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SUBSONIC AND SUPERSONIC STEADY-STATES OF BIPOLAR HYDRODYNAMIC MODEL OF SEMICONDUCTORS WITH SONIC BOUNDARY*

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Abstract. In this paper, we investigate the well-posedness/ill-posedness of the stationary solutions to the isothermal bipolar hydrodynamic model of semiconductors driven by Euler-Poisson equations. Here, the density of electrons is proposed with sonic boundary and considered in interiorly subsonic case or interiorly supersonic case, while the density of holes is considered in fully subsonic case or fully supersonic case. With the developed technique based on the topological degree method, the following four kinds of stationary solutions under some conditions are proved to exist: the interiorly-subsonic-vs-fully-subsonic solution, the interiorly-supersonic-vs-fully-subsonic solution, the interiorly-subsonic-vs-fully-supersonic solution, and the interiorly-supersonic-vs-fully-supersonic solution. The non-existence of the above four kinds of solutions under some conditions is also technically proved. For the existence of these physical solutions, different from the previous studies, where traditional fixed-point argument via energy estimates is used, we recognize that such an approach fails for our cases, due to that the effect of boundary degeneracy for the electrons causes difficulty in estimating the upper and lower bounds for the holes. Instead of it, we use the topological degree method to prove the existence of physical solutions.

Keywords. Bipolar hydrodynamic model of semiconductors; Euler-Poisson equations; sonic boundary; steady-states; subsonic/supersonic solutions; topological degree method.

AMS subject classifications. 35Q35; 35J70; 76N10.

1. Introduction

The bipolar hydrodynamic (HD) model of semiconductors is usually used to describe the flow of electrons and holes in semiconductor devices [4, 19, 24], which is written as the following system of Euler-Poisson equations:

$$\begin{cases} \rho_t + J_{1x} = 0, \\ J_{1t} + \big(\dfrac{J_1^2}{\rho} + P_1(\rho)\big)_x = \rho\Phi_x - \dfrac{J_1}{\tau}, \\ n_t + J_{2x} = 0, \\ J_{2t} + \big(\dfrac{J_2^2}{n} + P_2(n)\big)_x = -n\Phi_x - \dfrac{J_2}{\tau}, \\ \Phi_{xx} = \rho - n - b(x). \end{cases} \tag{1.1}$$

In semiconductor devices, the unknowns $\rho(x,t)$, $n(x,t)$, $J_1(x,t)$, $J_2(x,t)$ and $\Phi(x,t)$ represent the electron density, the hole density, the current density of electrons, the current density of holes and the electrostatic potential, respectively. $P_1(\cdot)$, $P_2(\cdot)$, $b(x)$ are given functions which represent the pressure of electrons, the pressure of holes, and the doping profile standing for the density of impurities in semiconductor devices. The

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given constant τ represents the momentum relaxation time. The global existence and uniqueness of the solutions as well as the asymptotic behaviors and classic limits for such a model subjected to different IVP or IBVPs has been intensively studied, for example, see [7,10,13,15–17,25] and the references therein.

The corresponding steady-state equations of (1.1) can be written by

$$\begin{cases} J_1 = constant_1, \\ \big(\frac{J_1^2}{\rho} + P_1(\rho)\big)_x = \rho\Phi_x - \frac{J_1}{\tau}, \\ J_2 = constant_2, \\ \big(\frac{J_2^2}{n} + P_2(n)\big)_x = -n\Phi_x - \frac{J_2}{\tau}, \\ \Phi_{xx} = \rho - n - b(x). \end{cases} \tag{1.2}$$

Our main purpose in this paper is to study the well/ill-posedness of the steady-state solutions of (1.2). There is limited research on the bipolar steady-state Equations (1.2). For example, in [31], Li and Zhou studied the existence of stationary solutions to a Dirichlet problem in 1-D, but they assumed that the doping profile is zero. Then, Tsuge [29] studied the 1-D bipolar HD model with Ohmic contact boundary, and obtained the existence and uniqueness of the subsonic stationary solution with the assumption that the electrostatic potential is small enough. In [30], Yu obtained the existence and uniqueness of the subsonic stationary solution with insulating boundary conditions by the calculus of variations in N-D, where $N = 1,2$.

The target in this paper is to investigate the existence/non-existence of the stationary solutions of (1.2) with sonic boundary to electrons, and make a classification of these solutions. Setting

$$(v,u) = \big(\frac{|J_1|}{\rho}, \frac{|J_2|}{n}\big),$$

then v, u are the absolute velocities of electrons and holes, respectively. By the terminology in gas dynamics, $c_e := \sqrt{P_1'(\rho)}$ is called the sound speed of electrons, and $c_h := \sqrt{P_2'(n)}$ is called the sound speed of holes. For the stationary solution (J_1, J_2, ρ, n, E) of (1.2), the corresponding electron velocity v is said to be subsonic/sonic/supersonic if

$$v \lesseqqgtr c_e = \sqrt{P_1'(\rho)}: \text{ sound speed of electrons.} \tag{1.3}$$

Meanwhile, the corresponding hole velocity u is said to be subsonic/sonic/supersonic if

$$u \lesseqqgtr c_h = \sqrt{P_2'(n)}: \text{ sound speed of holes.} \tag{1.4}$$

For convenience, we assume that the pressures are in accordance with the isothermal case:

$$P_1(s) = P_2(s) = P(s) = Ts,$$

for some constant temperature $T > 0$. We denote:

$$\Phi_x = E,$$

where E is called the electric field of the semiconductor model. We are concerned about the current driven flow, i.e, with the given current densities J_1 and J_2, we seek for ρ, n and E from (1.2). Without loss of generality, we assume that $J_1=1$, $J_2=-1$ and $T=1$, and we use $\Omega=(0,1)$ to denote the bounded semiconductor device domain. Then (1.2) reads:

$$\begin{cases} \Big(\dfrac{1}{\rho}-\dfrac{1}{\rho^3}\Big)\rho_x = E-\dfrac{1}{\tau\rho}, \\ \Big(\dfrac{1}{n}-\dfrac{1}{n^3}\Big)n_x = -E+\dfrac{1}{\tau n}, \qquad x\in(0,1), \\ E_x=\rho-n-b(x). \end{cases} \tag{1.5}$$

From (1.3) and (1.4), the flow of electrons is said to be subsonic/sonic/supersonic if

$$v \lesseqqgtr 1, \text{ or equivalently, } \rho \gtreqqless 1,$$

and the flow of holes is said to be subsonic/sonic/supersonic if

$$u \lesseqqgtr 1, \text{ or equivalently, } n \gtreqqless 1.$$

We impose (1.5) with the sonic boundary condition to electrons:

$$\rho(0)=\rho(1)=1, \tag{1.6}$$

and a given boundary condition to holes

$$n(0)=\sigma_0, \tag{1.7}$$

where the size of σ_0 will be discussed later. Differentiating the first and the second equations with respect to x in (1.5), and collecting the boundary conditions in (1.6) and (1.7), we obtain

$$\begin{cases} \Big[\Big(\dfrac{1}{\rho}-\dfrac{1}{\rho^3}\Big)\rho_x\Big]_x + \Big(\dfrac{1}{\tau\rho}\Big)_x = \rho-n-b(x), \\ \Big[\Big(\dfrac{1}{n}-\dfrac{1}{n^3}\Big)n_x\Big]_x - \Big(\dfrac{1}{\tau n}\Big)_x = n+b(x)-\rho, \qquad x\in(0,1), \\ \rho(0)=\rho(1)=1,\ n(0)=\sigma_0. \end{cases} \tag{1.8}$$

Throughout this paper, we assume that $b(x)\in L^\infty(0,1)$, and denote

$$\underline{b}=\operatorname*{essinf}_{x\in[0,1]} b(x), \quad \text{and} \quad \bar{b}=\operatorname*{esssup}_{x\in[0,1]} b(x).$$

We use C to denote positive constants, which may take different values in each appearance. Since the system (1.8) is degenerate at the sonic boundary $\rho(0)=\rho(1)=1$, the solutions of (1.8) will not possess a certain regularity, and have to be in the weak form. So, we give the following definition for weak solutions.

Definition 1.1. *Assume that $\rho(0)=\rho(1)=1$, $n(0)=\sigma_0$, $(\rho(x)-1)^2\in H_0^1(0,1)$, $n(x)\in W^{2,\infty}(0,1)$, and for any $\varphi\in H_0^1(0,1)$ it holds that*

$$\int_0^1 \Big(\frac{1}{\rho}-\frac{1}{\rho^3}\Big)\rho_x\varphi_x dx+\frac{1}{\tau}\int_0^1\frac{\varphi_x}{\rho}dx+\int_0^1(\rho-n-b)\varphi dx=0, \tag{1.9}$$

and

$$\int_0^1 (\frac{1}{n}-\frac{1}{n^3})n_x\varphi_x dx-\frac{1}{\tau}\int_0^1 \frac{\varphi_x}{n}dx+\int_0^1 (n+b-\rho)\varphi dx=0. \tag{1.10}$$

Furthermore, we define that:

(1) If $\rho>1$ over $(0,1)$ and $n>1$ over $[0,1]$, then (ρ,n) is called a pair of interiorly-subsonic solutions $\rho(x)$ coupled with fully-subsonic solution $n(x)$ of (1.8), simply, the subsonic-vs-subsonic solution;

(2) If $0<\rho<1$ over $(0,1)$ and $n>1$ over $[0,1]$, then (ρ,n) is called a pair of interiorly-supersonic solutions $\rho(x)$ coupled with fully-subsonic solution $n(x)$ of (1.8), simply, the supersonic-vs-subsonic solution;

(3) If $\rho>1$ over $(0,1)$ and $0<n<1$ over $[0,1]$, then (ρ,n) is called a pair of interiorly-subsonic solutions $\rho(x)$ coupled with fully-supersonic solution $n(x)$ of (1.8), simply, the subsonic-vs-supersonic solution;

(4) If $0<\rho<1$ over $(0,1)$ and $0<n<1$ over $[0,1]$, then (ρ,n) is called a pair of interiorly-supersonic solutions $\rho(x)$ coupled with fully-supersonic solution $n(x)$ of (1.8), simply, the supersonic-vs-supersonic solution.

We will prove the well-posedness/ill-posedness of the above four kinds of solutions. First, to obtain the solutions of (1.8), we should give an extra boundary condition. However, the condition imposed on (1.8) can not be arbitrary since we have to ensure that the solution of (1.8) is also the one of (1.5). In the following sections of this paper, (1.8) will be regarded as a free boundary value problem by assuming that $n(1)=\sigma_1$ for some σ_1 to be determined.

When $\rho(x)>1$ or $0<\rho(x)<1$ for $x\in(0,1)$, the first equation in (1.8) is elliptic but degenerate at the sonic boundary. The degeneracy of the electron equation in (1.8) will bring us some difficulties. To handle the degenerate boundary, we will use the technical compactness method in [20], in which the authors investigated the well-posedness and regularity of stationary solutions to the unipolar HD model with sonic boundary. However, in proving the well-posedness of the system (1.8), the main difficulty occurs in the second equation. We will seek for the solutions of (1.8) such that the hole density is fully subsonic or fully supersonic, i.e, $n>1$ or $0<n<1$ over $[0,1]$, hence the second equation in (1.8) is uniformly elliptic on $[0,1]$. For this purpose, we have to carefully estimate the lower and upper bounds of n. Notice that the traditional energy estimate method after linearization of the system (1.8) can not be applied here since some key estimates for the bounds of n could not be obtained from the linearized system. For this reason, remarkably, we shall develop a new technique based on the topological degree method to transfer the problem for proving the well-posedness of the system (1.8) into finding the suitable bounds of the solutions to some new systems which have the same second and first order terms as (1.8). We will see in the following sections that these new systems have some "good properties" similar to (1.8) which could be used to obtain the desired estimates of the bounds of n, and as a result, the well-posedness of the above four kinds of stationary solutions could be proved. Finally, the ill-posedness of the solutions to (1.8) under some conditions is proved using standard energy estimates and some refined analysis based on the original ODE system (1.5).

REMARK 1.1.

(1) The identity (1.9) is well-defined for $(\rho(x)-1)^2\in H_0^1(0,1)$ and $\varphi\in H_0^1(0,1)$ since it

is equivalent to

$$\frac{1}{2}\int_0^1 \frac{\rho+1}{\rho^3}\left[(\rho-1)^2\right]_x\varphi_x dx+\frac{1}{\tau}\int_0^1 \frac{\varphi_x}{\rho}dx+\int_0^1 (\rho-n-b)\varphi dx=0;$$

(2) Once $\rho(x)$ is obtained from (1.8), we could solve the electric field $E(x)$ by taking

$$E(x)=(\frac{1}{\rho}-\frac{1}{\rho^3})\rho_x+\frac{1}{\tau\rho}=\frac{(\rho+1)[(\rho-1)^2]_x}{2\rho^3}+\frac{1}{\tau\rho}. \tag{1.11}$$

The following sections will be devoted to investigating the well-posedness/ill-posedness of the interior-subsonic-fully-subsonic flow, the interior-supersonic vs fully-subsonic flow, the interior-subsonic vs fully-supersonic flow and the interior-supersonic vs fully-supersonic flow, respectively. For the main results of this paper, we refer to Theorem 2.1, Theorem 2.2, Theorem 3.1, Theorem 3.2, Theorem 4.1, Theorem 4.2, Theorem 5.1 and Theorem 5.2 in the following sections.

In the end of this section, we mention some results of the stationary solutions to the unipolar HD model. For example, in 1990, Degond and Markowich [5] showed the existence of the completely subsonic flow, and proved the uniqueness with $|J|\ll 1$. And then a lot of attention has been paid to the existence of the subsonic flow for the steady-state equations with different boundary conditions and higher dimensions, see [2, 3, 6, 14, 18, 26] and the references therein. In 2006, Peng and Violet [27] obtained the existence and uniqueness of supersonic solution with a strong supersonic boundary condition. For the transonic flow, we refer to [1, 11, 12, 20, 22, 23, 28]. Recently, Li-Mei-Zhang-Zhang [20, 21] thoroughly studied the unipolar HD model of Euler-Poisson equations with sonic boundary, and made a classification of the solutions. In [20], the authors showed that when the relaxation time τ is small enough and the subsonic doping b is a constant, the system admits infinitely many C^1 smooth transonic solutions, but has no transonic shock solutions. And when $\tau\gg 1$ and the subsonic doping b is flat, the system admits infinitely many transonic shock solutions. The results showed that the semiconductor effect cannot be ignored. For the hydrodynamic system with quantum effect, the subsonic steady-states were investigated in [8, 9].

2. Interiorly-subsonic-vs-fully-subsonic flow

Consider the system

$$\begin{cases} \left(\frac{1}{\rho}-\frac{1}{\rho^3}\right)\rho_x=E-\frac{1}{\tau\rho}, \\ \left(\frac{1}{n}-\frac{1}{n^3}\right)n_x=-E+\frac{1}{\tau n}, \qquad x\in(0,1), \\ E_x=\rho-n-b(x). \end{cases} \tag{2.1}$$

Let $\sigma_0>1$. In this section, we investigate the existence and non-existence of solutions of (2.1) such that

$$\rho(0)=\rho(1)=1,\ \rho(x)>1,\ x\in(0,1);\ n(0)=\sigma_0,\ n(x)>1,\ x\in[0,1]. \tag{2.2}$$

Differentiating the first and the second equations in (2.1) with respect to x, we obtain

$$\begin{cases} \left[\left(\frac{1}{\rho}-\frac{1}{\rho^3}\right)\rho_x\right]_x+\left(\frac{1}{\tau\rho}\right)_x=\rho-n-b(x), \\ \left[\left(\frac{1}{n}-\frac{1}{n^3}\right)n_x\right]_x-\left(\frac{1}{\tau n}\right)_x=n+b(x)-\rho, \qquad x\in(0,1), \\ \rho(0)=\rho(1)=1,\ n(0)=\sigma_0. \end{cases} \tag{2.3}$$

Observing that (2.3) is a coupled PDE system of (ρ,n) of second order degenerate elliptic equations, we should give an extra boundary condition. However, as mentioned in the previous section, the condition could not be arbitrarily imposed since we have to ensure that the solutions of (2.3) also solve (2.1). Assume that (ρ,n) is a solution of (2.3) with $n(1)=\sigma_1$. Combining the equations in (2.3) we obtain

$$(\frac{1}{\rho}-\frac{1}{\rho^3})\rho_x+\frac{1}{\tau\rho}=\frac{1}{\tau n}-(\frac{1}{n}-\frac{1}{n^3})n_x+c, \tag{2.4}$$

where c is a constant. It is easy to verify that (ρ,n,E) with E defined in (1.11) solves (2.1) if and only if $c=0$. Integrating (2.4) over $(0,1)$, we obtain

$$f_{\sigma_0}(\sigma_1):=\big(\ln\sigma_0+\frac{1}{2\sigma_0^2}\big)-\big(\ln\sigma_1+\frac{1}{2\sigma_1^2}\big)=\frac{1}{\tau}\int_0^1[\frac{1}{\rho(x)}-\frac{1}{n(x)}]dx-c. \tag{2.5}$$

Hence $c=0$ if and only if

$$f_{\sigma_0}(n(1))=f_{\sigma_0}(\sigma_1)=\frac{1}{\tau}\int_0^1[\frac{1}{\rho(x)}-\frac{1}{n(x)}]dx. \tag{2.6}$$

On the contrary, if (ρ,n,E) is a solution of (2.1)-(2.2), then (2.6) must hold.

