rho_0\|_2 + \|\nabla\rho_0\|_q$, and

$$\sup_{0 \le t \le T} \|\sqrt{t}\nabla^2 u\|_2^2 \le C \sup_{0 \le t \le T} (\Psi^{10} + \Psi\|\sqrt{t}\sqrt{\rho}u_t\|_2^2),$$

where α_2 is the number in Proposition 2.5.

   Proof.   The estimate of $\|\rho - \rho_\infty\|_2$ follows in the same way as that for $\|P - P_\infty\|_2$ in Proposition 2.3, while those for $\|\nabla\rho\|_2$ and $\|\nabla\rho\|_q$ can be proved similarly as in Proposition 2.7. The conclusion for $\|\sqrt{t}\nabla^2 u\|_2^2$ follows from combining Propositions 2.2 and 2.4.   □

   We end up this section with the following proposition on the lifespan estimate and a priori estimates.

PROPOSITION 2.9.   *Assume in addition to the conditions in Proposition 2.1 that $\rho_0 \ge \underline{\rho}$ for some positive constant $\underline{\rho}$.*

   Then, there are two positive constants $\mathcal{T}$ and C depending only on μ, λ, a, γ, q, and the upper bound of $\psi_0 := \|\rho_0\|_\infty + \|\rho_0 - \rho_\infty\|_2 + \|\nabla\rho_0\|_{L^2 \cap L^q} + \|\nabla u_0\|_2$, and, in particular, independent of $\underline{\rho}$ and $\|\nabla^2 u_0\|_2$, such that system (1.1)–(1.2), subject to (1.3), has a unique solution (ρ, u) on $\mathbb{R}^3 \times (0, \mathcal{T})$, enjoying the regularities stated in Proposition 2.1, with T_* there replaced by $\mathcal{T}$, and the following a priori estimates

$$\sup_{0 \le t \le \mathcal{T}} (\|\rho\|_\infty + \|\rho - \rho_\infty\|_2 + \|\nabla\rho\|_2 + \|\nabla\rho\|_q + \|\rho_t\|_2) \le C,$$

$$\sup_{0 \le t \le \mathcal{T}} \|\nabla u\|_2^2 + \int_0^{\mathcal{T}} (\|\nabla^2 u\|_2^2 + \|\sqrt{\rho}u_t\|_2^2)dt \le C,$$

$$\sup_{0 \le t \le \mathcal{T}} (\|\sqrt{t}\sqrt{\rho}u_t\|_2^2 + \|\sqrt{t}\nabla^2 u\|_2^2) + \int_0^{\mathcal{T}} (\|\sqrt{t}\nabla u_t\|_2^2 + \|\sqrt{t}\nabla^2 u\|_q^2)dt \le C.$$

   Proof.   Define the maximal time $T_{\max}$ as

$$T_{\max} := \max\{T \in \mathfrak{T}\},$$

where

$$\mathfrak{T} := \{T \in [T_*, \infty) \mid \text{There is a solution } (\rho, u) \text{ in the class } \mathfrak{X}_T$$
$$\text{to system } (1.1) - (1.2), \text{ subject to } (1.3), \text{ on } \mathbb{R}^3 \times (0, T)\},$$

where $\mathfrak{X}_T$ is the class of (ρ, u) enjoying the regularities as stated in Proposition 2.1, with T_* there replaced with T. By Proposition 2.1, it is clear that $T^{\max}$ is well defined and $T^{\max} \ge T_*$. Moreover, by the uniqueness result, see the proof of the uniqueness part of Theorem 1.1 in the next section, one can easily show that any two solutions $(\bar{\rho}, \bar{u})$ and $(\tilde{\rho}, \tilde{u})$ to system (1.1)–(1.2), subject to (1.3), on $\mathbb{R}^3 \times (0, \bar{T})$ and on $\mathbb{R}^3 \times (0, \tilde{T})$, respectively, coincide on $\mathbb{R}^3 \times (0, \min\{\bar{T}, \tilde{T}\})$. Choose $T_k \in \mathfrak{T}$ with $T_k \uparrow T^{\max}$ as $k \uparrow \infty$. By definition of $\mathfrak{T}$, there is a solution (ρ_k, u_k) to system (1.1)–(1.2), subject to (1.3), on $\mathbb{R}^3 \times (0, T_k)$. Define (ρ, u) on $\mathbb{R}^3 \times (0, T^{\max})$ as

$$(\rho, u)(x, t) = (\rho_k, u_k)(x, t), \quad x \in \mathbb{R}^3, t \in (0, T_k), k = 1, 2, \cdots.$$

Applying the uniqueness result again, the definition of (ρ, u) is independent of the choice of the sequence $\{T_k\}_{k=1}^\infty$. By the construction of (ρ, u), one can verify that (ρ, u) is a solution to (1.1)–(1.2), subject to (1.3), on $\mathbb{R}^3 \times (0, T^{\max})$, and $(\rho, u) \in \mathfrak{X}_T$, for any $T \in (0, T^{\max})$.

