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ON THE FREE BOUNDARY PROBLEM OF 1D COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH HEAT CONDUCTIVITY DEPENDENT OF TEMPERATURE*

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Abstract. The free boundary problem of one-dimensional heat conducting compressible Navier-Stokes equations with large initial data is investigated. We obtain the global existence of strong solution under stress-free boundary condition along the free surface, where the heat conductivity depends on temperature ($\kappa=\overline{\kappa}\theta^b$, $b\in(0,\infty)$) and the viscosity coefficient depends on density ($\mu=\overline{\mu}(1+\rho^a)$, $a\in[0,\infty)$). Moreover, the large-time behavior of the free boundary for the full compressible Navier-Stokes equations is also considered when the viscosity is constant and it is first shown that the interfaces which separate the gas from vacuum will expand outwards at an algebraic rate in time for all $\gamma>1$.

Keywords. Compressible Navier-Stokes equations; temperature-dependent heat conductivity; free boundary; global strong solution, large-time behavior.

AMS subject classifications. 35Q30; 35R35; 76N10.

1. Introduction

The free boundary problem of one dimensional compressible heat-conducting Navier-Stokes equations can be described in the Eulerian coordinates as follows:

$$\begin{cases}\rho_\tau+(\rho v)_y=0,\\ (\rho v)_\tau+(\rho v^2+P)_y=(\mu v_y)_y,\\ (\rho(e+\frac{1}{2}v^2))_\tau+(\rho(e+\frac{1}{2}v^2)v)_y+(Pv)_y=(\mu v v_y)_y+(\kappa e_y)_y,\end{cases}\tag{1.1}$$

for $0\le y\le a(\tau)$, $\tau>0$, with the initial condition

$$(\rho,v,e)\big|_{\tau=0}=(\rho_0,v_0,e_0)\text{ for } y\in[0,a],\tag{1.2}$$

and boundary condition

$$e_y(d,\tau)=0,\quad d=0,a(\tau);\quad (P-\mu v_y)(a(\tau),\tau)=0,\quad v(0,\tau)=0,\quad \tau\ge 0,\tag{1.3}$$

where ρ, v, e and P denote the density, the fluid velocity, the internal energy and the pressure respectively; μ and κ are the viscosity coefficient and the heat conductivity coefficient. In this paper, we focus on ideal polytropic gas and the constitution relation reads

$$P(\rho,\theta)=R\rho\theta=Ae^{\frac{S}{c_v}}\rho^\gamma,\quad e=c_v\theta,\quad c_v=\frac{R}{\gamma-1},\quad \gamma>1,\tag{1.4}$$

where R is the ratio of the ideal fluid constant over the heat capacity, S is the specific entropy and c_v is the heat capacity. $a(\tau)$ is the free boundary defined by

$$\begin{cases}\frac{da(\tau)}{d\tau}=v(a(\tau),\tau),\quad \tau>0\\ a(0)=a,\end{cases}\tag{1.5}$$

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which is the interface separating the gas from the vacuum.

To solve the free boundary problem (1.1)-(1.3), it is convenient to convert the free boundaries to the fixed boundaries by using Lagrangian mass coordinates. This means, let

$$x=\int_0^y \rho(z,\tau)dz, \quad t=\tau, \tag{1.6}$$

then the free boundary $y=a(\tau)$ becomes $x=\int_0^{a(\tau)}\rho(z,\tau)dz=\int_0^a \rho_0(z)dz$ by the conservation of mass. Without loss of generality, we assume $\int_0^a \rho_0(z)dz=1$. Let $u=\frac{1}{\rho}$ be specific volume, then $P=\frac{R\theta}{u}$. Thus the free boundary problem (1.1)-(1.3) becomes

$$\begin{cases} u_t-v_x=0,\\ v_t+P_x=[\frac{\mu v_x}{u}]_x,\\ (e+\frac{1}{2}v^2)_t+(Pv)_x=[\frac{\kappa\theta_x+\mu v v_x}{u}]_x, \end{cases} \tag{1.7}$$

for $(x,t)\in[0,1]\times(0,+\infty)$, with the initial condition

$$(u,v,\theta)\big|_{t=0}=(u_0,v_0,\theta_0) \text{ for } x\in[0,1], \tag{1.8}$$

and the boundary condition

$$\theta_x(d,t)=0,\ d=0,1,\ \Big(P-\frac{\mu v_x}{u}\Big)(1,t)=0,\ v(0,t)=0, \quad t\geq 0. \tag{1.9}$$

when μ and κ are positive constants, the existence of strong solutions to (1.1) has been successfully studied by many mathematicians. The local theories were established long ago, see [9, 11, 15]. Kazhikhov and Shelukhin [10] first obtained global existence and uniqueness of smooth solution for arbitrarily large and smooth initial data under Dirichlet boundary condition. Such results have been further generalized to nonlinear thermoviscoelasticity by [2, 3], and to viscous heat-conductive real gases by [6, 8, 12, 13]. It is noted that in the above results, μ is independent of θ, and heat conductivity is allowed to depend on temperature in a special way with a positive lower bound and balanced with corresponding constitution relations.

When one derives the compressible Navier-Stokes Equations (1.1) from the celebrated Boltzmann equation for the monatomic gas with a slab symmetry by using the Chapman-Enskog expansion, then the viscosity coefficient μ and the heat conductivity coefficient κ are functions of temperature. The functional dependence is the same for both coefficients as

$$\mu=\overline{\mu}\theta^b, \quad \kappa=\overline{\kappa}\theta^b, \qquad b\in(\frac{1}{2},\infty). \tag{1.10}$$

where $\overline{\mu}$ and $\overline{\kappa}$ are positive constants. See Chapman and Cowling [1] for a thorough discussion of these issues. With some smallness assumptions, considering the one-dimensional full compressible Navier-Stokes equations for ideal polytropic gas whose viscosity coefficient and heat conductivity coefficient satisfy $\mu=\widetilde{\mu}h(u)\theta^\alpha, \kappa=\widetilde{\kappa}h(u)\theta^\alpha$, Liu and Yang et al. in [17] obtained the global non-vacuum classical solution with smallness mechanism (i.e., $\gamma-1$ small), and later Wang and Zhao in [19] obtained the global non-vacuum classical solution with smallness assumptions for $|\alpha|$. However, if both viscosity coefficients and heat conductivity coefficient depend on temperature, the well-posedness of solutions to (1.1) is still open. Note that, if the viscosity is a positive constant and only the heat conductivity coefficient depends on temperature, the framework of Kazhikhov and Shelukhin [10] works. As an example of this direction,

- Jenssen and Karper [7] proved the global existence of a weak solution to initial-boundary value problem (IBVP) (1.1) under the assumption
$$\mu=\overline{\mu},\quad \kappa=\overline{\kappa}\theta^b,\qquad b\in[0,\frac{3}{2}).$$
- Wen and Zhu [18] proved the existence of global classical solutions of (1.1) with vacuum under the boundary condition $\theta_x(d,t)=0, v(d,t)=0,\ d=0,1$, where $\kappa\in C^2[0,\infty)$ satisfies $C_6(1+\theta^q)\le\kappa(\theta)\le C_7(1+\theta^q),\ \ q\ge 2$.
- Pan and Zhang [14] proved the existence of global strong solutions of (1.7) with $b\in[0,\infty)$ under the boundary condition $\theta_x(d,t)=0, v(d,t)=0,\ d=0,1.$ where boundary condition $v(d,t)=0,\ d=0,1$ offers conservative quality $\int_0^1 u(x,t)dx=\int_0^1 u_0(x)dx$ which plays an important role when getting the bound of u.
- Duan, Guo and Zhu [4] studied the same problem of (1.7) with $b\in(0,\infty)$ under the boundary $\theta_x(d,t)=0,\ \left(P-\frac{\mu v_x}{u}\right)(d,t)=0, d=0,1.$ Although the boundary condition $\left(P-\frac{\mu v_x}{u}\right)(d,t)=0, d=0,1$ does not yield $\int_0^1 u(x,t)dx=\int_0^1 u_0(x)dx$, but gives $\int_0^1 v(x,t)dx=\int_0^1 v_0(x)dx$ which yields $\int_0^1 u(x,t)dx\le C$. This estimate together with $C\le u(x,t)$ implies $C^{-1}\le\int_0^1 u(x,t)dx\le C$.