Due to the discussions above, we could reformulate our problem into the following: **P1**: Finding the solution (ρ,n) of (2.3) such that (2.2) and (2.6) hold, where $\sigma_1=n(1)$.

2.1. Well-posedness. We are going to prove the existence of subsonic-vs-subsonic solutions to **P1** as well as their regularities.

THEOREM 2.1. *For any $b(x)\in L^\infty(0,1)$, $\tau>0$ and $\underline{n}>1$, there exists a constant $\sigma^*=\sigma^*(\bar{b},\tau,\underline{n})>1$ which only depends on $\bar{b}$, τ and $\underline{n}$, such that for any $\sigma_0\geq\sigma^*$,* **P1** *admits the pair of subsonic-vs-subsonic solution $(\rho,n)\in C^{\frac{1}{2}}[0,1]\times W^{2,\infty}(0,1)$ and $n\geq\underline{n}$ over $[0,1]$.*

Since the first equation of (2.3) is degenerate at the boundary, we consider the following approximate system:

$$\begin{cases}[(\frac{1}{\rho_j}-\frac{j}{\rho_j^3})(\rho_j)_x]_x+(\frac{1}{\tau\rho_j})_x=\rho_j-n_j-b(x),\\ [(\frac{1}{n_j}-\frac{1}{n_j^3})(n_j)_x]_x-(\frac{1}{\tau n_j})_x=n_j+b(x)-\rho_j, \qquad x\in(0,1),\\ \rho_j(0)=\rho_j(1)=1,\ n_j(0)=\sigma_0,\end{cases} \tag{2.7}$$

where $j\in(0,1)$ is a constant. For simplicity of notations, we omit the subscript j and denote the solution of (2.7) by (ρ,n).

LEMMA 2.1. *For any $b(x)\in L^\infty(0,1)$, $\tau>0$, $j\in(0,1)$ and $\underline{n}>1$, there exists a constant $\sigma^*=\sigma^*(\bar{b},\tau,\underline{n})>1$ which only depends on $\bar{b}$, τ and $\underline{n}$, such that for any $\sigma_0\geq\sigma^*$, (2.7) admits a solution $(\rho,n)\in W^{2,\infty}(0,1)\times W^{2,\infty}(0,1)$ which satisfies (2.2), (2.6) and $n\geq\underline{n}$ over $[0,1]$.*

Proof. Taking $\bar{n}>1$ such that $\bar{n}=f_{\sigma_0}^{-1}(-\frac{2}{\tau})$, where $f_{\sigma_0}^{-1}$ is the inverse function of f_{σ_0} defined in (2.5). It is easy to check that $\bar{n}>\sigma_0$. Define

$$X=\big\{(\rho,n)\in C[0,1]\times C[0,1]\big\},$$
$$D=\big\{(\rho,n)\in X,\ m<\rho<M,\ \lambda<n<\Lambda\big\},$$

where

$$m=\frac{\sqrt{j}+1}{2},\ M=\bar{n}+\bar{b}+2,\ \lambda=\frac{\underline{n}+1}{2},\ \Lambda=\bar{n}+1.$$

It is easy to verify that D is a bounded and open subset of X, and

$$\partial D=\big\{(\rho,n)\in X,\ m\le\rho\le M,\ \lambda\le n\le\Lambda,\ and\ \exists x\in[0,1],\\ s.t:\ \rho(x)=m\ or\ \rho(x)=M\ or\ n(x)=\lambda\ or\ n(x)=\Lambda\big\}.$$

For any $(\tilde{\rho},\tilde{n})\in\bar{D}$, we have

$$\frac{1}{\tau}\int_0^1\frac{1}{\tilde{\rho}(x)}-\frac{1}{\tilde{n}(x)}dx\le\frac{1}{\tau}\int_0^1\frac{1}{\tilde{\rho}(x)}dx\le\frac{2}{\tau(\sqrt{j}+1)}\le\frac{2}{\tau}.$$

Therefore for any given $\tau>0$ and $\underline{n}>1$, there is a unique $\sigma_1^*>1$ such that

$$f_{\sigma_1^*}(\underline{n})=\frac{2}{\tau}\ge\frac{1}{\tau}\int_0^1\frac{1}{\tilde{\rho}(x)}-\frac{1}{\tilde{n}(x)}dx.$$

Assuming that $\sigma_0\ge\sigma_1^*$, then $f_{\sigma_0}(\underline{n})\ge f_{\sigma_1^*}(\underline{n})=\frac{2}{\tau}$. On the other hand, since

$$\frac{1}{\tau}\int_0^1\frac{1}{\tilde{\rho}(x)}-\frac{1}{\tilde{n}(x)}dx\ge-\frac{1}{\tau}\int_0^1\frac{1}{\tilde{n}(x)}dx\ge-\frac{1}{\tau}>-\frac{2}{\tau}=f_{\sigma_0}(\bar{n}),$$

we conclude that for any $(\tilde{\rho},\tilde{n})\in\bar{D}$, there exists a unique $\tilde{\sigma}_1\in[\underline{n},\bar{n}]$ such that

$$f_{\sigma_0}(\tilde{\sigma}_1)=\frac{1}{\tau}\int_0^1\frac{1}{\tilde{\rho}(x)}-\frac{1}{\tilde{n}(x)}dx. \tag{2.8}$$

Now we define the operator $\Gamma:\bar{D}\to X$, $(\tilde{\rho},\tilde{n})\mapsto(\rho,n)$ by solving

$$\begin{cases}\big[(\frac{1}{\tilde{\rho}}-\frac{j}{\tilde{\rho}^3})\rho_x\big]_x-\frac{1}{\tau\tilde{\rho}^2}\rho_x=\tilde{\rho}-\tilde{n}-b(x),\\ \big[(\frac{1}{\tilde{n}}-\frac{1}{\tilde{n}^3})n_x\big]_x+\frac{1}{\tau\tilde{n}^2}n_x=\tilde{n}+b(x)-\tilde{\rho}, \qquad x\in(0,1),\\ \rho(0)=\rho(1)=1,\ n(0)=\sigma_0,\ n(1)=\tilde{\sigma}_1,\end{cases} \tag{2.9}$$

where $\tilde{\sigma}_1$ is defined in (2.8). Then from the linear theory of elliptic equations, $\Gamma:\bar{D}\to X$ is a compact and continuous operator. Set $\tilde{\psi}=(\tilde{\rho},\tilde{n})$, and $\psi=(\rho,n)=\Gamma\tilde{\psi}$.

Take $G:\bar{D}\times[0,1]\to X$, $G(\tilde{\psi},\epsilon)=\tilde{\psi}-\epsilon\Gamma\tilde{\psi}=\tilde{\psi}-\epsilon\psi$. It is obvious that if $G(\tilde{\psi},1)=0$, then $\tilde{\psi}$ is also a solution of (2.7), and (2.6) holds for $\tilde{\psi}$. Take $q=(1,\sigma_0)\in D$, and set $p(\epsilon)=(1-\epsilon)q$ for $\epsilon\in[0,1]$. If $p(\epsilon)\notin G(\partial D,\epsilon)$ for any $\epsilon\in[0,1]$, then due to the topological degree theory,

$$deg\big(G(\cdot,1),D,0\big)=deg\big(G(\cdot,0),D,q\big)=deg\big(id,D,q\big)=1, \tag{2.10}$$

and therefore $G(\tilde{\psi},1)=0$ admits a solution $\tilde{\psi}\in D$.

On the contrary, if there are $\epsilon\in[0,1]$ and $\tilde{\psi}\in\partial D$ such that

$$p(\epsilon)=(1-\epsilon)q=G(\tilde{\psi},\epsilon)=\tilde{\psi}-\epsilon\psi,$$

then if $\epsilon=0$, we obtain that $\tilde{\psi}=q$. This is impossible since $q\in D$, $\tilde{\psi}\in\partial D$ and D is an open set in X. If $\epsilon\in(0,1]$, then

$$\psi=\frac{1}{\epsilon}\tilde{\psi}-\frac{1-\epsilon}{\epsilon}q. \tag{2.11}$$

Introducing (2.11) into (2.9), we conclude that $\tilde{\psi}\in\partial D$ satisfies the following equations

$$\begin{cases}\big[(\frac{1}{\tilde{\rho}}-\frac{j}{\tilde{\rho}^3})\tilde{\rho}_x\big]_x+(\frac{1}{\tau\tilde{\rho}})_x=\epsilon\big(\tilde{\rho}-\tilde{n}-b(x)\big),\\ \big[(\frac{1}{\tilde{n}}-\frac{1}{\tilde{n}^3})\tilde{n}_x\big]_x-(\frac{1}{\tau\tilde{n}})_x=\epsilon\big(\tilde{n}+b(x)-\tilde{\rho}\big), \qquad x\in(0,1),\\ \tilde{\rho}(0)=\tilde{\rho}(1)=1,\ \tilde{n}(0)=\sigma_0,\ \tilde{n}(1)=\sigma_0+\epsilon(\tilde{\sigma}_1-\sigma_0)\triangleq\hat{\sigma}_1,\end{cases} \tag{2.12}$$

where $f_{\sigma_0}(\tilde{\sigma}_1)=\frac{1}{\tau}\int_0^1\frac{1}{\tilde{\rho}(x)}-\frac{1}{\tilde{n}(x)}dx$ and as a result $\hat{\sigma}_1\in[\underline{n},\bar{n}]$. In the following proof, we will show that for any $\tilde{\psi}\in\partial D$, $\tilde{\psi}$ can not be a solution of (2.12) under some conditions, hence (2.10) holds. We will estimate the bounds of the solutions of (2.12) directly. Notice that (2.12) has the same second and first order terms as (2.7).

First, taking $w_1=\tilde{\rho}-1$ and $w_2=\tilde{\rho}-(\bar{n}+\bar{b}+1)$ respectively into the first equation of (2.12), and using the standard maximum principle, we obtain that $w_1\geq 0$ and $w_2\leq 0$, hence

$$m<1\leq\tilde{\rho}(x)\leq\bar{n}+\bar{b}+1<M,\ \forall x\in[0,1].$$

Now we show that $\lambda<\tilde{n}<\Lambda$ when σ_0 keeps away from 1. From (2.12) we have

$$(\frac{1}{\tilde{\rho}}-\frac{j}{\tilde{\rho}^3})\tilde{\rho}_x+\frac{1}{\tau\tilde{\rho}}=\frac{1}{\tau\tilde{n}}-(\frac{1}{\tilde{n}}-\frac{1}{\tilde{n}^3})\tilde{n}_x+c, \tag{2.13}$$

where c is a constant. Integrating (2.13) over $[0,1]$, we have

$$\frac{1}{\tau}\int_0^1\frac{1}{\tilde{\rho}(x)}-\frac{1}{\tilde{n}(x)}dx=\big(\ln\sigma_0+\frac{1}{2\sigma_0^2}\big)-\big(\ln\hat{\sigma}_1+\frac{1}{2\hat{\sigma}_1^2}\big)+c=f_{\sigma_0}(\hat{\sigma}_1)+c,$$

hence

$$c=\frac{1}{\tau}\int_0^1\frac{1}{\tilde{\rho}(x)}-\frac{1}{\tilde{n}(x)}dx-f_{\sigma_0}(\hat{\sigma}_1)=f_{\sigma_0}(\tilde{\sigma}_1)-f_{\sigma_0}(\hat{\sigma}_1).$$

Since f_{σ_0} is monotonically decreasing and $\hat{\sigma}_1=\sigma_0+\epsilon(\tilde{\sigma}_1-\sigma_0)$ is between σ_0 and $\tilde{\sigma}_1$, we have

$$\begin{aligned}|c|&=|f_{\sigma_0}(\tilde{\sigma}_1)-f_{\sigma_0}(\hat{\sigma}_1)|\leq|f_{\sigma_0}(\tilde{\sigma}_1)-f_{\sigma_0}(\sigma_0)|\\&=|f_{\sigma_0}(\tilde{\sigma}_1)|=|\frac{1}{\tau}\int_0^1\frac{1}{\tilde{\rho}(x)}-\frac{1}{\tilde{n}(x)}dx|\leq\frac{1}{\tau},\end{aligned}$$

therefore $c\in[-\frac{1}{\tau},\frac{1}{\tau}]$. For any $x\in(0,1]$, integrating (2.13) over $(0,x)$, and setting $F_j(\tilde{\rho})=\ln\tilde{\rho}+\frac{j}{2\tilde{\rho}^2}$, $F(\tilde{n})=\ln\tilde{n}+\frac{1}{2\tilde{n}^2}$, we have

$$F_j(\tilde{\rho})=F(\sigma_0)+\frac{j}{2}-\frac{1}{\tau}\int_0^x\frac{1}{\tilde{\rho}(\xi)}-\frac{1}{\tilde{n}(\xi)}d\xi+cx-F(\tilde{n}), \tag{2.14}$$

Based on the discussions above, we observe that $F_j(\tilde\rho)+F(\tilde n)$ is bounded from both above and below. That is

$$F(\sigma_0)+\frac{j}{2}-\frac{2}{\tau}\le F_j(\tilde\rho)+F(\tilde n)\le F(\sigma_0)+\frac{j}{2}+\frac{2}{\tau}. \tag{2.15}$$

The above inequalities reveal that $\tilde\rho$ and $\tilde n$ are almost negatively correlated. This property will play an important role in the following proof. Furthermore, since $\tilde\rho\ge 1$, we have that $F_j(\tilde\rho)\ge\frac{j}{2}$. From (2.15) we have

$$F(\tilde n)\le F(\sigma_0)+\frac{2}{\tau},$$

hence

$$f_{\sigma_0}(\tilde n)=\big(\ln\sigma_0+\frac{1}{2\sigma_0^2}\big)-\big(\ln\tilde n+\frac{1}{2\tilde n^2}\big)\ge-\frac{2}{\tau},$$

and as a result $\tilde n\le f_{\sigma_0}^{-1}(-\frac{2}{\tau})=\bar n<\Lambda$.