By Propositions 2.3, 2.4, 2.6, and 2.7, it is clear

$$\Psi(t) \leq C_m \exp\left(C_m \int_0^t \Psi^{\alpha_2} d\tau\right), \quad t \in (0, T^{\max}),$$

where C_m is a positive constant depending only on μ, λ, a, γ, q, and the upper bound of ψ_0. Here we have used the fact that Ψ_0 can be controlled by ψ_0. One can easily derive from the above inequality that

$$\Psi(t) \leq 2^{\frac{1}{\alpha_2}} C_m, \quad \forall t \in \left(0, \min\left\{T^{\max}, (2^{\alpha_2} C_m^{\alpha_2+1})^{-1}\right\}\right). \tag{2.8}$$

Thanks to the above estimate, one can get by applying Propositions 2.5 and 2.8 that

$$(\|\rho - \rho_\infty\|_2 + \|\nabla\rho\|_2 + \|\nabla\rho\|_q + \|\sqrt{t}\nabla^2 u\|_2^2)(t) + \int_0^t \|\nabla u\|_\infty d\tau \leq C, \tag{2.9}$$

for any $t \in \left(0, \min\left\{T^{\max}, (2^{\alpha_2} C_m^{\alpha_2+1})^{-1}\right\}\right)$, and for a positive constant C depending only on μ, λ, a, γ, q, and the upper bound of ψ_0. Thanks to (2.8)–(2.9) and using (1.2) one can further obtain

$$\begin{aligned}
\|\rho_t\|_2 &= \|u \cdot \nabla\rho + \operatorname{div} u\rho\|_2 \\
&\leq \|u\|_6 \|\nabla\rho\|_3 + \|\nabla u\|_2 \|\rho\|_\infty \\
&\leq C(1 + \|\nabla\rho\|_{L^2 \cap L^q}) \|\nabla u\|_2 \leq C,
\end{aligned} \tag{2.10}$$

for any $0 < t < \min\{T^{\max}, (2^{\alpha_2} C_m^{\alpha_2+1})^{-1}\}$. Using the estimate $\int_0^t \|\nabla u\|_\infty d\tau \leq C$ in (2.9) and recalling (2.6), it is clear that

$$\rho(x,t) \geq e^{-C} \underline{\rho}, \quad x \in \mathbb{R}^3, \quad 0 < t < \min\left\{T^{\max}, (2^{\alpha_2} C_m^{\alpha_2+1})^{-1}\right\}. \tag{2.11}$$

We claim that $T^{\max} > (2^{\alpha_2} C_m^{\alpha_2+1})^{-1}$. Assume in contradiction that this does not hold. Then, all the estimates in (2.8)–(2.11) hold for any $t \in (0, T^{\max})$. Estimates (2.8)–(2.11), holding on time interval $(0, T^{\max})$, guarantee that $(\rho, u)(\cdot, t)$ can be uniquely extended to time $T^{\max}$, with $(\rho, u)(\cdot, T^{\max})$ defined as the limit of $(\rho, u)(\cdot, t)$ as $t \uparrow T^{\max}$, and that

$$(\rho - \rho_\infty)(\cdot, T^{\max}) \in H^1 \cap W^{1,q}, \quad u(\cdot, T^{\max}) \in D_0^1 \cap D^2.$$

Thanks to this and recalling (2.11), it is clear that the compatibility condition holds at time $T^{\max}$. Therefore, by the local well-posedness result, i.e., Proposition 2.1, one can further extend solution (ρ, u) beyond the time $T^{\max}$, which contradicts the definition of $T^{\max}$. This contradiction proves the claim that $T^{\max} > (2^{\alpha_2} C_m^{\alpha_2+1})^{-1}$. As a result, one obtains a solution (ρ, u) on time interval $(0, (2^{\alpha_2} C_m^{\alpha_2+1})^{-1})$ satisfying all the a priori estimates in (2.8)–(2.10), except $\int_0^{\mathcal{T}} \|\sqrt{t}\nabla^2 u\|_q^2 dt \leq C$, on the same time interval.