However, compared with results in [4] and [14], stress-free boundary condition $\left(P-\frac{\mu v_x}{u}\right)(1,t)=0, v(0,t)=0$ can neither yield $\int_0^1 u(x,t)dx=\int_0^1 u_0(x)dx$ nor imply $\int_0^1 v(x,t)dx=\int_0^1 v_0(x)dx$. Hence, our main goal in this paper is to establish the global existence of strong solutions and consider the large-time behavior of the free boundary to free boundary problems (1.1)-(1.3) with assumptions
$$\mu=\overline{\mu}(1+u^{-a}),\qquad a\in[0,\infty),\quad \kappa=\overline{\kappa}\theta^b,\qquad b\in(0,\infty). \tag{1.11}$$
Where $\overline{\mu}$ and $\overline{\kappa}$ are positive constants.

Notations:

(1) $I=[0,1]$, $\partial I=\{0,1\}$, $Q_T=I\times[0,T]$ for $T>0$.

(2) For $p\ge 1$, $L^p=L^p(I)$ denotes the L^p space with the norm $\|\cdot\|_{L^p}$. For $k\ge 1$ and $p\ge 1$, $W^{k,p}=W^{k,p}(I)$ denotes the Sobolev space, whose norm is denoted as $\|\cdot\|_{W^{k,p}}$, $H^k=W^{k,2}(I)$.

(3) Throughout this paper, the same letter C (sometimes used as $C(X)$ to emphasize the dependence of C on X) denotes various generic positive constants.

The following are the main results of this paper.

THEOREM 1.1. *Suppose that μ and κ satisfy (1.11) for some positive constants $\overline{\mu}$ and $\overline{\kappa}$. If the initial data $(u_0,v_0,\theta_0)(x)$ is compatible with the boundary conditions, satisfying*
$$(u_0,v_0,\theta_0)(x)\in H^1\times H^2\times H^2, \tag{1.12}$$
and there are constants $\underline{u}$, $\overline{u}$, $\underline{\theta}$, $\overline{\theta}$ such that
$$0<\underline{u}\le u_0(x)\le\overline{u},\quad 0<\underline{\theta}\le\theta_0(x)\le\overline{\theta}. \tag{1.13}$$
Then for any $T>0$, there exists a unique global strong solution (u,v,θ) to the initial-boundary value problems (1.7)-(1.9) satisfying
$$\begin{cases} C^{-1}\le u(x,t)\le C,\quad C^{-1}\le\theta(x,t)\le C,\\ \|(u,v,\theta)(.,t)\|_{H^1}^2+\int_0^t\|(u,v,\theta)(.,s)\|_{H^1}^2ds\le C,\\ \|(v,\theta)(.,t)\|_{H^2}^2+\int_0^t\|(u_{xt},v_{xt},v_{xx},\theta_{xt},\theta_{xx})(.,s)\|_{L^2}^2ds\le C,\end{cases} \tag{1.14}$$

where $C>0$ is some finite constant depending on initial data and T.

THEOREM 1.2. *In addition to the assumptions of Theorem 1.1, if the viscosity $\mu=$ constant >0 and the initial total entropy satisfies*

$$\int_0^a \rho_0 S_0 dy \doteq k_0 > 0,$$

where $S(x,t)$ is the entropy of the fluid and $S_0=S(x,0)$. Then we have

$$M(t)=\max_{s\in[0,t]}(a(s)-0)\geq \begin{cases} C(1+t)^{1-\frac{1}{\gamma}}, & 1<\gamma<2,\\ C(1+t)^{\frac{1}{\gamma}}, & \gamma\geq 2. \end{cases} \tag{1.15}$$

REMARK 1.1. It should be noted that the Theorem 1.1 also holds for constant viscosity.

REMARK 1.2. The additional assumption in Theorem 1.2 where we require the initial entropy has a positive bound is natural due to the the second law of thermodynamics, and these expanding rates also hold for the constant heat conductivity and other various free boundary conditions just with some small modifications in the proof.

REMARK 1.3. To our best knowledge, although the large-time behavior of the free boundary for the isentropic compressible Navier-Stokes equations has been studied by many authors (see [5], [16]), however the similar results for the full compressible Navier-Stokes equations are very few, our result in Theorem 1.2 is the first one to give the expanding rate of the free boundary by using the entropy variable.

The existence and uniqueness of local-in-time solution can be obtained by a standard Banach fixed point argument due to the contraction of the solution operators defined by the linearized problem, c.f. [9,11] and [15]. As a special case of the result in [15], the following lemma gives the local existence for the purpose of our problem.

LEMMA 1.1. *If (1.11)-(1.13) hold, and the initial data is compatible with boundary conditions, then there exists a unique local strong solution (u,v,θ) to (1.7) on $[0,1]\times[0,T_*]$, for T_* depending on the initial data satisfying*

$$\begin{cases} C^{-1}\leq u(x,t)\leq C, \quad C^{-1}\leq \theta(x,t)\leq C,\\ \|(u,v,\theta)(.,t)\|_{H^1}^2+\int_0^t\|(u,v,\theta)(.,s)\|_{H^1}^2 ds\leq C,\\ \|(v,\theta)(.,t)\|_{H^2}^2+\int_0^t\|(u_{xt},v_{xt},v_{xx},\theta_{xt},\theta_{xx})(.,s)\|_{L^2}^2 ds\leq C. \end{cases} \tag{1.16}$$

The rest of the paper is organized as follows. In Section 2, we give some a priori estimates. In Section 3, first, based on the local existence of the solutions and the a priori estimates in Section 2, we prove Theorem 1.1 by a standard continuity argument and then we give the proof of Theorem 1.2.

2. A priori estimates

In this section, we will perform a sequence of estiamtes which are stated in the following as lemmas to prove Theorem 1.1. Furthermore, we get a unique global strong solution of (1.7)-(1.9) by using some *a priori* estimates of the solution based on the local existence. We now assume that $(u,v,\theta)(x,t)$ is the unique global strong solution of (1.7) defined on $[0,1]\times[0,T]$ for any $T>0$. For simplicity of presentation, we will denote $\overline{\mu}=\overline{\kappa}=c_v=1$.

