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NONLINEAR STABILITY OF COMPOSITE WAVES FOR ONE-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES EQUATIONS FOR A REACTING MIXTURE*

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Abstract. In this paper, we study the long-time behavior of the solutions for the initial-boundary value problem to a one-dimensional Navier-Stokes equations for a reacting mixture in a half line $\mathbb{R}_+ := (0,\infty)$. We give the asymptotic stability of not only stationary solution for the impermeability problem but also the composite waves consisting of the subsonic BL-solution, the contact wave, and the rarefaction wave for the inflow problem of Navier-Stokes equations for a reacting mixture under some smallness conditions. The proofs are based on basic energy method.

Keywords. Compressible Navier-Stokes equations; reacting mixture; composite waves; nonlinear stability.

AMS subject classifications. 35Q35; 35B40; 76N10.

1. Introduction

In this paper, we investigate the existence and asymptotic behavior of global solutions to the compressible Navier-Stokes equations for a reacting mixture, which describe dynamic combustion. These equations in the Euler coordinates are of the following form, cf. [31]

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = (\mu u_x)_x, \\ (\rho E)_t + (u(\rho E + p))_x = (\mu u u_x)_x + (\kappa\theta_x)_x + (qd\rho Z_x)_x, \\ (\rho Z)_t + (\rho u Z)_x = -K\rho\phi(\theta)Z + (d\rho Z_x)_x, \end{cases} \tag{1.1}$$

where $(x,t) \in [0,+\infty) \times [0,+\infty)$ and ρ, u, E, θ, and Z are the density, the fluid velocity, the total specific energy, temperature and mass fraction of the reactant, respectively. While the positive constants μ, κ, q, d, and K are the coefficients of bulk viscosity, heat conduction, species diffusion, difference in the internal energy of the reactant and the product, the product of Boltzmann's gas constant and the molecular weight, and the rate of reaction, respectively. The total energy has the form

$$E = e + \frac{u^2}{2} + qZ, \tag{1.2}$$

where e is the specific internal energy and q is the difference in the heats of formation of the reactant and the product. The thermodynamic variables ρ, p, e, s and θ are related by the Gibbs equation $de = ds - pd\rho^{-1}$, where s is the specific entropy. The presence of the reaction rate function $\phi(\theta)$ is a main distinctive feature of the above system. Here

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$\phi(\theta)$ is typically determined by the Arrhenius' law (see [31]):

$$\phi(\theta)=\theta^{\alpha}e^{-\frac{A}{\theta}},\ \theta>0, \tag{1.3}$$

where $A>0$, α denote the activation energy, thermodynamic constants, respectively. The Boyle-Charles law gives that

$$p=R\rho\theta,$$

where $R>0$ is a gas constant. From the second law of thermodynamics we have

$$e=c_v\theta,$$

where c_v denotes the specific heat constant.

In 1963, Williams introduced the model of compressible Navier-Stokes system for a reacting mixture in [31], which had received considerable attention in the last few years. Many mathematicians studied this model and have made much progress about existence, uniqueness, and asymptotic behavior of the solutions. For the compressible Navier-Stokes equations for a reacting gas, Gardner in [6] and Wagner in [29] studied the existence and behavior of steady plane wave solutions. Some theoretical and computational properties regarding the structure are analyzed in [31] and the references cited therein. The local-, global-in-time existence theorem and large-time behavior of solutions for the initial/initial-boundary value problems of system (1.1) was proved by Chen in [1], Chen et al. in [2]. In addition to the above results, Chen in [3] also obtained the global entropy solutions for this model with his co-workers. [4, 5, 30] discussed the similar results.

Recently, Li in [18] employed the methods by Kazhikhov-Shelukhin in [15, 16], Jiang in [13, 14] and [17] got the global existence of weak solution to this model on one-dimensional unbounded domains with large initial data in H^1 and the large-time behaviour of the weak solution. Zhao et al. in [19] showed the global existence and large-time behavior for this model with viscous radiative and reactive gas. However, to the best of our knowledge, the large-time behavior of solutions to the compressible Navier-Stokes equations for a reacting mixture in a half line is still open. In this paper, we will partly give some positive answers for this problem.

The initial data for system (1.1) are given by

$$(\rho,u,\theta,Z)(x,0)=(\rho_0,u_0,\theta_0,Z_0)(x),\qquad \inf_{x\in\mathbb{R}_+}\rho_0(x)>0,\qquad \inf_{x\in\mathbb{R}_+}\theta_0(x)>0, \tag{1.4}$$

and the boundary values for ρ, u, θ, Z at $x=0$ are given by

$$(\rho,u,\theta,Z)(0,t)=(\rho_-,u_-,\theta_-,Z_-),\qquad \forall\ t\geq 0, \tag{1.5}$$

where $\rho_->0$, $u_->0$, $\theta_->0$, $Z_->0$ are constants and the following far-field conditions hold

$$\lim_{x\to\infty}(\rho_0,u_0,\theta_0,Z_0)(x)=(\rho_+,u_+,\theta_+,Z_+), \tag{1.6}$$

and the following compatibility conditions hold

$$\rho_0(0)=\rho_-,\quad u_0(0)=u_-,\quad \theta_0(0)=\theta_-,\quad Z_0(0)=Z_-. \tag{1.7}$$

The assumption $u_-<0$ denotes that the gas blows away from the boundary $x=0$, thus the problem is called an outflow problem. The assumption $u_-=0$ means that the gas is impermeable at the boundary $x=0$, this problem is called an impermeable wall problem. The assumption $u_->0$ means that gases blow into the region through the boundary $x=0$ with the velocity u_- and hence this problem is called an inflow problem (see [23]). In this paper, we are devoted to investigating the impermeable wall problem and inflow problem. It is noted that in the case $u_-=0$ the condition on the density can't be imposed. But for the cases $u_->0$ the inflow boundary condition implies that the characteristic of the hyperbolic Equation $(1.1)_1$ for the density ρ is positive around the boundary such that the boundary conditions not only on u, θ and Z to parabolic Equations $(1.1)_2$, $(1.1)_3$ and $(1.1)_4$ but also on ρ to a hyperbolic Equation $(1.1)_1$ are necessary and sufficient for the well-posedness of this inflow problem.

Note that, when $Z=0$, the compressible Equation (1.1) becomes classical compressible Navier-Stokes equations. Then the system (1.1) is reduced to one dimensional Navier-Stokes system in the form of

$$\begin{cases} \rho_t+(\rho u)_x=0, \\ (\rho u)_t+(\rho u^2+p)_x=(\mu u_x)_x, \\ [\rho(e+\frac{u^2}{2})]_t+(\rho u(e+\frac{u^2}{2})+pu)_x=(\mu u u_x)_x+(\kappa\theta_x)_x, \end{cases} \tag{1.8}$$

Moreover, if we neglect the dissipation effects for the large-time behavior, Navier-Stokes system (1.8) turns to the following Euler equations

$$\begin{cases} \rho_t+(\rho u)_x=0, \\ (\rho u)_t+(\rho u^2+p)_x=0, \\ [\rho(e+\frac{u^2}{2})]_t+(\rho u(e+\frac{u^2}{2})+pu)_x=0. \end{cases} \tag{1.9}$$

As we known, the Riemann solutions for Euler system (1.9) consists of shock waves, contact discontinuity waves, rarefaction waves and their compositions. There are many mathematical analysis in the literature on the nonlinear stability of basic waves of the solutions to Cauchy problem of the compressible Navier-Stokes system. Interested readers may refer for some important results to [7,8,10,12,20,21,24,25] and the references cited therein. On the other hand, the initial-boundary value problem (IBVP) of Navier-Stokes system has drawn increasing interest because it has more physical background and of course produces some new mathematical trouble due to the boundary effect. Not only basic wave patterns but also a new wave, which is called the boundary layer solution (BL-solution) [23], may appear in the IBVP case. So far, there have been a large number of nice results about the BL-solution (to the outflow, inflow and impermeable wall problem) for Navier-Stokes system. Interested readers please refer to [9,11] and the references therein. In [26], Qin, Wang and Wang showed the global stability of the wave patterns with the superposition of viscous contact wave and rarefaction wave for the one-dimensional compressible Navier-Stokes equations with free boundary. In [28], Shi, Yong and Zhang investigated the vanishing viscosity limit for non-isentropic gas dynamics with interacting shocks. Recently, Yin in [22] gave the stability of composite wave for inflow problem on the planar magnetohydrodynamics. In this paper, inspired by the relationship between one-dimensional Navier-Stokes equations for a reacting mixture and Navier-Stokes equations, we pay attention to the stability of stationary solution for the impermeable problem and the stability of composite wave, which consists of the subsonic BL-solution, the contact discontinuity wave, and the rarefaction wave for the inflow problem on system (1.1) to the Riemann problem on Euler system in the setting of $Z(x,t)=0$. Compared with the classical Navier-Stokes system, the mass fraction of the reactant Z in this model $(1.10)_4$ leads to some trouble. In fact that $\phi(\theta)$ and the dissipation term $(\frac{d}{v^2}Z_x)_x$ is strongly nonlinear. To overcome these troubles, we use the boundedness of $\phi(\theta)$ and L^2-estimate to obtain exponential decay of $\|Z\|_{L^2(\mathbb{R}_+)}$. To investigate the long-time behavior of solutions to (1.1), (1.4), (1.5), (1.6) and (1.7), it is more convenient to use the Lagrangian coordinates transformation:

$$x \Rightarrow \int_{(0,0)}^{(x,t)} \rho(y,\tau)dy - \rho u(y,\tau)d\tau, \qquad t \Rightarrow t.$$

Thus the system (1.1) can be turned to the following free boundary value problem of Navier-Stokes for reacting mixture system in the Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \qquad x > \sigma_- t, \quad t > 0, \\ u_t + p_x = \mu\left(\frac{u_x}{v}\right)_x, \qquad x > \sigma_- t, \quad t > 0, \\ \left(e + \frac{u^2}{2}\right)_t + (pu)_x = \left(\frac{\kappa\theta_x}{v} + \frac{\mu u u_x}{v}\right)_x + qK\phi(\theta)Z, \qquad x > \sigma_- t, \quad t > 0, \\ Z_t + K\phi(\theta)Z = \left(\frac{d}{v^2}Z_x\right)_x, \qquad x > \sigma_- t, \quad t > 0, \\ (v,u,\theta,Z)(x = \sigma_- t, t) = (v_-, u_-, \theta_-, 0), \qquad u_- > 0, \\ (v,u,\theta,Z)(x,0) = (v_0,u_0,\theta_0,Z_0)(x) \to (v_+,u_+,\theta_+,0), \quad as \quad x \to +\infty, \end{cases} \tag{1.10}$$

where $v(x,t) = \frac{1}{\rho(x,t)} > 0$ denotes the specific volume and $\sigma_- = -\frac{u_-}{v_-}$ represents the boundary moving speed. Now for the perfect gas, we have

$$p = \frac{R\theta}{v}. \tag{1.11}$$

Let $\xi = x - \sigma_- t$. Then we have the half-space problem

$$\begin{cases} \partial_t v - \sigma_- \partial_\xi v - \partial_\xi u = 0, \qquad \xi > 0, \quad t > 0, \\ \partial_t u - \sigma_- \partial_\xi u + \partial_\xi p = \mu \partial_\xi\left(\frac{\partial_\xi u}{v}\right), \qquad \xi > 0, \quad t > 0, \\ \frac{R}{\gamma - 1}(\partial_t\theta - \sigma_-\partial_\xi\theta) + p\partial_\xi u = \kappa\partial_\xi\left(\frac{\partial_\xi\theta}{v}\right) + \frac{\mu(\partial_\xi u)^2}{v} + Kq\phi(\theta)Z, \qquad \xi > 0, \quad t > 0, \\ \partial_t Z - \sigma_-\partial_\xi Z + K\phi(\theta)Z = \partial_\xi\left(\frac{d}{v^2}\partial_\xi Z\right), \qquad \xi > 0, \quad t > 0, \\ (v,u,\theta,Z)(\xi = 0, t) = (v_-,u_-,\theta_-,Z_-), \qquad t \geq 0, \\ (v,u,\theta,Z)(\xi,0) = (v_0,u_0,\theta_0,Z_0)(\xi) \to (v_+,u_+,\theta_+,Z_+), \quad as \ \xi \to +\infty. \end{cases} \tag{1.12}$$

From the thermodynamics theory, the entropy s can be defined as follows:

$$s = R\ln v + \frac{R}{\gamma - 1}\ln\theta + 1, \tag{1.13}$$

which obeys the second law of thermodynamics

$$\theta ds = de + pdv.$$

Then from (1.10), the initial data $s(v_0(x),\theta_0(x))$ can be expressed by $(v_0(x),\theta_0(x))$ as follows:

$$s(v_0(x),\theta_0(x)) = R\ln v_0(x) + \frac{R}{\gamma-1}\ln\theta_0(x) + 1. \tag{1.14}$$

Thus $s_+ = \lim\limits_{x\to+\infty} s(v_0(x),\theta_0(x))$ satisfying

$$s_+ = s(v_+,\theta_+) = R\ln v_+ + \frac{R}{\gamma-1}\ln\theta_+ + 1. \tag{1.15}$$

The rest of the paper is organized as follows. Section 2 is devoted to some preliminaries which will be used in this paper. In Section 3, we reformulate the original system (1.1) and establish our main Theorem 3.2 including the global existence and asymptotic stability of solutions. The proof of Theorem 3.1 and 3.2 are given in Section 4.