Now we estimate the lower bound of $\tilde n$. Taking $\sigma_2^*>1$ such that

$$F(\sigma_2^*)+\frac{j}{2}-\frac{2}{\tau}-F(\underline{n})\ge F_j(\underline{n}+\bar b),$$

setting $\sigma^*=\max\{\sigma_1^*,\sigma_2^*\}$ and assuming that $\sigma_0\ge\sigma^*$, we have

$$F_j^{-1}\big(F(\sigma_0)+\frac{j}{2}-\frac{2}{\tau}-F(\underline{n})\big)\ge\underline{n}+\bar b>1. \tag{2.16}$$

Multiplying the second equation of (2.12) by $(\tilde n-\underline{n})^-:=\min\{\tilde n-\underline{n},0\}$ and integrating it by parts, and noting that

$$\begin{aligned}\frac{1}{\tau}\int_0^1\big(\frac{1}{\tilde n}\big)_x(\tilde n-\underline{n})^-dx&=-\frac{1}{\tau}\int_0^1\frac{1}{\tilde n^2}(\tilde n-\underline{n})^-\tilde n_x dx\\&=-\frac{1}{\tau}\int_0^1\frac{(\tilde n-\underline{n})^-}{\big((\tilde n-\underline{n})^-+\underline{n}\big)^2}\big[(\tilde n-\underline{n})^-\big]_x dx\\&=-\frac{1}{\tau}\int_0^1\Big[\ln\big((\tilde n-\underline{n})^-+\underline{n}\big)+\frac{\underline{n}}{(\tilde n-\underline{n})^-+\underline{n}}\Big]_x dx\\&=0,\end{aligned}$$

we have by (2.15) and (2.16) that

$$\begin{aligned}\int_0^1\big(\frac{1}{\tilde n}-\frac{1}{\tilde n^3}\big)\big|\big((\tilde n-\underline{n})^-\big)_x\big|^2dx&=-\epsilon\int_0^1(\tilde n+b-\tilde\rho)(\tilde n-\underline{n})^-dx\\&=-\epsilon\int_{\tilde\Omega}(\tilde n+b-\tilde\rho)(\tilde n-\underline{n})^-dx\\&\le-\epsilon\int_{\tilde\Omega}\big[\tilde n+b-F_j^{-1}\big(F(\sigma_0)+\frac{j}{2}-\frac{2}{\tau}-F(\tilde n)\big)\big](\tilde n-\underline{n})^-dx\\&\le-\epsilon\int_{\tilde\Omega}\big[\tilde n+b-F_j^{-1}\big(F(\sigma_0)+\frac{j}{2}-\frac{2}{\tau}-F(\underline{n})\big)\big](\tilde n-\underline{n})^-dx\\&\le-\epsilon\int_{\tilde\Omega}(\tilde n+b-\underline{n}-\bar b)(\tilde n-\underline{n})^-dx\\&\le 0,\end{aligned} \tag{2.17}$$

where $\tilde{\Omega}=\{x\in[0,1],\tilde{n}\le \underline{n}\}$. Therefore $(\tilde{n}-\underline{n})^-\equiv 0$ and $\tilde{n}\ge\underline{n}>\lambda$.

Now we have proved that $m<\tilde{\rho}<M$ and $\lambda<\tilde{n}<\Lambda$ when $\sigma_0\ge\sigma^*$, which contradicts $(\tilde{\rho},\tilde{n})\in\partial D$. Therefore $deg(G(\cdot,1),D,0)=1$ and (2.7) admits a solution $(\rho,n)\in D$, and obviously (2.6) holds for (ρ,n). Due to the standard regularity theory and the discussions above, we conclude that $(\rho,n)\in W^{2,+\infty}(0,1)\times W^{2,+\infty}(0,1)$ and $n\ge\underline{n}$.

Finally, using the same method as in [20] (Lemma 2.3), we can prove that

$$\rho(x)\ge 1+\varepsilon\sin(\pi x)>1,\ \forall x\in(0,1), \tag{2.18}$$

where $\varepsilon>0$ is a small constant independent of j. The proof is completed. □

REMARK 2.1. The main difficulty in the proof above is to obtain the desired lower bound of $\tilde{n}$. The key observation is the "almost negative correlation" between $\tilde{\rho}$ and $\tilde{n}$. This is an important property derived directly from (2.12).

Proof. **(Proof of Theorem 2.1.)** Now we use the compactness method in [20] to obtain the solution of (2.3). Assume that (ρ_j,n_j) is the solution of (2.7) obtained in Lemma 2.1. Multiplying the first equation in (2.7) by (ρ_j-1) and integrating it by parts, we obtain

$$\frac{4}{9}\int_0^1\frac{\rho_j+1}{\rho_j^3}|((\rho_j-1)^{\frac{3}{2}})_x|^2dx+(1-j)\int_0^1\frac{|(\rho_j)_x|^2}{\rho_j^3}dx=-\int_0^1(\rho_j-n_j-b)(\rho_j-1)dx.$$

Using the standard energy estimate and the uniform boundness of ρ_j, n_j in $L^\infty(0,1)$, we have

$$\|((\rho_j-1)^{\frac{3}{2}})_x\|_{L^2(0,1)}\le C,$$

here and after, C is independent of j. Noting that $((\rho_j-1)^2)_x=\frac{4}{3}(\rho_j-1)^{\frac{1}{2}}((\rho_j-1)^{\frac{3}{2}})_x$, we have by using the boundness of ρ_j that

$$\|(\rho_j-1)^2\|_{H_0^1(0,1)}\le C. \tag{2.19}$$

Next, multiplying the second equation of (2.7) by (n_j-n_{Bj}), where $n_{Bj}(x)=\sigma_0+x(n_j(1)-\sigma_0)$, and integrating the resultant equation by parts, we obtain

$$\begin{aligned}\int_0^1\frac{n_j^2-1}{n_j^3}|(n_j)_x|^2dx=&\int_0^1\frac{n_j^2-1}{n_j^3}(n_j)_x(n_{Bj})_xdx+\frac{1}{\tau}\int_0^1\frac{1}{n_j}(n_j-n_{Bj})_xdx\\&-\int_0^1(n_j+b-\rho_j)(n_j-n_{Bj})dx.\end{aligned}$$

Noting that $m\le\rho_j\le M$ and $1<\underline{n}\le n_j\le\Lambda$, we have

$$\int_0^1|(n_j)_x|^2dx\le C\int_0^1|(n_j)_x|dx+C\le\frac{1}{2}\int_0^1|(n_j)_x|^2+C.$$

Hence $\|(n_j)_x\|_{L^2(0,1)}\le C$, and therefore $\|n_j\|_{H^1(0,1)}\le C$. As a result, we conclude that there is a subsequence of j (which is still denoted by j) such that as $j\to 1^-$,

$$\begin{aligned}&(\rho_j-1)^2\rightharpoonup(\rho-1)^2\ \text{ weakly in } H_0^1(0,1),\\&n_j\rightharpoonup n\ \text{ weakly in } H^1(0,1),\end{aligned}$$

and

$$\|(\rho-1)^2\|_{H_0^1(0,1)}\leq C,\quad \|n\|_{H^1(0,1)}\leq C.$$

In similar fashion to [20], we can prove that $(\rho,n)\in C^{\frac{1}{2}}[0,1]\times W^{2,+\infty}(0,1)$ and (ρ,n) is a weak solution of (2.3). Meanwhile, since the imbedding from $H^1(0,1)$ to $C[0,1]$ is compact, we obtain in view of (2.18) and $n_j\geq \underline{n}$ that $\rho>1$ over $(0,1)$, $n\geq \underline{n}$ over $[0,1]$ and (2.6) hold. ∎

2.2. Ill-posedness. This subsection is devoted to the proof of non-existence of subsonic-vs-subsonic solutions when the semiconductor effect disappears.

THEOREM 2.2. *There is no pair of subsonic-vs-subsonic solutions to (2.1) if $b(x)\not\equiv 0$, $\tau=+\infty$ and $\sigma_0>1$ but close to 1.*

Proof. Assume that (ρ,n,E) is a subsonic-vs-subsonic solution of (2.1) with $\tau=+\infty$, namely, $\rho(0)=\rho(1)=1$, $n(0)=\sigma_0$, $\rho(x)>1$, $n>1$ over $(0,1)$. Due to (2.1) we have

$$\ln\rho+\frac{1}{2}\Big(\frac{1}{\rho^2}-1\Big)=\Big(\ln\sigma_0+\frac{1}{2\sigma_0^2}\Big)-\Big(\ln n+\frac{1}{2n^2}\Big).$$

Hence $n(1)=\sigma_0$, $\rho<\sigma_0$, $n<\sigma_0$ in $(0,1)$. Obviously, n keeps decreasing in $(0,\varepsilon_1)$ and increasing in $(\varepsilon_2,1)$ for some $\varepsilon_1,\varepsilon_2\in(0,1)$, and $E(0)>0$, $E(1)<0$.

Let $x_1\in(0,1)$ be the first point such that n reaches its minimum, and $x_2\in(0,1)$ the last point. Then $n(x_1)\geq\underline{n}$, $n(x_2)\geq\underline{n}$, where $\underline{n}=\inf_{x\in[0,1]}n(x)>1$. And $E(x_1)=E(x_2)=0$, $n_x<0$, $E>0$ over $(0,x_1)$, $n_x>0$, $E<0$ over $(x_2,1)$. Combining the second and the third equations of (2.1), we have

$$\frac{n^2-1}{n^3}(n+b-\rho)n_x=EE_x. \tag{2.20}$$

Integrating (2.20) over $(0,x_1)$, we have

$$\begin{aligned}\frac{1}{2}E^2(0)&=\int_0^{x_1}\frac{n^2-1}{n^3}(n+b-\rho)(-n_x)dx\\&\leq\int_0^{x_1}\frac{n^2-1}{n^3}(n+\bar{b})(-n_x)dx\\&=\int_{n(x_1)}^{\sigma_0}\frac{n^2-1}{n^3}(n+\bar{b})dn\\&\leq\int_{\underline{n}}^{\sigma_0}\frac{n^2-1}{n^3}(n+\bar{b})dn\\&\leq\int_1^{\sigma_0}\frac{n^2-1}{n^3}(n+\bar{b})dn.\end{aligned}\tag{2.21}$$

Similarly, integrating (2.20) over $(x_2,1)$ gives

$$\begin{aligned}\frac{1}{2}E^2(1)&=\int_{x_2}^{1}\frac{n^2-1}{n^3}(n+b-\rho)n_xdx\\&\leq\int_{x_2}^{1}\frac{n^2-1}{n^3}(n+\bar{b})n_xdx\end{aligned}$$

$$
\begin{aligned}
&= \int_{n(x_2)}^{\sigma_0} \frac{n^2-1}{n^3}(n+\bar{b})dn \\
&\le \int_{\underline{n}}^{\sigma_0} \frac{n^2-1}{n^3}(n+\bar{b})dn \\
&\le \int_{1}^{\sigma_0} \frac{n^2-1}{n^3}(n+\bar{b})dn. \qquad (2.22)
\end{aligned}
$$

According to the third equation of (2.1), it is easy to obtain that

$$E(0)+(-E(1)) = \int_0^1 b+n-\rho dx \ge \int_0^1 bdx + (1-\sigma_0).$$

For the given $b(x) \not\equiv 0$, assuming that $\sigma_0 - 1 \ll 1$ such that $E(0)+(-E(1)) \ge \frac{1}{2}\int_0^1 bdx > 0$, then we have

$$E^2(0)+E^2(1) = E^2(0)+(-E(1))^2 \ge \frac{1}{2}\big(E(0)+(-E(1))\big)^2 \ge \frac{1}{8}\Big(\int_0^1 bdx\Big)^2. \qquad (2.23)$$

Combining (2.21), (2.22) with (2.23), we obtain

$$\int_1^{\sigma_0} \frac{n^2-1}{n^3}(n+\bar{b})dn \ge \frac{1}{4}\big(E^2(0)+E^2(1)\big) \ge \frac{1}{32}\int_0^1 bdx. \qquad (2.24)$$

Since $\frac{n^2-1}{n^3}(n+\bar{b}) \le \sigma_0 + \bar{b}$, we could draw a contradictory conclusion to (2.24) if $\sigma_0 - 1 \ll 1$. Hence there is no interior-subsonic-fully-subsonic solution to (2.1) if $b(x) \not\equiv 0$, $\tau = +\infty$ and $\sigma_0 > 1$ is close to 1. □

3. Interiorly-supersonic-vs-fully-subsonic flow

In this section, we consider the supersonic-vs-subsonic flow of (1.5). For convenience, we use the velocity equation of electron, namely, taking $v(x) = \frac{1}{\rho(x)}$, then (1.5) is equivalent to the following

$$
\begin{cases}
\big(v - \frac{1}{v}\big)v_x = E - \frac{1}{\tau}v, \\
\big(\frac{1}{n} - \frac{1}{n^3}\big)n_x = -E + \frac{1}{\tau n}, \qquad x \in (0,1), \\
E_x = \frac{1}{v} - n - b(x).
\end{cases} \qquad (3.1)
$$

Let $\sigma_0 > 1$. We investigate the existence and non-existence of solutions of (3.1) such that

$$v(0)=v(1)=1,\ v(x)>1,\ x\in(0,1);\ n(0)=\sigma_0,\ n(x)>1,\ x\in[0,1]. \qquad (3.2)$$

Similar to the analysis in the previous section, we could reformulate the above problem into the following:

P2: Finding the solution (v,n) of

$$
\begin{cases}
\big[\big(v-\dfrac{1}{v}\big)v_x\big]_x + \dfrac{1}{\tau}v_x = \dfrac{1}{v} - n - b(x), \\
\big[\big(\dfrac{1}{n}-\dfrac{1}{n^3}\big)n_x\big]_x - \big(\dfrac{1}{\tau n}\big)_x = n + b(x) - \dfrac{1}{v}, \qquad x \in (0,1), \\
v(0)=v(1)=1,\ n(0)=\sigma_0,
\end{cases} \qquad (3.3)
$$

such that (3.2) and

$$f_{\sigma_0}(n(1)) = f_{\sigma_0}(\sigma_1) = \frac{1}{\tau}\int_0^1 v(x) - \frac{1}{n(x)}dx, \qquad (3.4)$$

hold, where $\sigma_1 = n(1)$ and f_{σ_0} is defined in (2.5).

3.1. Well-posedness. In this subsection, we prove that **P2** admits a supersonic-vs-subsonic solution for some $b(x)\in L^\infty(0,1)$, $\tau>0$ and $\sigma_0>1$. First we give some notations:

Notations. The following definitions will be used in this section:

(1) $n=N(Y)$ is the inverse function of $Y(n)=\ln n+\frac{1}{2n^2}$ for $n>1$;

(2) $y(x;\underline{Y})=-\ln\left[\cos^2\left(\sqrt{\frac{e^{\underline{Y}}}{2}}(x-\frac{1}{2})\right)\right]+\underline{Y}$;

(3) $\underline{Y}^*=\frac{1}{2}(\frac{1}{2}+\ln 2\pi^2)$, $\underline{Y}^{**}=\frac{1}{8}(3+2\ln 2\pi^2)$, $\underline{Y}^{***}=\frac{1}{16}(7+2\ln 2\pi^2)$;

(4) $\sigma^*=N\big(y(0;\underline{Y}^*)\big)$.

REMARK 3.1. The notations defined above will be illustrated in the proof of Lemma 3.1. Note that $\underline{Y}^*>\underline{Y}^{**}>\underline{Y}^{**}>\frac{1}{2}$, $\underline{Y}^{**}=\frac{1}{2}(\frac{1}{2}+\underline{Y}^*)$ and $\underline{Y}^{***}=\frac{1}{2}(\frac{1}{2}+\underline{Y}^{**})$.

THEOREM 3.1. *Assuming that $b(x)\in L^\infty(0,1)$, $\frac{1}{8}\bar{b}\le\frac{1}{4}(\underline{Y}^*-\frac{1}{2})$ and $\sigma_0\ge\sigma^*$, then there is a constant $\tau^*=\tau^*(\sigma_0)>0$ which only depends on σ_0, such that for any $\tau\ge\tau^*$,* **P2** *admits a supersonic-vs-subsonic solution $(v,n)\in C^{\frac{1}{2}}[0,1]\times W^{2,\infty}(0,1)$ and $n\ge N(\underline{Y}^{***})>1$ over $[0,1]$.*

Since the first equation of (3.3) is degenerate at the boundary, we consider the following approximate system

$$\begin{cases}\left[\left(kv_k-\dfrac{1}{v_k}\right)(v_k)_x\right]_x+\dfrac{1}{\tau}(v_k)_x=\dfrac{1}{v_k}-n_k-b(x),\\ \left[\left(\dfrac{1}{n_k}-\dfrac{1}{n_k^3}\right)(n_k)_x\right]_x-\left(\dfrac{1}{\tau n_k}\right)_x=n_k+b(x)-\dfrac{1}{v_k},\qquad x\in(0,1),\\ v_k(0)=v_k(1)=1,\ n_k(0)=\sigma_0,\end{cases}\tag{3.5}$$

where $1<k<2$ is a constant. For simplicity of notations, we omit the subscript k and denote the solution of (3.5) by (v,n).