LEMMA 2.1. *There exists a constant C such that*

$$\int_I (\theta+\frac{1}{2}v^2)dx=\int_I (\theta_0+\frac{1}{2}v_0^2)dx=E_0, \tag{2.1}$$

and there exists $\xi(t)\in(0,1)$ and a constant C, such that

$$\theta(\xi(t),t)\le C. \tag{2.2}$$

Proof. Integrating $(1.7)_3$ over Q_T, using integration by parts and (1.9), we get (2.1). From (2.1), we have $\int_I \theta dx\le C$, then by the mean value theorem, there exists $\xi(t)\in(0,1)$, such that

$$\theta(\xi(t),t)=\int_I \theta dx\le C.$$

This completes the proof of Lemma 2.1. ∎

LEMMA 2.2. *There exists a constant C such that*

$$C^{-1}\le u(x,t)\le C\left(1+\int_0^t \theta(x,s)ds\right), \tag{2.3}$$

and

$$\int_0^1 |\ln u|dx\le C. \tag{2.4}$$

Proof. Using the mass equation, one can rewrite the momentum equation as

$$v_t+\left(\frac{\theta}{u}\right)_x=\left(\mu(u)\frac{u_t}{u}\right)_x. \tag{2.5}$$

Integrating this equation in space from x to 1, and then integrating this equation in time over $[0,t]$ for any $t\in[0,T]$, we get

$$\ln u-\frac{u^{-a}}{a}=\ln u_0-\frac{u_0^{-a}}{a}+\int_0^t \frac{\theta}{u}(x,s)ds-\int_x^1 (v(y,t)-v_0(y))dy, \tag{2.6}$$

taking exponential on both sides of (2.6), then we have

$$\frac{e^{\int_0^t \frac{\theta}{u}(x,s)ds}}{u(x,t)}=\frac{1}{u_0(x)}\frac{B_1(x,t)B_3(x)}{B_2(x,t)}, \tag{2.7}$$

where

$$B_1(x,t)=\exp\left(\int_x^1 (v(y,t)-v_0(y))dy\right), B_2(x,t)=\exp\left(\frac{u^{-a}}{a}\right), B_3(x)=\exp\left(\frac{u_0^{-a}}{a}\right). \tag{2.8}$$

Mutiplying (2.7) with θ and integrating in time, we get

$$e^{\int_0^t \frac{\theta}{u}(x,s)ds}=1+\frac{B_3(x)}{u_0(x)}\int_0^t \frac{B_1(x,s)}{B_2(x,s)}\theta(x,s)ds, \tag{2.9}$$

plugging (2.9) into (2.7), we get

$$u(x,t)=\frac{u_0(x)B_2(x,t)}{B_1(x,t)B_3(x)}\left(1+\frac{B_3(x)}{u_0(x)}\int_0^t \frac{B_1(x,s)}{B_2(x,s)}\theta(x,s)ds\right). \tag{2.10}$$

Note

$$\left|\int_x^1 (v_0(y)-v(y,t))dy\right| \leq 2\left(\int_0^1 v^2(y,t)dy\right)^{\frac{1}{2}} + 2\left(\int_0^1 v_0^2(y)dy\right)^{\frac{1}{2}} \leq C, \tag{2.11}$$

therefore, from (2.8) and (1.13), there exists a constant $C>0$ such that

$$C^{-1} \leq B_1(x,t), B_3(x) \leq C, \; 1 \leq B_2(x,t) \tag{2.12}$$

which together with (2.10), (1.13) and nonnegative of $\theta(x,s)$ implies that

$$0 < C^{-1} \leq u, \; 1 \leq B_2(x,t) \leq C, \tag{2.13}$$

consequently, we have

$$C^{-1} \leq u \leq C\left(1 + \int_0^t \theta(x,s)ds\right).$$

Moreover, in combination with (2.1), (2.6), (2.11) and (2.13), we have

$$\int_0^1 |\ln u| dx \leq C.$$

The proof of Lemma 2.2 is completed. ∎

The following lemma shows that the absolute temperature θ stays positive all the time. The proof of Lemma 2.3 is almost exactly the same as Lemma 2.2 in [4]. Here for the convenience of the reader and the completeness of the paper, we state the details.

Lemma 2.3. *There exists a constant C such that for any $p>2$*

$$\left\|\frac{1}{\theta}\right\|_{L^\infty(L^p)} + \int_{Q_T}\left(\frac{\mu(u)v_x^2}{u\theta^{p+1}} + (p-1)\frac{\theta^b\theta_x^2}{u\theta^{p+2}}\right)dxdt \leq C. \tag{2.14}$$

In particular, we have

$$0 < C \leq \theta(x,t), \quad \textit{for any} \;\; (x,t) \in Q_T. \tag{2.15}$$

Proof. Using $(1.7)_1$ and $(1.7)_2$, we rewrite $(1.7)_3$ as

$$\theta_t = \left(\frac{\theta^b\theta_x}{u}\right)_x + \frac{\mu(u)v_x^2}{u} - \frac{\theta v_x}{u}. \tag{2.16}$$

Multiplying above equation by $-p\frac{1}{\theta^{p+1}}$, then integrating it over $[0,1]$, using integration by parts and the Cauchy inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\int\frac{1}{\theta^p}dx + p(p+1)\int\frac{\theta^b\theta_x^2}{u\theta^{p+2}}dx + p\int\frac{\mu(u)v_x^2}{\theta^{p+1}u}dx = \int p\frac{v_x}{\theta^p u}\\
\leq &\varepsilon p\int\frac{v_x^2}{\theta^{p+1}u}dx + Cp\int\frac{\theta^2}{\theta^{p+1}u}dx\\
\leq &\varepsilon p\int\frac{v_x^2}{\theta^{p+1}u}dx + Cp\left(\int\frac{1}{\theta^p}dx\right)^{\frac{p-1}{p}},
\end{aligned} \tag{2.17}$$

due to the Grönwall inequality, which yields

$$\left\|\frac{1}{\theta}\right\|_{L^\infty(L^p)} \le C,$$

where C is uniform of index p, so letting $p\to+\infty$, we have

$$\left\|\frac{1}{\theta}\right\|_{L^\infty(L^\infty)} \le C \Rightarrow \theta(x,t)\ge C>0, \quad \text{for any} \quad (x,t)\in Q_T.$$

Then the proof of Lemma 2.3 is completed. ∎

LEMMA 2.4. *For any* $0<\varepsilon<\min\{1,b\}$, *the following estimates hold:*

$$\int_{Q_T} \frac{\theta^b\theta_x^2}{u\theta^{1+\varepsilon}}dxdt \le C, \tag{2.18}$$

$$\int_{Q_T} \theta^{b+3-\varepsilon}dxdt + \int_0^T \max_{[0,1]}\theta^{b+2-\varepsilon}dt \le C, \tag{2.19}$$

in particular, when $\varepsilon=1$, *it also holds that*

$$\int_{Q_T} \frac{\theta^b\theta_x^2}{u\theta^2}dxdt + \int_{Q_T} \frac{\mu(u)v_x^2}{u\theta}dxdt \le C. \tag{2.20}$$

Proof. Multiplying (2.16) by $\frac{1}{\theta^\varepsilon}$, integrating it over Q_T, and using integration by parts, it turns out

$$\varepsilon\int_{Q_T} \frac{\theta^b\theta_x^2}{u\theta^{\varepsilon+1}}dxdt + \int_{Q_T} \frac{\mu(u)v_x^2}{u\theta^\varepsilon}dxdt = \frac{1}{1-\varepsilon}\int_I (\theta^{1-\varepsilon}-\theta_0^{1-\varepsilon})dx + \int_{Q_T} \frac{\theta^{1-\varepsilon}v_x}{u}dxdt. \tag{2.21}$$

Using the fact that $1-\varepsilon\in(0,1)$, Young's inequality and (2.1), it is clear that

$$\int_I \theta^{1-\varepsilon}dx \le \int_I \theta dx + C \le C.$$

This, together with (2.21), (2.3) and Young's inequality, gives

$$\begin{aligned}
\varepsilon\int_{Q_T} \frac{\theta^b\theta_x^2}{u\theta^{\varepsilon+1}}dxdt + \int_{Q_T} \frac{\mu(u)v_x^2}{u\theta^\varepsilon}dxdt &\le C + \int_{Q_T} \frac{\mu(u)\theta^{1-\varepsilon}v_x}{\mu(u)u}dxdt \\
&\le \frac{1}{2}\int_{Q_T} \frac{\mu(u)v_x^2}{u\theta^\varepsilon}dxdt + 2\int_{Q_T} \frac{\theta^{2-\varepsilon}}{\mu(u)u}dxdt + C \\
&\le \frac{1}{2}\int_{Q_T} \frac{\mu(u)v_x^2}{u\theta^\varepsilon}dxdt + C\int_0^T \max_{0\le x\le 1}\theta dt + C.
\end{aligned}$$

