COMMUN. MATH. SCI.
Vol. 18, No. 7, pp. 2059–2074

ON THE INTEGRAL EQUATION WITH THE AXIS-SYMMETRIC KERNEL*

ZHONG TAN†, YONG WANG‡, AND JIANKAI XU§

Abstract. In this paper, we study some properties of positive solutions of nonlinear integral equations with axis-symmetric kernels, which arise from weak-type convolution-Young's inequality and the stationary magnetic compressible fluid stars. With the help of the method of moving planes and regularity lifting lemma, we show that all of the positive solutions in certain functional spaces are symmetric and monotonically decreasing on the axis of symmetry, and the integrable interval of positive solutions is also obtained. In addition, by analyzing the decay rates of positive solutions in different directions, we prove that no radial solution is allowed in some weighted functional space.

Keywords. Axis-symmetric kernel; Integral equation; Regularity lifting lemma; Non-existence of radial solutions.

AMS subject classifications. 45E10; 45G05.

1. Introduction

In this paper, we consider some properties of positive solutions of the following integral equation

$$u(x)=\int_{\mathbb{R}^3}\frac{r^2(x-y)u^q(y)\,dy}{[r^2(x-y)+z^2(x-y)]^{\frac{3}{2}}}. \tag{1.1}$$

Here $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, $r(x)=\sqrt{x_1^2+x_2^2}$, $z(x)=x_3$ and $q>3$.

The motivations for studying this kind of integral equation come from the related sharp constant of the weak-type convolution-Young's inequality and the stationary axis-symmetric solutions of magnetic compressible fluid stars. To figure out the origin of the integral Equation (1.1), we introduce some background. For every positive function $h\in L^{p,\infty}(\mathbb{R}^n)$, the weak-type convolution-Young's inequality states that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)h(x-y)g(y)\,dxdy\leqslant C(n,s,r)\|h\|_{p,\infty}\|f\|_r\|g\|_s \tag{1.2}$$

for all $f\in L^r(\mathbb{R}^n)$ and $g\in L^s(\mathbb{R}^n)$, where $1<s,\ p,\ r<\infty$ and $1/s+1/p+1/r=2$.

To prove the existence of the sharp maximizing pair (f,g) in (1.2) and explicitly compute the best constant $C(n,s,r)$ and (f,g), we maximize the functional:

$$J(f,g)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)h(x-y)\,dxdy$$

*Received: May 15, 2019; Accepted (in revised form): June 17, 2020. Communicated by Yaguang Wang.
The second author was partially supported by the NSF of China (No. 11701264, 11971179) the Zhujiang River Talent Project of Guangdong Province (No. 2017GC010407) and Basic and Applied Basic Research Foundation of Guangdong Province (No. 2020B1515310002). The third author was partially supported by the NSF of China (No. 11671086, 11871208) and the NSF of Hunan Province of China (No. 2018JJ2159).
†School of Mathematical Sciences, Xiamen University, Fujian, Xiamen 361005, China (ztan85@163.com).
‡Corresponding author. South China Research Center for Applied Mathematics and Interdisciplinary Studies, South China Normal University, Guangzhou, Guangdong 510631, China (wangyongxmu@163.com).
§Corresponding author. College of Information Science and Technology, Hunan Agriculture University, Changsha, Hunan 410128, China (jiankaixu@126.com).

under the constraint conditions:

$$\|f\|_r = \|g\|_s = 1 \quad \text{and} \quad f,\ g > 0.$$

It is easy to check that the corresponding Euler-Lagrange equations satisfy the following integral equations

$$\begin{cases} \lambda_1\ r\ f^{r-1}(x) = \displaystyle\int_{\mathbb{R}^n} h(x-y) g(y)\, dy, \\ \lambda_2\ s\ g^{s-1}(x) = \displaystyle\int_{\mathbb{R}^n} h(x-y) f(y)\, dy, \end{cases} \tag{1.3}$$

where λ_1, λ_2 are two constants such that $\lambda_1 r = \lambda_2 s = J(f,g)$. Let $u = c_1 f^{r-1}$, $v = c_2 g^{s-1}$, $\tau = (r-1)^{-1}$, $q = (s-1)^{-1}$ with c_1 and c_2 being proper constants. The corresponding system (1.3) can be rewritten as

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} h(x-y) v^q(y)\, dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} h(x-y) u^\tau(y)\, dy. \end{cases} \tag{1.4}$$

When $\tau = q$ and $u = v$, the system (1.4) can be reduced to the following single equation:

$$u(x) = \int_{\mathbb{R}^n} h(x-y) u^q(y)\, dy, \qquad \forall\ x \in \mathbb{R}^n. \tag{1.5}$$

In particular, if $n = 3$ and the kernel function $h(x)$ is $r^2(x)/[r^2(x)+z^2(x)]^{\frac{3}{2}}$, the Equation (1.5) becomes the Equation (1.1).