LEMMA 3.1. *Assuming that $b(x)\in L^\infty(0,1)$, $\frac{1}{8}\bar{b}\le\frac{1}{4}(\underline{Y}^*-\frac{1}{2})$ and $\sigma_0\ge\sigma^*$, then there is a constant $\tau^*=\tau^*(\sigma_0)>0$ which only depends on σ_0, such that for any $\tau\ge\tau^*$ and $k\in(1,2)$, (3.5) admits a solution $(v,n)\in W^{2,\infty}(0,1)\times W^{2,\infty}(0,1)$ which satisfies (3.4) and $v\ge 1$, $n\ge N(\underline{Y}^{***})>1$ over $[0,1]$.*

Proof. We split the proof into four steps.

Step 1. In this step, we reformulate our problem by using topological degree method. For $\sigma_0\ge\sigma^*$, define

$$X=\big\{(v,n)\in C[0,1]\times C[0,1]\big\},$$
$$D=\big\{(v,n)\in X,\ m<v<M,\ \lambda<n<\Lambda\big\},$$

where

$$\Lambda=\sigma_0+1,\ m=\max\{\frac{1}{\lambda},\frac{\sqrt{\frac{1}{k}}+1}{2}\},\ M=2+\frac{1}{\tau}+\sqrt{(1+\frac{1}{\tau})^2+4(\sigma_0+\bar{b})},\tag{3.6}$$

and λ is a given constant with a size in $(1,\frac{3}{2})\bigcap(1,N(\underline{Y}^{***}))$. Then D is a bounded and open subset of X, and

$$\partial D=\big\{(v,n)\in X,\ m\le v\le M,\ \lambda\le n\le\Lambda,\ \textit{and}\ \exists x\in[0,1],$$
$$s.t:\ v(x)=m\ \textit{or}\ v(x)=M\ n(x)=\lambda\ \textit{or}\ n(x)=\Lambda\big\}.$$

Taking $\tau_1^*>0$ such that $\frac{M}{\tau_1^*}\le f_{\sigma_0}(\frac{3}{2})$, then for any $(\tilde v,\tilde n)\in\bar D$ we have

$$0<\frac{1}{\tau}\int_0^1 \tilde v(x)-\frac{1}{\tilde n(x)}dx\le\frac{1}{\tau}\int_0^1\tilde v(x)dx\le\frac{M}{\tau}\le\frac{M}{\tau_1^*}\le f_{\sigma_0}(\frac{3}{2})<f_{\sigma_0}(\lambda),$$

for any $\tau\ge\tau_1^*$. Assume that $\tau\ge\tau_1^*$, then for any $(\tilde v,\tilde n)\in\bar D$, there exists a unique $\tilde\sigma_1\in(\lambda,\sigma_0)$, such that

$$f_{\sigma_0}(\tilde\sigma_1)=\frac{1}{\tau}\int_0^1\tilde v(x)-\frac{1}{\tilde n(x)}dx. \tag{3.7}$$

Define the operator $\Gamma:\bar D\to X$, $(\tilde v,\tilde n)\mapsto(v,n)$ by solving

$$\begin{cases}\left[\left(k\tilde v-\frac{1}{\tilde v}\right)v_x\right]_x+\frac{1}{\tau}v_x=\frac{1}{\tilde v}-\tilde n-b(x),\\ \left[\left(\frac{1}{\tilde n}-\frac{1}{\tilde n^3}\right)n_x\right]_x+\frac{1}{\tau\tilde n^2}n_x=\tilde n+b(x)-\frac{1}{\tilde v}, \qquad x\in(0,1),\\ v(0)=v(1)=1,\ n(0)=\sigma_0,\ n(1)=\tilde\sigma_1,\end{cases}$$

where $\tilde\sigma_1$ is defined in (3.7). Then $\Gamma:\bar D\to X$ is a compact and continuous operator.

The following proof in this step is similar to the proof in Lemma 2.1: by the theory of topological degree, we prove that for any $\epsilon\in(0,1]$ and $(v,n)\in\partial D$, (v,n) can not be a solution of

$$\begin{cases}\left[\left(kv-\frac{1}{v}\right)v_x\right]_x+\frac{1}{\tau}v_x=\epsilon\left(\frac{1}{v}-n-b(x)\right),\\ \left[\left(\frac{1}{n}-\frac{1}{n^3}\right)n_x\right]_x-\left(\frac{1}{\tau n}\right)_x=\epsilon\left(n+b(x)-\frac{1}{v}\right), \qquad x\in(0,1),\\ v(0)=v(1)=1,\ n(0)=\sigma_0,\ n(1)=\sigma_0-\epsilon(\sigma_0-\sigma_1)\triangleq\hat\sigma_1,\end{cases} \tag{3.8}$$

where $f_{\sigma_0}(\sigma_1)=\frac{1}{\tau}\int_0^1 v(x)-\frac{1}{n(x)}dx$ and obviously $\hat\sigma_1\in[\sigma_1,\sigma_0]$. Due to the standard maximum principle of elliptic equations, it is easy to check that

$$n\le\max\{\sigma_0,\hat\sigma_1\}\le\sigma_0<\Lambda,\text{ and } v\ge1>m.$$

Step 2. In this step, we estimate the upper bound of v in (3.8). Setting $(kv-\frac{1}{v})v_x+\frac{1}{\tau}v=E$, then we have

$$\begin{cases}\left(kv-\frac{1}{v}\right)v_x=E-\frac{v}{\tau},\\ E_x=\epsilon\left(\frac{1}{v}-n-b(x)\right), \qquad x\in(0,1),\\ v(0)=v(1)=1.\end{cases} \tag{3.9}$$

Since $v\ge1$ and $n\ge\lambda>1$, we have $\frac{1}{v}<n$. Therefore $E_x<0$ over $(0,1)$. By a simple monotonicity analysis, we can verify that there exists a unique $T\in(0,1)$, such that $v(T)=\max_{x\in(0,1)}v(x)\triangleq\bar v$, $v_x>0$ over $(0,T)$, $v_x<0$ over $(T,1)$, and $E(T)=\frac{1}{\tau}\bar v$. Indeed, if there is a $\hat x\in(0,1)$ where v assumes its minimum, then there are ε_1, $\varepsilon_2\in(0,1)$ such that $E<\frac{1}{\tau}v$ over $(\varepsilon_1,\hat x)$, $E>\frac{1}{\tau}v$ over $(\hat x,\varepsilon_2)$, $E(\hat x)=\frac{1}{\tau}v(\hat x)$ and $v_x(\hat x)=0$. Since $E_x<0$ over $(0,1)$, there exists a $\varepsilon_3\in(\hat x,\varepsilon_2)$, such that $E(x)<\frac{1}{\tau}v(x)$, $x\in(\hat x,\varepsilon_3)$, which contradicts $E>\frac{1}{\tau}v$ over $(\hat x,\varepsilon_2)$. Hence there is no minimum point of v over $(0,1)$. As a result, there is only one critical point $T\in(0,1)$ and v assumes its maximum at T.

Integrating the first equation of (3.9) over $(0,T)$ we obtain

$$\frac{1}{2}k\bar{v}^2-\ln\bar{v}-\frac{k}{2}=\int_0^T E(x)-\frac{v(x)}{\tau}dx\le E(0)T\le E(0). \tag{3.10}$$

Due to the second equation of (3.9) we have

$$\begin{aligned}E(0)&=E(T)+\int_T^0 E_x(x)dx\\&=E(T)+\int_0^T \epsilon\big(n(x)+b(x)-\frac{1}{v(x)}\big)dx\\&\le\frac{\bar{v}}{\tau}+\bar{n}+\bar{b}\\&\le\frac{\bar{v}}{\tau}+\sigma_0+\bar{b}.\end{aligned} \tag{3.11}$$

Combining (3.11) and (3.10) we further have

$$\frac{1}{2}k\bar{v}^2\le\ln\bar{v}+\frac{k}{2}+\frac{\bar{v}}{\tau}+\sigma_0+\bar{b}\le\bar{v}-1+\frac{k}{2}+\frac{\bar{v}}{\tau}+\sigma_0+\bar{b},$$

therefore

$$\bar{v}\le\frac{1+\frac{1}{\tau}+\sqrt{(1+\frac{1}{\tau})^2+2k(\sigma_0+\bar{b}+\frac{k}{2}-1)}}{k}<2+\frac{1}{\tau}+\sqrt{(1+\frac{1}{\tau})^2+4(\sigma_0+\bar{b})}=M.$$

Step 3. In this step, we consider the second equation in (3.8) without damping, which will be used to estimate the lower bound of n. Consider

$$\begin{cases}\big[\big(\frac{1}{w}-\frac{1}{w^3}\big)w_x\big]_x=\epsilon\big(w+b(x)-\frac{1}{v}\big), & x\in(0,1),\\ w(0)=\sigma_0,\ w(1)=\hat{\sigma}_1,\end{cases} \tag{3.12}$$

where ϵ, $v(x)$, $b(x)$, σ_0, $\hat{\sigma}_1$ are from (3.8). We prove that (3.12) admits a solution which has a lower bound $\underline{w}>1$, topological method will be employed.

First, we define $X^*=\{w\in C[0,1]\}$, $D^*=\{w\in X^*,\ \mu<w<\bar{w}\}$, where $\bar{w}=\sigma_0+1$ and $\mu\in(1,N(\underline{Y}^{**}))$ is a constant. Then D^* is a bounded and open subset of X^*, and

$$\partial D^*=\{w\in X^*,\ \mu\le w\le\bar{w},\ \text{and}\ \exists x\in[0,1]\ s.t{:}\ w(x)=\mu\ \text{or}\ w(x)=\bar{w}\}.$$

Define $\Gamma^*:\overline{D^*}\to X^*$, $\tilde{w}\mapsto w$ by solving

$$\begin{cases}\big[\big(\frac{1}{\tilde{w}}-\frac{1}{\tilde{w}^3}\big)w_x\big]_x=\epsilon\big(\tilde{w}+b(x)-\frac{1}{v}\big), & x\in(0,1),\\ w(0)=\sigma_0,\ w(1)=\hat{\sigma}_1,\end{cases}$$

then Γ^* is a compact and continuous operator.

Similar to the analysis in the previous section, it suffices to prove that for any $t\in(0,1]$ and $w\in\partial D^*$, w can not be a solution of

$$\begin{cases}\big[\big(\frac{1}{w}-\frac{1}{w^3}\big)w_x\big]_x=t\epsilon\big(w+b(x)-\frac{1}{v}\big), & x\in(0,1),\\ w(0)=\sigma_0,\ w(1)=\bar{\sigma}_1,\end{cases} \tag{3.13}$$

where $\bar{\sigma}_1 = \sigma_0 - t(\sigma_0 - \hat{\sigma}_1) = \sigma_0 - t\epsilon(\sigma_0 - \sigma_1) \in [\sigma_1, \sigma_0]$.

By the standard maximum principle, we have that $w \le \sigma_0 < \bar{w}$. Now we estimate the lower bound of w and prove that $w > \mu$. Setting $Z = \ln w + \frac{1}{2w^2}$, then there is an inverse function $w = N(Z)$ since $w \ge \mu > 1$. We also have that $w \le e^Z$. Hence

$$\begin{cases} Z_{xx} = t\epsilon\big(N(Z) + b(x) - \frac{1}{v}\big) \le e^Z + \bar{b}, & x \in (0,1), \\ Z(0) = Z_0, \ Z(1) = Z_1, \end{cases}$$

where $Z_0 = \ln \sigma_0 + \frac{1}{2\sigma_0^2}$, $Z_1 = \ln \bar{\sigma}_1 + \frac{1}{2\bar{\sigma}_1^2}$. It suffices to prove that $Z \ge \underline{Z} > \ln \mu + \frac{1}{2\mu^2}$.

Setting $\tilde{Z}(x) = Z(x) + x(Z_0 - Z_1) - \frac{1}{2}\bar{b}x(x-1)$, $x \in (0,1)$, then

$$\begin{cases} \tilde{Z}_{xx} \le e^{\tilde{Z} - x(Z_0 - Z_1) + \frac{1}{2}\bar{b}x(x-1)} \le e^{\tilde{Z}}, & x \in (0,1), \\ \tilde{Z}(0) = \tilde{Z}(1) = Z_0. \end{cases} \tag{3.14}$$

Assume that $Y(x)$ is the solution of

$$\begin{cases} Y_{xx} = e^Y, & x \in (0,1), \\ Y(0) = Y(1) = Z_0. \end{cases} \tag{3.15}$$

Applying the comparison principle to (3.15) and (3.14), we obtain that $Y \le \tilde{Z}$ over $[0,1]$.

Since (3.15) is a symmetric system, the solution $Y(x)$ must assume its minimum value $\underline{Y}$ at $x = \frac{1}{2}$, and $Y_x(\frac{1}{2}) = 0$. On the contrary, if there exists a $\underline{Y}$ such that the solution $y(x)$ of the initial problem

$$\begin{cases} y_{xx} = e^y, & x < \frac{1}{2}, \\ y(\frac{1}{2}) = \underline{Y}, \ y_x(\frac{1}{2}) = 0, \end{cases} \tag{3.16}$$

is well-defined at $x = 0$ and $y(0) = Z_0$, then $y(x)$ is also a solution of (3.15). Nevertheless, we can write the solution of (3.16) as

$$y(x) = -\ln\left[\cos^2\left(\sqrt{\frac{e^{\underline{Y}}}{2}}(x - \frac{1}{2})\right)\right] + \underline{Y} \triangleq y(x; \underline{Y}).$$

To ensure that $y(x;\underline{Y})$ is well-defined at $x = 0$, we require that $\frac{1}{2}\sqrt{\frac{e^{\underline{Y}}}{2}} < \frac{\pi}{2}$, i.e, $\underline{Y} < \ln 2\pi^2$. Since $y(0;\underline{Y}) \to +\infty$ as $\underline{Y} \to (\ln 2\pi^2)^-$, we conclude that for any $Z_0 > 1$ large enough, (3.15) admits a unique solution $Y(x)$ and the minimum value $\underline{Y} < \ln 2\pi^2$. And due to the comparison principle, $\underline{Y}$ is monotone increasing with respect to Z_0. Now taking $\underline{Y}^* = \frac{1}{2}(\frac{1}{2} + \ln 2\pi^2)$, $Z_0^* = y(0;\underline{Y}^*)$, and $\sigma^* = N(Z_0^*)$, then once $\sigma_0 \ge \sigma^*$, we have $Z_0 \ge Z_0^*$ and then (3.15) admits a solution $Y(x)$ which has a minimum $\underline{Y} = Y(\frac{1}{2}) \in [\underline{Y}^*, \ln 2\pi^2)$.

Finally, since $Y(x) \le \tilde{Z}(x) = Z(x) + x(Z_0 - Z_1) - \frac{1}{2}\bar{b}x(x-1)$, we have that

$$\begin{aligned} Z(x) &= \tilde{Z}(x) - x(Z_0 - Z_1) + \frac{1}{2}\bar{b}x(x-1) \\ &\ge \underline{Y} - (Z_0 - Z_1) - \frac{1}{8}\bar{b} \\ &\ge \underline{Y}^* - (Z_0 - Z_1) - \frac{1}{8}\bar{b}. \end{aligned}$$

Since $\bar\sigma_1\in[\sigma_1,\sigma_0]$, we have

$$\begin{aligned}Z_0-Z_1&=\big(\ln\sigma_0+\frac{1}{2\sigma_0^2}\big)-\big(\ln\bar\sigma_1+\frac{1}{2\bar\sigma_1^2}\big)\\&\le\big(\ln\sigma_0+\frac{1}{2\sigma_0^2}\big)-\big(\ln\sigma_1+\frac{1}{2\sigma_1^2}\big)\\&=f_{\sigma_0}(\sigma_1)\\&\le\frac{M}{\tau},\end{aligned}$$

where M is defined in (3.6). Therefore, if $\frac18\bar b\le\frac14(\underline{Y}^*-\frac12)$, then there exists a τ_2^* such that for any $\tau\ge\max\{\tau_1^*,\tau_2^*\}$, we have $(Z_0-Z_1)+\frac18\bar b\le\frac{M}{\tau}+\frac18\bar b\le\frac12(\underline{Y}^*-\frac12)$, hence

$$Z(x)\ge\underline{Y}^*-(Z_0-Z_1)-\frac18\bar b\ge\frac12(\frac12+\underline{Y}^*)\triangleq\underline{Y}^{**}=\frac18(3+2\ln2\pi^2)>\frac12.$$

Since $\mu\in(1,N(\underline{Y}^{**}))$, we have $\ln\mu+\frac{1}{2\mu^2}\in(\frac12,\underline{Y}^{**})$. Now we have proved that when $\sigma_0\ge\sigma^*$, $\tau\ge\max\{\tau_1^*,\tau_2^*\}$ and $\frac18\bar b\le\frac14(\underline{Y}^*-\frac12)$, the solution of (3.13) satisfies $\mu<N(\underline{Y}^{**})\le w\le\sigma_0<\bar w$, which contradicts $w\in\partial D^*$. Hence (3.12) admits a solution $w\in D^*$, and $w\ge N(\underline{Y}^{**})$.