Notation: To simplify the presentation, we denote positive constants and constants independent of t by C and c, respectively. In addition, the character "C" and "c" may take different values in different places. $L^p = L^p(\mathbb{R}_+)$ $(1\le p\le\infty)$ denotes the usual Lebesgue space on $[0,\infty)$ with its norm $\|\cdot\|_{L^p}$, and when $p=2$, we write $\|\cdot\|_{L^2(\mathbb{R}_+)} = \|\cdot\|$. $H^s = H^s(\mathbb{R}_+)$ denotes the usual s-th order Sobolev space with its norm $\|f\|_{H^s(\mathbb{R}_+)} = (\sum\limits_{i=0}^{s}\|\partial^i f\|^2)^{\frac12}$.

2. Some preliminaries

In the following, we first define stationary solutions $(\tilde v,\tilde u,\tilde\theta,\tilde Z)(\xi)$ by

$$\begin{cases} -\sigma_-\partial_\xi\tilde v - \partial_\xi\tilde u = 0, \qquad \xi>0,\\ -\sigma_-\partial_\xi\tilde u + \partial_\xi\tilde p = \mu\partial_\xi(\frac{\partial_\xi\tilde u}{\tilde v}), \qquad \xi>0,\\ -\frac{R}{\gamma-1}\sigma_-\partial_\xi\tilde\theta + \tilde p\partial_\xi\tilde u = \kappa\partial_\xi(\frac{\partial_\xi\tilde\theta}{\tilde v}) + \frac{\mu(\partial_\xi\tilde u)^2}{\tilde v} + Kq\phi(\tilde\theta)\tilde Z, \qquad \xi>0,\\ -\sigma_-\partial_\xi\tilde Z + K\phi(\tilde\theta)\tilde Z = \partial_\xi(\frac{d}{\tilde v^2}\partial_\xi\tilde Z), \qquad \xi>0,\\ (\tilde v,\tilde u,\tilde\theta,\tilde Z)(\xi=0) = (v_-,u_-,\theta_-,Z_-),\\ (\tilde v,\tilde u,\tilde\theta,\tilde Z)(\xi=+\infty) = (v_+,u_+,\theta_+,Z_+). \end{cases} \tag{2.1}$$

LEMMA 2.1. *If $(\tilde v,\tilde u,\tilde\theta,\tilde Z)$ is the solution to (2.1), satisfying $\tilde v\ge \underline{v} = \inf\limits_{\mathbb{R}_+}\tilde v > 0$ and $\tilde\theta\ge\underline{\theta} = \inf\limits_{\mathbb{R}_+}\tilde\theta>0$, then we have $Z_+=0$.*

Proof. Assume that Z_+ is a positive and finite constant. Then there exists a constant ξ_0 such that $\tilde Z(\xi) > \frac{Z_+}{2}$ for all $\xi>\xi_0$ due to $\tilde Z(+\infty) = Z_+$. Then for all $\xi>\xi_0$, by $(2.1)_4$ we have

$$\begin{aligned}\left(\frac{d}{\tilde v^2}\partial_\xi\tilde Z + \sigma_-\tilde Z\right)_\xi &= K\phi(\tilde\theta)\tilde Z\\ &\ge K\phi(\underline{\theta})\frac{Z_+}{2}.\end{aligned} \tag{2.2}$$

Here we have used $\phi'(\theta) = (\alpha\theta + A)\theta^{\alpha-2}e^{-\frac{A}{\theta}} > 0$ for $\theta>0$.

Integrating (2.2) over (ξ_0,ξ), we have

$$\frac{d}{\tilde{v}^2}\partial_\xi \tilde{Z}+\sigma_-\tilde{Z}\geq\left(\frac{d}{\tilde{v}^2}\partial_\xi \tilde{Z}+\sigma_-\tilde{Z}\right)(\xi_0)+C(\xi-\xi_0), \tag{2.3}$$

where $C=\frac{1}{2}K\phi(\underline{\theta})Z_+>0$. Let ξ be large enough, we get

$$\partial_\xi \tilde{Z}\geq 1, \tag{2.4}$$

which is in contradiction with $\tilde{Z}(+\infty)=Z_+$. This proves Lemma 2.1. ◻

Now we divide the steady-state solution of (2.1) into three cases: Case 1: $u_-=0$; Case 2: $u_->0, Z_-=0$; Case 3: $u_->0, Z_->0$. In this paper, we discuss only Case 1 and Case 2, and Case 3 will be left to the forthcoming paper in the future.

Case 1: $u_-=0$, which implies $\sigma_-=0$.

Under Case 1, the stationary equations corresponding to system (1.12) is the following ODE system:

$$\begin{cases}\tilde{u}_\xi=0,\\ \left(\frac{R\tilde{\theta}}{\tilde{v}}\right)_\xi=\left(\frac{\mu\tilde{u}_\xi}{\tilde{v}}\right)_\xi,\\ \left(\frac{R\tilde{u}\tilde{\theta}}{\tilde{v}}\right)_\xi=\left(\frac{\mu\tilde{u}\tilde{u}_\xi+\kappa\tilde{\theta}_\xi}{\tilde{v}}\right)_\xi+qK\phi(\tilde{\theta})\tilde{Z},\\ K\phi(\tilde{\theta})\tilde{Z}=\left(\frac{d}{\tilde{v}^2}\tilde{Z}_\xi\right)_\xi,\\ \tilde{v}(0)=v_-,\quad \tilde{v}(+\infty)=v_+,\\ \tilde{u}(0)=u_-,\quad \tilde{u}(+\infty)=u_+,\\ \tilde{\theta}(0)=\theta_-,\quad \tilde{\theta}(+\infty)=\theta_+,\\ \tilde{Z}(0)=Z_-,\quad \tilde{Z}(+\infty)=Z_+.\end{cases} \tag{2.5}$$

We claim that the stationary solution of (2.5) is trivial.

PROPOSITION 2.1. *If $(\tilde{v},\tilde{u},\tilde{\theta},\tilde{Z})$ is a solution of (2.5), then $v_-=v_+$, $u_-=u_+=0$, $\theta_-=\theta_+$, $Z_-=Z_+=0$, $\frac{\theta_+}{v_+}=\frac{\theta_-}{v_-}$, and $\tilde{v}(\xi)=v_-$, $\tilde{u}(\xi)=0$, $\tilde{\theta}(\xi)=\theta_-$, $\tilde{Z}(\xi)=0$.*

Proof. By $(2.5)_1$, we have $\tilde{u}(\xi)\equiv constant$, which implies $\tilde{u}(\xi)=u_-=u_+=0$. Substituting $(2.5)_1$ into $(2.5)_2$, we obtain $\tilde{p}_\xi=0$, which implies $\tilde{p}(\xi)=p_+$ and $\frac{\tilde{\theta}(\xi)}{\tilde{v}(\xi)}=\frac{\theta_+}{v_+}=\frac{\theta_-}{v_-}$. Thus we have the following equations from the $(2.5)_3$:

$$\begin{cases}(\frac{\kappa\tilde{\theta}_\xi}{\tilde{v}})_\xi+qK\phi(\tilde{\theta})\tilde{Z}=0,\\ K\phi(\tilde{\theta})\tilde{Z}=(\frac{d}{\tilde{v}^2}Z_\xi)_\xi.\end{cases} \tag{2.6}$$

Substituting $(2.6)_2$ into $(2.6)_1$, we obtain

$$\kappa a_+\left(\frac{\tilde{v}_\xi}{\tilde{v}}\right)_\xi+qd\left(\frac{\tilde{Z}_\xi}{\tilde{v}^2}\right)_\xi=0, \tag{2.7}$$

where $a_+=\frac{\theta_+}{v_+}$. Integrating the resulting equation on $(+\infty,\xi)$, we have

$$\kappa a_+\frac{\tilde{v}_\xi}{\tilde{v}}+qd\frac{\tilde{Z}_\xi}{\tilde{v}^2}=0, \tag{2.8}$$

which is equal to

$$\left(\frac{1}{2}\kappa a_+\tilde{v}^2+qd\tilde{Z}\right)_\xi=0. \tag{2.9}$$

Thus

$$\frac{1}{2}\kappa a_+\tilde{v}^2+qd\tilde{Z}=b_+\triangleq\frac{1}{2}\kappa a_+v_+^2+qdZ_+, \tag{2.10}$$

which implies

$$\tilde{v}=\frac{2(b_+-qd\tilde{Z})}{\kappa a_+}^{\frac{1}{2}}>0. \tag{2.11}$$

Then we have

$$\tilde{\theta}=a_+\tilde{v}=a_+\left(\frac{2(b_+-qd\tilde{Z})}{\kappa a_+}\right)^{\frac{1}{2}}. \tag{2.12}$$

Substituting (2.11), (2.12) into $(2.6)_2$, we have

$$K\phi\left(a_+\left(\frac{2(b_+-qd\tilde{Z})}{\kappa a_+}\right)^{\frac{1}{2}}\right)\tilde{Z}=\left(\frac{d\kappa a_+}{2(b_+-qd\tilde{Z})}\tilde{Z}_\xi\right)_\xi. \tag{2.13}$$

Multiplying the (2.13) by $2\frac{d\kappa a_+}{2(b_+-qd\tilde{Z})}\tilde{Z}_\xi$, we have

$$\frac{d}{d\xi}\left(\frac{d\kappa a_+}{2(b_+-qd\tilde{Z})}\tilde{Z}_\xi\right)^2=K\phi\left(a_+\left(\frac{2(b_+-qd\tilde{Z})}{\kappa a_+}\right)^{\frac{1}{2}}\right)\tilde{Z}\frac{d\kappa a_+}{b_+-qd\tilde{Z}}\tilde{Z}_\xi. \tag{2.14}$$

Let $F(s)=K\phi(a_+(\frac{2(b_+-qds)}{\kappa a_+})^{\frac{1}{2}})\frac{d\kappa a_+s}{b_+-qds}$ and $G'(s)=F(s)$. Then we have

$$\frac{d}{d\xi}\left(\frac{d\kappa a_+\tilde{Z}_\xi}{2(b_+-qd\tilde{Z})}\right)^2=G(\tilde{Z})_\xi, \tag{2.15}$$

which implies

$$\left(\frac{d\kappa a_+}{2(b_+-qd\tilde{Z})}\tilde{Z}_\xi\right)^2=G(\tilde{Z})-G(Z_+). \tag{2.16}$$

By (2.16), we have $G(\tilde{Z})\geq G(Z_+)$, which implies $\tilde{Z}(\xi)\geq Z_+$ due to $G'(s)=F(s)>0$. Thus $Z_-\geq Z_+$.

Next we will prove $Z_- = Z_+ = 0$. Otherwise, let $Z_- > Z_+$. Then we get the following boundary value problem:

$$\begin{cases} \tilde{Z}_\xi = -\frac{2(b_+ - qd\tilde{Z})}{d\kappa a_+}\sqrt{G(\tilde{Z}) - G(Z_+)}, \\ \tilde{Z}(0) = Z_-, \qquad \tilde{Z}(+\infty) = Z_+, \qquad \text{for} \quad Z_- > Z_+. \end{cases} \tag{2.17}$$

Let $g(\tilde{Z}) = \frac{2(b_+ - qd\tilde{Z})(\sqrt{G(\tilde{Z}) - G(Z_+)})}{d\kappa a_+} > 0$. Then (2.17) can be rewritten as

$$\begin{cases} \tilde{Z}_\xi = -g(\tilde{Z}), \\ \tilde{Z}(0) = Z_-, \quad \tilde{Z}(+\infty) = Z_+, \quad Z_- > Z_+, \end{cases} \tag{2.18}$$

and we have

$$Z_+ \le \tilde{Z}(\xi) \le Z_-, \quad \tilde{Z}_\xi < 0. \tag{2.19}$$

Multiplying the first equation of (2.17) with $G'(Z)$, we have

$$\frac{d}{d\xi}(\sqrt{G(Z) - G(Z_+)}) = -\frac{b_+ - qd\tilde{Z}}{d\kappa a_+} G'(Z). \tag{2.20}$$

Integrating the resulting equation on $(0,\xi)$, we obtain

$$\sqrt{G(Z) - G(Z_+)} - \sqrt{G(Z_-) - G(Z_+)} = -\int_0^\xi \frac{b_+ - qd\tilde{Z}}{d\kappa a_+} G'(\tilde{Z}) d\xi. \tag{2.21}$$

Since $Z_+ < \tilde{Z} < Z_-$, the integral is divergent as $\xi \to +\infty$, which is in contradiction with the left-hand side of (2.21). This implies $Z_- = Z_+ = 0$, and $Z(\xi) \equiv 0$. Combining $\tilde{Z}$ with (2.11), (2.12), we have that $\tilde{\theta}$ and $\tilde{v}$ are constants and $\tilde{v}(\xi) = v_-$, $\tilde{\theta}(\xi) = \theta_-$. □

Case 2: $u_- > 0$ and $Z_- = 0$. We have the following proposition.

PROPOSITION 2.2. *Under the Case 2 ($u_- > 0$ and $Z_- = 0$), if $(\tilde{v}, \tilde{u}, \tilde{\theta}, \tilde{Z})(\xi)$ is a solution of (2.1), then $\tilde{Z}(\xi) \equiv 0$.*

Proof. If $\tilde{Z}(\xi) \not\equiv 0$, then $\tilde{Z}(\xi)$ has at least a positive maximum or negative minimum on $(0,+\infty)$. Without loss of generality, we assume that $\tilde{Z}(\xi)$ has a positive maximum at $\xi_0 \in (0,+\infty)$, then $\tilde{Z}(\xi_0) > 0$, $\tilde{Z}'(\xi_0) = 0$, $\tilde{Z}''(\xi_0) \le 0$, which contradicts with $(2.1)_4$. □

To sum up, we know that the steady-state solution $(\tilde{v}, \tilde{u}, \tilde{\theta}, \tilde{Z})(\xi)$ of (2.1) satisfies $\tilde{Z}(\xi) \equiv 0$. Thus we expect that the large-time behavior of system (1.1) becomes the same as that of Navier-Stokes equations.