It remains to verify $\int_0^{\mathcal{T}} \|\sqrt{t}\nabla^2 u\|_q^2 dt \leq C$. To this end, recalling (2.2) and (2.3), it follows from the elliptic estimate, the estimates just obtained, and the Young inequality that

$$\begin{aligned}
\|\nabla^2 u\|_q &\leq C(\|\rho u_t\|_q + \|\rho(u \cdot \nabla)u\|_q + \|\nabla P\|_q) \\
&\leq C(\|\rho\|_\infty^{\frac{5q-6}{4q}} \|\sqrt{\rho} u_t\|_2^{\frac{6-q}{2q}} \|\nabla u_t\|_2^{\frac{3q-6}{2q}} + \|\rho\|_\infty \|\nabla u\|_2^{\frac{3}{q}} \|\nabla^2 u\|_2^{\frac{2q-3}{q}} + \|\nabla\rho\|_q) \\
&\leq C(1 + \|\nabla^2 u\|_2^2 + \|\sqrt{\rho} u_t\|_2 + \|\nabla u_t\|_2),
\end{aligned}$$

and further that

$$\int_0^{\mathcal{T}} \|\sqrt{t}\nabla^2 u\|_q^2 dt \leq C \int_0^{\mathcal{T}} (1 + \|\nabla^2 u\|_2^2 \|\sqrt{t}\nabla^2 u\|_2^2 + \|\sqrt{\rho} u_t\|_2^2 + \|\sqrt{t}\nabla u_t\|_2^2) dt \leq C,$$

proving the conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1.

The following lemma, proved in [34], will be used in proving the uniqueness.

LEMMA 3.1.   *Given a positive time T and nonnegative functions f, g, G on $[0,T]$, with f and g being absolutely continuous on $[0,T]$. Suppose that*

$$
\begin{cases}
\frac{d}{dt} f(t) \leq \delta(t) f(t) + A\sqrt{G(t)}, \\
\frac{d}{dt} g(t) + G(t) \leq \alpha(t) g(t) + \beta(t) f^2(t), \\
f(0) = 0,
\end{cases}
$$

where α, β and δ are three nonnegative functions, with $\alpha, \delta, t\beta \in L^1((0,T))$.

Then, the following estimates hold

$$
f(t) \leq AB\sqrt{tg(0)} \exp\left(\frac{1}{2} \int_0^t (\alpha(s) + A^2 B^2 s\beta(s)) ds\right),
$$

$$
g(t) + \int_0^t G(s) ds \leq g(0) \exp\left(\int_0^t (\alpha(s) + A^2 B^2 s\beta(s)) ds\right),
$$

where $B = 1 + e^{\int_0^T \delta(\tau) d\tau}$. In particular, if $g(0) = 0$, then $f \equiv g \equiv 0$ on $(0,T)$.

We are now ready to prove Theorem 1.1.

Proof. **(Proof of Theorem 1.1.)** We first prove the uniqueness and then the existence.

Uniqueness: Let $(\check{\rho}, \check{u})$, $(\hat{\rho}, \hat{u})$ be two solutions of system (1.1)-(1.2), subject to (1.3), on $\mathbb{R}^3 \times (0,T)$, satisfying the regularities stated in the theorem. For $u \in \{\hat{u}, \check{u}\}$, by the Gagliardo-Nirenberg and Hölder inequalities, one has

$$
\int_0^T \|\nabla u\|_\infty dt \leq C \int_0^T \|\nabla u\|_2^{1-\theta} \|\nabla^2 u\|_q^\theta dt \leq C \int_0^T \|\sqrt{t}\nabla^2 u\|_q^\theta t^{-\frac{\theta}{2}} dt
$$

$$
\leq C \left(\int_0^T \|\sqrt{t}\nabla^2 u\|_q^2 dt\right)^{\frac{\theta}{2}} \left(\int_0^T t^{-\frac{\theta}{2-\theta}} dt\right)^{1-\frac{\theta}{2}} < \infty,
$$

where $\theta = \frac{3q}{5q-6} \in (0,1)$. Therefore, $\nabla \hat{u}, \nabla \check{u} \in L^1(0,T;L^\infty)$.

Denote

$$
\sigma = \check{\rho} - \hat{\rho}, \quad W = \check{P} - \hat{P}, \quad v = \check{u} - \hat{u}.
$$

Then, straightforward calculations show

$$
\sigma_t + v \cdot \nabla \hat{\rho} + \check{u} \cdot \nabla \sigma + \operatorname{div} \check{u}\sigma + \operatorname{div} v\hat{\rho} = 0, \tag{3.1}
$$

$$
\hat{\rho}(v_t + \hat{u} \cdot \nabla v) - \mu\Delta v - (\lambda + \mu)\nabla\operatorname{div} v + \nabla W = -\sigma\check{u}_t - \sigma\check{u} \cdot \nabla\check{u} - \hat{\rho}v \cdot \nabla\check{u}, \tag{3.2}
$$

$$
W_t + v \cdot \nabla \hat{P} + \check{u} \cdot \nabla W + \gamma\operatorname{div} \check{u}W + \gamma\operatorname{div} v\hat{P} = 0. \tag{3.3}
$$