Step 4. In this step, we estimate the lower bound of n in (3.8). We prove that $n>\lambda$ when $\tau\gg1$ by perturbation. Assume that $\sigma_0\ge\sigma^*$, $\tau\ge\max\{\tau_1^*,\tau_2^*\}$, $\frac18\bar b\le\frac14(\underline{Y}^*-\frac12)$, w is the solution of (3.12) and n is from (3.8). Taking $z_1=\ln n+\frac{1}{2n^2}$, $z_2=\ln w+\frac{1}{2w^2}$, then $n=N(z_1)$, $w=N(z_2)$ and

$$\begin{cases}(z_1)_{xx}-\big(\frac{1}{\tau N(z_1)}\big)_x=\epsilon\big(N(z_1)+b-\frac1v\big),\qquad x\in(0,1),\\ z_1(0)=Z_0,\ z_1(1)=\hat Z_1,\end{cases}$$

$$\begin{cases}(z_2)_{xx}=\epsilon\big(N(z_2)+b-\frac1v\big),\qquad x\in(0,1),\\ z_2(0)=Z_0,\ z_2(1)=\hat Z_1,\end{cases}$$

where $Z_0=\ln\sigma_0+\frac{1}{2\sigma_0^2}$ and $\hat Z_1=\ln\hat\sigma_1+\frac{1}{2\hat\sigma_1^2}$. Taking $z=z_1-z_2$, then

$$\begin{cases}z_{xx}-\big(\frac{1}{\tau N(z_1)}\big)_x=\epsilon\big(N(z_1)-N(z_2)\big),\qquad x\in(0,1),\\ z(0)=z(1)=0.\end{cases}$$

Multiplying the above equation by z and integrating by parts, we have

$$\int_0^1z_x^2dx=-\epsilon\int_0^1\big(N(z_1)-N(z_2)\big)zdx-\frac1\tau\int_0^1\big(\frac{1}{N(z_1)}\big)_xzdx.$$

Since $N(z)$ is increasing about z, we have that

$$\begin{aligned}\int_0^1z_x^2dx&\le-\frac1\tau\int_0^1\big(\frac{1}{N(z_1)}\big)_xzdx\\&=\frac1\tau\int_0^1\frac{1}{N(z_1)}z_xdx\\&\le\frac12\int_0^1z_x^2dx+\frac{1}{2\tau^2},\end{aligned}$$

hence $|z|\le\frac{1}{\tau}$ by Poincare's inequality. Therefore, there is a $\tau_3^*>0$ such that for any $\tau\ge\max\{\tau_1^*,\tau_2^*,\tau_3^*\}\triangleq\tau^*$, $|z|=|z_1-z_2|\le\frac{1}{2}(\underline{Y}^{**}-\frac{1}{2})$.

Since $w\ge N(\underline{Y}^{**})$, we have $z_2=\ln w+\frac{1}{2w^2}\ge\underline{Y}^{**}$. Hence

$$z_1\ge z_2-\frac{1}{2}(\underline{Y}^{**}-\frac{1}{2})\ge\frac{1}{2}(\underline{Y}^{**}+\frac{1}{2})\triangleq\underline{Y}^{***}=\frac{1}{16}(7+2\ln 2\pi^2)>\frac{1}{2},$$

as a result $n\ge N(\underline{Y}^{***})>\lambda>1$. Similar to the discussions in Step 1, we conclude that (3.5) admits a solution $(v,n)\in D$ which satisfies $v\ge1$, $n\ge N(\underline{Y}^{***})$ over $(0,1)$. Due to the standard regularity theory, we have $(v,n)\in W^{2,+\infty}(0,1)\times W^{2,+\infty}(0,1)$. □

REMARK 3.2. Different form the proof of Lemma 2.1, where a large enough boundary data σ_0 could bring us a desired lower bound of $\tilde{n}$, in this section, the magnitude of the right side of the second equation in (3.8) mainly depends on the magnitude of n for fixed b and ϵ since $\frac{1}{v}\le1$. Therefore, for large boundary data σ_0, n decreases dramatically around the boundary, hence it is not an easy task to obtain the desired lower bound of n. Due to this reason, we adopt a different and tedious proof and the conclusion of Theorem 3.1 is quite different from that in Theorem 2.1.

Proof. **(Proof of Theorem 3.1.)** Now we use the compactness method in [20] to obtain the solution of (3.3). Let (v_k,n_k) be a solution of (3.5) obtained in Lemma 3.1. Multiplying (v_k-1) to the first equation in (3.5) and integrating by parts, we obtain

$$\frac{4}{9}k\int_0^1\frac{v_k+1}{v_k}|((v_k-1)^{\frac{3}{2}})_x|^2dx+(k-1)\int_0^1\frac{|(v_k)_x|^2}{v_k}dx=-\int_0^1(\frac{1}{v_k}-n_k-b)(v_k-1)dx.$$

By the standard energy estimate and the uniform boundness of v_k, n_k in $L^\infty(0,1)$ we have

$$\|((v_k-1)^{\frac{3}{2}})_x\|_{L^2(0,1)}\le C,$$

where (and in the following proof) C is independent of k. Noting that $((v_k-1)^2)_x=\frac{4}{3}(v_k-1)^{\frac{1}{2}}((v_k-1)^{\frac{3}{2}})_x$, we have by using the boundness of v_k that

$$\|(v_k-1)^2\|_{H_0^1(0,1)}\le C. \tag{3.17}$$

Next, similar to the proof in Theorem 2.1 we could obtain that

$$\|n_k\|_{H^1(0,1)}\le C. \tag{3.18}$$

The estimates in (3.17) and (3.18) imply that there is a subsequence of k (which is still denoted by k) such that as $k\to1^+$,

$$(v_k-1)^2\rightharpoonup(v-1)^2\ \text{ weakly in } H_0^1(0,1),$$
$$n_k\rightharpoonup n\ \text{ weakly in } H^1(0,1),$$

and

$$\|(v-1)^2\|_{H_0^1(0,1)}\le C,\quad \|n\|_{H^1(0,1)}\le C.$$

In similar fashion to [20], we could prove that $(v,n)\in C^{\frac{1}{2}}[0,1]\times W^{2,+\infty}(0,1)$, $v(x)>1$ over $(0,1)$ and (v,n) is a weak solution of (3.3). Meanwhile, duo to the fact that the imbedding from $H^1(0,1)$ to $C[0,1]$ is compact, we conclude that $n\ge N(\underline{Y}^{***})$ over $[0,1]$ and (3.4) hold. □

3.2. Ill-posedness. This subsection is devoted to the proof of non-existence of supersonic-vs-subsonic solution to (3.1) when the semiconductor effect is very weak.

THEOREM 3.2. *There is no supersonic-vs-subsonic solution to* (3.1) *if* $b(x)\not\equiv 0$, $\tau\gg 1$ *and* $\sigma_0>1$ *is close to* 1.

Proof. Assume that (v,n,E) is a supersonic-vs-subsonic solution of (3.1) which satisfies (3.2). We have shown that $n(1)=\sigma_1<\sigma_0$. Since E keeps decreasing in $(0,1)$, we can verify that there is only one point $T\in(0,1]$ such that $E(T)=\frac{1}{\tau n(T)}$, and n reaches its minimum at T. Since the case $T=1$ is trivial in the following proof, we assume that $T\in(0,1)$. Set $\underline{n}=n(T)$. We have that $n_x<0$ over $(0,T)$, $n_x>0$ over $(T,1)$ and $n(x)\le\sigma_0$ over $[0,1]$.

From (3.1) we have

$$\frac{(n^2-1)(n+b-\frac{1}{v})}{n^3}n_x=(E-\frac{1}{\tau n})E_x. \tag{3.19}$$

Integrating (3.19) over $(0,T)$ we obtain that

$$\begin{aligned}\frac{1}{2}E^2(0)-\frac{1}{2}E^2(T)+\frac{1}{\tau}\int_0^T\frac{1}{n}E_xdx&=\int_0^T\frac{(n^2-1)(n+b-\frac{1}{v})}{n^3}(-n_x)dx\\&\le\int_0^T\frac{(n^2-1)(n+\bar{b})}{n^3}(-n_x)dx\\&=\int_{\underline{n}}^{\sigma_0}\frac{(n^2-1)(n+\bar{b})}{n^3}dn\\&\le\int_1^{\sigma_0}\frac{(n^2-1)(n+\bar{b})}{n^3}dn.\end{aligned} \tag{3.20}$$

Similarly, integrating (3.19) over $(T,1)$ we have

$$\begin{aligned}\frac{1}{2}E^2(1)-\frac{1}{2}E^2(T)-\frac{1}{\tau}\int_T^1\frac{1}{n}E_xdx&=\int_T^1\frac{(n^2-1)(n+b-\frac{1}{v})}{n^3}n_xdx\\&\le\int_T^1\frac{(n^2-1)(n+\bar{b})}{n^3}n_xdx\\&=\int_{\underline{n}}^{\sigma_1}\frac{(n^2-1)(n+\bar{b})}{n^3}dn\\&\le\int_1^{\sigma_0}\frac{(n^2-1)(n+\bar{b})}{n^3}dn.\end{aligned} \tag{3.21}$$

And due to the third equation of (3.1), we further have

$$E(0)+\big(-E(1)\big)=\int_0^1 n+b-\frac{1}{v}dx\ge\int_0^1 bdx>0,$$

hence

$$E^2(0)+E^2(1)\ge\frac{1}{2}\big(E(0)+(-E(1))\big)^2\ge\frac{1}{2}\Big(\int_0^1 bdx\Big)^2. \tag{3.22}$$

Combining (3.20), (3.21) with (3.22) we obtain

$$
\begin{aligned}
\frac{1}{2}\Big(\int_0^1 b(x)dx\Big)^2 \le\, & E^2(0)+E^2(1)\\
&\le 4\int_1^{\sigma_0}\frac{(n^2-1)(n+\bar{b})}{n^3}dn+2E^2(T)-\frac{2}{\tau}\int_0^T\frac{1}{n}E_x dx+\frac{2}{\tau}\int_T^1\frac{1}{n}E_x dx\\
&\le 4\int_1^{\sigma_0}\frac{(n^2-1)(n+\bar{b})}{n^3}dn+\frac{2}{\tau^2}+\frac{2}{\tau}\big(E(0)-E(1)\big)\\
&\le 4\int_1^{\sigma_0}\frac{(n^2-1)(n+\bar{b})}{n^3}dn+\frac{2}{\tau^2}+\frac{2}{\tau}\big(\sigma_0+\bar{b}\big).
\end{aligned}
\tag{3.23}
$$

Since $\frac{(n^2-1)(n+\bar{b})}{n^3}\le\sigma_0+\bar{b}$, we could draw a contradictory conclusion to (3.23) if $\sigma_0-1\ll 1$ and $\tau\gg 1$. So there is no interior-supersonic-fully-subsonic solution to (3.1) if $b(x)\not\equiv 0$, $\tau\gg 1$ and $\sigma_0>1$ is close to 1. ∎

4. Interiorly-subsonic-vs-fully-supersonic flow

In this section, we consider the subsonic-vs-supersonic flow of (1.5). First we assume that $n(0)=\sigma_0\in(0,1)$. We use the velocity equation of holes. Taking $u(x)=\frac{1}{n(x)}$, then (1.5) is equivalent to

$$
\begin{cases}
\big(\frac{1}{\rho}-\frac{1}{\rho^3}\big)\rho_x=E-\frac{1}{\tau\rho},\\
\big(u-\frac{1}{u}\big)u_x=-E+\frac{u}{\tau}, \qquad x\in(0,1),\\
E_x=\rho-\frac{1}{u}-b(x),
\end{cases}
\tag{4.1}
$$

and $u(0)=\frac{1}{\sigma_0}\triangleq a_0>1$. We investigate the existence and non-existence of solutions of (4.1) such that

$$
\rho(0)=\rho(1)=1,\ \rho(x)>1,\ \forall x\in(0,1);\ u(0)=a_0,\ u(x)>1,\ \forall x\in[0,1]. \tag{4.2}
$$

As the previous sections, this problem is equivalent to:

P3: Finding the solution (ρ,u) of

$$
\begin{cases}
\big[\big(\frac{1}{\rho}-\frac{1}{\rho^3}\big)\rho_x\big]_x+\big(\frac{1}{\tau\rho}\big)_x=\rho-\frac{1}{u}-b(x),\\
\big[\big(u-\frac{1}{u}\big)u_x\big]_x-\frac{1}{\tau}u_x=\frac{1}{u}+b(x)-\rho, \qquad x\in(0,1),\\
\rho(0)=\rho(1)=1,\ u(0)=a_0,
\end{cases}
\tag{4.3}
$$

such that (4.2) and

$$
g_{a_0}(u(1))=g_{a_0}(a_1):=\Big(\frac{1}{2}a_0^2-\ln a_0\Big)-\Big(\frac{1}{2}a_1^2-\ln a_1\Big)=\frac{1}{\tau}\int_0^1\frac{1}{\rho(x)}-u(x)dx, \tag{4.4}
$$

hold, where $a_1=u(1)$.

4.1. Well-posedness. In this subsection, we are going to prove the existence of subsonic-vs-supersonic solution to **P3**, as well as their regularities.

THEOREM 4.1. *For any $b(x)\in L^\infty(0,1)$, $b\geq 1$, $\tau>0$ and $\underline{u}>1$, there exists a constant $a^*>\underline{u}$, which only depends on $b(x)$, τ and $\underline{u}$, such that for any $a_0\geq a^*$,* **P3** *admits a subsonic-vs-supersonic solution $(\rho,u)\in C^{\frac12}[0,1]\times W^{2,\infty}(0,1)$ such that $u\geq\underline{u}$ over $[0,1]$.*

Consider the following approximate system

$$\begin{cases}\left[\left(\frac{1}{\rho_j}-\frac{j}{\rho_j^3}\right)(\rho_j)_x\right]_x+\left(\frac{1}{\tau\rho_j}\right)_x=\rho_j-\frac{1}{u_j}-b(x),\\ \left[\left(u_j-\frac{1}{u_j}\right)(u_j)_x\right]_x-\frac{1}{\tau}(u_j)_x=\frac{1}{u_j}+b(x)-\rho_j, \qquad x\in(0,1),\\ \rho_j(0)=\rho_j(1)=1,\ u_j(0)=a_0,\end{cases} \tag{4.5}$$

where $0<j<1$ is a constant. Also, we omit the subscript j and denote the solution of (4.5) by (ρ,u).