Then, as time $t \to +\infty$, we only consider the following Navier-Stokes system

$$\begin{cases} v_t - u_x = 0, \qquad x > \sigma_- t, \quad t > 0, \\ u_t + p_x = \mu\left(\frac{u_x}{v}\right)_x, \qquad x > \sigma_- t, \quad t > 0, \\ \left(e + \frac{u^2}{2}\right)_t + (pu)_x = \left(\frac{\kappa\theta_x}{v} + \frac{\mu u u_x}{v}\right)_x, \qquad x > \sigma_- t, \quad t > 0, \\ (v,u,\theta)(x = \sigma_- t, t) = (v_-, u_-, \theta_-), \qquad u_- > 0, \\ (v,u,\theta)(x,0) = (v_0,u_0,\theta_0)(x) \to (v_+,u_+,\theta_+), \quad \text{as } x \to +\infty, \end{cases} \tag{2.22}$$

and

$$\begin{cases} \partial_t v - \sigma_- \partial_\xi v - \partial_\xi u = 0, & \xi > 0, \quad t > 0, \\ \partial_t u - \sigma_- \partial_\xi u + \partial_\xi p = \mu \partial_\xi (\frac{\partial_\xi u}{v}), & \xi > 0, \quad t > 0, \\ \frac{R}{\gamma-1}(\partial_t \theta - \sigma_- \partial_\xi \theta) + p \partial_\xi u = \kappa \partial_\xi (\frac{\partial_\xi \theta}{v}) + \frac{(\partial_\xi u)^2}{v}, & \xi > 0, \quad t > 0, \\ (v,u,\theta)(\xi=0,t) = (v_-,u_-,\theta_-), & u_- > 0, \\ (v,u,\theta)(\xi,0) = (v_0,u_0,\theta_0)(\xi) \to (v_+,u_+,\theta_+), & as \ \xi \to +\infty. \end{cases} \tag{2.23}$$

When the right end state $(v_+,u_+,\theta_+) \in \mathbb{M}$ ($\mathbb{M}$ denotes a center-stable manifold defined in Section 2.1), Qin and Wang in [27] proved the existence of the boundary layer solution for inflow problem on the Navier-Stokes Equations (2.22) and (2.23). They also obtained the asymptotic stability of the composite wave including the subsonic BL-solution, the contact wave and the rarefaction wave. Now, let us review some known results of Navier-Stokes equations in [27].

If the right state (v_+,u_+,θ_+) is known, the wave curves (BL-solution curve, contact wave curve and 3-rarefaction wave curve) in terms of (v,u,θ) with $v>0$ and $\theta>0$ can be defined in the phase space as follows:

$$BL(v_+,u_+,\theta_+) \equiv \left\{ (v,u,\theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ : \frac{u}{v} = -\sigma_- = \frac{u_+}{v_+}, \ (u,\theta) \in \mathcal{M}(u_+,\theta_+) \right\},$$

$$CD(v_+,u_+,\theta_+) \equiv \{ (v,u,\theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ : p = p_+, u = u_+, v \neq v_+ \},$$

$$R_3(v_+,u_+,\theta_+)$$
$$\equiv \left\{ (v,u,\theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ : s(v,\theta) = s_+, u = u_+ - \int_{v_+}^{v} \lambda_3(z,s_+) dz, \ v > v_+, u < u_+ \right\},$$

where $p_+ = \frac{R\theta_+}{v_+}$, and $\lambda_3 = \lambda_3(v,s)$ is the third characteristic speed which is given in (2.24).

In this paper, we are devoted to proving that if the left end state $(v_-,u_-,\theta_-) \in$ BL-CD-R_3 (v_+,u_+,θ_+) then there exist a unique state $(v_*,u_*,\theta_*) \in \Omega^+_{sub}$ and a unique state (v^*,u^*,θ^*) such that $(v_-,u_-,\theta_-) \in BL(v_*,u_*,\theta_*)$, $(v_*,u_*,\theta_*) \in CD(v^*,u^*,\theta^*)$ and $(v^*,u^*,\theta^*) \in R_3(v_+,u_+,\theta_+)$, and the superposition of the BL-solution, the viscous contact wave and the 3-rarefaction wave for the inflow problem on the equations is asymptotically stable provided that the conditions in Theorem 3.2 hold. It is remarked that we require the BL-solution and the viscous contact wave must be weak but the rarefaction wave is not necessarily weak. Moreover, Ω^+_{sub} is defined in Section 2.1.

2.1. BL-solutions. As we know, the Euler equations corresponding to the Navier-Stokes Equations (2.22) have three characteristic speeds,

$$\lambda_1 = -\sqrt{\frac{\gamma p}{v}}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{\frac{\gamma p}{v}}. \tag{2.24}$$

The sound speed $C(v,\theta)$ and the Mach number $M(v,u,\theta)$ can be defined by

$$C(v,\theta) = v\sqrt{\frac{\gamma p}{v}} = \sqrt{R\gamma\theta}$$

and

$$M(v,u,\theta) = \frac{|u|}{C(v,\theta)} = \frac{|u|}{\sqrt{R\gamma\theta}}.$$

Let $C_+ = C(v_+,\theta_+) = \sqrt{R\gamma\theta_+}$ and $M_+ = \frac{|u_+|}{C_+}$ be the sound speed and the Mach number at the far-field $x = +\infty$, respectively. We can divide the phase plane $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ of (v,u,θ) into three subsections:

$$\Omega_{sub} := \{(v,u,\theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ :\ M(v,u,\theta) < 1\},$$

$$\Gamma_{trans} := \{(v,u,\theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ :\ M(v,u,\theta) = 1\},$$

$$\Omega_{super} := \{(v,u,\theta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ :\ M(v,u,\theta) > 1\},$$

where Ω_{sub}, Γ_{trans} and Ω_{super} denote the subsonic, transonic and supersonic regions, respectively. If we consider the alternative condition $u > 0$ or $u < 0$, then we obtain six connected subsets $\Omega^{\pm}_{sub}$, $\Gamma^{\pm}_{trans}$ and $\Omega^{\pm}_{super}$.

When $(v_-,u_-,\theta_-) \in \Omega^{+}_{sub} := \{(u,\theta)|0 \le u \le \sqrt{R\gamma\theta_+}\}$, we have $\lambda_1(v_-,u_-,\theta_-) < \sigma_- < 0$, hence the existence of the traveling wave solution

$$\begin{cases} (V^B,U^B,\Theta^B)(\xi), \quad \xi = x - \sigma_- t \\ (V^B,U^B,\Theta^B)(0) = (v_-,u_-,\theta_-), \quad (V^B,U^B,\Theta^B)(+\infty) = (v_+,u_+,\theta_+) \end{cases} \tag{2.25}$$

to (2.1) or the stationary solution (BL-solution) to (2.2) is expected. Recalling (2.25), we know that BL-solution satisfies the following ODE equations:

$$\begin{cases} -\sigma_-\partial_\xi V^B - \partial_\xi U^B = 0, \qquad \xi > 0, \\ -\sigma_-\partial_\xi U^B + \partial_\xi P^B = \mu\partial_\xi(\frac{\partial_\xi U^B}{V^B}), \qquad \xi > 0, \\ -\sigma_-\partial_\xi(\frac{R}{\gamma-1}\Theta^B + \frac{(U^B)^2}{2}) + \partial_\xi(P^B U^B) = \mu\partial_\xi(\frac{U^B\partial_\xi U^B}{V^B}) + \kappa\partial_\xi(\frac{\partial_\xi\Theta^B}{V^B}), \qquad \xi > 0, \\ (V^B,U^B,\Theta^B)(0) = (v_-,u_-,\theta_-), \quad (V^B,U^B,\Theta^B)(+\infty) = (v_+,u_+,\theta_+), \end{cases} \tag{2.26}$$

where $P^B = p(V^B,\Theta^B) = \frac{R\Theta^B}{V^B}$.

Integrating the system $(2.26)_1$ over $(\xi,+\infty)$, and then letting $\xi = 0$ in the resulting equality, we see that

$$\sigma_- = -\frac{u_-}{v_-} = -\frac{U^B}{V^B} = -\frac{u_+}{v_+}. \tag{2.27}$$

The existence and uniqueness of the solution to the ODE system (2.26) are established as follows. Here, we shall show some useful results of solutions for (2.26).

Proposition 2.3 (see [27]). *Suppose that $v_\pm > 0$, $u_- > 0$, $\theta_\pm > 0$ and define $\delta^B = |(u_+ - u_-, \theta_+ - \theta_-)|$. If $u_+ \le 0$, then (2.26) has no solution. If $u_+ > 0$, then there exists a suitably small constant $\delta > 0$ such that if $0 < \delta^B \le \delta$, then have the following cases:*

Case I. Supersonic case: $M_+ > 1$. Then there is no solution to (2.26).

Case II. Transonic case: $M_+ = 1$. Then there exists a unique trajectory Γ tangent to the line

$$u_+(U^B - u_+) - \kappa(\gamma-1)(\Theta^B - \theta_+) = 0$$

at the point (u_+,θ_+). For each $(u_-,\theta_-) \in \Gamma$, there exists a unique solution (U^B,Θ^B) satisfying

$$U^B_\xi > 0, \quad \Theta^B_\xi > 0,$$

and

$$\left|\frac{d^n}{d\xi^n}(U^B-u_+,\Theta^B-\theta_+)\right|\le C\frac{(\delta^B)^{n+1}}{(1+\delta^B\xi)^{n+1}},\quad n=0,1,2,\cdots. \tag{2.28}$$

Case III. Subsonic case: $M_+<1$. *Then there exists a center-stable manifold* $\mathcal{M}$ *tangent to the line*

$$(1+a_2c_2u_+)(U^B-u_+)-a_2(\Theta^B-\theta_+)=0$$

on the opposite directions at the point (u_+,θ_+), *where* a_2 *and* c_2 *are some positive constants which are defined in [27]. Only when* $(u_-,\theta_-)\in M$, *there exists a unique solution* $(U^B,\Theta^B)\subset M$ *satisfying the following inequality*

$$\left|\frac{d^n}{d\xi^n}(U^B-u_+,\Theta^B-\theta_+)\right|\le C\delta^B e^{-c\xi},\quad n=0,1,2,\cdots. \tag{2.29}$$

2.2. Viscous contact wave. If $(v_-,u_-,\theta_-)\in CD(v_+,u_+,\theta_+)$, i.e.,

$$u_-=u_+,\quad p_-=p_+, \tag{2.30}$$

then the following Riemann problem of the Euler system

$$\begin{cases}\partial_t v-\partial_x u=0, & x\in\mathbb{R},\quad t>0,\\ \partial_t u+\partial_x p=0, & x\in\mathbb{R},\quad t>0,\\ \partial_t(e+\frac{u^2}{2})+\partial_x(pu)=0, & x\in\mathbb{R},\quad t>0,\\ (v,u,\theta)(x,0)=\begin{cases}(v_-,u_-,\theta_-), & x<0,\\ (v_+,u_+,\theta_+), & x>0,\end{cases}\end{cases} \tag{2.31}$$

admits a single contact discontinuity solution

$$(v,u,\theta)(x,t)=\begin{cases}(v_-,u_-,\theta_-), & x<0,\\ (v_+,u_+,\theta_+), & x>0.\end{cases} \tag{2.32}$$

From [10], we know that the viscous version of the above contact discontinuity, called viscous contact wave $(V^{CD},U^{CD},\Theta^{CD})(x,t)$, could be defined by

$$\begin{cases}\Theta^{CD}(x,t)=\Theta^{Sim}(\frac{x}{\sqrt{1+t}}),\\ V^{CD}(x,t)=\dfrac{R\Theta^{CD}(x,t)}{p_+},\\ U^{CD}(x,t)=u_++\dfrac{\kappa(\gamma-1)}{R\gamma}\dfrac{\partial_x\Theta^{CD}(x,t)}{\Theta^{CD}(x,t)},\end{cases} \tag{2.33}$$

where $\Theta^{Sim}(\eta)(\eta=\frac{x}{\sqrt{1+t}})$ is the unique self-similar solution of the nonlinear diffusion equation

$$\partial_t\Theta=\frac{\kappa(\gamma-1)p_+}{R^2\gamma}\partial_x\left(\frac{\partial_x\Theta}{\Theta}\right),\quad \Theta(\pm\infty,t)=\theta_\pm. \tag{2.34}$$