Thus, we have

$$\int_{Q_T} \frac{\theta^b\theta_x^2}{u\theta^{\varepsilon+1}}dxdt + \int_{Q_T} \frac{v_x^2}{u\theta^\varepsilon}dxdt \le C\int_0^T \max_{0\le x\le 1}\theta dt + C. \tag{2.22}$$

On the other hand, using (2.2), (2.3), (2.22) and Hölder's inequality, we have

$$\begin{aligned}\int_{Q_T}\theta^{b+3-\varepsilon}dxdt &\le C\int_0^T \max_{0\le x\le 1}\theta^{b+2-\varepsilon}dt\\ &\le C+C\int_0^T \max_{0\le x\le 1}\left(\theta^{\frac{2+b-\varepsilon}{2}}(x,t)-\theta^{\frac{2+b-\varepsilon}{2}}(\xi(t),t)\right)^2 dt\\ &\le C+C\int_0^T\left(\int_I \theta^{\frac{b-\varepsilon}{2}}|\theta_x|dx\right)^2 dt\\ &\le C+C\int_0^T\left(\int_I \frac{\theta^b\theta_x^2}{u\theta^{\varepsilon+1}}dx\right)\left(\int_I u\theta dx\right)dt\\ &\le C+C\max_{Q_T}u\int_{Q_T}\frac{\theta^b\theta_x^2}{u\theta^{\varepsilon+1}}dxdt\\ &\le C+C\left(\int_0^T \max_{0\le x\le 1}\theta dt\right)^2\\ &\le C+C\int_0^T \max_{0\le x\le 1}\theta^2 dt.\end{aligned}$$

For any ε small enough, such that $b-\varepsilon>0$, Young's inequality yields

$$\int_0^T \max_{0\le x\le 1}\theta^{b+2-\varepsilon}dt \le C+\int_0^T \max_{0\le x\le 1}\theta^2 dxdt \le C+\frac{1}{2}\int_0^T \max_{0\le x\le 1}\theta^{b+2-\varepsilon}dt,$$

and therefore

$$\int_0^T \max_{0\le x\le 1}\theta^{b+2-\varepsilon}dxdt\le C. \tag{2.23}$$

Multiplying (2.16) by $\frac{1}{\theta}$, integrating it over Q_T, and using integration by parts, (1.13), (2.4) and (2.1), we have

$$\int_{Q_T}\frac{\theta^b\theta_x^2}{u\theta^2}dxdt+\int_{Q_T}\frac{\mu(u)v_x^2}{u\theta}dxdt=\int_I(\ln u+\ln\theta)dx-\int_I(\ln u_0+\ln\theta_0)dx\le C.$$

Using (2.22)-(2.23) and Hölder's inequality, we can complete the proof of Lemma 2.4. □

Combining (2.3) with (2.19), we easily get the following lemma.

Lemma 2.5. *There exists a constant C such that*

$$\max_{Q_T}u\le C, \tag{2.24}$$

$$\int_{Q_T}\theta^{b-1-\varepsilon}\theta_x^2dxdt\le C. \tag{2.25}$$

Proof. Combining (2.3) with (2.19), we easily get (2.24). The estimate (2.25) can be obtained directly by (2.18) and (2.24). This proves Lemma 2.5. □

Lemma 2.6. *There exists a constant C such that*

$$\sup_{0\le t\le T}\int_I v^2dx+\int_{Q_T}v_x^2dxdt\le C. \tag{2.26}$$

Proof. Multiplying $(1.7)_2$ by v, integrating it over Q_T, using integration by parts, Cauchy inequality, (2.3) and (2.19), we have

$$\begin{aligned}\sup_{0\le t\le T}\int_I v^2dx+\int_{Q_T}v_x^2dxdt&\le C+C\int_{Q_T}P^2dxdt\\&\le C+C\int_{Q_T}\theta^2dxdt\\&\le C.\end{aligned}\tag{2.27}$$

This completes the proof of Lemma 2.6. □

Lemma 2.7. *There exists a constant C such that*

$$\sup_{0\le t\le T}\int_I u_x^2dx\le C.$$

Proof. One can rewrite (2.5) as

$$((\ln u-\frac{1}{a}u^{-a})_x-v)_t=\left(\frac{\theta}{u}\right)_x.\tag{2.28}$$

We multiply (2.28) by $2((\ln u-\frac{1}{a}u^{-a})_x-v)$ and integrate the resulting equation over Q_T, yields

$$\begin{aligned}&\int_I\left((\ln u-\frac{1}{a}u^{-a})_x-v\right)^2dx-\int_I\left((\ln u_0-\frac{1}{a}u_0^{-a})_x-v_0\right)^2dx\\=&2\int_{Q_T}\left(\frac{\theta_x}{u}-\frac{\theta}{u}(\ln u)_x\right)\left((\ln u-\frac{1}{a}u^{-a})_x-v\right)dxdt.\end{aligned}\tag{2.29}$$

By the lower bound of θ and (2.25), the right-hand side can be controlled as follows,

$$\begin{aligned}&\int_{Q_T}\frac{\theta_x}{u}\left((\ln u-\frac{1}{a}u^{-a})_x-v\right)dxdt\\\le&C\int_{Q_T}\theta^{b-1-\varepsilon}\theta_x^2dxdt+C\int_0^T\max_{0\le x\le1}\theta^{1+\varepsilon}\int_I\left((\ln u-\frac{1}{a}u^{-a})_x-v\right)^2dxdt\\\le&C+C\int_0^T\max_{0\le x\le1}\theta^{1+\varepsilon}\int_I\left((\ln u-\frac{1}{a}u^{-a})_x-v\right)^2dxdt.\end{aligned}\tag{2.30}$$

On the other hand, using (2.1) and (2.19), we have

$$\begin{aligned}&\int_{Q_T}-\frac{\theta}{u}(\ln u)_x\left((\ln u-\frac{1}{a}u^{-a})_x-v\right)dxdt\\=&-\int_{Q_T}\frac{\theta}{u}(\ln u)_x^2dxdt+\int_{Q_T}\frac{\theta}{au}(\ln u)_x(u^{-a})_xdxdt+\int_{Q_T}\frac{\theta}{u}(\ln u)_xvdxdt\\\le&-\frac{1}{2}\int_{Q_T}\frac{\theta}{u}(\ln u)_x^2dxdt-\int_{Q_T}\frac{\theta(u_x)^2}{u^{3+a}}dxdt+C\int_0^T\max_{0\le x\le1}\theta\int_I v^2dxdt\\\le&-\frac{1}{2}\int_{Q_T}\frac{\theta}{u}(\ln u)_x^2dxdt-\int_{Q_T}\frac{\theta(u_x)^2}{u^{3+a}}dxdt+C.\end{aligned}\tag{2.31}$$