LEMMA 4.1. *For any $b(x)\in L^\infty(0,1)$, $b\geq1$, $j\in(0,1)$, $\tau>0$ and $\underline{u}>1$, there exists a constant $a^*>\underline{u}$, which only depends on $b(x)$, τ and $\underline{u}$, such that for any $a_0\geq a^*$, (4.5) admits a solution $(\rho,u)\in W^{2,\infty}(0,1)\times W^{2,\infty}(0,1)$ which satisfies (4.2), (4.4) and $u\geq\underline{u}$ over $[0,1]$.*

Proof. For the given $\tau>0$, set $h(u)=\frac12u^2-\ln u-\frac{2}{\tau}u$ for $u\geq\frac{1}{\tau}+\sqrt{\frac{1}{\tau^2}+1}$. It is easy to verify that h admits an inverse function $u=h^{-1}(\cdot)$. Define

$$X=\left\{(\rho,u)\in C[0,1]\times C[0,1]\right\},$$
$$D=\left\{(\rho,u)\in X,\ m<\rho<M,\ \ \lambda<u<\Lambda\right\},$$

where

$$\lambda=\frac{\underline{u}+1}{2},\ m=\max\left\{\frac{1}{\lambda},\ \frac{\sqrt{j}+1}{2}\right\},\ M=\bar{b}+2,$$
$$\Lambda=\max\left\{\bar{a}_1,h^{-1}\left(\frac12a_0^2-\ln a_0+2(1+\bar{b})\right)\right\}+1,$$

and $\bar{a}_1>a_0$ is defined by

$$g_{a_0}(\bar{a}_1)=-\frac{1}{\tau}\bar{a}_1. \tag{4.6}$$

Since $g_{a_0}(a)$ is convex about a, $\bar{a}_1$ is well-defined. It is easy to check that D is a bounded and open subset of X, and

$$\partial D=\left\{(\rho,u)\in X,\ m\leq\rho\leq M,\ \lambda\leq u\leq\Lambda,\ \textit{and}\ \exists x\in[0,1],\right.$$
$$\left.s.t\!:\ \rho(x)=m\ \textit{or}\ \rho(x)=M\ \textit{or}\ u(x)=\lambda\ \textit{or}\ u(x)=\Lambda\right\}.$$

For any $(\tilde\rho,\tilde u)\in\bar{D}$, since

$$\frac{1}{\tau}\int_0^1\frac{1}{\tilde\rho(x)}-\tilde u(x)dx<0,$$

there is a unique $\tilde a_1>a_0$, such that

$$g_{a_0}(\tilde a_1)=\frac{1}{\tau}\int_0^1\frac{1}{\tilde\rho(x)}-\tilde u(x)dx. \tag{4.7}$$

Define $\Gamma:\bar{D}\to X$, $(\tilde{\rho},\tilde{u})\mapsto(\rho,u)$ by solving

$$\begin{cases}\big[\big(\frac{1}{\tilde{\rho}}-\frac{j}{\tilde{\rho}^3}\big)\rho_x\big]_x-\frac{1}{\tau\tilde{\rho}^2}\rho_x=\tilde{\rho}-\frac{1}{\tilde{u}}-b(x),\\ \big[\big(\tilde{u}-\frac{1}{\tilde{u}}\big)u_x\big]_x-\frac{1}{\tau}u_x=\frac{1}{\tilde{u}}+b(x)-\tilde{\rho}, \qquad x\in(0,1),\\ \rho(0)=\rho(1)=1,\ u(0)=a_0,\ u(1)=\tilde{a}_1,\end{cases}$$

where $\tilde{a}_1>a_0$ is defined in (4.7). Then $\Gamma:\bar{D}\to X$ is a compact and continuous operator. Due to the same analysis as Lemma 2.1, it suffices to prove that for any $\epsilon\in(0,1]$ and $(\rho,u)\in\partial D$, (ρ,u) can not be a solution of

$$\begin{cases}\big[\big(\frac{1}{\rho}-\frac{j}{\rho^3}\big)\rho_x\big]_x+\big(\frac{1}{\tau\rho}\big)_x=\epsilon\big(\rho-\frac{1}{u}-b(x)\big),\\ \big[\big(u-\frac{1}{u}\big)u_x\big]_x-\frac{1}{\tau}u_x=\epsilon\big(\frac{1}{u}+b(x)-\rho\big), \qquad x\in(0,1),\\ \rho(0)=\rho(1)=1,\ u(0)=a_0,\ u(1)=a_0+\epsilon(a_1-a_0)\triangleq\hat{a}_1,\end{cases}$$

where $g_{a_0}(a_1)=\frac{1}{\tau}\int_0^1\frac{1}{\rho(x)}-u(x)dx$ and obviously $\hat{a}_1\in(a_0,a_1]$. Using the standard maximum principle of elliptic equations, it is easy to verify that

$$m<1\le\rho(x)\le 1+\bar{b}<M,\quad x\in[0,1].$$

Now we estimate the lower bound of u. Setting $(u-\frac{1}{u})u_x-\frac{1}{\tau}u=-E$, then we have

$$\begin{cases}\big(u-\frac{1}{u}\big)u_x=-E+\frac{u}{\tau},\\ E_x=\epsilon\big(\rho-\frac{1}{u}-b(x)\big), \qquad x\in(0,1),\\ u(0)=a_0,\ u(1)=\hat{a}_1.\end{cases} \tag{4.8}$$

Set $u(T)=\inf\limits_{x\in(0,1)}u(x)$. If $T=0$, then $u(x)\ge a_0\ge\underline{u}>\lambda$. If $T\in(0,1)$, then $u_x(T)=0$ and hence $E(T)=\frac{1}{\tau}u(T)\le\frac{1}{\tau}a_0$. For any $x\in(0,1]$, integrating the first equation of (4.8) over $(0,x)$ we have

$$\begin{aligned}\frac{1}{2}u^2(x)-\ln u(x)&=\frac{1}{2}a_0^2-\ln a_0-\int_0^x E(\xi)-\frac{1}{\tau}u(\xi)d\xi\\ &\ge\frac{1}{2}a_0^2-\ln a_0-\int_0^x E(\xi)d\xi.\end{aligned} \tag{4.9}$$

Since

$$\begin{aligned}|E(x)|&=|E(T)+\int_T^x E_x(\xi)d\xi|\\ &=|\frac{1}{\tau}u(T)+\int_T^x\epsilon\big(\rho(\xi)-\frac{1}{u(\xi)}-b(\xi)\big)d\xi|\\ &\le\frac{1}{\tau}a_0+2(1+\bar{b}),\end{aligned} \tag{4.10}$$

we have

$$\frac{1}{2}u^2(x)-\ln u(x)\geq \frac{1}{2}a_0^2-\ln a_0-\int_0^x E(\xi)d\xi$$
$$\geq \frac{1}{2}a_0^2-\ln a_0-\frac{1}{\tau}a_0-2(1+\bar{b}). \tag{4.11}$$

Taking $a_1^*>1$ such that

$$\frac{1}{2}(a_1^*)^2-\ln a_1^*-\frac{1}{\tau}a_1^*-2(1+\bar{b})=\frac{1}{2}\underline{u}^2-\ln \underline{u},$$

and assuming that $a_0\geq a_1^*$, then

$$\frac{1}{2}u(x)^2-\ln u(x)\geq \frac{1}{2}a_0^2-\ln a_0-\frac{1}{\tau}a_0-2(1+\bar{b})$$
$$\geq \frac{1}{2}(a_1^*)^2-\ln a_1^*-\frac{1}{\tau}a_1^*-2(1+\bar{b})$$
$$=\frac{1}{2}\underline{u}^2-\ln \underline{u}, \tag{4.12}$$

therefore $u\geq \underline{u}>\lambda$. Set $a^*=\max\left\{a_1^*,\frac{1}{\tau}+\sqrt{\frac{1}{\tau^2}+1}\right\}$ and assume that $a_0\geq a^*$.

Now we estimate the upper bound of u. Set $\bar{u}=u(T^*)=\max\limits_{x\in[0,1]}u(x)$. If $T^*=1$, then $\bar{u}\leq \hat{a}_1\leq a_1$, hence

$$g_{a_0}(a_1)=\frac{1}{\tau}\int_0^1\frac{1}{\rho}-udx\geq -\frac{1}{\tau}a_1.$$

A simple analysis of $g_{a_0}(a_1)+\frac{1}{\tau}a_1$ shows that $a_1\leq \bar{a}_1$, where $\bar{a}_1$ is defined in (4.6). Hence $\bar{u}\leq a_1\leq \bar{a}_1<\Lambda$. If $T^*\in(0,1)$, then $u_x(T^*)=0$ and hence $E(T^*)=\frac{1}{\tau}u(T^*)=\frac{1}{\tau}\bar{u}$.

Combining the first identity in (4.9) with (4.10) substituted T by T^*, we have

$$\frac{1}{2}u(x)^2-\ln u(x)=\frac{1}{2}a_0^2-\ln a_0-\int_0^x E(\xi)-\frac{1}{\tau}u(\xi)d\xi$$
$$\leq \frac{1}{2}a_0^2-\ln a_0+\int_0^1|E(\xi)|d\xi+\frac{1}{\tau}\bar{u}$$
$$\leq \frac{1}{2}a_0^2-\ln a_0+\frac{1}{\tau}\bar{u}+2(1+\bar{b})+\frac{1}{\tau}\bar{u}, \tag{4.13}$$

hence

$$h(\bar{u})\leq \frac{1}{2}a_0^2-\ln a_0+2(1+\bar{b}). \tag{4.14}$$

Since $\bar{u}\geq a_0\geq a^*\geq \frac{1}{\tau}+\sqrt{\frac{1}{\tau^2}+1}$, we have

$$\bar{u}\leq h^{-1}\big(\frac{1}{2}a_0^2-\ln a_0+2(1+\bar{b})\big)<\Lambda. \tag{4.15}$$

Based on the analysis above, we obtain that $m<\rho<M$, $\lambda<u<\Lambda$ when $a_0\geq a^*$, which contradicts $(\rho,u)\in\partial D$. Similar to Lemma 2.1, we conclude that (4.5) admits a solution $(\rho,u)\in D$ which satisfies (4.4). Due to the regularity theory, we have that

$(\rho,u)\in W^{2,+\infty}(0,1)\times W^{2,+\infty}(0,1)$ and $u\geq \underline{u}$ over $[0,1]$. Finally, using the same method in [20] to estimate the lower bound of ρ, we have

$$\rho(x)\geq 1+\varepsilon\sin(\pi x)>1,\ \forall x\in(0,1),$$

where $\varepsilon>0$ is a small constant independent of j. □

Proof. **(Proof of Theorem 4.1.)** The proof is similar to that of the proofs of Theorem 2.1 and Theorem 3.1. First, the following estimates could be obtained

$$\|(\rho_j-1)^2\|_{H_0^1(0,1)}\leq C,\quad \|u_j\|_{H^1(0,1)}\leq C. \tag{4.16}$$

Therefore, there is a subsequence of j (which is still denoted by j) such that as $j\to 1^-$,

$$\begin{aligned}&(\rho_j-1)^2\rightharpoonup(\rho-1)^2\ \text{ weakly in } H_0^1(0,1),\\ &u_j\rightharpoonup u\ \text{ weakly in } H^1(0,1),\end{aligned}$$

and

$$\|(\rho-1)^2\|_{H_0^1(0,1)}\leq C,\quad \|u\|_{H^1(0,1)}\leq C.$$

Then, in similar fashion to [20], we could prove that $(\rho,u)\in C^{\frac12}[0,1]\times W^{2,+\infty}(0,1)$ and (ρ,u) is a weak solution of (4.3). Also, we could verify that $\rho>1$ over $(0,1)$, $u\geq\underline{u}$ over $[0,1]$, and (4.4) hold. □

4.2. Ill-posedness. This subsection is devoted to the proof of non-existence of subsonic-vs-supersonic solution of (4.1) in certain cases.

THEOREM 4.2. *There is no subsonic-vs-supersonic solution of* (4.1) *in the following three cases: (i)* $\forall\tau>0$, $\bar b<1$ *and* $a_0\gg1$*; (ii)* $\forall\tau>0$, $b(x)\equiv0$ *and* $\forall a_0>1$*; (iii)* $\tau=+\infty$, $b(x)\not\equiv0$ *and* $a_0-1\ll1$.

Proof. Assume that (ρ,u,E) is a solution of (4.1) such that $1<\rho(x)\leq1+\bar b$, $n>1$ for $x\in(0,1)$. Since (ii) is a direct corollary of Theorem 1.3 in [21], we omit the proof here.

First we prove (i). Due to (4.1) we have

$$\Big(\frac12u^2(x)-\ln u(x)\Big)-\Big(\frac12a_0^2-\ln a_0\Big)=-\int_0^x E(\xi)-\frac1\tau u(\xi)d\xi\geq-\int_0^x E(\xi)d\xi.$$

Set $\underline{E}=E(T)=\inf\limits_{x\in[0,1]}E(x)$. Since $\rho(x)>1$ for $x\in(0,1)$ and $\rho(1)=1$, there must be a point $\hat x$ close to 1^- such that $\underline{E}\leq E(\hat x)<\frac{1}{\tau\rho(\hat x)}<\frac1\tau$ in view of (4.1). Hence

$$E(x)=\underline{E}+\int_T^x E_x(\xi)d\xi\leq\frac1\tau+\int_T^x\rho-\frac1u-bd\xi\leq\frac1\tau+2(1+\bar b),$$

and

$$\frac12u^2(x)-\ln u(x)\geq\frac12a_0-\ln a_0-\Big(\frac1\tau+2(1+\bar b)\Big).$$

Therefore, when $\bar b<1$ and $a_0\gg1$, we have $u\gg1$ and hence $\frac1u+\bar b<1\leq\rho$. Multiplying the first equation of (4.3) by $(\rho-1)$ and integrating by parts, we have

$$\int_0^1\Big(\frac1\rho-\frac1{\rho^3}\Big)|\rho_x|^2dx=-\int_0^1\Big(\rho-\frac1u-b\Big)(\rho-1)dx. \tag{4.17}$$

So if $\bar{b}<1$ and $a_0\gg 1$ such that $\frac{1}{u}+\bar{b}<1\le\rho$, then the right side of (4.17) is negative, which contradicts the left side.

Now we prove (iii). When $\tau=+\infty$, we have by combining the first and the second equations in (4.1) that

$$\ln\rho+\frac{1}{2\rho^2}=\ln u-\frac{1}{2}u^2+\frac{1}{2}a_0^2-\ln a_0+\frac{1}{2}, \tag{4.18}$$

hence

$$\ln\rho\le-\frac{1}{2}+\frac{1}{2}a_0^2-\ln a_0+\frac{1}{2}=\frac{1}{2}a_0^2-\ln a_0, \tag{4.19}$$

and $u(1)=a_0$, $\rho(x)\le e^{\frac{1}{2}a_0^2-\ln a_0}$, $u(x)<a_0$ over $(0,1)$.

Let $x_1\in(0,1)$ be the first point such that u reaches its minimum, and $x_2\in(0,1)$ the last point. Then $u(x_1)\ge\underline{u}$, $u(x_2)\ge\underline{u}$, where $\underline{u}=\inf_{x\in[0,1]}u(x)>1$. And $E(x_1)=E(x_2)=0$, $u_x<0$, $E>0$ over $(0,x_1)$, $u_x>0$, $E<0$ over $(x_2,1)$. From (4.1) we have

$$\frac{(u^2-1)(\frac{1}{u}+b-\rho)}{u}u_x=EE_x. \tag{4.20}$$

Integrating (4.20) over $(0,x_1)$, we have

$$\begin{aligned}\frac{1}{2}E^2(0)&=\int_0^{x_1}\frac{(u^2-1)(\frac{1}{u}+b-\rho)}{u}(-u_x)dx\\&\le\int_0^{x_1}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}(-u_x)dx\\&=\int_{u(x_1)}^{a_0}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}du\\&\le\int_1^{a_0}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}du.\end{aligned} \tag{4.21}$$

Then integrating (4.20) over $(x_2,1)$, we have

$$\begin{aligned}\frac{1}{2}E^2(1)&=\int_{x_2}^{1}\frac{(u^2-1)(\frac{1}{u}+b-\rho)}{u}u_xdx\\&\le\int_{x_2}^{1}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}u_xdx\\&=\int_{u(x_2)}^{a_0}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}du\\&\le\int_1^{a_0}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}du.\end{aligned} \tag{4.22}$$

Similar to the previous sections, we obtain

$$E(0)+\big(-E(1)\big)=\int_0^1 b+\frac{1}{u}-\rho dx=\int_0^1 bdx-\int_0^1\rho-\frac{1}{u}dx. \tag{4.23}$$

Using again the identity (4.18), we have

$$\begin{aligned}\big(\ln\rho+\frac{1}{2\rho^2}\big)-\big(\ln a_0+\frac{1}{2a_0^2}\big)&=\ln u-\frac{1}{2}u^2+\frac{1}{2}a_0^2-2\ln a_0+\frac{1}{2}-\frac{1}{2a_0^2}\\&\le -\frac{1}{2}+\frac{1}{2}a_0^2-2\ln a_0+\frac{1}{2}-\frac{1}{2a_0^2}\\&=\frac{1}{2}a_0^2-2\ln a_0-\frac{1}{2a_0^2}.\end{aligned}$$