Thus the viscous contact wave defined in (2.34) satisfies the following property:

$$\begin{aligned}&(1+t)^{\frac{3}{2}}|\partial_x^3\Theta^{CD}|+(1+t)|\partial_x^2\Theta^{CD}|+(1+t)^{\frac{1}{2}}|\partial_x\Theta^{CD}|+|\Theta^{CD}-\theta_\pm|\\ &=O(1)\delta^{CD}e^{-\frac{c_0x^2}{1+t}},\quad as\ \ x\to\pm\infty,\end{aligned} \tag{2.35}$$

where $\delta^{CD}=|\theta_+-\theta_-|$ is the amplitude of the viscous contact wave and c_0 is some positive constant. Note that $\xi=x-\sigma_-t$, then the viscous contact wave $(V^{CD},U^{CD},\Theta^{CD})(\xi,t)$ satisfies

$$\begin{cases}\partial_t V^{CD}-\sigma_-\partial_\xi V^{CD}-\partial_\xi U^{CD}=0,\\ \partial_t U^{CD}-\sigma_-\partial_\xi U^{CD}+\partial_\xi P^{CD}=\mu\partial_\xi(\frac{\partial_\xi U^{CD}}{V^{CD}})+\bar{Q}_1,\\ \frac{R}{\gamma-1}(\partial_t\Theta^{CD}-\sigma_-\partial_\xi\Theta^{CD})+P^{CD}\partial_\xi U^{CD}=\mu\frac{(\partial_\xi U^{CD})^2}{V^{CD}}+\kappa\partial_\xi(\frac{\partial_\xi\Theta^{CD}}{V^{CD}})+\bar{Q}_2,\end{cases}\tag{2.36}$$

where $P^{CD}:=p(V^{CD},\Theta^{CD})=\frac{R\Theta^{CD}}{V^{CD}}$ and the error terms $\bar{Q}_1$, $\bar{Q}_2$ are given by

$$\begin{aligned}\bar{Q}_1&=\partial_t U^{CD}-\sigma_-\partial_\xi U^{CD}-\partial_\xi\left(\frac{\partial_\xi U^{CD}}{V^{CD}}\right)\\ &=O(1)(|\partial_\xi\Theta^{CD}|^3+|\partial_\xi^3\Theta^{CD}|+|\partial_\xi^2\Theta^{CD}||\partial_\xi\Theta^{CD}|)\\ &=O(1)\delta^{CD}(1+t)^{-\frac{3}{2}}e^{-\frac{c_0(\xi+\sigma_-t)^2}{1+t}},\quad as\ \ |\xi+\sigma_-t|\to+\infty,\end{aligned}\tag{2.37}$$

$$\begin{aligned}\bar{Q}_2&=-\mu\frac{(\partial_\xi U^{CD})^2}{V^{CD}}=O(1)(|\partial_\xi\Theta^{CD}|^4+|\partial_\xi^2\Theta^{CD}|^2)\\ &=O(1)\delta^{CD}(1+t)^{-2}e^{-\frac{c_0(\xi+\sigma_-t)^2}{1+t}},\quad as\ \ |\xi+\sigma_-t|\to+\infty.\end{aligned}\tag{2.38}$$

2.3. Rarefaction wave. If $(v_-,u_-,\theta_-)\in R_3(v_+,u_+,\theta_+)$, then there exists a 3-rarefaction wave $(v^r,u^r,\theta^r)(\frac{x}{t})$ which is the global (in time) weak solution of the following Riemann problem

$$\begin{cases}\partial_t v^r-\partial_x u^r=0,\quad x\in\mathbb{R},\quad t>0,\\ \partial_t u^r+\partial_x p(v^r,\theta^r)=0,\quad x\in\mathbb{R},\quad t>0,\\ \frac{R}{\gamma-1}\partial_t\theta^r+p(v^r,\theta^r)\partial_x u^r=0,\quad x\in\mathbb{R},\quad t>0,\\ (v^r,u^r,\theta^r)(x,0)=\begin{cases}(v_-,u_-,\theta_-),\quad x<0,\\ (v_+,u_+,\theta_+),\quad x>0.\end{cases}\end{cases}\tag{2.39}$$

In order to construct the smooth approximated rarefaction wave, we consider the Riemann problem on the Burgers equation

$$\begin{cases}\partial_t\bar{\omega}+\bar{\omega}\partial_x\bar{\omega}=0,\\ \bar{\omega}(x,0)=\bar{\omega}_0(x)=\begin{cases}\omega_-,\quad x<0,\\ \omega_+,\quad x>0\end{cases}\end{cases}\tag{2.40}$$

for $\omega_-<\omega_+$. It is well-known that the Riemann problem (2.40) admits a continuous weak solution $\omega^r(\frac{x}{t})$ connecting ω_- and ω_+, taking the form of

$$\omega^r\left(\frac{x}{t}\right)=\begin{cases}\omega_-,\quad x\le\omega_-t\\ \frac{x}{t},\quad \omega_-t<x<\omega_+t,\\ \omega_+,\quad \omega_+t\le x.\end{cases}\tag{2.41}$$

Moreover, $\omega^r(\frac{x}{t})$ is approximated by a smooth function $\omega(x,t)$ satisfying

$$\begin{cases}\partial_t\omega+\omega\partial_x\omega=0,\\ \omega(x,0)=\omega_0(x)=\begin{cases}\omega_-,\quad x<0,\\ \omega_-+C_q\delta^r\int_0^{\epsilon x}y^qe^{-y}dy,\quad x>0,\end{cases}\end{cases}\tag{2.42}$$

where $\delta^r := \omega_+ - \omega_-$, $q \geq 16$ is a constant, C_q is a constant such that $C_q \int_0^\infty y^q e^{-y} dy = 1$, and $\epsilon \leq 1$ is a small positive constant to be determined later. Then the solution $\omega(x,t)$ of the Burgers Equation (2.42) has the following properties:

LEMMA 2.2 ([8]). *Let $0 < \omega_- < \omega_+$, then the Cauchy problem (2.42) has a unique global smooth solution $\omega(x,t)$ satisfying the following properties:*

(1) $\omega_- < \omega(x,t) < \omega_+$, $\omega_x(x,t) > 0$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$.

(2) For any p $(1 \leq p \leq +\infty)$, there exists a constant $C_{p,q}$ such that for $t \geq 0$

$$\begin{cases} |\omega_x(t)|_p \leq C(p) \min\{\widetilde{\omega}\varepsilon^{1-\frac{1}{p}}, \widetilde{\omega}^{\frac{1}{p}} t^{-1+\frac{1}{p}}\}, \\ |\omega_{xx}(t)|_p \leq C(p) \min\{\widetilde{\omega}\varepsilon^{2-\frac{1}{p}}, \varepsilon^{1-\frac{1}{p}} t^{-1}\}. \end{cases} \tag{2.43}$$

(3) When $x \leq \omega_- t$, $\omega(x,t) \equiv \omega_-$.

(4) $\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left|\omega(x,t) - \omega^r(\frac{x}{t})\right| = 0$.

Here $\widetilde{\omega} = \omega_+ - \omega_- > 0$.

Then the approximate rarefaction waves $(V^R, U^R, \Theta^R)(x,t)$ are defined by

$$\begin{cases} \lambda_3(V^R(t,x), s_+) = -\sqrt{-\widetilde{p}_v(V^R(t,x), \bar{s})} = \omega(x, 1+t), \\ U^R(x,t) = u_+ - \int_{v_+}^{V^R(x,t)} \lambda_3(z, s_+) dz, \\ S^R(x,t) = s(V^R(x,t), \Theta^R(x,t)) = s_+, \\ \Theta^R(t,x) = \frac{a}{R}[V^R(t,x)]^{1-\gamma} \exp\left(\frac{\bar{s}}{c_v}\right). \end{cases} \tag{2.44}$$

Note that $\xi = x - \sigma_- t$, then the smooth 3-rarefaction wave $(V^R, U^R, \Theta^R)(\xi, t)$ defined above satisfies

$$\begin{cases} \partial_t V^R - \sigma_- \partial_\xi V^R - \partial_\xi U^R = 0, \quad \xi > 0, \quad t > 0, \\ \partial_t U^R - \sigma_- \partial_\xi U^R + \partial_\xi P^R = 0, \quad \xi > 0, \quad t > 0, \\ \frac{R}{\gamma - 1}(\partial_t \Theta^R - \sigma_- \partial_\xi \Theta^R) + P^R \partial_\xi U^R = 0, \quad \xi > 0, \quad t > 0, \\ (V^R, U^R, \Theta^R)(\xi = 0, t) = (v_-, u_-, \theta_-), \\ (V^R, U^R, \Theta^R)(\xi = +\infty, t) = (v_+, u_+, \theta_+), \end{cases} \tag{2.45}$$

where $P^R := p(V^R, \Theta^R) = \frac{R\Theta^R}{V^R}$.

LEMMA 2.3. *Let $\delta^R = |(v_+, u_+, \theta_+) - (v_-, u_-, \theta_-)|$, then the smooth approximate rarefaction wave $(V^R, U^R, \Theta^R)(\xi, t)$ satisfies the following properties:*

(i) $\partial_\xi U^R \geq 0$ for $\xi \in \mathbb{R}^+$ and $t \geq 0$.

(ii) For any $1 \leq p \leq +\infty$, there exists a constant $C_{p,q}$ such that for $t \geq 0$,

$$\|\partial_\xi (V^R, U^R, \Theta^R)\|_{L^p(\mathbb{R}_+)} \leq C_{p,q} \min\{\delta^R \epsilon^{1-\frac{1}{p}}, (\delta^R)^{\frac{1}{p}} (1+t)^{-1+\frac{1}{p}}\},$$

$$\|\partial_\xi^2 (V^R, U^R, \Theta^R)\|_{L^p(\mathbb{R}_+)} \leq C_{p,q} \min\{\delta^R \epsilon^{2-\frac{1}{p}}, [(\delta^R)^{\frac{1}{p}} + (\delta^R)^{\frac{1}{q}}](1+t)^{-1+\frac{1}{q}}\}.$$

(iii) If $\xi + \sigma_- t \leq \lambda_3(v_-, u_-, \theta_-)(1+t)$, then $(V^R, U^R, \Theta^R)(\xi, t) \equiv (v_-, u_-, \theta_-)$.

(iv) $\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left|(V^R, U^R, \Theta^R)(\xi, t) - (v^r, u^r, \theta^r)(\frac{\xi}{1+t})\right| = 0$.

3. Reformulation of the problem and main results

Case 1: $u_-=0$. There exist constant stationary solutions $(\tilde{v},\tilde{u},\tilde{\theta},\tilde{Z})(\xi)=(\tilde{v},0,\tilde{\theta},0)$ of (2.5). Define the perturbation as

$$(\varphi,\psi,\zeta,Z)(\xi,t)=(v-\tilde{v},u-0,\theta-\tilde{\theta},Z-0)(\xi,t),$$

then $(\varphi,\psi,\zeta,Z)(\xi,t)$ satisfies

$$\begin{cases}\partial_t\varphi-\partial_\xi\psi=0, \qquad \xi>0, \quad t>0,\\ \partial_t\psi+\partial_\xi p=\mu\partial_\xi(\frac{\partial_\xi\psi}{v}), \qquad \xi>0, \quad t>0,\\ \frac{R}{\gamma-1}(\partial_\xi\zeta)+p\partial_\xi\psi=\kappa\partial_\xi(\frac{\partial_\xi\zeta}{v})+\mu(\frac{(\partial_\xi\psi)^2}{v})+qK\phi(\theta)Z, \qquad \xi>0, \ \ t>0,\\ \partial_t Z+K\phi(\theta)Z=\partial_\xi(\frac{d}{v^2}\partial_\xi Z), \qquad \xi>0, \quad t>0,\\ (\varphi,\psi,\zeta,Z)(\xi,0)=(\varphi_0,\psi_0,\zeta_0,Z_0)(\xi)\\ \quad =(v_0(\xi)-\tilde{v},u_0(\xi),\theta_0(\xi)-\tilde{\theta},Z_0(\xi))\to(0,0,0,0), \qquad as\ \xi\to+\infty,\\ (\varphi,\psi,\zeta,Z)(0,t)=(\varphi,\psi,\zeta,Z)(+\infty,t)=(0,0,0,0).\end{cases} \tag{3.1}$$

THEOREM 3.1 (Case 1: $u_-=0$). *Let $(\tilde{v},\tilde{u},\tilde{\theta},\tilde{Z})(\xi)=(\tilde{v},0,\tilde{\theta},0)$ be the stationary solution. There exists a positive constant $\delta_0>0$, such that if*

$$(\varphi_0,\psi_0,\zeta_0,Z_0)(\xi)\in H^1(\mathbb{R}_+)$$

and

$$\|(\varphi_0,\psi_0,\zeta_0,Z_0)(\xi)\|_{H^1(\mathbb{R}_+)}\le\delta_0, \tag{3.2}$$

then the global solution to the impermeable problem (1.10) or to the half-space problem (1.12) $(v,u,\theta,Z)(\xi,t)$ can be obtained. Moreover the solution satisfies

$$(v-\tilde{v},u,\theta-\tilde{\theta},Z)\in C(0,+\infty;H^1(\mathbb{R}_+))$$

and

$$\lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}_+}|(v-\tilde{v},u,\theta-\tilde{\theta},Z)|=0. \tag{3.3}$$

Proof. This proof is simpler than that of Theorem 3.2, we omit the details. □

Case 2: $u_->0, Z_-=0$.

There exists the stationary solution $(\tilde{v},\tilde{u},\tilde{\theta},\tilde{Z})(\xi)$ satisfying $\tilde{Z}(\xi)\equiv0$. Thus we expect that the solution of (1.1) time-asymptotically becomes $(V,U,\Theta,0)(\xi)$, where $(V,U,\Theta)(\xi)$ is the composite wave consisting of the subsonic BL-solution, the contact wave, and the rarefaction wave of Euler equations corresponding to Navier-Stokes Equations (2.22) for the inflow problem.