Testing (3.1) with σ and using the Hölder inequality, we have

$$
\frac{d}{dt} \int |\sigma|^2 dx \leq C \int (|v \cdot \nabla\hat{\rho}||\sigma| + |\operatorname{div}\check{u}||\sigma|^2 + |\operatorname{div} v\hat{\rho}||\sigma|)
$$

$$
\leq C(\|\nabla\hat{\rho}\|_3\|\sigma\|_2\|v\|_6 + \|\nabla\check{u}\|_\infty\|\sigma\|_2^2) + C\|\hat{\rho}\|_\infty\|\nabla v\|_2\|\sigma\|_2
$$

$$\leq C||\nabla \check{u}||_\infty ||\sigma||_2^2 + C(||\hat{\rho}||_\infty + ||\nabla \hat{\rho}||_3)||\nabla v||_2 ||\sigma||_2,$$

and, thus,

$$\frac{d}{dt}||\sigma||_2 \leq C||\nabla \check{u}||_\infty ||\sigma||_2 + C(||\hat{\rho}||_\infty + ||\nabla \hat{\rho}||_3)||\nabla v||_2. \tag{3.4}$$

Similarly, by testing (3.3) with W yields

$$\frac{d}{dt}||W||_2 \leq C||\nabla \check{u}||_\infty ||W||_2 + C(||\nabla \hat{P}||_3 + ||\hat{P}||_\infty)||\nabla v||_2. \tag{3.5}$$

Testing (3.2) with v and using the Hölder and Young inequalities, we have

$$\frac{1}{2}\frac{d}{dt}\int \hat{\rho}|v|^2 dx + \int [\mu|\nabla v|^2 + (\lambda+\mu)(\mathrm{div}\,v)^2]dx$$

$$\leq \int (|W||\nabla v| + |\sigma||\check{u}_t||v| + |\sigma||\check{u}||\nabla \check{u}||v| + |\hat{\rho}(v\cdot\nabla \check{u})\cdot v|)dx =: RHS. \tag{3.6}$$

We proceed with the proof separately for the cases $\rho_\infty = 0$ and $\rho_\infty > 0$.

Case I: $\rho_\infty = 0$.

By the Hölder, Sobolev, and Young inequalities, we can control RHS as

$$RHS \leq ||W||_2||\nabla v||_2 + ||\sigma||_{\frac{3}{2}}||\check{u}_t||_6||v||_6 + ||\sigma||_2||\check{u}||_6||\nabla \check{u}||_6||v||_6 + ||\nabla \check{u}||_\infty ||\sqrt{\hat{\rho}}v||_2^2$$

$$\leq C(||W||_2||\nabla v||_2 + ||\sigma||_{\frac{3}{2}}||\nabla \check{u}_t||_2||\nabla v||_2 + ||\sigma||_2||\nabla \check{u}||_2||\nabla^2 \check{u}||_2||\nabla v||_2 + ||\nabla \check{u}||_\infty ||\sqrt{\hat{\rho}}v||_2^2)$$

$$\leq \frac{\mu}{2}||\nabla v||_2^2 + C(||W||_2^2 + ||\nabla \check{u}_t||_2^2||\sigma||_{\frac{3}{2}}^2 + ||\nabla \check{u}||_2^2||\nabla^2 \check{u}||_2^2||\sigma||_2^2 + ||\nabla \check{u}||_\infty ||\sqrt{\hat{\rho}}v||_2^2),$$

which plugged into (3.6) leads to

$$\frac{d}{dt}||\sqrt{\hat{\rho}}v||_2^2 + \mu||\nabla v||_2^2 \leq C||\nabla \check{u}||_\infty ||\sqrt{\hat{\rho}}v||_2^2 + C(1 + ||\nabla \check{u}_t||_2^2 + ||\nabla \check{u}||_2^2|\nabla^2 \check{u}||_2^2)$$

$$\times (||W||_2^2 + ||\sigma||_2^2 + ||\sigma||_{\frac{3}{2}}^2). \tag{3.7}$$

The appearance of $||\sigma||_{\frac{3}{2}}$ in the above inequality requires the energy estimate for $||\sigma||_{\frac{3}{2}}$ given below.