The integral Equation (1.1) is also closely related to the stationary equations of the magnetic compressible fluid stars, which were widely studied in [6–9]. The Euler-Poisson system of compressible fluids coupled to a magnetic field is given by

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = -\rho \nabla \Phi + \dfrac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \\ \dfrac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \\ \operatorname{div} \mathbf{B} = 0, \\ \Delta \Phi = 4\pi G \rho, \end{cases} \tag{1.6}$$

where ρ, $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{B}$, $P(\rho)$ and Φ represent the density, the velocity, the magnetic field, the pressure and the gravitational potential, respectively. $G > 0$ is the gravity constant.

It is significant to find an axis-symmetric solution of (1.6) with $\mathbf{u} = \mathbf{0}$, see [6]:

$$\begin{cases} \rho(x) = \rho(r,z), \quad \Phi(x) = \Phi(r,z), \\ \mathbf{B}(x) = B^r(r,z)\mathbf{e}_r + B^\theta(r,z)\mathbf{e}_\theta + B^z(r,z)\mathbf{e}_3. \end{cases}$$

Here $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3\}$ is the normal orthogonal basis in cylindrical coordinates defined by

$$\mathbf{e}_r = (\frac{x_1}{r},\ \frac{x_2}{r},\ 0)^T, \quad \mathbf{e}_\theta = (-\frac{x_2}{r},\ \frac{x_1}{r},\ 0)^T, \quad \mathbf{e}_3 = (0,\ 0,\ x_3)^T.$$

And $\mathbf{J}$ is the magnetic current given by as follows

$$\begin{aligned}\mathbf{J} &\triangleq \nabla \times \mathbf{B}\\ &= (\frac{x_2}{r(x)} g - \frac{x_1}{r(x)} \partial_z B^\theta)\ \mathbf{i} - (\frac{x_1}{r(x)} g + \frac{x_2}{r(x)} \partial_z B^\theta)\ \mathbf{j} + (\frac{B^\theta}{r(x)} + \partial_r B^\theta)\ \mathbf{k},\end{aligned}$$

where

$$g(r,z) = (\partial_r B^z - \partial_z B^r).$$

With these assumptions $B^\theta = B^r \Omega'(r) = 0$ and $\mathbf{J} = \rho\ (x_2,\ -x_1,\ 0)$, the Equations (1.6) can be reduced to the following form

$$\nabla(i(\rho) + \mathbf{J} + \Phi - \beta\psi) = 0, \qquad \text{whenever} \quad \rho > 0.$$

Here $i(\rho)$ is defined by

$$i(\rho) = \int_0^\rho \frac{P'(s)}{s}\, ds,$$

and ψ is a magnetic potential satisfying

$$\operatorname{div}(\frac{1}{r^2} \nabla \psi) = -4\pi\beta\rho.$$

After basic calculations, it is easy to verify that

$$k(x) = \frac{r^2(x)}{(r^2(x) + z^2(x))^{\frac{3}{2}}}$$

is a solution of $\operatorname{div}(\frac{1}{r^2} \nabla \psi) = 0$.