Substituting (2.30)-(2.31) into (2.29), we have

$$\begin{aligned}&\int_I\left((\ln u-\frac{1}{a}u^{-a})_x-v\right)^2dx+\frac{1}{2}\int_{Q_T}\frac{\theta}{u}(\ln u)_x^2dxdt+\int_{Q_T}\frac{(u_x)^2}{u^{3+2a}}dxdt\\ \le& C\int_0^T\max_{0\le x\le 1}\theta^{1+\varepsilon}\int_I\left((\ln u-\frac{1}{a}u^{-a})_x-v\right)^2dxdt+C,\end{aligned}\tag{2.32}$$

which along with the Grönwall inequality and (2.19) gives

$$\int_I\left((\ln u-\frac{1}{a}u^{-a})_x-v\right)^2dx\le C\quad\text{for}\quad\varepsilon\le\frac{1+b}{2},$$

which together with (2.1) implies that

$$\int_I\left[\frac{u_x^2}{u^2}+2\frac{u_x^2}{u^{a+2}}+\frac{u_x^2}{u^{2(a+1)}}\right]dx\le C\int_I\left((\ln u-\frac{1}{a}u^{-a})_x-v\right)^2dx\le C,$$

this together with (2.24) gives the proof of Lemma 2.7. □

In order to obtain higher norms and the upper bound of θ, similar to [14], we introduce two useful functionals as follows:

$$Z=\sup_{0\le t\le T}\int_I v_{xx}^2dx,\quad Y=\sup_{0\le t\le T}\int_I\theta^{2b}\theta_x^2dx.\tag{2.33}$$

LEMMA 2.8. *There exists a constant C such that*

$$\max_{Q_T}\theta\le C+CY^{\frac{1}{2b+3}},\tag{2.34}$$

$$\max_{Q_T}|v_x|\le C+CZ^{\frac{3}{8}}.\tag{2.35}$$

Proof. Using $W^{1,1}\hookrightarrow L^\infty$ and the Young inequality, we get

$$\begin{aligned}\max_{0\le x\le 1}\theta^{2b+2}\le& C\int_I\theta^{2b+2}dx+C\int_I\theta^{2b+1}|\theta_x|dx\\ \le& C\max_{0\le x\le 1}\theta^{2b+1}+C\left(\int_I\theta^{2b}\theta_x^2dx\right)^{\frac{1}{2}}\left(\int_I\theta^{2b+2}dx\right)^{\frac{1}{2}}\\ \le&\frac{1}{2}\max_{0\le x\le 1}\theta^{2b+2}+CY^{\frac{1}{2}}\max_{0\le x\le 1}\theta^{b+\frac{1}{2}}+C,\end{aligned}$$

which implies

$$\max_{0\le x\le 1}\theta^{2b+2-b-\frac{1}{2}}\le C+CY^{\frac{1}{2}}.$$

Hence

$$\max_{0\le x\le 1}\theta\le C+CY^{\frac{1}{2b+3}}.$$

For the second estimate, using $W^{1,1}\hookrightarrow L^\infty$ and the interpolation inequality, we have

$$\max_{0\le x\le 1}|v_x|^2\le C\int_I v_x^2dx+C\int_I|v_xv_{xx}|dx$$

$$\leq C\int_I v_x^2 dx + C\left(\int_I v_x^2 dx\right)^{\frac{1}{2}}\left(\int_I v_{xx}^2 dx\right)^{\frac{1}{2}}$$
$$\leq C + CZ^{\frac{3}{4}},$$

where we have used the fact that

$$\int_I v_x^2 dx \leq C\int_I v^2 dx + C\left(\int_I v^2 dx\right)^{\frac{1}{2}}\left(\int_I v_{xx}^2 dx\right)^{\frac{1}{2}}.$$

Then the proof of Lemma 2.8 is completed. □

Lemma 2.9. *There exists a constant C and $0<C(b)<1$ such that*

$$Y + \int_{Q_T}\theta^b\theta_t^2 dxdt \leq C\left(1+Z^{C(b)}\right). \tag{2.36}$$

Proof. Multiplying (2.16) by $\theta^b\theta_t$, integrating over Q_T, and using integration by parts, we have

$$\begin{aligned}&\int_{Q_T}\theta^b\theta_t^2 dxdt + \int_I \frac{\theta^{2b}\theta_x^2}{2u}dx\\ =&\int_I \frac{\theta_0^{2b}\theta_{0x}^2}{2u_0}dx - \int_{Q_T}\frac{\theta^{2b}\theta_x^2 v_x}{2u^2}dxdt + \int_{Q_T}\left(\frac{\mu(u)v_x^2}{u} - \frac{\theta v_x}{u}\right)\theta^b\theta_t dxdt.\end{aligned} \tag{2.37}$$

Where we have used

$$\begin{aligned}\int_{Q_T}\left(\frac{\theta^b\theta_x}{u}\right)_x\theta^b\theta_t dxdt =& -\int_{Q_T}\frac{\theta^b\theta_x}{u}(\theta^b\theta_t)_x dxdt\\ =& -\int_{Q_T}\frac{\theta^b\theta_x}{u}(\theta^b\theta_x)_t dxdt\\ =& -\int_I \frac{\theta^{2b}\theta_x^2}{2u}dx + \int_I \frac{\theta_0^{2b}\theta_{0x}^2}{2u_0}dx - \int_{Q_T}\frac{\theta^{2b}\theta_x^2 v_x}{2u^2}dxdt.\end{aligned}$$

Hence, (2.37) implies that

$$\begin{aligned}&\frac{1}{2}\int_I \frac{\theta^{2b}\theta_x^2}{u}dx + \int_I \theta^b\theta_t^2 dxdt\\ \leq& C + C\int_{Q_T}\theta^{2b}\theta_x^2|v_x|dxdt + C\int_{Q_T}v_x^2\theta^b|\theta_t|dxdt + C\int_{Q_T}\theta|v_x|\theta^b|\theta_t|dxdt\\ \doteq& C + I_1 + I_2 + I_3.\end{aligned}$$

It follows from (2.25) and (2.34)-(2.35), the fact $\frac{b+1+\varepsilon}{2b+3}<\frac{1}{2}$ and Young's inequality that

$$I_1 \leq \max_{Q_T}(\theta^{b+1+\varepsilon}|v_x|)\int_{Q_T}\theta^{b-1-\varepsilon}\theta_x^2 dxdt \leq C(1+Y^{\frac{b+1+\varepsilon}{2b+3}})(1+Z^{\frac{3}{8}}) \leq C + \frac{\delta}{3}Y + CZ^{\frac{3}{4}}. \tag{2.38}$$

To estimate I_2, we divide the proof into two cases:

Case 1: When $b\le 1$, using (2.26), (2.34)-(2.35), and Young's inequality, we have

$$\begin{aligned}
I_2 &\le C\int_{Q_T} v_x^2\theta^b|\theta_t|dxdt \le \frac{1}{4}\int_{Q_T}\theta^b\theta_t^2dxdt + C\int_{Q_T}\theta^b v_x^4 dxdt\\
&\le \frac{1}{4}\int_{Q_T}\theta^b\theta_t^2dxdt + C\max_{Q_T}(\theta^b v_x^2)\int_{Q_T} v_x^2dxdt\\
&\le \frac{1}{4}\int_{Q_T}\theta^b\theta_t^2dxdt + C(1+Y^{\frac{b}{2b+3}})(1+Z^{\frac{3}{4}})\\
&\le \frac{1}{4}\int_{Q_T}\theta^b\theta_t^2dxdt + C+\frac{\delta}{3}Y + CZ^{\frac{3(2b+3)}{4(b+3)}},
\end{aligned}\tag{2.39}$$

where $\frac{3(2b+3)}{4(b+3)}<1$, for $b\le 1$.