Testing (3.1) with $sign(\sigma)|\sigma|^{\frac{1}{2}}$ and using the Hölder and Sobolev inequalities that

$$\frac{d}{dt}\int |\sigma|^{\frac{3}{2}}dx \leq C\int (|v\cdot\nabla \hat{\rho}||\sigma|^{\frac{1}{2}} + |\mathrm{div}\check{u}||\sigma|^{\frac{3}{2}} + |\mathrm{div}\,v\hat{\rho}||\sigma|^{\frac{1}{2}})$$

$$\leq C(||\nabla \hat{\rho}||_2||\sigma||_{\frac{3}{2}}^{\frac{1}{2}}||v||_6 + ||\nabla \check{u}||_\infty ||\sigma||_{\frac{3}{2}}^{\frac{3}{2}}) + C||\hat{\rho}||_6||\nabla v||_2||\sigma||_{\frac{3}{2}}^{\frac{1}{2}}$$

$$\leq C||\nabla \check{u}||_\infty ||\sigma||_{\frac{3}{2}}^{\frac{3}{2}} + C||\nabla \hat{\rho}||_2||\nabla v||_2||\sigma||_{\frac{3}{2}}^{\frac{1}{2}},$$

which gives

$$\frac{d}{dt}||\sigma||_{\frac{3}{2}} \leq C||\nabla \check{u}||_\infty ||\sigma||_{\frac{3}{2}} + C||\nabla \hat{\rho}||_2||\nabla v||_2. \tag{3.8}$$

Denote

$$f_1(t) = (||\sigma||_{\frac{3}{2}} + ||\sigma||_2 + ||W||_2)(t), \quad g_1(t) = ||\sqrt{\hat{\rho}}v||_2^2(t), \quad G_1(t) = \mu||\nabla v||_2^2(t),$$

$$\delta_1(t) = C||\nabla \check{u}||_\infty(t), \quad A_1 = C\sup_{0\leq t\leq T}(||\hat{\rho}||_\infty + ||\hat{P}||_\infty + ||\nabla \hat{\rho}||_{L^2\cap L^3} + ||\nabla \hat{P}||_3)(t),$$

$$\alpha_1(t) = C\|\nabla\check{u}\|_\infty(t), \quad \beta_1(t) = C(1 + \|\nabla\check{u}_t\|_2^2 + \|\nabla\check{u}\|_2^2\|\nabla^2\check{u}\|_2^2)(t),$$

then, it follows from (3.4), (3.5), (3.7), and (3.8) that

$$\begin{cases} \frac{d}{dt}f_1(t) \le \delta_1(t)f_1(t) + A_1\sqrt{G_1(t)}, \\ \frac{d}{dt}g_1(t) + G_1(t) \le \alpha_1(t)g_1(t) + \beta_1(t)f_1^2(t), \\ f_1(0) = 0. \end{cases}$$

By the regularities of $(\hat{\rho}, \hat{u})$ and $(\check{\rho}, \check{u})$, and recalling $\nabla\check{u} \in L^1(0, T; L^\infty)$, one can easily verify that $\alpha_1, \delta_1, t\beta_1 \in L^1((0, T))$. Therefore, one can apply Lemma 3.1 to get $f_1(t) = g_1(t) = G_1(t) = 0$, on $(0, T)$, which implies the uniqueness for Case I.

Case II: $\rho_\infty > 0.$

By the Hölder and Sobolev inequalities, it follows for $(\rho, u) \in \{(\hat{\rho}, \hat{u}), (\check{\rho}, \check{u})\}$ that

$$\begin{aligned} \rho_\infty^4 \int |u_t|^2 dx &= \int |\rho_\infty - \rho + \rho|^4 |u_t|^2 dx \\ &\le C \int (|\rho - \rho_\infty|^4 + \rho^4)|u_t|^2 dx \\ &\le C(\|\rho - \rho_\infty\|_6^4 \|u_t\|_6^2 + \|\rho\|_\infty^3 \|\sqrt{\rho}u_t\|_2^2) \\ &\le C(\|\nabla\rho\|_2^4 \|\nabla u_t\|_2^2 + \|\rho\|_\infty^3 \|\sqrt{\rho}u_t\|_2^2), \end{aligned}$$

and, thus,

$$\int_0^T t\|u_t\|_2^2 dt \le C \sup_{0 \le t \le T} (\|\nabla\rho\|_2^4 + \|\rho\|_\infty^3) \int_0^T (\|\sqrt{t}\nabla u_t\|_2^2 + \|\sqrt{\rho}u_t\|_2^2) dt < \infty,$$

that is, $\sqrt{t}u_t \in L^2(0, T; L^2)$, for $u \in \{\hat{u}, \check{u}\}$.