We recall some related results. When the kernel function in (1.2) is $h(x) = |x|^{-\lambda}$ $(0 < \lambda < n)$, the inequality (1.2) becomes the well-known Hardy-Littlewood-Sobolev inequality

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x) g(y)}{|x-y|^\lambda}\, dy dx \leqslant C(n,s,r) \|f\|_r \|g\|_s,$$

where $r,\ s > 1$ satisfying $1/r + 1/s + \lambda/n = 2$. The existence of the sharp constants and extremal functions of the Hardy-Littlewood-Sobolev inequality were shown by Lieb [16]. Furthermore, in the special case where $f = g$ and $r = s$, by a proper choice of parameters, the corresponding Euler-Lagrange equation becomes

$$u(x) = \int_{\mathbb{R}^n} \frac{u^q(y)\, dy}{|x-y|^\lambda}, \qquad \forall\ x \in \mathbb{R}^n. \tag{1.7}$$

Under the condition of conformal invariance, *i.e.*, $q = 2n/\lambda - 1$ and $\lambda \in (0,n)$, Lieb obtained the specific form of positive solutions to (1.7) by using the method of stereographic projection. Meanwhile, Lieb observed that the function $|x|^{-\lambda/2}$ is also a positive solution of (1.7) and raised an open problem on the classification of positive solutions of (1.7) (see [16, p. 361]). Subsequently, Chen, Li and Ou [4] solved this open problem by using the method of moving planes. Precisely, they showed that the solution u, up to translation, scaling, and Kelvin transformation, is unique in the class of $L^{2n/\lambda}_{loc}(\mathbb{R}^n)$

and radially symmetric and monotonically decreasing about some point in $\mathbb{R}^n$. On the other hand, more general equations, including those which are conformally invariant or not, were considered by Li [15]. With the method of moving spheres, Li showed that the Equation (1.7) does not have any non-negative Lebesgue-measurable solution when $\lambda<0$ and $2n/\lambda-1<q<0$. However, if the integral Equation (1.7) is conformally invariant, namely, $\lambda<0$ and $q=2n/\lambda-1$, it admits a positive solution. Later, Xu [19] studied the case of $\lambda<0$ and $q<2n/\lambda-1$. In fact, under the condition of $\lambda<0$, Xu proved that there exists a positive solution to (1.7) if and only if $q=2n/\lambda-1$. Recently, Xu, Wu and Tan [18] considered the following integral equation:

$$u(x,b)=\int_{\mathbb{R}^n}\frac{u^q(y,b)}{(b+|x-y|)^\lambda}\,dy. \tag{1.8}$$

The authors showed that for $\lambda\in(-\infty,0)\cup(0,n)$ and $q=2n/\lambda-1$, the integral Equation (1.8) has no positive solution, which is distinct from the conformally invariant integral Equation (1.7) and simultaneously implies that the maximizing pair of the weak-type convolution-Young's inequality with kernel function $(b+|x|)^{-\lambda}$ does not exist. For more information on positive solutions of integral equations, the readers can refer to [1,5,10–14,17].

As one can see, for the inequality (1.2) with the specific kernel function $h(x)=|x|^{-\lambda}$ or $h(x)=(|x|+b)^{-\lambda}$, the corresponding Euler-Lagrange functional is radially symmetric under some integrability conditions. From an analytical point of view, an interesting question that arises from above results is whether the positive solution of the integral Equation (1.1) is radially symmetric if the kernel function takes the axis-symmetric kernel $h(x)=r^2(x)/[(r^2(x)+z^2(x))^{\frac{3}{2}}]$. Here, we can show that the positive solution is symmetric about the axis $z=x_3$, which is stated in the following Theorem 1.1. On the other hand, noting that $u(x)=0$, which can be regarded as the special radial function, is a solution of (1.1). What is more, in an integral operator, if the kernel function is not radial, acting on a radial function, we get a radial function. Indeed, for example,

$$\int_{-\infty}^{+\infty}\cos[2(xt+\frac{\pi}{2})]e^{-t^2}\chi_{[0,\ +\infty)}(t)\,dt=-\frac{\sqrt{2\pi}}{2}e^{-x^2}.$$

A more natural and interesting problem is whether (1.1) admits a non-zero radial positive solution. To solve this problem, we heuristically study the decay rates of positive solutions of the integral Equation (1.1) in different directions. We discover that the decay rates in the horizontal and vertical directions are totally different, which implies that the integral Equation (1.1) does not admit a radially symmetric positive solution.