Define the composite wave $(V,U,\Theta)(\xi,t)$ by

$$\begin{pmatrix}V\\U\\\Theta\end{pmatrix}(\xi,t)=\begin{pmatrix}V^B+V^{CD}+V^R\\U^B+U^{CD}+U^R\\\Theta^B+\Theta^{CD}+\Theta^R\end{pmatrix}(\xi,t)-\begin{pmatrix}v_*+v^*\\u_*+u^*\\\theta_*+\theta^*\end{pmatrix}, \tag{3.4}$$

where $(V^B,U^B,\Theta^B)(\xi,t)$ is the subsonic BL-solution (Case III) defined in Proposition 2.1 with end states $(\upsilon_-,u_-,\theta_-)$ and $(\upsilon_*,u_*,\theta_*)$, $(V^{CD},U^{CD},\Theta^{CD})(\xi,t)$ is the viscous contact wave defined in (2.12) with the end states $(\upsilon_-,u_-,\theta_-)$ and $(\upsilon_+,u_+,\theta_+)$ replaced by $(\upsilon_*,u_*,\theta_*)$ and $(\upsilon^*,u^*,\theta^*)$, respectively, and $(V^R,U^R,\Theta^R)(\xi,t)$ is the smoothed 3-rarefaction wave defined in (2.22) with the end states $(\upsilon^*,u^*,\theta^*)$ and $(\upsilon_+,u_+,\theta_+)$. From (2.5), (2.15), (2.23) and (3.1), by a careful calculation, we have

$$\begin{cases} \partial_t V-\sigma_-\partial_\xi V-\partial_\xi U=0, \qquad \xi>0, \quad t>0,\\ \partial_t U-\sigma_-\partial_\xi U+\partial_\xi P=\mu\partial_\xi(\frac{\partial_\xi U}{V})+Q_1, \qquad \xi>0, \quad t>0,\\ \frac{R}{\gamma-1}(\partial_t\Theta-\sigma_-\partial_\xi\Theta)+P\partial_\xi U=\mu\frac{(\partial_\xi U)^2}{V}+\kappa\partial_\xi(\frac{\partial_\xi\Theta}{V})+Q_2, \qquad \xi>0, \quad t>0,\\ (V,U,\Theta)(\xi=0,t)=(\upsilon_-+V^{CD}-\upsilon_*,u_-+U^{CD}-u_*,\theta_-+\Theta^{CD}-\theta_*)(\xi=0,t),\\ (V,U,\Theta)(\xi=+\infty,t)=(\upsilon_+,u_+,\theta_+), \end{cases} \tag{3.5}$$

where $P:=p(V,\Theta)=\frac{R\Theta}{V}$ and the error terms Q_1,Q_2 are given by

$$Q_1=\partial_\xi(P-P^B-P^{CD}-P^R)-\left[\partial_\xi\left(\frac{\partial_\xi U}{V}\right)-\partial_\xi\left(\frac{\partial_\xi U^B}{V^B}\right)-\partial_\xi\left(\frac{\partial_\xi U^{CD}}{V^{CD}}\right)\right]+\bar{Q}_1,$$

$$\begin{aligned} Q_2=&(P\partial_\xi U-P^B\partial_\xi U^B-P^{CD}\partial_\xi U^{CD}-P^R\partial_\xi U^R)+\bar{Q}_2\\ &-\left[\frac{(\partial_\xi U)^2}{V}-\frac{(\partial_\xi U^B)^2}{V^B}-\frac{(\partial_\xi U^{CD})^2}{V^{CD}}\right]-\kappa\left[\partial_\xi(\frac{\partial_\xi\Theta}{V})-\partial_\xi(\frac{\partial_\xi\Theta^B}{V^B})-\partial_\xi(\frac{\partial_\xi\Theta^{CD}}{V^{CD}})\right], \end{aligned}$$

where $\bar{Q}_1$ and $\bar{Q}_2$ are the error terms defined in (2.37) and (2.38) of the viscous contact wave.

By a complicated calculation, the error terms Q_1 and Q_2 can be estimated as follows:

$$\begin{aligned} Q_1=&O(1)\big[|(\partial_\xi U^B,\partial_\xi V^B,\partial_\xi\Theta^B,\partial_\xi^2U^B)|\times|(V-V^B,\Theta-\Theta^B,\partial_\xi V^{CD},\partial_\xi U^{CD},\partial_\xi V^R,\partial_\xi U^R)|\\ &+|(\partial_\xi U^{CD},\partial_\xi V^{CD},\partial_\xi\Theta^{CD},\partial_\xi^2U^{CD})|\times|(V-V^{CD},\Theta-\Theta^{CD},\partial_\xi V^R,\partial_\xi U^R)|\\ &+|(\partial_\xi V^R,\partial_\xi\Theta^R)|\times|(V-V^R,\Theta-\Theta^R)|+|\partial_\xi^2U^R,\partial_\xi U^R\partial_\xi V^R|\big]+\bar{Q}_1\\ =&O(1)(\delta^B+\delta^{CD})e^{-c(\xi+t)}+O(1)|(\partial_\xi^2U^R,\partial_\xi U^R\partial_\xi V^R)|+\bar{Q}_1 \end{aligned} \tag{3.6}$$

$$\begin{aligned} Q_2=&O(1)\big[|(\partial_\xi U^B,\partial_\xi V^B,\partial_\xi\Theta^B,\partial_\xi^2\Theta^B)|\\ &\qquad\times|(V-V^B,\Theta-\Theta^B,\partial_\xi V^{CD},\partial_\xi U^{CD},\partial_\xi V^R,\partial_\xi U^R,\partial_\xi\Theta^R)|\\ &+|(\partial_\xi U^{CD},\partial_\xi V^{CD},\partial_\xi\Theta^{CD},\partial_\xi^2\Theta^{CD})|\times|(V-V^{CD},\Theta-\Theta^{CD},\partial_\xi V^R,\partial_\xi U^R,\partial_\xi\Theta^R)|\\ &+|(\partial_\xi V^R,\partial_\xi\Theta^R)|\times|(V-V^R,\Theta-\Theta^R)|+|(\partial_\xi^2\Theta^R,\partial_\xi\Theta^R\partial_\xi V^R,|\partial_\xi U^R|^2)|\big]+\bar{Q}_2\\ =&O(1)(\delta^B+\delta^{CD})e^{-c(\xi+t)}+O(1)|(\partial_\xi^2\Theta^R,\partial_\xi\Theta^R\partial_\xi V^R,|\partial_\xi U^R|^2)|+\bar{Q}_2, \end{aligned} \tag{3.7}$$

where c is some positive constant independent of ξ and t.

Let us define the perturbation as

$$(\varphi,\psi,\zeta,Z)(\xi,t)=[\upsilon-V,u-U,\theta-\Theta,Z-0](\xi,t).$$

Then we get the initial boundary value problem of $[\varphi,\psi,\zeta,Z](\xi,t)$ as follows:

$$\begin{cases} \partial_t\varphi-\sigma_-\partial_\xi\varphi-\partial_\xi\psi=0, \qquad \xi>0, \quad t>0, \\ \partial_t\psi-\sigma_-\partial_\xi\psi+\partial_\xi(p-P)=\mu\partial_\xi(\frac{\partial_\xi u}{v}-\frac{\partial_\xi U}{V})-Q_1, \qquad \xi>0, \quad t>0, \\ \frac{R}{\gamma-1}(\partial_t\zeta-\sigma_-\partial_\xi\zeta)+(p\partial_\xi u-P\partial_\xi U)=\kappa\partial_\xi(\frac{\partial_\xi\theta}{v}-\frac{\partial_\xi\Theta}{V})+\mu(\frac{(\partial_\xi u)^2}{v}-\frac{(\partial_\xi U)^2}{V}) \\ \qquad\qquad\qquad\qquad +qK\phi(\theta)Z-Q_2, \qquad \xi>0, \quad t>0, \\ \partial_t Z-\sigma_-\partial_\xi Z+K\phi(\theta)Z=\partial_\xi(\frac{d}{v^2}\partial_\xi Z), \qquad \xi>0, \quad t>0, \\ (\varphi,\psi,\zeta,Z)(\xi,0)=(\varphi_0,\psi_0,\zeta_0,Z_0)(\xi) \\ \quad =(v_0(\xi)-V(\xi,0),u_0(\xi)-U(\xi,0),\theta_0(\xi)-\Theta(\xi,0),Z_0(\xi)) \\ \quad \to(0,0,0,0), \quad as\ \xi\to+\infty, \\ (\varphi,\psi,\zeta,Z)(0,t)=(v_--V,u_--U,\theta_--\Theta,Z_-)(0,t). \end{cases} \tag{3.8}$$

THEOREM 3.2. *Assume that $u_\pm>0$, $Z_\pm=0$ and $(v_-,u_-,\theta_-)\in BL\text{-}CD\text{-}R_3(v_+,u_+,\theta_+)$. Let $(V,U,\Theta)(\xi,t)$ be the composite wave of the subsonic BL-solution, the viscous contact wave, and the rarefaction wave which is defined in (3.4) with the BL-solution amplitude δ^B and the contact wave amplitude δ^{CD}. Then if*

$$(\varphi_0,\psi_0,\zeta_0,Z_0)(\xi)\in H^1(\mathbb{R}_+)$$

and

$$\|(\varphi_0,\psi_0,\zeta_0,Z_0)(\xi)\|_{H^1(\mathbb{R}_+)}+\delta^B+\delta^{CD}+\epsilon^{\frac{1}{9}}\le\delta_0, \tag{3.9}$$

where ϵ is a small positive constant defined in (2.21), then there exist positive constants $\delta_0>0$ and $C_0>0$, such that a unique global solution $(v,u,\theta,Z)(\xi,t)$ to the inflow problem (1.10) or to the half-space problem (1.12) can be obtained. Moreover the solution satisfies

$$(v-V,u-U,\theta-\Theta,Z)\in C(0,+\infty;H^1(\mathbb{R}_+))$$

and

$$\lim_{t\to+\infty}\sup_{\xi\in\mathbb{R}_+}|(v-V,u-U,\theta-\Theta,Z)|=0. \tag{3.10}$$

PROPOSITION 3.1. *Let (v,u,θ,Z) be a weak solution on $[0,T]\times\mathbb{R}$. If θ satisfies $(1.3)_2$, then there holds*

$$\|Z(\cdot,t)\|_{L^2(\mathbb{R}_+)}\le\|Z_0\|_{L^2(\mathbb{R}_+)}e^{-\alpha_1 t}, \tag{3.11}$$

where $\alpha_1:=K\alpha_0$, $\alpha_0:=\inf_{\xi\in\mathbb{R}_+}\phi(\theta)$.

Proof. From the Equation $(1.10)_4$, we have

$$Z_t-\sigma_-\partial_\xi Z+K\alpha_0 Z\le\partial_\xi\left(\frac{d}{v^2}\partial_\xi Z\right). \tag{3.12}$$

Multiplying the inequality (3.12) by $Ze^{2\alpha_1 t}$ and integrating the resulting inequality in $\mathbb{R}_+$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+}(Ze^{\alpha_1 t})^2 d\xi + \int_{\mathbb{R}_+}\frac{d}{v^2}\left(Z_\xi e^{\alpha_1 t}\right)^2 d\xi \le 0. \tag{3.13}$$

Then a direct calculation shows that

$$\|Z(\cdot,t)\|_{L^2(\mathbb{R}_+)} \le \|Z_0\|_{L^2(\mathbb{R}_+)} e^{-\alpha_1 t}. \tag{3.14}$$

□

4. Global existence and large-time behavior

In this section, we will prove our main stability results of Theorems 3.1 and 3.2. We are devoted to the proof of Theorem 3.2, i.e., the stability of the superposition wave. The proof of Theorem 3.1 is almost along the same lines as, but simpler than, that of Theorem 3.2, so we will omit it for brevity. We will obtain the uniform *a priori* bounds for solutions to the IBVP problem (3.8), then the global existence part of Theorem 3.2 can be derived by the classical continuation method. We first give *a priori* assumption as follows:

$$\sup_{0\le\tau\le t}\|[\varphi,\psi,\zeta,Z](\tau)\|_{H^1(\mathbb{R}_+)} \le \varepsilon_1, \tag{4.1}$$

where ε_1 is a small positive constant.