By the Hölder, Sobolev, and Cauchy inequalities, we deduce

$$\begin{aligned} RHS &\le \|W\|_2\|\nabla v\|_2 + \|\sigma\|_2\|\check{u}_t\|_3\|v\|_6 + \|\sigma\|_2\|\check{u}\|_6\|\nabla\check{u}\|_6\|v\|_6 + \|\nabla\check{u}\|_\infty\|\sqrt{\hat{\rho}}v\|_2^2 \\ &\le \|W\|_2\|\nabla v\|_2 + \|\nabla\check{u}\|_\infty\|\sqrt{\hat{\rho}}v\|_2^2 + (\|\check{u}_t\|_2^{\frac{1}{2}}\|\nabla\check{u}_t\|_2^{\frac{1}{2}} + \|\nabla\check{u}\|_2\|\nabla^2\check{u}\|_2)\|\nabla v\|_2\|\sigma\|_2 \\ &\le \frac{\mu}{2}\|\nabla v\|_2^2 + C\|\nabla\check{u}\|_\infty\|\sqrt{\hat{\rho}}v\|_2^2 + C(1 + \|\check{u}_t\|_2\|\nabla\check{u}_t\|_2 + \|\nabla\check{u}\|_2^2\|\nabla^2\check{u}\|_2^2)(\|\sigma\|_2^2 + \|W\|_2^2). \end{aligned}$$

Plugging this into (3.6) leads to

$$\begin{aligned} \frac{d}{dt}\|\sqrt{\hat{\rho}}v\|_2^2 + \mu\|\nabla v\|_2^2 &\le C\|\nabla\check{u}\|_\infty\|\sqrt{\hat{\rho}}v\|_2^2 + C(1 + \|\check{u}_t\|_2\|\nabla\check{u}_t\|_2 \\ &\quad + \|\nabla\check{u}\|_2^2\|\nabla^2\check{u}\|_2^2)(\|\sigma\|_2^2 + \|W\|_2^2). \end{aligned} \tag{3.9}$$

Denote

$$f_2(t) = (\|\sigma\|_2 + \|W\|_2)(t), \quad g_2(t) = \|\sqrt{\hat{\rho}}v\|_2^2(t), \quad G_2(t) = \mu\|\nabla v\|_2^2(t),$$
$$\delta_2(t) = C\|\nabla\check{u}\|_\infty(t), \quad A_2 = C \sup_{0 \le t \le T} (\|\hat{\rho}\|_\infty + \|\hat{P}\|_\infty + \|\nabla\hat{\rho}\|_2 + \|\nabla\hat{P}\|_3)(t),$$
$$\alpha_2(t) = C\|\nabla\check{u}\|_\infty(t), \quad \beta_2(t) = C(1 + \|\check{u}_t\|_2\|\nabla\check{u}_t\|_2 + \|\nabla\check{u}\|_2^2\|\nabla^2\check{u}\|_2^2)(t),$$

then, it follows from (3.4), (3.5), and (3.9) that

$$\begin{cases} \frac{d}{dt}f_2(t) \le \delta_2(t)f_2(t) + A_2\sqrt{G_2(t)}, \\ \frac{d}{dt}g_2(t) + G_2(t) \le \alpha_2(t)g_2(t) + \beta_2(t)f_2^2(t), \\ f_2(0) = 0. \end{cases}$$

By the regularities of $(\hat{\rho}, \hat{u})$ and $(\check{\rho}, \check{u})$, and recalling $\nabla u \in L^1(0, T; L^\infty)$ and $\sqrt{t} u_t \in L^2(0, T; L^2)$, for $u \in \{\hat{u}, \check{u}\}$, one can easily verify that $\alpha_2, \delta_2, t\beta_2 \in L^1((0, T))$. Therefore, one can apply Lemma 3.1 to get $f_2(t) = g_2(t) = G_2(t) = 0$, on $(0, T)$, which implies the uniqueness for Case II.

Existence: Set $\rho_{0n} = \rho_0 + \frac{1}{n}$, $\rho_{n\infty} = \rho_\infty + \frac{1}{n}$, and choose $u_{0n} \in D_0^1 \cap D^2$, such that $u_{0n} \to u_0$ in D_0^1, as $n \to \infty$. Denote

$$\psi_0 = \|\rho_0\|_\infty + \|\rho_0 - \rho_\infty\|_2 + \|\nabla\rho_0\|_{L^2 \cap L^q} + \|\nabla u_0\|_2,$$
$$\psi_{0n} = \|\rho_{0n}\|_\infty + \|\rho_{0n} - \rho_{n\infty}\|_2 + \|\nabla\rho_{0n}\|_{L^2 \cap L^q} + \|\nabla u_{0n}\|_2.$$