Notation. Let $\mathbb{R}^n$ $(n\geqslant 2)$ be the n dimensional Euclidean space and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $dm=dm(\cdot)$. For a measurable function f on $\mathbb{R}^n$, we denote the distribution of f by

$$\lambda_f(t)=m(\{x\in\mathbb{R}^n:|f(x)|>t\}).$$

For $0<p<\infty$, the spaces $L^p(\mathbb{R}^n)$ and $L^{p,\infty}(\mathbb{R}^n)$ will denote the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(\mathbb{R}^n)}=\left(\int_{\mathbb{R}^n}|f(x)|^p\,dx\right)^{1/p}=\left(p\int_0^\infty t^{p-1}\lambda_f(t)\,dt\right)^{1/p}<\infty$$

and

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)}=\sup_{\gamma>0}\{\gamma[\lambda_f(\gamma)]^{1/p}\}<\infty,$$

respectively. For simplicity, we denote $\|\cdot\|_{L^p(\mathbb{R}^n)}$ and $\|\cdot\|_{L^{p,\infty}(\mathbb{R}^n)}$ by $\|\cdot\|_p$ and $\|\cdot\|_{p,\infty}$, respectively.

Throughout this paper, we always use the letter C to denote a generic positive constant that may vary from line to line.

Now, we state our main results as follows:

THEOREM 1.1. *Assume that $u(x)\in L^t(\mathbb{R}^3)(t=[3(q-1)]/2,\ q>3)$ is a positive solution of (1.1). Then the following results hold.*

- *Symmetry and monotonicity. For some $\lambda_0\in\mathbb{R}$,*

$$u(x_1,x_2,x_3)=u(x_1,x_2,2\lambda_0-x_3),\qquad \forall\ x=(x_1,x_2,x_3)\in\mathbb{R}^3.$$

Moreover, $u(x)$ is monotonically decreasing along x_3 direction about $x_3=\lambda_0$.

- *Integrability. It holds that*

$$u\in L^s(\mathbb{R}^3),\qquad s\in(3,\ \ \infty].$$

THEOREM 1.2. *Let $q>3$. Set*

$$\mathfrak{F}=\left\{f\ \ \Big|\ \ f\in L^{[3(q-1)]/2}(\mathbb{R}^3)\quad and\quad \int_{\mathbb{R}^3} r^2(x)|f(x)|^q\,dx<\infty\right\}.$$

Then the integral Equation (1.1) does not have any radially symmetric positive solutions in $\mathfrak{F}$.

There are some remarks about our main results in Theorems 1.1-1.2.

REMARK 1.1. Our results in Theorems 1.1–1.2 show that all the positive solutions of the integral Equation (1.1) are axis-symmetric but not radially symmetric. However, the positive solutions of (1.7)–(1.8) in [4,18] are radially symmetric. The essential reasons lie in the different structure of kernel functions. Moreover, the essential difference results in that the existence of positive solutions of (1.1) is different from that of the Equation (1.7). Precisely, the different structures between (1.1) and (1.7)–(1.8) reveal that the methods of moving spheres and weak derivatives used in [15,19] to obtain the existence of positive solutions to (1.7) do not apply for the Equation (1.1). Therefore, we have to look for a new way to study the existence of positive solutions of (1.1). In this paper, we decompose the whole space $\mathbb{R}^3$ into three parts and make use of the a priori estimates to obtain the different asymptotic behaviors in different directions, which implies that the integral Equation (1.1) does not admit a radially symmetric positive solution.

REMARK 1.2. Similar results as in Theorems 1.1–1.2 hold for the dimension $n>3$, since the key proof steps depending on Lemma 2.1 and Lemma 3.2 are available for the dimension $n>3$.

REMARK 1.3. Note that it is not clear for us whether the integral Equation (1.1) is equivalent to the partial differential equation $\mathrm{div}(\frac{1}{r^2(x)}u(x))=u^q(x)$ or not. This is an interesting problem, which we will study in the future.

The rest of this paper is organized as follows. In Section 2, we will give the proof of Theorem 1.1, where we first build up a special convolution-Young's inequality and then obtain the symmetry and integrability of positive solutions of (1.1). In Section 3, we establish the sharp decay estimates of positive solutions in the horizontal and vertical directions and thus prove Theorem 1.2.

2. Proof of Theorem 1.1

In this section, we will give a complete proof of Theorem 1.1. First, we establish some necessary ingredients needed in the following proofs.

2.1. A weak-type convolution-Young's inequality. The following lemma state a special weak-type Young's inequality with an axis-symmetric kernel, which is totally different from the classical Young's inequality with the Riesz's kernel [16].