From *a priori* assumption (4.1), it is easy to get

$$\|[\varphi,\psi,\zeta,Z](\tau)\|_{L^\infty} \le \sqrt{2}\varepsilon_1, \tag{4.2}$$

where we used the following fact

$$\|h(\xi)\|_{L^\infty} \le \sqrt{2}\|h\|^{\frac{1}{2}}\|h_\xi\|^{\frac{1}{2}} \quad for \quad h(\xi)\in H^1(\mathbb{R}_+). \tag{4.3}$$

Proposition 4.1 (A Priori Estimates). *Suppose all the conditions in Theorem 3.2 hold. Let $(\varphi,\psi,\zeta,Z)(\xi,t)$ be a solution to the problem (3.8) on $\mathbb{R}_+\times(0,T]$. There exist constants δ_0 and $C>0$, such that if $(\varphi,\psi,\zeta,Z)\in C(0,T;H^1(\mathbb{R}_+))$ and*

$$\|(\varphi_0,\psi_0,\zeta_0,Z_0)(\xi)\|_{H^1(\mathbb{R}_+)} + \delta^B + \delta^{CD} + \epsilon^{\frac{1}{9}} \le \delta_0, \tag{4.4}$$

then the following estimate holds:

$$\begin{aligned}&\|(\varphi,\psi,\zeta,Z)(\tau)\|^2_{H^1(\mathbb{R}_+)} + \int_0^t \left(\|\partial_\xi\varphi(\tau)\|^2 + \|\partial_\xi(\psi,\zeta,Z)(\tau)\|^2_{H^1(\mathbb{R}_+)}\right)d\tau\\ \le& C\|(\varphi_0,\psi_0,\zeta_0,Z_0)\|^2_{H^1(\mathbb{R}_+)} + C(\epsilon^{\frac{1}{9}} + \delta^B + \delta^{CD}).\end{aligned} \tag{4.5}$$

Lemma 4.1 (Boundary Estimates, see [22, 27, Lemma 4.1]). *There exists a positive constant C such that for any $t>0$,*

$$\int_0^t [|(\varphi,\psi,\zeta)|^2 + |\partial_\tau(\varphi,\psi,\zeta)|^2](0,\tau)d\tau \le C(\delta^{CD})^2, \tag{4.6}$$

$$\int_0^t \left[\left(\frac{\partial_\xi u}{\upsilon}-\frac{\partial_\xi U}{V}\right)\psi\right](0,\tau)d\tau \leq \eta\int_0^t (\|\partial_\xi\psi\|^2+\|\partial_\xi^2\psi\|^2)d\tau + C_\eta(\delta^{CD})^2, \tag{4.7}$$

$$\int_0^t \left[\kappa\frac{\xi}{\theta}\left(\frac{\partial_\xi\theta}{\upsilon}-\frac{\partial_\xi\Theta}{V}\right)\right](0,\tau)d\tau \leq \eta\int_0^t (\|\partial_\xi\zeta\|^2+\|\partial_\xi^2\zeta\|^2)d\tau + C_\eta(\delta^{CD})^2, \tag{4.8}$$

$$\int_0^t [(\partial_\xi\psi)^2+(\partial_\xi\varphi)^2](0,\tau)d\tau \leq \eta\int_0^t \|\partial_\xi^2\psi\|^2 d\tau + C_\eta\int_0^t \|\partial_\xi\psi\|^2 d\tau + C(\delta^{CD})^2, \tag{4.9}$$

where η is a positive small constant to be determined later, and C_η is a positive constant depending on η.

LEMMA 4.2. *Assume the conditions in Theorem 3.2 hold, then we have the following inequality for all $t\in[0,T]$,*

$$\begin{aligned}
&\|[\varphi,\psi,\zeta,Z]\|^2+\int_0^t \|\partial_\xi[\psi,\zeta,Z]\|^2 d\tau+\int_0^t \|\sqrt{\partial_\xi U^R}[\varphi,\zeta]\|^2 d\tau\\
&\leq C\|[\psi_0,\zeta_0,\varphi_0,Z_0]\|^2+C\|Z_0\|_{L^2(\mathbb{R}_+)}+C(\epsilon^{\frac{1}{9}}+\delta^B+\delta^{CD})+C(\epsilon^{\frac{1}{9}}+\delta^B)\int_0^t \|\partial_\xi\varphi\|^2 d\tau\\
&+C\delta^{CD}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\zeta^2)d\xi d\tau + C\|Z_0\|_{L^2(\mathbb{R})}\int_0^t e^{-\alpha_1\tau}\int_{\mathbb{R}_+}\zeta^2 d\xi d\tau.
\end{aligned} \tag{4.10}$$

Proof. Multiplying $(3.8)_1$, $(3.8)_2$, $(3.8)_3$ and $(3.8)_4$ by $-R\Theta(\frac{1}{\upsilon}-\frac{1}{V})$, ψ, $\frac{\zeta}{\theta}$ and Z, respectively, then taking the summation of the resulting equations, we obtain

$$\begin{aligned}
&\partial_t\left(\frac{1}{2}\psi^2+R\Theta\Phi\left(\frac{\upsilon}{V}\right)+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)+\frac{Z^2}{2}\right)+\partial_\xi H_1+\mu\frac{\Theta(\partial_\xi\psi)^2}{\upsilon\theta}+\kappa\frac{\Theta}{\upsilon\theta^2}(\partial_\xi\zeta)^2\\
&\quad+K\phi(\theta)Z^2+\frac{d}{\upsilon^2}Z_\xi^2+P\partial_\xi U^R\left[\Phi\left(\frac{\theta V}{\upsilon\Theta}\right)+\gamma\Phi\left(\frac{\upsilon}{V}\right)\right]\\
&=Q_3-Q_1\psi-\frac{\zeta}{\theta}Q_2+\frac{\zeta}{\theta}Kq\phi(\theta)Z,
\end{aligned} \tag{4.11}$$

where

$$\Phi(s)=s-1-\ln s,$$

$$\begin{aligned}
H_1=&-\sigma_-\left(\frac{1}{2}\psi^2+R\Theta\Phi\left(\frac{\upsilon}{V}\right)+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)+\frac{Z^2}{2}\right)+(p-P)\psi-\mu\left(\frac{\partial_\xi u}{\upsilon}-\frac{\partial_\xi U}{V}\right)\psi\\
&-\kappa\frac{\zeta}{\theta}\left(\frac{\partial_\xi\theta}{\upsilon}-\frac{\partial_\xi\Theta}{V}\right)-\frac{d}{\upsilon^2}Z_\xi Z,
\end{aligned}$$

$$\begin{aligned}
Q_3=&-P(\partial_\xi U^B+\partial_\xi U^{CD})\left[\Phi\left(\frac{\theta V}{\upsilon\Theta}\right)+\gamma\Phi\left(\frac{\upsilon}{V}\right)\right]\\
&+\left[\mu\frac{(\partial_\xi U)^2}{V}+\kappa\partial_\xi\left(\frac{\partial_\xi\Theta}{V}\right)+Q_2\right]\left[(\gamma-1)\Phi\left(\frac{\upsilon}{V}\right)-\Phi\left(\frac{\Theta}{\theta}\right)\right]\\
&+\kappa\frac{\partial_\xi\Theta}{\theta^2\upsilon}\zeta\partial_\xi\zeta+\kappa\frac{\Theta\varphi\partial_\xi\zeta}{\theta^2\upsilon V}\partial_\xi\Theta-\kappa\frac{\zeta\varphi}{\theta^2\upsilon V}(\partial_\xi\Theta)^2+\mu\frac{\partial_\xi U}{\upsilon V}\varphi\partial_\xi\psi-\mu\frac{(\partial_\xi U)^2}{\upsilon\theta V}\varphi\zeta+2\mu\frac{\partial_\xi U}{\upsilon\theta}\zeta\partial_\xi\psi.
\end{aligned}$$

By integrating (4.11) with respect to ξ and τ over $\mathbb{R}_+\times[0,t]$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\left(\frac{1}{2}\psi^2+R\Theta\Phi\left(\frac{\upsilon}{V}\right)+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)+\frac{Z^2}{2}\right)d\xi+\mu\int_0^t\int_{R_+}\frac{\Theta(\partial_\xi\psi)^2}{\upsilon\theta}d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}K\phi(\theta)Z^2\\
&+\int_0^t\int_{\mathbb{R}_+}\frac{d}{\upsilon^2}Z_\xi^2+\kappa\int_0^t\int_{\mathbb{R}_+}\frac{\Theta(\partial_\xi\zeta)^2}{\upsilon\theta^2}d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}P\partial_\xi U^R\left[\Phi\left(\frac{\theta V}{\upsilon\Theta}\right)+\Phi\left(\frac{\upsilon}{V}\right)\right]d\xi d\tau\\
&=\int_{\mathbb{R}_+}\left(\frac{1}{2}\psi^2+R\Theta\Phi\left(\frac{\upsilon}{V}\right)+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)+\frac{Z^2}{2}\right)(\xi,0)d\xi+\int_0^t H_1(0,\tau)d\tau\\
&+\int_0^t\int_{\mathbb{R}_+}Q_3d\xi d\tau-\int_0^t\int_{\mathbb{R}_+}Q_1\psi d\xi d\tau-\int_0^t\int_{\mathbb{R}_+}\frac{\zeta}{\theta}Q_2d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}\frac{\zeta}{\theta}Kq\phi(\theta)Zd\xi d\tau. \qquad (4.12)
\end{aligned}$$

Owning to the smallness of perturbation solutions $[\varphi,\psi,\zeta,Z]$ and the definition of $\Phi(\cdot)$, we obtain

$$\frac{1}{2}\psi^2+R\Theta\Phi\left(\frac{\upsilon}{V}\right)+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)+\frac{Z^2}{2}=O(1)(\varphi^2+\psi^2+\zeta^2+Z^2), \qquad (4.13)$$

$$\Phi\left(\frac{\theta V}{\upsilon\Theta}\right)+\Phi\left(\frac{\upsilon}{V}\right)=O(1)(\varphi^2+\zeta^2). \qquad (4.14)$$

Since $\partial_\xi U^R\geq 0$, we have

$$\int_0^t\int_{\mathbb{R}_+}P\partial_\xi U^R\left[\Phi\left(\frac{\theta V}{\upsilon\Theta}\right)+\Phi\left(\frac{\upsilon}{V}\right)\right]d\xi d\tau\geq c\int_0^t\int_{\mathbb{R}_+}\partial_\xi U^R(\varphi^2+\zeta^2)d\xi d\tau, \qquad (4.15)$$

where (4.14) is used. ∎

By using the *a priori* assumption (4.1), Cauchy-Schwarz's inequality with $0<\eta<1$, the Boundary estimate of Lemma 4.1 and Sobolev's inequality, we derive the estimates for the right-hand side of (4.12) as follows:

$$\int_{\mathbb{R}_+}\left(\frac{1}{2}\psi^2+R\Theta\Phi\left(\frac{\upsilon}{V}\right)+\frac{R\Theta}{\gamma-1}\Phi\left(\frac{\theta}{\Theta}\right)+\frac{Z^2}{2}\right)(\xi,0)d\xi\leq c\|[\varphi_0,\psi_0,\zeta_0,Z_0]\|^2, \qquad (4.16)$$

$$\int_0^t H_1(0,\tau)d\tau\leq\eta\int_0^t(\|\partial_\xi[\psi,\zeta]\|^2+\|\partial_\xi^2[\psi,\zeta]\|^2)d\tau+C_\eta(\delta^{CD})^2 \qquad (4.17)$$

and

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}_+}Q_3d\xi d\tau\leq(C_\eta+C)\int_0^t\int_{\mathbb{R}_+}\Big|\{&\partial_\xi^2\Theta^B,\partial_\xi\Theta^B,\partial_\xi V^B,\partial_\xi U^B,\\
&\partial_\xi^2\Theta^R,(\partial_\xi\Theta^R)^2,(\partial_\xi V^R)^2,(\partial_\xi U^R)^2,\partial_\xi^2\Theta^{CD},(\partial_\xi\Theta^{CD})^2\}\Big|(\varphi^2+\zeta^2)d\xi d\tau\\
&+C\int_0^t\int_{\mathbb{R}_+}|Q_2|(\varphi^2+\zeta^2)d\xi d\tau+\eta\int_0^t\|\partial_\xi[\psi,\zeta]\|^2d\tau. \qquad (4.18)
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}|(V^B_\xi,U^B_\xi,\Theta^B_\xi,\Theta^B_{\xi\xi})|(\varphi^2+\zeta^2)d\xi d\tau\\
&\le C\delta^B\int_0^t\int_{\mathbb{R}_+}e^{-c\xi}(|(\varphi,\zeta)|^2(0,\tau)+\xi\|\partial_\xi[\varphi,\zeta]\|^2)d\xi d\tau\\
&\le C\delta^B\int_0^t|(\varphi,\zeta)|^2(0,\tau)d\tau+C\delta^B\int_0^t\int_{\mathbb{R}_+}\|\partial_\xi[\varphi,\zeta]\|^2d\xi d\tau\\
&\le C\delta^B(\delta^{CD})^2+C\delta^B\int_0^t\int_{\mathbb{R}_+}\|\partial_\xi[\varphi,\zeta]\|^2d\xi d\tau,
\end{aligned}\tag{4.19}$$

where we have used (2.29), (4.6) and the inequality

$$|f(\xi)|=\left|f(0)+\int_0^\xi\partial_\xi f dy\right|\le|f(0)|+\sqrt{\xi}\|\partial_\xi f\|.\tag{4.20}$$

Similar to [22], Lemma 2.3 gives

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}(|(V^R_\xi,U^R_\xi,\Theta^R_\xi)|^2,|\Theta^R_{\xi\xi}|)(\varphi^2+\zeta^2)d\xi d\tau\\
&\le\int_0^t(\|\partial_\xi[V^R,U^R,\Theta^R]\|^2+\|\partial_\xi^2\Theta^R\|_{L^1})\|[\varphi,\zeta]\|^2_{L^\infty}d\tau\\
&\le C\epsilon^{\frac19}\int_0^t(1+\tau)^{-\frac56}\|[\varphi,\zeta]\|\|\partial_\xi[\varphi,\zeta]\|d\tau\\
&\le C\epsilon^{\frac19}+C\epsilon^{\frac19}\int_0^t\|\partial_\xi[\varphi,\zeta]\|^2d\tau,
\end{aligned}\tag{4.21}$$

where we have used

$$\|\partial_\xi[V^R,U^R,\Theta^R]\|^2\le C\epsilon^{\frac19}(1+t)^{-\frac89}$$

and

$$\|\partial_\xi^2\Theta^R\|_{L^1}\le C\epsilon^{\frac19}(1+t)^{-\frac56}.$$

Using the properties of the viscous contact wave, we have

$$\int_0^t\int_{\mathbb{R}_+}(|\Theta^{CD}_\xi|^2,|\Theta^{CD}_{\xi\xi}|)(\phi^2+\zeta^2)\le C\delta^{CD}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\zeta^2)d\xi d\tau.\tag{4.22}$$