Then, one can easily check that $\psi_{0n} \leq \psi_0 + 1$, for sufficiently large n. By Proposition 2.9, there are two positive constants $\mathcal{T}$ and C depending only on μ, λ, a, γ, q, and ψ_0, such that system (1.1)–(1.2), subject to (1.3), has a unique solution (ρ_n, u_n), on $\mathbb{R}^3 \times (0, \mathcal{T})$, satisfying

$$\sup_{0 \leq t \leq \mathcal{T}} (\|\rho_n\|_\infty + \|\rho_n - \rho_{n\infty}\|_2 + \|\nabla\rho_n\|_2 + \|\nabla\rho_n\|_q + \|\partial_t\rho_n\|_2) \leq C, \tag{3.10}$$

$$\sup_{0 \leq t \leq \mathcal{T}} \|\nabla u_n\|_2^2 + \int_0^{\mathcal{T}} (\|\nabla^2 u_n\|_2^2 + \|\sqrt{\rho_n}\partial_t u_n\|_2^2) dt \leq C, \tag{3.11}$$

$$\sup_{0 \leq t \leq \mathcal{T}} \|\sqrt{t}\nabla^2 u_n\|_2^2 + \int_0^{\mathcal{T}} (\|\sqrt{t}\partial_t\nabla u_n\|_2^2 + \|\sqrt{t}\nabla^2 u_n\|_q^2) dt \leq C. \tag{3.12}$$

Thanks to (3.10)–(3.12), there is a subsequence, still denoted by (ρ_n, u_n), and a pair (ρ, u), satisfying

$$\rho - \rho_\infty \in L^\infty(0, \mathcal{T}; H^1 \cap W^{1,q}), \quad \rho_t \in L^\infty(0, \mathcal{T}; L^2), \tag{3.13}$$
$$u \in L^\infty(0, \mathcal{T}; D_0^1) \cap L^2(0, \mathcal{T}; D^2), \tag{3.14}$$
$$\sqrt{t}\nabla^2 u \in L^\infty(0, \mathcal{T}; L^2), \quad \sqrt{t}\nabla u_t \in L^2(0, \mathcal{T}; L^2), \quad \sqrt{t}\nabla^2 u \in L^2(0, \mathcal{T}; L^q), \tag{3.15}$$

such that

$$\rho_n - \rho_{n\infty} \overset{*}{\rightharpoonup} \rho - \rho_\infty, \text{ in } L^\infty(0, \mathcal{T}; H^1 \cap W^{1,q}), \tag{3.16}$$

$$\partial_t\rho_n \overset{*}{\rightharpoonup} \rho_t, \text{ in } L^\infty(0, \mathcal{T}; L^2), \tag{3.17}$$

$$u_n \overset{*}{\rightharpoonup} u, \text{ in } L^\infty(0, \mathcal{T}; D_0^1), \tag{3.18}$$

$$u_n \rightharpoonup u, \text{ in } L^2(0, \mathcal{T}; D^2), \tag{3.19}$$

$$\partial_t u_n \rightharpoonup u_t, \text{ in } L^2(\delta, \mathcal{T}; D_0^1), \tag{3.20}$$

for any $\delta \in (0, \mathcal{T})$. Note that $W^{1,q} \hookrightarrow\hookrightarrow C(\overline{B_k})$, for any positive integer k. With the aid of (3.16)–(3.20), by the Aubin-Lions lemma, and using the Cantor's diagonal argument, there is a sequence, still denoted by (ρ_n, u_n), such that

$$\rho_n \to \rho, \text{ in } C([0, \mathcal{T}]; C(\overline{B_k})), \tag{3.21}$$

$$u_n \to u, \text{ in } L^2(\delta, \mathcal{T}; H^1(B_k)) \cap C([\delta, \mathcal{T}]; L^2(B_k)), \tag{3.22}$$

for any positive integer k, and for any $\delta \in (0, \mathcal{T})$, where B_k is the ball in $\mathbb{R}^3$ centered at the origin of radius k. By the aid of (3.20), (3.21) and (3.22), one has

$$\rho_n u_n \to \rho u, \text{ in } L^2(B_k \times (0, \mathcal{T})), \tag{3.23}$$

$$\rho_n \partial_t u_n \rightharpoonup \rho u_t, \text{ in } L^2(B_k \times (\delta, T)), \tag{3.24}$$

$$\rho_n(u_n \cdot \nabla)u_n \to \rho(u \cdot \nabla)u, \text{ in } L^1(B_k \times (\delta, T)), \tag{3.25}$$

$$a\rho_n^\gamma \to a\rho^\gamma, \text{ in } C(\overline{B_k} \times [0, \mathcal{T}]), \tag{3.26}$$

for any $\delta \in (0, \mathcal{T})$, and for any positive integer k.