If $\rho(x_0)>a_0$ at some points $x_0\in(0,1)$, then using differential mean value theorem at these points we have

$$\big(\ln\rho+\frac{1}{2\rho^2}\big)-\big(\ln a_0+\frac{1}{2a_0^2}\big)=\frac{\xi^2-1}{\xi^3}(\rho-a_0)\le\frac{1}{2}a_0^2-2\ln a_0-\frac{1}{2a_0^2},$$

where $\xi\in(a_0,\rho)$. Hence

$$\begin{aligned}\rho&\le a_0+\big(\frac{1}{2}a_0^2-2\ln a_0-\frac{1}{2a_0^2}\big)\frac{\xi^3}{\xi^2-1}\\&\le a_0+\big(\frac{1}{2}a_0^2-2\ln a_0-\frac{1}{2a_0^2}\big)\frac{\rho^3}{a_0^2-1}\\&\le a_0+\frac{\frac{1}{2}a_0^2-2\ln a_0-\frac{1}{2a_0^2}}{a_0^2-1}e^{\frac{3}{2}a_0^3-3\ln a_0}.\end{aligned}\tag{4.24}$$

If $x\in(0,1)$ such that $\rho(x)\le a_0$, then (4.24) still holds at these x. Applying L'Hospital's rule to (4.24) we obtain that

$$\frac{\frac{1}{2}a_0^2-2\ln a_0-\frac{1}{2a_0^2}}{a_0^2-1}\to 0,\qquad as\ a_0\to 1^+,$$

hence $\rho\to 1^+$ as $a_0\to 1^+$ in view of (4.24) and

$$0\le\rho-\frac{1}{u}\le\rho-\frac{1}{a_0}\to 0^+,\qquad as\ a_0\to 1^+.$$

Therefore, when $a_0-1>0$ is small enough, we have by (4.23) that

$$E(0)+\big(-E(-1)\big)\ge\frac{1}{2}\int_0^1 bdx>0.\tag{4.25}$$

Combining (4.25), (4.21) and (4.22), we obtain

$$\begin{aligned}\int_1^{a_0}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}du&\ge\frac{1}{4}\big(E^2(0)+E^2(1)\big)\\&\ge\frac{1}{8}\big(E(0)+(-E(1))\big)^2\\&\ge\frac{1}{32}\big(\int_0^1 bdx\big)^2.\end{aligned}\tag{4.26}$$

Since $\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}\le\frac{(a_0^2-1)(1+\bar{b})}{a_0}$, we could draw a contradictory conclusion to (4.26) if $\sigma_0-1\ll 1$. ∎

5. Interiorly-supersonic-vs-fully-supersonic flow

This section is devoted to the supersonic-vs-supersonic flow. Assume that $n(0)=\sigma_0\in(0,1)$. We use the velocity equations both to electrons and holes by taking $v(x)=\frac{1}{\rho(x)}$ and $u(x)=\frac{1}{n(x)}$, then (1.5) is equivalent to

$$\begin{cases}(v-\dfrac{1}{v})v_x=E-\dfrac{v}{\tau},\\ (u-\dfrac{1}{u})u_x=-E+\dfrac{u}{\tau}, \qquad x\in(0,1),\\ E_x=\frac{1}{v}-\frac{1}{u}-b(x),\end{cases} \tag{5.1}$$

and $u(0)=\frac{1}{\sigma_0}\triangleq a_0>1$. We prove the existence and non-existence of solutions of (5.1) such that

$$v(0)=v(1)=1,\ v(x)>1,\ \forall x\in(0,1);\ u(0)=a_0,\ u(x)>1,\ \forall x\in[0,1]. \tag{5.2}$$

Also, the problem is equivalent to:

P4: Finding the solution (v,u) of

$$\begin{cases}\big[(v-\dfrac{1}{v})v_x\big]_x+\dfrac{1}{\tau}v_x=\dfrac{1}{v}-\dfrac{1}{u}-b(x),\\ \big[(u-\dfrac{1}{u})u_x\big]_x-\dfrac{1}{\tau}u_x=\dfrac{1}{u}+b(x)-\dfrac{1}{v}, \qquad x\in(0,1),\\ v(0)=v(1)=1,\ u(0)=a_0,\end{cases} \tag{5.3}$$

such that (5.2) and

$$g_{a_0}(u(1))=g_{a_0}(a_1)=\frac{1}{\tau}\int_0^1 v(x)-u(x)dx, \tag{5.4}$$

hold, where $a_1=u(1)$ and g_{a_0} is defined in (4.4).

5.1. Well-posedness. This subsection is devoted to the proof of the existence of supersonic-vs-supersonic solutions for **P4**, as well as their regularities.

THEOREM 5.1. *For any $b(x)\in L^\infty(0,1)$, $b\geq 1$, $\tau>0$ and $\underline{u}>1$, there exists a constant $a^*>\underline{u}$, which only depends on $b(x)$, τ and $\underline{u}$, such that for any $a_0\geq a^*$, **P4** admits a supersonic-vs-supersonic solution $(v,u)\in C^{\frac{1}{2}}[0,1]\times W^{2,\infty}(0,1)$ and $u\geq\underline{u}$ over $[0,1]$.*

Consider the following approximate system

$$\begin{cases}\big[(kv_k-\dfrac{1}{v_k})(v_k)_x\big]_x+\dfrac{1}{\tau}(v_k)_x=\dfrac{1}{v_k}-\dfrac{1}{u_k}-b(x),\\ \big[(u_k-\dfrac{1}{u_k})(u_k)_x\big]_x-\dfrac{1}{\tau}(u_k)_x=\dfrac{1}{u_k}+b(x)-\dfrac{1}{v_k}, \qquad x\in(0,1),\\ v_k(0)=v_k(1)=1,\ u_k(0)=a_0,\end{cases} \tag{5.5}$$

where $1<k<2$ is a constant. Also, we omit the subscript k and denote the solution of (5.5) by (v,u).

LEMMA 5.1. *For any $b(x)\in L^\infty(0,1)$, $b\geq 1$, $k\in(1,2)$, $\tau>0$ and $\underline{u}>1$, there exists a constant $a^*>\underline{u}$, which only depends on $b(x)$, τ and $\underline{u}$, such that for any $a_0\geq a^*$, (5.5)* *admits a solution* $(v,u)\in W^{2,\infty}(0,1)\times W^{2,\infty}(0,1)$ *which satisfies* (5.4) *and* $v\ge 1$, $u\ge \underline{u}$ *over* $[0,1]$.

Proof. Define

$$X=\{(v,u)\in C[0,1]\times C[0,1]\},$$
$$D=\{(v,u)\in X,\ m<v<M,\ \lambda<u<\Lambda\},$$

where

$$\lambda=\frac{1+\underline{u}}{2},\ \Lambda=\bar a_1+1,\ m=\frac{\frac{1}{\sqrt{k}}+1}{2},$$
$$M=\frac{1}{\tau}+2+\sqrt{(\frac{1}{\tau}+1)^2+2(3+\bar b)},$$

and $\bar a_1>1$ is determined by

$$g_{a_0}(\bar a_1)=-\frac{1}{\tau}\bar a_1. \tag{5.6}$$

It is easy to verify that $\bar a_1>a_0$ and D is a bounded and open subset of X, and

$$\partial D=\{(v,u)\in X,\ m\le v\le M,\ \lambda\le u\le\Lambda \text{ and } \exists x\in[0,1],$$
$$s.t:\ v(x)=m \text{ or } v(x)=M \text{ or } u(x)=\lambda \text{ or } u(x)=\Lambda\}.$$

For any $(\tilde v,\tilde u)\in\bar D$, take $\tilde a_1>1$ such that

$$g_{a_0}(\tilde a_1)=\frac{1}{\tau}\int_0^1\tilde v(x)-\tilde u(x)dx. \tag{5.7}$$

Take $a_1^*>1$ such that $\frac{1}{2}(a_1^*)^2-\ln a_1^*-\frac{1}{\tau}M=1$, and assume that $a_0\ge a_1^*$. Since

$$\frac{1}{\tau}\int_0^1\tilde v(x)-\tilde u(x)dx\le\frac{1}{\tau}M, \tag{5.8}$$

we have

$$\begin{aligned}\frac{1}{2}\tilde a_1^2-\ln\tilde a_1&=\frac{1}{2}a_0^2-\ln a_0-\frac{1}{\tau}\int_0^1\tilde v(x)-\tilde u(x)dx\\&\ge\frac{1}{2}a_0^2-\ln a_0-\frac{1}{\tau}M\\&\ge\frac{1}{2}(a_1^*)^2-\ln a_1^*-\frac{1}{\tau}M\\&=1>\frac{1}{2},\end{aligned}$$

hence $\tilde a_1$ in (5.7) is well-defined and $\tilde a_1>1$.

Define $\Gamma:\bar D\to X$, $(\tilde v,\tilde u)\mapsto(v,u)$ by solving

$$\begin{cases}\big[\big(k\tilde v-\frac{1}{\tilde v}\big)v_x\big]_x+\frac{1}{\tau}v_x=\frac{1}{\tilde v}-\frac{1}{\tilde u}-b(x),\\ \big[\big(\tilde u-\frac{1}{\tilde u}\big)u_x\big]_x-\frac{1}{\tau}u_x=\frac{1}{\tilde u}+b(x)-\frac{1}{\tilde v}, & x\in(0,1),\\ v(0)=v(1)=1,\ u(0)=a_0,\ u(1)=\tilde a_1,\end{cases}$$

then $\Gamma:\bar{D}\to X$ is a compact and continuous operator. Similar to the proof of Lemma 2.1, it suffices to prove that for any $\epsilon\in(0,1]$ and $(v,u)\in\partial D$, (v,u) can not be a solution of

$$\begin{cases}\big[\big(kv-\frac{1}{v}\big)v_x\big]_x+\frac{1}{\tau}v_x=\epsilon\big(\frac{1}{v}-\frac{1}{u}-b(x)\big),\\ \big[\big(u-\frac{1}{u}\big)u_x\big]_x-\frac{1}{\tau}u_x=\epsilon\big(\frac{1}{u}+b(x)-\frac{1}{v}\big), \qquad x\in(0,1),\\ v(0)=v(1)=1,\ u(0)=a_0,\ u(1)=a_0+\epsilon(a_1-a_0)\triangleq\hat{a}_1,\end{cases} \tag{5.9}$$

where $a_1>1$ is determined by

$$g_{a_0}(a_1)=\frac{1}{\tau}\int_0^1 v(x)-u(x)dx. \tag{5.10}$$

First, due to the maximum principle, we have that $v\geq 1>m$. Now we estimate the upper bound of v. Setting $E_1=(kv-\frac{1}{v})v_x+\frac{1}{\tau}v$, then

$$\begin{cases}\big(kv-\frac{1}{v}\big)v_x=E_1-\frac{1}{\tau}v,\\ E_{1x}=\epsilon\big(\frac{1}{v}-\frac{1}{u}-b(x)\big), \qquad x\in(0,1),\\ v(0)=v(1)=1.\end{cases} \tag{5.11}$$

Let $\bar{v}=v(T_1)=\sup\limits_{x\in[0,1]}v(x)$. If $T_1=0$ or $T_1=1$, then $\bar{v}=1<M$. If $T_1\in(0,1)$, then $v_x(T_1)=0$ and therefore $E_1(T_1)=\frac{1}{\tau}v(T_1)=\frac{1}{\tau}\bar{v}$. Hence for any $x\in(0,1)$ we have

$$E_1(x)=\int_{T_1}^x E_{1x}(\xi)d\xi+E_1(T_1)=\int_{T_1}^x \epsilon\big(\frac{1}{v(\xi)}-\frac{1}{u(\xi)}+b(\xi)\big)d\xi+E_1(T_1)\leq 2+\bar{b}+\frac{1}{\tau}\bar{v}.$$

Integrating the first equation of (5.11) over $(0,x)$, we have

$$\frac{k}{2}v^2(x)-\ln v(x)-\frac{k}{2}=\int_0^x E_1(\xi)-\frac{1}{\tau}v(\xi)d\xi\leq\int_0^1|E_1(\xi)|d\xi\leq 2+\bar{b}+\frac{1}{\tau}\bar{v}.$$

Due to the arbitrariness of $x\in(0,1)$, we obtain

$$\frac{1}{2}\bar{v}^2-\bar{v}-\frac{1}{\tau}\bar{v}\leq\frac{k}{2}\bar{v}^2-\ln\bar{v}-\frac{1}{\tau}\bar{v}\leq 2+\bar{b}+\frac{k}{2}\leq 3+\bar{b},$$

therefore

$$\bar{v}\leq 1+\frac{1}{\tau}+\sqrt{(1+\frac{1}{\tau})^2+2(3+\bar{b})}<M.$$

Now we estimate the upper bound of u. By the maximum principle, we have that $u\leq\max\{a_0,\ \hat{a}_1\}$. If $a_1\leq a_0$, then $\max\{a_0,\hat{a}_1\}=a_0$, hence $u\leq a_0<\Lambda$. If $a_1>a_0$, then $\max\{a_0,\hat{a}_1\}=\hat{a}_1$, hence $u\leq\hat{a}_1\leq a_1$. Since

$$g_{a_0}(a_1)=\frac{1}{\tau}\int_0^1 v(x)-u(x)dx\geq-\frac{1}{\tau}a_1,$$

we have that $u \le a_1 \le \bar{a}_1 < \Lambda$, where $\bar{a}_1$ is defined in (5.6).

Finally, we estimate the lower bound of u. Setting $E_2 = -(u - \frac{1}{u})u_x + \frac{1}{\tau}u$, then

$$\begin{cases} (u - \dfrac{1}{u})u_x = -E_2 + \dfrac{1}{\tau}u, \\ E_{2x} = \epsilon\big(\dfrac{1}{v} - \dfrac{1}{u} - b(x)\big), \qquad x \in (0,1), \\ u(0) = a_0,\ u(1) = \hat{a}_1. \end{cases} \tag{5.12}$$

Assume that $\dot{u} = u(T_2) = \inf_{x\in[0,1]} u(x)$. If $T_2 = 0$, then $\dot{u} = a_0 \ge \underline{u} > \lambda$. If $T_2 = 1$, then $a_1 < a_0$ and $\dot{u} = \hat{a}_1 \ge a_1$. Since

$$g_{a_0}(a_1) = \frac{1}{\tau}\int_0^1 v(x) - u(x)dx \le \frac{1}{\tau}M,$$

we have

$$\frac{1}{2}a_1^2 - \ln a_1 \ge \frac{1}{2}a_0^2 - \ln a_0 - \frac{1}{\tau}M.$$

Taking $a_1^* > 1$ such that $\frac{1}{2}(a_1^*)^2 - \ln a_1^* - \frac{1}{\tau}M = \frac{1}{2}\underline{u}^2 - \ln \underline{u}$, assuming that $a_0 \ge a_1^*$, then

$$\frac{1}{2}a_1^2 - \ln a_1 \ge \frac{1}{2}a_0^2 - \ln a_0 - \frac{1}{\tau}M \ge \frac{1}{2}(a_1^*)^2 - \ln a_1^* - \frac{1}{\tau}M = \frac{1}{2}\underline{u}^2 - \ln \underline{u},$$

hence $\dot{u} \ge a_1 \ge \underline{u} > \lambda$. If $T_2 \in (0,1)$, then $u_x(T_2) = 0$ and hence $E_2(T_2) = \frac{1}{\tau}u(T_2) = \frac{1}{\tau}\dot{u}$. For any $x \in (0,1)$, we have

$$E_2(x) = \int_{T_2}^x E_{2x}(\xi)d\xi + E_2(T_2) = \int_{T_2}^x \epsilon\big(\frac{1}{v(\xi)} - \frac{1}{u(\xi)} - b(\xi)\big)d\xi + E_2(T_2) \le 2 + \bar{b} + \frac{1}{\tau}\dot{u}.$$

Integrating the first equation in (5.12) over $(0,x)$, we have

$$\begin{aligned} \frac{1}{2}u^2(x) - \ln u(x) &= \int_0^x -E_2(\xi) + \frac{1}{\tau}u(\xi)d\xi + \frac{1}{2}a_0^2 - \ln a_0 \\ &\ge -\int_0^1 |E_2(\xi)|d\xi + \frac{1}{2}a_0^2 - \ln a_0 \\ &\ge -(2 + \bar{b} + \frac{1}{\tau}\dot{u}) + \frac{1}{2}a_0^2 - \ln a_0. \end{aligned}$$