Case 2: When $b>1$, we re-estimate the I_2 as follows

$$\begin{aligned}
I_2 &\le C\int_{Q_T} v_x^2\theta^b|\theta_t|dxdt \le \frac{1}{4}\int_{Q_T}\theta^b\theta_t^2dxdt + C\int_{Q_T}\theta^b v_x^4 dxdt\\
&\le \frac{1}{4}\int_{Q_T}\theta^b\theta_t^2dxdt + C\max_{Q_T}\theta^b\int_{Q_T} v_x^4dxdt\\
&\le \frac{1}{4}\int_{Q_T}\theta^b\theta_t^2dxdt + C(1+Y^{\frac{b}{2b+3}})\int_{Q_T} v_x^4dxdt\\
&\le \frac{1}{4}\int_{Q_T}\theta^b\theta_t^2dxdt + C+\frac{\delta}{3}Y.
\end{aligned}$$

Next, we claim that $\int_{Q_T}|v_x|^4dxdt\le C$, when $b>1$.

Set $h(x,t)=\int_x^1 v(y,t)dy$, then Equation $(1.7)_2$ yields

$$h_t-\frac{\mu(u)}{u}h_{xx}=p=\frac{\theta}{u}$$

with the following initial-boundary conditions

$$h(y,0)=\int_x^1 v_0(y)dy,\quad h(0,t)=\int_0^1 v(y,t)dy,\quad h(1,t)=0.$$

Hence the standard L^p-estimate for solution to above linear parabolic problem yields

$$\int_{Q_T}|v_x|^4dxdt=\int_{Q_T}|h_{xx}|^4dxdt\le C+C\int_{Q_T}|p|^4dxdt\le C+C\int_{Q_T}|\theta|^4dxdt\le C,\tag{2.40}$$

where we have used (2.19) and required $b>1$.

$$\begin{aligned}
I_3=\int_{Q_T}\theta|v_x|\theta^b|\theta_t|dxdt &\le \frac{1}{4}\int_{Q_T}\theta^b\theta_t^2dxdt + C\int_{Q_T}\theta^{2+b}v_x^2dxdt\\
&\le \frac{1}{4}\int_{Q_T}\theta^b\theta_t^2dxdt + C\max_{Q_T}|v_x|^2\int_{Q_T}\theta^{2+b}dxdt\\
&\le \frac{1}{4}\int_{Q_T}\theta^b\theta_t^2dxdt + CZ^{\frac{3}{4}}+C.
\end{aligned}$$

Substituting the above estimate and (2.38)-(2.40) into (2.37) and using (2.24), it holds that for δ suitably small

$$Y+\int_{Q_T}\theta^b\theta_t^2dxdt\le C\left(1+Z^{C(b)}\right).$$

This completes the proof of Lemma 2.9. □

Finally, we are ready to give the estimate on Z.

LEMMA 2.10. *There exists a constant C such that*

$$\sup_{[0,t]}\int_I v_t^2dx+\int_{Q_T}v_{xt}^2\le C\left(1+Z^{C(b)}\right),\tag{2.41}$$

and

$$Z\le C.\tag{2.42}$$

Proof. Differentiating $(1.7)_2$ with respect to t, we have

$$v_{tt}+P_{xt}=\left[\frac{\mu(u)v_x}{u}\right]_{xt}.\tag{2.43}$$

Multiplying it with v_t, and integrating it over $[0,1]$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_I v_t^2dx+\int_I\frac{\mu(u)v_{xt}^2}{u}dx\\&=\int_I\left(\frac{(\mu(u)-\mu'(u)u)v_x^2}{u^2}+\frac{\theta_t}{u}-\frac{\theta v_x}{u^2}\right)v_{xt}dx\\&\le\frac{1}{2}\int_I\frac{\mu(u)v_{xt}^2}{u}dx+C\int_I\left(v_x^4+\theta_t^2+v_x^2\theta^2\right)dx\\&\le\frac{1}{2}\int_I\frac{\mu(u)v_{xt}^2}{u}dx+C\int_I\theta^b\theta_t^2dx+C\max_{[0,1]}v_x^2\int_I(\theta^2+v_x^2)dx.\end{aligned}$$

we integrate the above inequality in time and use (2.19) and (2.26) to obtain

$$\begin{aligned}\sup_{0\le t\le T}\int_I v_t^2dx+\int_{Q_T}v_{xt}^2dxdt&\le C\left(1+Z^{C(b)}\right)+CZ^{\frac{3}{4}}\int_{Q_T}(\theta^2+v_x^2)dxdt\\&\le C\left(1+Z^{C(b)}\right).\end{aligned}$$

We rewrite the momentum equation $(1.7)_2$ as

$$\frac{\mu(u)v_{xx}}{u}=v_t+\left(\frac{\theta}{u}\right)_x+\frac{\mu(u)v_xu_x}{u^2}-\frac{\mu'(u)v_xu_x}{u},\tag{2.44}$$

which implies that

$$\begin{aligned}Z&\le\sup_{0\le t\le T}\left(\int_I v_t^2dx+\int_I v_x^2u_x^2dx+\int_I\theta_x^2dx+\int_I\theta^2u_x^2dx\right)\\&\le C\left(1+Z^{C(b)}+\max_{Q_T}(v_x^2+\theta^2)\int_I u_x^2dx+\int_I\theta^{2b}\theta_x^2dx\right)\\&\le C\left(1+Z^{C(b)}\right)+C\left(Y^{\frac{2}{2b+3}}+Z^{\frac{3}{4}}\right)\\&\le C\left(1+Z^{C(b)}\right)+CY\le C\left(1+Z^{C(b)}\right).\end{aligned}$$

Since $0<C(b)<1$, with the help of Young's inequality, we obtain

$$Z\le C.$$

The proof of Lemma 2.10 is completed. □

From a sequence of estimates which are stated in above lemmas, we easily obtain the following corollary.

COROLLARY 2.1. *There exists a constant C such that*

$$\begin{cases} \max_{Q_T}\theta\le C, \quad \max_{Q_T}|v_x|\le C, \\ \sup_{0\le t\le T}\int_I v_t^2 dx+\int_{Q_T} v_{xt}^2\le C, \\ \sup_{0\le t\le T}\int_I \theta_x^2 dx+\int_{Q_T}\theta_t^2 dxdt\le C. \end{cases} \tag{2.45}$$

LEMMA 2.11. *There exists a constant C such that*

$$\sup_{0\le t\le T}\int_I \theta_t^2+\theta_{xx}^2 dx+\int_{Q_T}\theta_{xt}^2 dxdt\le C. \tag{2.46}$$

Proof. Differentiating the temperature Equation (2.16) with respect to t, then multiplying the resultant by θ_t, and integrating it over I, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int\theta_t^2 dx+\int\frac{\theta^b\theta_{xt}^2}{u}dx=&-\int\frac{b\theta^{b-1}\theta_t u-u_t\theta^b}{u^2}\theta_x\theta_{xt}+\int\frac{\mu'(u)uu_t-u_t\mu(u)}{u^2}v_x^2\theta_t dx\\ &+\int\frac{\mu(u)2v_xv_{xt}\theta_t}{u}dx-\int\frac{\theta_t^2uv_x-u_t\theta\theta_tv_x+uv_{xt}\theta\theta_t}{u^2}dx\\ \le&\varepsilon\int\frac{\theta^b\theta_{xt}^2}{u}dx+C\left(\|\theta_t\theta_x u+v_x\theta_x+\theta_t+v_{xt}+\theta u_t v_x\|_{L^2}^2\right)\\ \le&\varepsilon\int\frac{\theta^b\theta_{xt}^2}{u}dx+C\left(\|\theta_t\|_{L^\infty}^2+1+\|\theta_t\|_{L^2}^2+\|v_{xt}\|_{L^2}^2\right)\\ \le&\varepsilon\int\frac{\theta^b\theta_{xt}^2}{u}dx+C\left(\|\theta_t\|_{L^2}^2+\|v_{xt}\|_{L^2}^2+1\right), \end{aligned} \tag{2.47}$$

where we have used the Hölder inequality and the following fact

$$\begin{aligned} \|\theta_t\|_{L^\infty}\le& C(\|\theta_t\|_{L^2}+\|\theta_t\|_{L^2}^{\frac{1}{2}}\|\theta_{xt}\|_{L^2}^{\frac{1}{2}})\\ \le&\varepsilon\|\frac{\sqrt{\theta^b}\theta_{xt}}{\sqrt{u}}\|_{L^2}+C\|\theta_t\|_{L^2}. \end{aligned} \tag{2.48}$$