Due to (3.17), (3.19), and (3.23)–(3.26), one can take the limit to the system of (ρ_n, u_u) to show that (ρ, u) is a strong solution to system (1.1)–(1.2), on $\mathbb{R}^3 \times (0, \mathcal{T})$, satisfying the regularities (3.13)–(3.15). The convergence (3.21) implies that the initial value of ρ is ρ_0. The regularity of $\rho - \rho_\infty \in C([0, \mathcal{T}]; L^2)$ follows from (3.13).

The regularity $\sqrt{\rho}u_t \in L^2(0, \mathcal{T}; L^2)$ is verified as follows. It follows from (3.20) and (3.21) that $\sqrt{\rho_n}\partial_t u_n \rightharpoonup \sqrt{\rho}u_t$ in $L^2(0, \mathcal{T}; L^2(B_k))$, for any positive integer k. Then, the weakly lower semi-continuity of the norms implies

$$\int_0^\mathcal{T} \|\sqrt{\rho}u_t\|_{L^2(B_k)}^2 dt \leq \liminf_{n \to \infty} \int_0^\mathcal{T} \|\sqrt{\rho_n}\partial_t u_n\|_{L^2(B_k)}^2 dt \leq C,$$

for a positive constant C independent of k. Taking $k \to \infty$ in the above inequality yields $\sqrt{\rho}u_t \in L^2(0, \mathcal{T}; L^2)$.

Finally, we show that $\rho u \in C([0, \mathcal{T}]; L^2)$ and $\rho u|_{t=0} = \rho_0 u_0$. By (1.2) and (3.13)–(3.14), and noticing that $\|u\|_\infty \leq C\|\nabla u\|_2^{\frac{1}{2}}\|\nabla^2 u\|_2^{\frac{1}{2}}$, guaranteed by the Gagliardo-Nirenberg and Sobolev embedding inequalities, it follows

$$\int_0^\mathcal{T} \|\partial_t(\rho u)\|_2^2 dt$$

$$= \int_0^\mathcal{T} \|-(u \cdot \nabla \rho + \operatorname{div} u\rho)u + \rho u_t\|_2^2 dt$$

$$\leq \int_0^\mathcal{T} \left(\|u\|_\infty^2 \|\nabla \rho\|_2 + \|u\|_\infty \|\nabla u\|_2 \|\rho\|_\infty + \|\rho\|_\infty^{\frac{1}{2}} \|\sqrt{\rho}u_t\|_2\right)^2 dt$$

$$\leq C \int_0^\mathcal{T} \left(\|\nabla u\|_2^2 \|\nabla^2 u\|_2^2 + \|\nabla u\|_2^3 \|\nabla^2 u\|_2 + \|\sqrt{\rho}u_t\|_2^2\right) dt$$

$$\leq C \int_0^\mathcal{T} (1 + \|\nabla^2 u\|_2^2 + \|\sqrt{\rho}u_t\|_2^2) dt \leq C. \tag{3.27}$$

Similarly, it follows from (3.10)–(3.11) that $\int_0^\mathcal{T} \|\partial_t(\rho_n u_n)\|_2^2 dt \leq C$, for a positive constant C independent of n. Thanks to these, we deduce by the Hölder inequality that

$$\|(\rho u)(\cdot, t) - \rho_0 u_0\|_{L^2(B_R)}$$

$$\leq \|\rho u - \rho_n u_n\|_{L^2(B_R)} + \|\rho_n u_n - \rho_{0n} u_0\|_{L^2(B_R)} + \|\rho_{0n} u_0 - \rho_0 u_0\|_{L^2(B_R)}$$

$$\leq \|\rho u - \rho_n u_n\|_{L^2(B_R)} + \int_0^t \|\partial_t(\rho_n u_n)\|_{L^2(B_R)} d\tau + \frac{C}{n}\|u_0\|_{L^2(B_R)}$$

$$\leq \|\rho u - \rho_n u_n\|_{L^2(B_R)} + C\sqrt{t} + \frac{C}{n}\|u_0\|_{L^2(B_R)},$$

for a positive constant C independent of n and R. Noticing that $\rho_n u_n \to \rho u$ in $C([\delta, \mathcal{T}]; L^2(B_R))$, for any $\delta \in (0, \mathcal{T})$, guaranteed by (3.21)–(3.22), one can pass the limits $n \to \infty$ first and then $R \to \infty$ to the above inequality, and end up with $\|(\rho u)(\cdot, t) - \rho_0 u_0\|_2 \leq C\sqrt{t}$. This implies $\rho u \in L^\infty(0, \mathcal{T}; L^2)$ and $\rho u|_{t=0} = \rho_0 u_0$. Thank to these and recalling (3.27), one gets further that $\rho u \in C([0, \mathcal{T}]; L^2)$. This completes the proof. □

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