Then substituting (4.19)-(4.22) into (4.18), we get

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}_+}Q_3d\xi d\tau\le&[\eta+(C_\eta+C)(\epsilon^{\frac19}+\delta^B)]\int_0^t\|\partial_\xi[\varphi,\psi,\zeta]\|^2d\tau+C(\epsilon^{\frac19}+\delta^B+\delta^{CD})\\
&+(C_\eta+C)\delta^{CD}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\zeta^2)d\xi d\tau.
\end{aligned}\tag{4.23}$$

Finally, we turn to estimate the remaining three terms as follows:

$$\int_0^t\int_{\mathbb{R}_+}Q_1\psi d\xi d\tau\le C\int_0^t\|\psi\|_{L^\infty}\|Q_1\|_{L^1}d\tau$$

$$\begin{aligned}&\le C\int_0^t \|\psi\|^{\frac12}\|\partial_\xi\psi\|^{\frac12}[(\delta^B+\delta^{CD})e^{-c\tau}+\delta^{CD}(1+\tau)^{-1}+\epsilon^{\frac19}(1+\tau)^{-\frac56}]d\tau\\ &\le C(\epsilon^{\frac19}+\delta^B+\delta^{CD})+C(\epsilon^{\frac19}+\delta^B+\delta^{CD})\int_0^t\|\partial_\xi\psi\|^2d\tau, \qquad (4.24)\end{aligned}$$

and

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_+}Q_2\frac{\zeta}{\theta}d\xi d\tau &\le C\int_0^t\|\zeta\|_{L^\infty}\|Q_2\|_{L^1}d\tau\\ &\le C\int_0^t\|\zeta\|^{\frac12}\|\partial_\xi\zeta\|^{\frac12}[(\delta^B+\delta^{CD})e^{-c\tau}+(\delta^{CD})^2(1+\tau)^{-\frac32}+\epsilon^{\frac19}(1+\tau)^{-\frac56}]d\tau\\ &\le C(\epsilon^{\frac19}+\delta^B+\delta^{CD})+C(\epsilon^{\frac19}+\delta^B+\delta^{CD})\int_0^t\|\partial_\xi\zeta\|^2d\tau. \qquad (4.25)\end{aligned}$$

Owing to Proposition 3.1, we have

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_+}\frac{\zeta}{\theta}Kq\phi(\theta)Zd\xi d\tau &\le C\int_0^t\|Z\|_{L^2(\mathbb{R}_+)}d\tau+C\int_0^t\|Z\|_{L^2(\mathbb{R}_+)}\int_{\mathbb{R}_+}\zeta^2d\xi d\tau\\ &\le C\|Z_0\|_{L^2(\mathbb{R}_+)}+C\|Z_0\|_{L^2(\mathbb{R}_+)}\int_0^t e^{-\alpha\tau}\int_{\mathbb{R}_+}\zeta^2d\xi d\tau. \qquad (4.26)\end{aligned}$$

Substituting (4.13)-(4.26) into (4.12), choosing suitably small η, ϵ, δ^B, δ^{CD} and ε_1, we obtain (4.10) and thus the proof of Lemma 4.2 is completed.

LEMMA 4.3. *Assume the conditions in Theorem 3.2 hold, then we have for* $t\in[0,T]$,

$$\begin{aligned}&\|\partial_\xi\varphi\|^2+\int_0^t\|\partial_\xi\varphi\|^2d\tau\\ \le& C\left(\|[\psi_0,\zeta_0,Z_0]\|^2+\|Z_0\|_{L^2(\mathbb{R}_+)}\right)+C\|\varphi_0\|^2_{H^1}+C(\epsilon^{\frac19}+\delta^B+\delta^{CD})+\eta\int_0^t\|\partial_\xi^2\psi\|^2d\tau\\ &+C\delta^{CD}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau\\ &+C\|Z_0\|_{L^2(\mathbb{R}_+)}\int_0^t e^{-\alpha_1 t}\int_{\mathbb{R}_+}\zeta^2d\xi d\tau. \qquad (4.27)\end{aligned}$$

Proof. Differentiating $(3.8)_1$ with respect to ξ and then we obtain

$$\partial_t\partial_\xi\varphi-\sigma_-\partial_\xi^2\varphi-\partial_\xi^2\psi=0. \qquad (4.28)$$

Multiplying $(3.8)_2$ and (4.28) by $-v\partial_\xi\varphi$ and $\mu\partial_\xi\varphi$, respectively, then integrating the resulting identities over $\mathbb{R}_+\times[0,t]$, we have

$$\begin{aligned}&-\int_0^t\int_{\mathbb{R}_+}\partial_t\psi v\partial_\xi\varphi d\xi d\tau+\sigma_-\int_0^t\int_{\mathbb{R}_+}\partial_\xi\psi v\partial_\xi\varphi d\xi d\tau\\ &\quad-\int_0^t\int_{\mathbb{R}_+}\partial_\xi(p-P)v\partial_\xi\varphi d\xi d\tau+\mu\int_0^t\int_{\mathbb{R}_+}\partial_\xi^2\psi\partial_\xi\varphi d\xi d\tau\\ &=-\mu\int_0^t\int_{\mathbb{R}_+}\partial_\xi(v)^{-1}\partial_\xi uv\partial_\xi\varphi d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}\partial_\xi^2U\left(\frac1V-\frac1v\right)v\partial_\xi\varphi d\xi d\tau\end{aligned}$$

$$+\int_0^t\int_{\mathbb{R}_+}\partial_\xi U\partial_\xi(V)^{-1}v\partial_\xi\varphi d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}Q_1 v\partial_\xi\varphi d\xi d\tau \tag{4.29}$$

and

$$\mu\int_0^t\int_{\mathbb{R}_+}(\partial_t\partial_\xi\varphi-\sigma_-\partial_\xi^2\varphi-\partial_\xi^2\psi)\partial_\xi\varphi d\xi d\tau=0. \tag{4.30}$$

Combining (4.29) with (4.30), we have

$$\begin{aligned}
&-\int_{\mathbb{R}_+}\psi v\partial_\xi\varphi d\xi+\frac{\mu}{2}\int_{\mathbb{R}_+}(\partial_\xi\varphi)^2 d\xi+\int_0^t\int_{\mathbb{R}_+}P(\partial_\xi\varphi)^2 d\xi d\tau\\
=&-\int_{\mathbb{R}_+}\psi_0(\xi)v_0(\xi)\partial_\xi\varphi_0(\xi)d\xi+\frac{\mu}{2}\int_{\mathbb{R}_+}(\partial_\xi\varphi_0(\xi))^2 d\xi+\frac{\mu|\sigma_-|}{2}\int_0^t(\partial_\xi\varphi)^2(0,\tau)d\tau\\
&-\int_0^t\int_{\mathbb{R}_+}\psi\partial_t v\partial_\xi\varphi d\xi d\tau-\int_0^t\int_{\mathbb{R}_+}\psi v\partial_t\partial_\xi\varphi d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}R\partial_\xi[\frac{\zeta}{v}]v\partial_\xi\varphi d\xi d\tau\\
&-\int_0^t\int_{\mathbb{R}_+}R\varphi\partial_\xi\left[\frac{\Theta}{vV}\right]v\partial_\xi\varphi d\xi d\tau-\sigma_-\int_0^t\int_{\mathbb{R}_+}\partial_\xi\psi v\partial_\xi\varphi d\xi d\tau-\mu\int_0^t\int_{\mathbb{R}_+}\partial_\xi(v^{-1})\partial_\xi u v\partial_\xi\varphi d\xi d\tau\\
&+\mu\int_0^t\int_{\mathbb{R}_+}\partial_\xi U\partial_\xi(V^{-1})v\partial_\xi\varphi d\xi d\tau+\mu\int_0^t\int_{\mathbb{R}_+}\partial_\xi^2 U(\frac{1}{V}-\frac{1}{v})v\partial_\xi\varphi d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}Q_1 v\partial_\xi\varphi d\xi d\tau\\
=&-\int_{\mathbb{R}_+}\psi_0(\xi)v_0(\xi)\partial_\xi\varphi_0(\xi)d\xi+\frac{\mu}{2}\int_{\mathbb{R}_+}(\partial_\xi\varphi_0(\xi))^2 d\xi+\sum_{i=5}^{14}I_i.
\end{aligned} \tag{4.31}$$

Combining the *a priori* assumption (4.1), Cauchy-Schwarz's inequality with $0<\eta<1$, Sobolev's inequality (4.5) with Lemma 4.1 [(4.6), (4.9)], we obtain the estimates I_i ($5\le i\le 14$) as follows:

$$|I_5|\le\eta\int_0^t\|\partial_\xi^2\psi\|^2 d\tau+C_\eta\int_0^t\|\partial_\xi\psi\|^2 d\tau+C(\delta^{CD})^2,$$

$$\begin{aligned}
|I_6|\le&C\int_0^t\int_{\mathbb{R}_+}|\psi\partial_\xi\psi\partial_\xi\varphi|d\xi d\tau+C\int_0^t\int_{\mathbb{R}_+}|\psi\partial_\xi U\partial_\xi\varphi|d\xi d\tau\\
&+C\int_0^t\int_{\mathbb{R}_+}|\psi(\partial_\xi\varphi)^2|d\xi d\tau+C\int_0^t\int_{\mathbb{R}_+}|\psi\partial_\xi V\partial_\xi\varphi|d\xi d\tau\\
\le&C_\eta(\delta^{CD}+\delta^B+\epsilon^{\frac{1}{2}})+C(\varepsilon_1+\eta+\delta^B)\int_0^t\|\partial_\xi[\psi,\varphi]\|^2 d\tau\\
&+C_\eta\delta^{CD}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}\psi^2 d\xi d\tau,
\end{aligned}$$

$$\begin{aligned}
|I_7|\le&\int_0^t|(\psi v\partial_\xi\psi)(0,\tau)+\sigma_-(\psi v\partial_\xi\varphi)(0,\tau)|d\tau+C\int_0^t\int_{\mathbb{R}_+}(\partial_\xi\psi)^2 d\xi d\tau\\
&+C\int_0^t\int_{\mathbb{R}_+}|\psi\partial_\xi v\partial_\xi[\varphi,\psi]|d\xi d\tau+C\int_0^t\int_{\mathbb{R}_+}|\partial_\xi\varphi v\partial_\xi\psi|d\xi d\tau\\
\le&C_\eta(\delta^B+(\delta^{CD})^2+\epsilon^{\frac{1}{2}})+\eta\int_0^t\|\partial_\xi[\psi,\varphi,\partial_\xi\psi]\|^2 d\tau+(C_\eta+C)\int_0^t\|\partial_\xi\psi\|^2 d\tau
\end{aligned}$$

$$+C_\eta\delta^{CD}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}\psi^2d\xi d\tau,$$

$$\begin{aligned}&|I_8|+|I_9|+|I_{10}|\\ \le&(\eta+C\varepsilon_1)\int_0^t\|\partial_\xi\varphi\|^2d\tau+C_\eta\int_0^t\|\partial_\xi[\zeta,\psi]\|^2d\tau+C_\eta\int_{R_+}(\varphi^2+\zeta^2)(\partial_\xi\Theta)^2d\xi d\tau\\ \le&C(\epsilon^{\frac19}+\delta^B)+(\eta+C\varepsilon_1+C\epsilon^{\frac19}+C\delta^B)\int_0^t\|\partial_\xi[\varphi,\zeta]\|^2d\tau+C_\eta\int_0^t\|\partial_\xi[\zeta,\psi]\|^2d\tau\\ &+C\delta^{CD}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\zeta^2)d\xi d\tau,\end{aligned}$$

$$\begin{aligned}|I_{11}+I_{12}+I_{13}|\le&C\int_0^t\int_{\mathbb{R}_+}(\partial_\xi\varphi)^2|\partial_\xi\psi|d\xi d\tau+C\int_0^t\int_{\mathbb{R}_+}(\partial_\xi\varphi)^2|\partial_\xi U|d\xi d\tau\\ &+C\int_0^t\int_{\mathbb{R}_+}(|\partial_\xi V\partial_\xi U|+|\partial_\xi^2U|)|\varphi\partial_\xi\varphi|d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}|\partial_\xi V\partial_\xi\psi\partial_\xi\varphi|d\xi d\tau\\ \le&C(\varepsilon_1+\delta^B+\delta^{CD}+\epsilon)\int_0^t\|\partial_\xi[\psi,\partial_\xi\psi,\varphi]\|^2d\tau+C\delta^{CD},\end{aligned}$$

and

$$|I_{14}|\le\eta\int_0^t\|\partial_\xi\varphi\|^2d\tau+C_\eta\int_0^t\|Q_1\|^2d\tau\le\eta\int_0^t\|\partial_\xi\varphi\|^2d\tau+C_\eta(\epsilon^{\frac19}+\delta^B+\delta^{CD}).\quad(4.32)$$

By substituting the estimates for I_i $(5\le i\le14)$ and (4.11) into (4.31), then letting η, ϵ, δ^B, δ^{CD} and ε_1 be small enough, and applying Cauchy-Schwarz's inequality, we derive (4.27). This completes the proof of Lemma 4.3. □