Then by the (2.45) and the Grönwall inequality, we have

$$\int\theta_t^2 dx+\int_{Q_T}\frac{\theta^b\theta_{xt}^2}{u}dxdt\le C.$$

Furthermore, the Equation (2.16) can be rewritten as

$$\theta_t=\frac{\theta^b\theta_{xx}}{u}+\frac{b\theta^{b-1}\theta_x u-u_x\theta^b}{u^2}\theta_x+\frac{\mu(u)v_x^2}{u}-\frac{\theta v_x}{u},$$

hence, we have

$$\begin{aligned}\|\theta_{xx}\|_{L^2} &\le \left(\|\theta_t+\theta_x^2u+u_x\theta^b\theta_x+v_x^2+\theta v_x\|_{L^2}\right)\\ &\le C\left(\|\theta_t\|_{L^2}+\|\theta_x\|_{L^2}^{\frac{3}{2}}\|\theta_{xx}\|_{L^2}^{\frac{1}{2}}+\|\theta_x\|_{L^2}^{\frac{1}{2}}\|\theta_{xx}\|_{L^2}^{\frac{1}{2}}\|u_x\|_{L^2}+1+\|v_x\|_{L^2}\right)\\ &\le \varepsilon\|\theta_{xx}\|_{L^2}+C, \end{aligned} \tag{2.49}$$

which implies

$$\|\theta_{xx}\|_{L^\infty(L^2)}\le C.$$

The proof of Lemma 2.11 is completed. □

3. Proof of Theorem 1.1 and Theorem 1.2

The proof of Theorem 1.1 are standard and similar to Theorem 1.1 in [14]. Here we present them for completeness.

Proof. **(Proof of Theorem 1.1.)** Then the proof of Theorem 1.1 follows from Lemma 1.1 which signifies the local existence of the strong solution and the global (in time) a priori estimates in Section 2. In fact, by Lemma 1.1, there exists a local strong solution (u,v,θ) on the time interval $(0,T_*]$ with $T_*>0$. Now let $T^*>0$ be the maximal existing time of the strong solution (u,v,θ) in Lemma 1.1. Then obviously one has $T^*\ge T_*$. Now we claim that $T^*\ge T$ with $T>0$ being any fixed positive constant given in Theorem 1.1. Otherwise, if $T^*<T$, then all the a priori estimates in Section 2 hold with T being replaced by T^*. Therefore, it follows from a priori estimates in Section 2 that $(u,v,\theta)(x,T^*)$ satisfy assumptions in Theorem 1.1, By using Lemma 1.1 again, there exists a $T_1^*>0$ such that the strong solution (u,v,θ) in Lemma 1.1 exists on $(0,T^*+T_1^*]$, which contradicts with T^* being the maximal existing time of the strong solution (u,v,θ). Thus it holds that $T^*>T$.

For (1.13) in Theorem 1.1: Equations (2.3) and (2.45) give the pointwise upper and lower bounds of u and θ. The H^1 estimates in (1.13) are given by Lemma 2.6-Lemma 2.8, Lemma 2.9. The H^2 estimates are given by Lemma 2.11, (2.45) and the boundedness of Z. The proof of Theorem 1.1 is completed. □

Proof. **(Proof of Theorem 1.2.)** To prove the Theorem 1.2, it is convenient to consider the free boundary problem in the Eulerian coordinates. First, let (ρ,v,θ) be any strong solution of (1.1), (1.2) and (1.3), we define an energy functional as follows

$$\begin{aligned}H(t)&=\int_0^{a(t)}(y-(1+t)v)^2\rho dy+\frac{2}{\gamma-1}(1+t)^2\int_0^{a(t)}Pdy\\ &=\int_0^{a(t)}\rho y^2dy-2(1+t)\int_0^{a(t)}\rho vydy+(1+t)^2\int_0^{a(t)}\left(\rho v^2+\frac{2}{\gamma-1}P\right)dy. \end{aligned}\tag{3.1}$$

A direct calculation and using Equations (1.1) gives

$$\begin{aligned}H'(t)=&\int_0^{a(t)}(\rho_ty^2-2\rho vy)dy+(1+t)^2\int_0^{a(t)}\left((\rho v^2)_t+\frac{2}{\gamma-1}P_t\right)dy\\ &+2(1+t)\int_0^{a(t)}\left(\rho v^2-(\rho v)_ty+\frac{2}{\gamma-1}P\right)dy\\ &+\left(\rho vy^2-2(1+t)\rho v^2y+(1+t)^2\rho v^3+\frac{2}{\gamma-1}(1+t)^2Pv\right)\Big|_{y=a(t)}\end{aligned}$$

$$\doteq I_1+I_2+I_3+I_{BD}, \tag{3.2}$$

where we have used the boundary conditions (1.3) and (1.5).

Now we estimate the terms I_1-I_3 as follows:

First, by $(1.1)_1$ and the boundary condition (1.3), we have

$$I_1=\int_0^{a(t)}\left(\rho_t y^2-2\rho v y\right)dy=-\int_0^{a(t)}(\rho v y^2)_y dy=-\rho v y^2|_{y=a(t)}. \tag{3.3}$$

Then by $(1.1)_3$, constitution relations (1.4) and (1.3), we have

$$\begin{aligned}
I_2&=(1+t)^2\int_0^{a(t)}\left((\rho v^2)_t+\frac{2}{\gamma-1}P_t\right)dy=2(1+t)^2\int_0^{a(t)}\left(\rho\left(\frac{v^2}{2}+e\right)\right)_t dy\\
&=2(1+t)^2\int_0^{a(t)}\left((\mu v v_y)_y+(\kappa e_y)_y-(\rho(e+\frac{1}{2}v^2)v)_y-(Pv)_y\right)dy\\
&=2(1+t)^2\left((\mu v v_y+\kappa e_y-Pv)-\frac{1}{\gamma-1}Pv-\frac{1}{2}\rho v^3\right)\Big|_{y=0}^{y=a(t)}\\
&=2(1+t)^2\left(-\frac{1}{\gamma-1}Pv-\frac{1}{2}\rho v^3\right)\Big|_{y=a(t)}.
\end{aligned} \tag{3.4}$$

By $(1.1)_2$, we can obtain

$$\begin{aligned}
I_3&=2(1+t)\int_0^{a(t)}\left(-(\rho v)_t y+\rho v^2+\frac{2}{\gamma-1}P\right)dy\\
&=2(1+t)\int_0^{a(t)}\left((\rho v^2+P-\mu v_y)_y y+\rho v^2+\frac{2}{\gamma-1}P\right)dy\\
&=2(1+t)\int_0^{a(t)}\left([(\rho v^2+P-\mu v_y)y]_y+\frac{3-\gamma}{\gamma-1}P+\mu v_y\right)dy\\
&=2(1+t)\rho v^2 y\Big|_{y=a(t)}+2(1+t)\int_0^{a(t)}\left(\frac{3-\gamma}{\gamma-1}P+\mu v_y\right)dy.
\end{aligned} \tag{3.5}$$

Substituting (3.3),(3.4) and (3.5) into (3.2), we have

$$\begin{aligned}
H'(t)&=2(1+t)\int_0^{a(t)}\left(\frac{3-\gamma}{\gamma-1}P+\mu v_y\right)dy\\
&=\frac{3-\gamma}{1+t}\frac{2}{\gamma-1}(1+t)^2\int_0^{a(t)}Pdy+2\mu(1+t)(v(a(t),t)-v(0,t))\\
&=\frac{3-\gamma}{1+t}\frac{2}{\gamma-1}(1+t)^2\int_0^{a(t)}Pdy+2\mu(1+t)a'(t)
\end{aligned} \tag{3.6}$$

Case 1: If $\gamma\geq 3$. By (3.6), we have

$$H'(t)\leq 2\mu(1+t)a'(t),$$

which yields

$$H(t)\leq H(0)+2\mu\int_0^t(1+s)a'(s)ds$$

$$
\begin{aligned}
&= H(0) + 2\mu(1+t)a(t) - 2\mu a - 2\mu \int_0^t a(s)ds \\
&\leq C(1 + (1+t)M(t)),
\end{aligned} \tag{3.7}
$$

where $M(t) = \max_{0\leq s\leq t}|a(s) - 0| \geq a > 0$.