Due to the arbitrariness of $x \in (0,1)$, we have

$$\frac{1}{2}\dot{u}^2 - \dot{u} + \frac{1}{\tau}\dot{u} \ge \frac{1}{2}a_0^2 - \ln a_0 - 2 - \bar{b},$$

therefore

$$\dot{u} \ge 1 - \frac{1}{\tau} + \sqrt{(1 - \frac{1}{\tau})^2 + a_0^2 - 2\ln a_0 - 4 - \bar{b}}.$$

As a result, there is a $a_2^* > 1$ such that for any $a_0 \ge a_2^*$,

$$\begin{aligned} \dot{u} &\ge 1 - \frac{1}{\tau} + \sqrt{(1 - \frac{1}{\tau})^2 + a_0^2 - 2\ln a_0 - 4 - \bar{b}} \\ &\ge 1 - \frac{1}{\tau} + \sqrt{(1 - \frac{1}{\tau})^2 + (a_2^*)^2 - 2\ln a_2^* - 4 - \bar{b}} \\ &\ge \underline{u} > \lambda. \end{aligned}$$

Take $a^*=\max\{a_1^*,a_2^*\}$ and assume that $a_0\geq a^*$. We have proved that if $(v,u)\in\partial D$ solves (5.9) for some $\epsilon\in(0,1]$, then

$$m<v(x)<M,\ \lambda<u(x)<\Lambda,\ x\in[0,1],$$

which contradicts $(v,u)\in\partial D$. Therefore, (5.5) admits a solution $(v,u)\in D$ which satisfies (5.4). Due to the regularity theory and the discussions above, we have that $(v,u)\in W^{2,+\infty}(0,1)\times W^{2,+\infty}(0,1)$ and $u\geq \underline{u}$ over $[0,1]$. ◻

Proof. **(Proof of Theorem 5.1.)** First, similar to the proofs in Theorem 2.1 and Theorem 3.1, we could obtain

$$\|(v_k-1)^2\|_{H_0^1(0,1)}\leq C,\quad \|u_k\|_{H^1(0,1)}\leq C. \tag{5.13}$$

Therefore, there is a subsequence of k (which is still denoted by k) such that as $k\to 1^+$,

$$\begin{aligned}&(v_k-1)^2\rightharpoonup (v-1)^2\ \text{ weakly in } H_0^1(0,1),\\ &u_k\rightharpoonup u\ \text{ weakly in } H^1(0,1),\end{aligned}$$

and

$$\|(v-1)^2\|_{H_0^1(0,1)}\leq C,\quad \|u\|_{H^1(0,1)}\leq C.$$

In similar fashion to [20], we could prove that $(v,u)\in C^{\frac12}[0,1]\times W^{2,+\infty}(0,1)$, $v>1$ over $(0,1)$ and (v,u) is a weak solution of (5.3). Also, we could verify that $u\geq\underline{u}$ over $[0,1]$ and (5.4) hold. ◻

5.2. Ill-posensess. This subsection is for the proof of non-existence of supersonic-vs-supersonic solution to (5.1) in certain cases as follows.

THEOREM 5.2. *There is no supersonic-vs-supersonic solution to (5.1) in the following two cases: (i) $\forall\tau>0$, $\bar b<\frac12$, and $a_0\gg1$; (ii) $b(x)\not\equiv0$, $\tau=+\infty$ and $a_0-1\ll1$.*

Proof. Assume that (v,u,E) is a supersonic-vs-supersonic solution of (5.1). First we prove (i). Assuming that $\bar v=v(T)=\sup\limits_{x\in[0,1]}v(x)$, then $E(T)=\frac{1}{\tau}\bar v$. Set $\underline{u}=\inf\limits_{x\in[0,1]}u(x)$. Multiplying the first equation of (5.1) by $((v-1)^2)_x$ and integrating over $(T,1)$ we have

$$\begin{aligned}\int_T^1\frac{v+1}{2v}|\big((v-1)^2\big)_x|^2dx&=\int_T^1\big(E-\frac{1}{\tau}v\big)\big((v-1)^2\big)_xdx\\ &=-\int_T^1(E-\frac{1}{\tau}v)_x(v-1)^2dx\\ &=\int_T^1(\frac1u+b-\frac1v)(v-1)^2dx+\frac1\tau\int_T^1(v-1)^2v_xdx\\ &=\int_T^1(\frac1u+b-\frac1v)(v-1)^2dx-\frac{(\bar v-1)^3}{3\tau}\\ &\leq\int_T^1(\frac{1}{\underline{u}}+\bar b)(v-1)^2dx\\ &\leq\int_T^1(\frac{1}{\underline{u}}+\bar b)^2dx+\frac14\int_T^1|\big((v-1)^2\big)_x|^2dx,\end{aligned}\tag{5.14}$$

where we have used Hölder's inequality and Poincare's inequality above. Since

$$\int_T^1 \frac{v+1}{2v}|\big((v-1)^2\big)_x|^2dx \geq \frac{1}{2}\int_T^1 |\big((v-1)^2\big)_x|^2dx, \tag{5.15}$$

we obtain by combining (5.15) with (5.14) that

$$\int_T^1 |\big((v-1)^2\big)_x|^2dx \leq 4\int_T^1 (\frac{1}{\underline{u}}+\bar{b})^2dx \leq 4(\frac{1}{\underline{u}}+\bar{b})^2.$$

Applying Sobolev's inequality we further have

$$v(x) \leq v(T) \leq 1+\sqrt{2(\frac{1}{\underline{u}}+\bar{b})}.$$

Setting $m=\frac{1}{\underline{u}}+\bar{b}$, then $v\leq 1+\sqrt{2m}$ and $\frac{1}{v}\geq \frac{1}{1+\sqrt{2m}}$ over $[T,1]$. By (5.14) we have

$$\int_T^1 \frac{v+1}{2v}|\big((v-1)^2\big)_x|^2dx \leq \int_T^1 (m-\frac{1}{1+\sqrt{2m}})(v-1)^2dx. \tag{5.16}$$

If $m<\frac{1}{2}$, then

$$m-\frac{1}{1+\sqrt{2m}}<0,$$

which contradicts (5.16) since the left side of (5.16) is nonnegative. Hence if $m=\frac{1}{\underline{u}}+\bar{b}<\frac{1}{2}$, there is no interior-supersonic-fully-supersonic solution to (5.1).

On the other hand, from (5.1) we have

$$\big(\frac{1}{2}u^2(x)-\ln u(x)\big)-\big(\frac{1}{2}a_0^2-\ln a_0\big)=-\int_0^x E(\xi)-\frac{1}{\tau}u(\xi)d\xi \geq -\int_0^x E(\xi)dx.$$

Since $v>1$ over $(0,1)$ and $v(1)=1$, we have that when x is close to 1^-, $E(x)<\frac{1}{\tau}v(x)\leq\frac{2}{\tau}$, hence $\underline{E}:=\min\limits_{x\in(0,1)} E(x)\leq\frac{2}{\tau}$. Assume that $\underline{E}=E(T^*)$, we have

$$E(x)=\underline{E}+\int_{T^*}^x E_x(\xi)d\xi<\frac{2}{\tau}+\bar{b}+2,$$

hence

$$\frac{1}{2}u^2(x)-\ln u(x)\geq \frac{1}{2}a_0^2-\ln a_0-(\frac{2}{\tau}+\bar{b}+2).$$

Therefore, for any $\tau>0$, $\bar{b}<\frac{1}{2}$ and $a_0\gg 1$ such that $m=\frac{1}{\underline{u}}+\bar{b}<\frac{1}{2}$, there is no interior-supersonic-fully-supersonic solution to (5.1).

Now we prove (ii). When $\tau=+\infty$, we have by (5.1) that

$$(\frac{1}{2}v^2-\ln v)-\frac{1}{2}=(\frac{1}{2}a_0^2-\ln a_0)-(\frac{1}{2}u^2-\ln u),$$

hence $u(1)=a_0$ and $u\leq a_0$ over $[0,1]$. Assuming that $x_1\in(0,1)$ is the first point such that u reaches its minimum and $x_2\in(0,1)$ the last point, then $u(x_1)\geq\underline{u}$, $u(x_2)\geq\underline{u}$ and $u_x<0$, $E>0$ over $(0,x_1)$, $u_x>0$, $E<0$ over $(x_2,1)$. From (5.1) we have

$$\frac{(u^2-1)(\frac{1}{u}+b-\frac{1}{v})}{u}u_x=EE_x. \tag{5.17}$$

Integrating (5.17) over $(0,x_1)$, we have

$$\begin{aligned}\frac{1}{2}E^2(0)&=\int_0^{x_1}\frac{(u^2-1)(\frac{1}{u}+b-\frac{1}{v})}{u}(-u_x)dx\\&\le\int_0^{x_1}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}(-u_x)dx\\&=\int_{u(x_1)}^{a_0}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}du\\&\le\int_1^{a_0}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}du.\end{aligned}$$

Similarly, integrating (5.17) over $(x_2,1)$, we get

$$\begin{aligned}\frac{1}{2}E^2(1)&=\int_{x_2}^{1}\frac{(u^2-1)(\frac{1}{u}+b-\frac{1}{v})}{u}u_x dx\\&\le\int_{x_2}^{1}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}u_x dx\\&=\int_{u(x_2)}^{a_0}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}du\\&\le\int_1^{a_0}\frac{(u^2-1)(\frac{1}{u}+\bar{b})}{u}du.\end{aligned}$$

If $a_0-1\ll 1$, then we have by (5.1) that

$$E(0)+\big(-E(1)\big)=\int_0^1 bdx-\int_0^1\frac{1}{v}-\frac{1}{u}dx\ge\int_0^1 bdx-(1-\frac{1}{a_0})\ge\frac{1}{2}\int_0^1 bdx>0.$$

Using the analysis similar to the previous sections, we could finish the proof. □

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REFERENCES

[1] U. Ascher, P. Markowich, P. Pietra, and C. Schmeiser, *A phase plane analysis of transonic solutions for the hydrodynamic semiconductor model*, Math. Mod. Meth. Appl. Sci., 1:347–376, 1991. 1

[2] M. Bae, B. Duan, and C. Xie, *Subsonic solutions for steady Euler-Poisson system in two-dimensional nozzles*, SIAM J. Math. Anal., 46:3455–3480, 2014. 1

[3] M. Bae, B. Duan, and C. Xie, *Subsonic flow for the multidimensional Euler-Poisson system*, Arch. Ration. Mech. Anal., 220:155–191, 2016. 1

[4] K. Bløtekjær, *Transport equations for electrons in two-valley semiconductors*, IEEE Trans. Electron Devices, 17:38–47, 1970. 1

[5] P. Degond and P. Markowich, *On a one-dimensional steady-state hydrodynamic model for semiconductors*, Appl. Math. Lett., 3:25–29, 1990. 1

[6] P. Degond and P. Markowich, *A steady state potential flow model for semiconductors*, Ann. Mat. Pura Appl., 4:87–98, 1993. 1

[7] F. Di Michele, M. Mei, B. Rubino, and R. Sampalmieri, *Thermal equilibrium solution to bipolar hybrid quantum hydrodynamical model*, J. Diff. Eqs., 263:1843–1873, 2017. 1

[8] F. Di Michele, M. Mei, B. Rubino, and R. Sampalmieri, *Stationary solutions to hybrid quantum hydrodynamical model of semiconductors in bounded domain*, Int. J. Numer. Anal. Model., 13(6):898–925, 2016. 1

[9] F. Di Michele, M. Mei, B. Rubino, and R. Sampalmieri, *Stationary solutions for a new hybrid quantum model for semiconductors with discontinuous pressure functional and relaxation time*, Math. Mech. Solids, 24(7):2096–2115, 2019. 1

[10] D. Donatelli, M. Mei, B. Rubino, and R. Sampalmieri, *Asymptotic behavior of solutions to Euler-Poisson equations for bipolar hydrodynamic model of semiconductors*, J. Diff. Eqs., 255:3150–3184, 2013. 1

[11] I.M. Gamba, *Stationary transonic solutions of a non-dimensional hydrodynamic model for semiconductors*, Comm. Part. Diff. Eqs., 17:553–577, 1992. 1

[12] I.M. Gamba and C.B. Morawetz, *A viscous approximation for a 2-D steady semiconductor or transonic gas dynamic flow: existence for potential flow*, Comm. Pure Appl. Math., 49:999–1049, 1996. 1

[13] I. Gasser, L. Hsiao, and H.-L. Li, *Large time behavior of solutions of the bipolar hydrodynamical model for semiconductors*, J. Diff. Eqs., 192:326–359, 2003. 1

[14] Y. Guo and W. Strauss, *Stability of semiconductor states with insulating and contact boundary condictions*, Arch. Ration. Mech. Anal., 179:1–30, 2005. 1

[15] H. Hu, M. Mei, and K. Zhang, *Relaxation limit in the bipolar semiconductor hydrodynamic model with non-constant doping profile*, J. Math. Anal. Appl., 448:1175–1203, 2017. 1

[16] F. Huang, M. Mei, and Y. Wang, *Large-time behavior of solutions to n-dimensional bipolar hydrodynamical model of semiconductors*, SIAM J. Math. Anal., 43:1595–1630, 2011. 1

[17] F. Huang, M. Mei, and Y. Wang, *Long-time behavior of solutions for bipolar hydrodynamic model of semiconductors with boundary effects*, SIAM J. Math. Anal., 44:1134–1164, 2012. 1

[18] J.W. Jerome, *Steady Euler-Poisson systems: a differential/integral equation formulation with general constitutive relations*, Nonlinear Anal., 71:e2188–e2193, 2009. 1

[19] A. Jüngel, *Quasi-hydrodynamic Semiconductor Equations*, Progr. Nonl. Diff. Eqs. Appl., Birkháuser Verlag, Besel, Boston, Berlin, 41, 2001. 1

[20] J. Li, M. Mei, G. Zhang, and K. Zhang, *Steady hydrodynamic model of semiconductors with sonic boundary: (I) subsonic doping profile*, SIAM J. Math. Anal., 49:4767–4811, 2017. 1, 1, 2.1, 2.1, 2.1, 3.1, 3.1, 4.1, 4.1, 5.1

[21] J. Li, M. Mei, G. Zhang, and K. Zhang, *Steady hydrodynamic model of semiconductors with sonic boundary: (II) supersonic doping profile*, SIAM J. Math. Anal., 50:718–734, 2018. 1, 4.2

[22] T. Luo, J. Rauch, C. Xie, and Z. Xin, *Stability of transonic solutions for one-dimensional Euler-Poisson equations*, Arch. Ration. Mech. Anal., 2002:787–827, 2011. 1

[23] T. Luo and Z. Xin, *Transonic shock solutions for a system of Euler-Poisson equations*, Commun. Math. Sci., 10:419–462, 2012. 1

[24] P. Markowich, C. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer, Wien, New York, 1989. 1

[25] M. Mei, B. Rubino, and R. Sampalmieri, *Asymptotic behavior of solutions to the bipolar hydrodynamic model of semiconductors in bounded domain*, Kinet. Relat. Model., 5:537–550, 2012. 1

[26] S. Nishibata and M. Suzuki, *Asymptotic stability of a stationary solution to a hydrodynamic model of semiconductors*, Osaka J. Math., 44:639–665, 2007. 1

[27] Y. Peng and I. Violet, *Example of supersonic solutions to a steady state Euler-Poisson system*, Appl. Math. Latt., 19:1335–1340, 2006. 1

[28] M.D. Rosini, *A phase analysis of transonic solutions for the hydrodynamic semiconductor model*, Quart. Appl. Math., 63:251–268, 2005. 1

[29] N. Tsuge, *Existence and uniqueness of stationary solutions to a one-dimensional bipolar hydrodynamic model of semiconductors*, Nonlinear Anal., 73:779–787, 2010. 1

[30] H. Yu, *On the stationary solutions of multi-dimensional bipolar hydrodynamic model of semiconductors*, Appl. Math. Latt., 64:108–112, 2017. 1

[31] F. Zhou and Y. Li, *Existence and some limits of stationary solutions to a one-dimensional bipolar Euler-Poisson system*, J. Math. Anal. Appl., 351:480–490, 2009. 1