LEMMA 4.4. *Assume the conditions in Theorem 3.2 hold, then we have the following energy estimate for* $t\in[0,T]$,

$$\begin{aligned}&\|\partial_\xi[\psi,\zeta,Z]\|^2+\int_0^t\|\partial_\xi^2[\psi,\zeta,Z]\|^2d\tau\\ \le&C(\|[\varphi_0,\psi_0,\zeta_0,Z_0]\|^2+\|Z_0\|)+C(\epsilon^{\frac19}+\delta^B+\delta^{CD})\\ &+C\delta^{CD}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau\\ &+C\|Z_0\|_{L^2(\mathbb{R}_+)}\int_0^te^{-\alpha_1t}\int_{\mathbb{R}_+}\zeta^2d\xi d\tau.\end{aligned}\quad(4.33)$$

Proof. Multiplying $(3.8)_2$ by $-\partial_\xi^2\psi$, and integrating the resulting identity over $\mathbb{R}_+\times[0,t]$, we have

$$\begin{aligned}&\frac12\int_{\mathbb{R}_+}(\partial_\xi\psi)^2d\xi+\mu\int_0^t\int_{\mathbb{R}_+}\frac{(\partial_\xi^2\psi)^2}{v}d\xi d\tau\\ =&\frac12\int_{\mathbb{R}_+}(\partial_\xi\psi_0)^2d\xi-\int_0^t(\partial_\xi\psi\partial_\tau\psi)(0,\tau)d\tau+\frac{\sigma_-}{2}\int_0^t(\partial_\xi\psi)^2(0,\tau)d\tau\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}_+}\partial_\xi(p-P)\partial_\xi^2\psi d\xi d\tau+\mu\int_0^t\int_{\mathbb{R}_+}\frac{\partial_\xi\psi\partial_\xi\varphi}{v^2}\partial_\xi^2\psi d\xi d\tau+\mu\int_0^t\int_{\mathbb{R}_+}\frac{\partial_\xi\psi\partial_\xi V}{v^2}\partial_\xi^2\psi d\xi d\tau\\
&-\mu\int_0^t\int_{\mathbb{R}_+}\partial_\xi(\frac{\partial_\xi U}{v}-\frac{\partial_\xi U}{V})\partial_\xi^2\psi d\xi d\tau+\int_0^t\int_{\mathbb{R}_+}Q_1\partial_\xi^2\psi d\xi d\tau\\
=&\frac{1}{2}\int_{\mathbb{R}_+}(\partial_\xi\psi_0)^2d\xi+\sum_{j=1}^7 J_j. \qquad (4.34)
\end{aligned}
$$

Now we turn to estimate J_j $(1\le j\le 7)$ term by term. It's similar as in [22]. For brevity, we only list the following estimates:

$$
|J_1|+|J_2|\le\eta\int_0^t\|\partial_\xi^2\psi\|^2d\tau+C_\eta\int_0^t\|\partial_\xi\psi\|^2d\tau+C(\delta^{CD})^2,
$$

$$
\begin{aligned}
|J_3|\le& C\int_0^t\int_{\mathbb{R}_+}|\partial_\xi[\zeta,\varphi]\partial_\xi^2\psi|d\xi d\tau+C\int_0^t\int_{\mathbb{R}_+}|[\zeta,\varphi]\partial_\xi[\varphi,V]\partial_\xi^2\psi|d\xi d\tau\\
\le& C(\epsilon^{\frac{1}{9}}+\delta^B)+(C\varepsilon_1+\eta)\int_0^t\|\partial_\xi[\varphi,\partial_\xi\psi]\|^2d\tau+[C_\eta+C(\epsilon^{\frac{1}{9}}+\delta^B)]\int_0^t\|\partial_\xi[\zeta,\varphi]\|^2d\tau\\
&+C\delta^{CD}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\zeta^2)d\xi d\tau,
\end{aligned}
$$

$$
|J_4|+|J_5|\le C(\epsilon+\delta^B+\delta^{CD}+\varepsilon_1)\int_0^t\|\partial_\xi[\psi,\partial_\xi\psi]\|^2d\tau
$$

$$
\begin{aligned}
|J_6|\le& C\int_0^t\int_{\mathbb{R}_+}|\partial_\xi^2U\varphi\partial_\xi^2\psi|d\xi d\tau+C\int_0^t\int_{\mathbb{R}_+}|\partial_\xi U\partial_\xi\varphi\partial_\xi^2\psi|d\xi d\tau\\
\le& C(\epsilon^{\frac{1}{9}}+\delta^B+\delta^{CD})+C(\epsilon+\delta^B+\delta^{CD})\int_0^t\|\partial_\xi[\varphi,\partial_\xi\psi]\|^2d\tau,
\end{aligned}
$$

and

$$
|J_7|\le\eta\int_0^t\|\partial_\xi^2\psi\|^2d\tau+C_\eta\int_0^t\|Q_1\|^2d\tau\le\eta\int_0^t\|\partial_\xi^2\psi\|^2d\tau+C_\eta(\epsilon^{\frac{1}{9}}+\delta^B+\delta^{CD}).
$$

Inserting the above estimations for J_j $(1\le j\le 7)$ into (4.34), and recalling (4.27) and (4.12), then choose $\epsilon>0,\delta^B>0,\delta^{CD}>0$ and $\eta>0$ small enough, to derive

$$
\begin{aligned}
\|\partial_\xi\psi\|^2+\int_0^t\|\partial_\xi^2\psi\|^2d\tau\le& C(\|[\zeta_0,Z_0]\|^2+\|Z_0\|)+C\|[\varphi_0,\psi_0]\|^2_{H^1(\mathbb{R}_+)}+C(\epsilon^{\frac{1}{9}}+\delta^B+\delta^{CD})\\
&+C\delta^{CD}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau\\
&+C\|Z_0\|_{L^2(\mathbb{R}_+)}\int_0^t e^{-\alpha_1 t}\int_{\mathbb{R}_+}\zeta^2d\xi d\tau. \qquad (4.35)
\end{aligned}
$$

Multiplying $(3.8)_3$ by $-\partial_\xi^2\zeta$, similar to the estimate of $\|\partial_\xi\psi\|^2(t)$, we have

$$\begin{aligned}\|\partial_\xi\zeta\|^2+\int_0^t\|\partial_\xi^2\zeta\|^2d\tau \le & C(\|[\varphi_0,\psi_0,\zeta_0,Z_0]\|^2_{H^1(\mathbb{R}_+)}+\|Z_0\|)+C(\epsilon^{\frac{1}{9}}+\delta^B+\delta^{CD})\\ &+C\delta^{CD}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau\\ &+C\|Z_0\|_{L^2(\mathbb{R}_+)}\int_0^t e^{-\alpha_1 t}\int_{\mathbb{R}_+}\zeta^2 d\xi d\tau.\end{aligned} \tag{4.36}$$

Multiplying $(3.8)_4$ by $-\partial_\xi^2 Z$, and integrating the resulting equality over $\mathbb{R}_+\times[0,t]$, we obtain

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}_+}|\partial_\xi Z|^2d\xi-\frac{1}{2}\sigma_-\int_0^t(\partial_\xi Z)^2(0,\tau)d\tau+\int_0^t\int_{\mathbb{R}_+}\frac{d}{\upsilon^2}|\partial_\xi^2 Z|^2d\xi d\tau\\ =&\frac{1}{2}\int_{\mathbb{R}_+}|\partial_\xi Z(\xi,0)|^2d\xi+\int_0^t\int_{\mathbb{R}_+}K\phi(\theta)ZZ_{\xi\xi}d\xi-\int_{\mathbb{R}_+}\frac{2d}{\upsilon^3}\upsilon_\xi Z_\xi Z_{\xi\xi}d\xi\\ \le & C\int_{\mathbb{R}_+}|\partial_\xi Z(\xi,0)|^2d\xi+(1+\|\varphi_\xi\|^{\frac{4}{3}}+\|V_\xi\|^2_{L^\infty})\int_0^t\int_{\mathbb{R}_+}\frac{d}{2\upsilon^2}|Z_{\xi\xi}|^2d\xi d\tau\\ &+C\int_0^t\int_{\mathbb{R}_+}K\phi(\theta)Z^2d\xi d\tau+C\int_0^t\int_{\mathbb{R}_+}|Z_\xi|^2d\xi d\tau.\end{aligned} \tag{4.37}$$

Combining (4.37) with (4.10), we have

$$\int_{\mathbb{R}_+}|Z_\xi|^2d\xi+\int_0^t\int_{\mathbb{R}_+}|Z_{\xi\xi}|^2d\xi d\tau\le\int_{\mathbb{R}_+}(|Z(\xi,0)|^2+|Z_\xi(\xi,0)|^2)d\xi. \tag{4.38}$$

Putting (4.38), (4.36) and (4.35) together, we get the desired estimate (4.33). Thus we complete the proof of Lemma 4.4. □

To close the energy, we need the following classical lemmas from [8, 27].

Lemma 4.5. *Assume that $f(\xi,t)$ satisfies*

$$f\in L^\infty(0,T;L^2(\mathbb{R}_+)),\quad \partial_\xi f\in L^2(0,T;L^2(\mathbb{R}_+)),\quad \partial_t f-\sigma_-\partial_\xi f\in L^2(0,T;H^{-1}(\mathbb{R}_+)),$$

then the following estimate holds:

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{\alpha(\xi+\sigma_-\tau)^2}{1+\tau}}f^2d\xi d\tau\le C_\alpha\Bigg[&\|f(\xi,0)\|^2+\int_0^t f^2(0,\tau)d\tau+\int_0^t\|\partial_\xi f\|^2d\tau\\ &+\int_0^t<\partial_t f-\sigma_-\partial_\xi f,fg^2>_{H^{-1}\times H^1}d\tau\Bigg],\end{aligned} \tag{4.39}$$

where

$$g(\xi,t)=-(1+t)^{-\frac{1}{2}}\int_{\xi+\sigma_-t}^{+\infty}e^{-\frac{\alpha x^2}{1+t}}dx,$$

and $\alpha>0$ is a constant to be determined later.

Now we devote to obtaining the delicate estimate concerning the term $\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau$.

LEMMA 4.6. *Under the conditions of Theorem 3.2, there exists a constant $C>0$ such that*

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}_+}(1+\tau)^{-1}e^{-\frac{c_0(\xi+\sigma_-\tau)^2}{1+\tau}}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau\\ \le &C+C\int_0^t(\|\partial_\xi[\varphi,\psi,\zeta]\|^2+\|\partial_\xi^2[\psi,\zeta]\|^2)d\tau\\ &+C\int_0^t e^{-\alpha_1 t}\int_{\mathbb{R}_+}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau. \end{aligned} \tag{4.40}$$

Proof. Compared with the non-isentropic Navier-Stokes equations, it suffices to estimate the terms including Z. We find that all the terms including Z can be controlled by $C\|Z_0\|+C\|Z_0\|\int_0^t e^{-\alpha_1 t}\int_{\mathbb{R}_+}(\varphi^2+\psi^2+\zeta^2)d\xi d\tau$. Then combining with the results in [22], we complete the proof of Lemma 4.6. □

Proof. **(Proof of Proposition 4.1.)** Now, we are ready to prove Proposition 4.1. Combining Lemmas 4.2-4.6 and using Gronwall's inequality, if the wave strength δ^B, δ^{CD} and the constants ϵ, ε_1 are small enough, then for all $t\in[0,T]$, we have

$$\begin{aligned}&\|[\varphi,\psi,\zeta,Z](t)\|^2_{H^1(\mathbb{R}_+)}+\int_0^t(\|\partial_\xi\varphi\|^2+\|\partial_\xi[\psi,\zeta,Z]\|^2_{H^1})d\tau\\ \le &C\|[\varphi_0,\psi_0,\zeta_0,Z_0]\|^2_{H^1(\mathbb{R}_+)}+C(\epsilon^{\frac19}+\delta^B+\delta^{CD}).\end{aligned} \tag{4.41}$$

This completes the proof of Proposition 4.1. □

Proof. **(Proof of Theorem 3.2.)** We are now devoted to completing the proof of Theorem 3.2. In view of the energy estimates obtained in Proposition 4.1, one has

$$\sup_{0\le\tau\le t}\|[\varphi,\psi,\zeta,Z](\tau)\|^2_{H^1(\mathbb{R}_+)}\le C\|[\varphi_0,\psi_0,\zeta_0,Z_0]\|^2_{H^1(\mathbb{R}_+)}+C(\epsilon^{\frac19}+\delta^B+\delta^{CD}). \tag{4.42}$$

Note that parameters ϵ, δ^B and δ^{CD} are independent of ε_1. Letting ϵ, δ^B and δ^{CD} be small enough, the global existence of solution to the half-space problem (3.8) then can be proved by using the standard continuation argument based on the local existence [1] and the *a priori* estimate (4.5). Our next goal is to prove the large-time behavior as (3.10). For this, from (3.8), (4.5), (2.35), (2.3) and (2.28), we have

$$\begin{aligned}\int_0^{+\infty}\left|\frac{d}{dt}\|\partial_\xi[\varphi,\psi,\zeta,Z]\|^2\right|dt=&2\int_0^{+\infty}|(\partial_t\partial_\xi[\varphi,\psi,\zeta,Z],\partial_\xi[\varphi,\psi,\zeta,Z])|dt\\ \le &C+C\int_0^{+\infty}\|\partial_\xi[\varphi,\psi,\zeta,Z,\partial_\xi[\psi,\zeta,Z]]\|^2dt<+\infty.\end{aligned} \tag{4.43}$$

Combining (4.43) with (4.42), we have the following limit:

$$\lim_{t\to+\infty}\|\partial_\xi[\varphi,\psi,\zeta,Z](t)\|^2_{L^2}=0. \tag{4.44}$$

This completes the proof of Theorem 3.2. □

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