Case 2: If $\gamma < 3$. By (3.6), we have

$$
H'(t) \leq \frac{3-\gamma}{1+t}H(t) + 2\mu(1+t)a'(t)
$$

by Grönwall inequality and integration by parts, we have

$$
\begin{aligned}
H(t) &\leq \exp\left(\int_0^t \frac{3-\gamma}{1+s}ds\right)\left(H(0) + \int_0^t \exp\left(-\int_0^s \frac{3-\gamma}{1+\tau}d\tau\right)2\mu(1+s)a'(s)ds\right) \\
&= (1+t)^{3-\gamma}\left(H(0) + \int_0^t \frac{1}{(1+s)^{2-\gamma}}2\mu a'(s)ds\right) \\
&\leq (1+t)^{3-\gamma}\left(H(0) + 2\mu\frac{a(t)}{(1+t)^{2-\gamma}} - 2\mu(\gamma-2)\int_0^t (1+s)^{\gamma-3}a(s)ds\right).
\end{aligned} \tag{3.8}
$$

Case 2.1: When $2 \leq \gamma < 3$, (3.8) directly gives

$$
\begin{aligned}
H(t) &\leq (1+t)^{3-\gamma}\left(H(0) + 2\mu\frac{a(t)}{(1+t)^{2-\gamma}}\right) \\
&\leq (1+t)^{3-\gamma}H(0) + 2\mu(1+t)M(t).
\end{aligned} \tag{3.9}
$$

Case 2.2: When $1 < \gamma < 2$, by integration in (3.8), we have

$$
\begin{aligned}
H(t) &\leq (1+t)^{3-\gamma}H(0) + 2\mu(1+t)M(t) + 2\mu(2-\gamma)(1+t)^{3-\gamma}M(t)\int_0^t (1+s)^{\gamma-3}ds \\
&\leq (1+t)^{3-\gamma}H(0) + 2\mu(1+t)M(t) + 2\mu(2-\gamma)(1+t)^{3-\gamma}M(t)\frac{[(1+t)^{\gamma-2}-1]}{\gamma-2} \\
&\leq (1+t)^{3-\gamma}H(0) + 2\mu(1+t)M(t) + 2\mu(1+t)^{3-\gamma}M(t).
\end{aligned} \tag{3.10}
$$

Thus, combining (3.7), (3.9) and (3.10), we have

$$
\begin{aligned}
\int_0^{a(t)} Pdy &\leq \frac{\gamma-1}{2(1+t)^2}H(t) \\
&\leq \begin{cases}
C\left(\frac{1}{(1+t)^2} + \frac{1}{1+t}M(t)\right), & 3 \leq \gamma \\
C\left(\frac{1}{(1+t)^{\gamma-1}} + \frac{1}{(1+t)}M(t)\right), & 2 \leq \gamma < 3, \\
C\left(\frac{1}{(1+t)^{\gamma-1}} + \frac{1}{(1+t)}M(t) + (1+t)^{1-\gamma}M(t)\right), & 1 < \gamma < 2.
\end{cases} \\
&\leq \begin{cases}
C\left(\frac{1}{(1+t)^2} + \frac{1}{1+t}\right)M(t), & 3 \leq \gamma \\
C\left(\frac{1}{(1+t)^{\gamma-1}} + \frac{1}{(1+t)}\right)M(t), & 2 \leq \gamma < 3, \\
C\left(\frac{1}{(1+t)^{\gamma-1}} + \frac{1}{(1+t)}\right)M(t), & 1 < \gamma < 2.
\end{cases} \\
&\leq \begin{cases}
C\frac{1}{(1+t)}M(t), & \gamma \geq 2, \\
C\frac{1}{(1+t)^{\gamma-1}}M(t), & 1 < \gamma < 2.
\end{cases}
\end{aligned} \tag{3.11}
$$

On the other hand, from (1.4) and (1.1), we have the entropy form of the energy equation:

$$(\rho S)_t+(\rho v S)_y=\left(\frac{\kappa\theta_y}{\theta}\right)_y+\frac{c_\nu\kappa\theta_y^2}{\theta^2}+\frac{\mu}{\theta}v_y^2.$$

Integrating it over $(0,a(t))$ and using the boundary condition (1.3), we have

$$\begin{aligned}\frac{d}{dt}\int_0^{a(t)}\rho S dy&=\int_0^{a(t)}(\rho S)_t dy+\int_0^{a(t)}(\rho v S)_y dy\\&=\int_0^{a(t)}\left(\frac{\kappa\theta_y}{\theta}\right)_y dy+\int_0^{a(t)}\left[\frac{c_\nu\kappa\theta_y^2}{\theta^2}+\frac{\mu}{\theta}v_y^2\right]dy\\&\geq\int_0^{a(t)}\left(\frac{\kappa\theta_y}{\theta}\right)_y dy=0\end{aligned}$$

Consequently, we have

$$\frac{d}{dt}\int_0^{a(t)}\rho S dy\geq 0,\quad \int_0^{a(t)}\rho S dy\geq\int_0^a\rho_0 S_0 dy\doteq k_0>0.$$

Using Hölder inequality and the fact $x<e^x$, we have

$$\begin{aligned}\frac{k_0}{\gamma c_v}\leq\int_0^{a(t)}\rho\frac{S}{\gamma c_v}dy&\leq\int_0^{a(t)}\rho e^{\frac{S}{\gamma c_v}}dy\\&\leq\left(\int_0^{a(t)}\rho^\gamma e^{\frac{S}{c_v}}dy\right)^{\frac{1}{\gamma}}\left(\int_0^{a(t)}dy\right)^{\frac{\gamma-1}{\gamma}}\\&=\left(\int_0^{a(t)}P dy\right)^{\frac{1}{\gamma}}M(t)^{\frac{\gamma-1}{\gamma}},\end{aligned}$$

which together with (3.11) yields

$$\begin{aligned}M(t)^{1-\gamma}&\leq\left(\frac{\gamma c_v}{k_0}\right)^\gamma\int_0^{a(t)}P dy\\&\leq\begin{cases}C\frac{1}{(1+t)}M(t), & \gamma\geq 2,\\ C\frac{1}{(1+t)^{\gamma-1}}M(t), & 1<\gamma<2.\end{cases}\end{aligned}$$

Consequently, which implies

$$M(t)\geq\begin{cases}C(1+t)^{\frac{1}{\gamma}}, & \gamma\geq 2,\\ C(1+t)^{1-\frac{1}{\gamma}}, & 1<\gamma<2.\end{cases}$$

Then the proof of Theorem 1.2 is completed. □

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