Lemma 2.1. *Let $R_K(f)(x)$ $(x\in\mathbb{R}^3)$ be a function defined by*

$$R_K(f)(x)\triangleq\int_{\mathbb{R}^3}K(x-y)f(y)\,dy,$$

where

$$K(x)=\frac{r^2(x)}{(r^2(x)+z^2(x))^{\frac{3}{2}}},\quad x=(x_1,x_2,x_3)\neq 0 \tag{2.1}$$

and

$$r(x)\triangleq\sqrt{x_1^2+x_2^2},\quad z(x)\triangleq x_3.$$

Then we have

$$K(x)\in L^{3,\infty}(\mathbb{R}^3) \tag{2.2}$$

and for $s>3$,

$$\|R_K(f)\|_{L^s(\mathbb{R}^3)}\leqslant C(s)\|f\|_{L^{\frac{3s}{2s+3}}(\mathbb{R}^3)}. \tag{2.3}$$

Proof. For every $t\in\mathbb{R}^+$, we set

$$\mathbb{G}_t\triangleq\left\{x\in\mathbb{R}^3\;\middle|\;|K(x)|>t\;\right\}$$

and

$$h(x_1,x_3)\triangleq\frac{x_1^2}{(x_1^2+x_3^2)^{\frac{3}{2}}},\qquad (x_1,\;x_3)\in\mathbb{R}^2. \tag{2.4}$$

Obviously, for every fixed $t>0$ and $h(x_1,x_3)=t$, we have

$$x_3(x_1)=\sqrt{t^{-\frac{2}{3}}x_1^{\frac{4}{3}}-x_1^2},\qquad 0\leqslant x_1\leqslant\;t^{-1}.$$

Now, we calculate the Lebesgue measure $m(\mathbb{G}_t)$ of the set $\mathbb{G}_t$ for any fixed $t>0$. We first construct a rotational body in $\mathbb{R}^3$ by rotating the planar domain $\{(x_1,x_3)\in\mathbb{R}^2|\;h(x_1,x_3)\geqslant t\}$ in x_1ox_3-plane with respect to the axis x_3. It is easy to verify that the volume of that rotational body is exactly equal to $m(\mathbb{G}_t)$. Therefore, we have

$$\begin{aligned}m(\mathbb{G}_t)&=\int_{\mathbb{G}_t}dx_1dx_2dx_3=\int_0^{2\pi}d\theta\int_0^{1/t}r\,dr\int_0^{\sqrt{t^{-2/3}r^{4/3}-r^2}}dx_3\\&=2\pi\int_0^{1/t}\sqrt{t^{-2/3}r^{4/3}-r^2}\,r\,dr=\left(2\pi\int_0^1\sqrt{1-s^{\frac{2}{3}}}s^{\frac{4}{3}}\,ds\right)t^{-3},\end{aligned}$$

which implies that

$$h(x_1,x_2,x_3)\in L^{3,\infty}(\mathbb{R}^3).$$

The estimate (2.3) follows from the weak-type convolution-Young's inequality with the kernel $r^2(x)/[(r^2(x)+z^2(x))^{\frac{3}{2}}]$ in $\mathbb{R}^3$. Hence, the proof of Lemma 2.1 is completed. ∎

2.2. Symmetry and monotonicity. To study the symmetry of positive solutions $u(x)$ of the integral Equation (1.1), we define

$$\Sigma_\lambda \triangleq \left\{ x=(x_1,x_2,x_3)\in\mathbb{R}^3 \;\middle|\; x_3=z(x)\geqslant\lambda \right\},$$

$$x^\lambda \triangleq (x_1,x_2,2\lambda-x_3) \quad \text{and} \quad u_\lambda(x)\triangleq u(x^\lambda),$$

where $\lambda\in\mathbb{R}$ is given. It is easy to check that

$$u(x)-u_\lambda(x)=\int_{\Sigma_\lambda}\left\{\frac{r^2(x-y)}{[r^2(x-y)+z^2(x-y)]^{\frac{3}{2}}} - \frac{r^2(x-y)}{[r^2(x-y)+z^2(x^\lambda-y)]^{\frac{3}{2}}}\right\}(u^q(y)-u_\lambda^q(y))\,dy. \tag{2.5}$$

Next, we will employ the method of moving planes in integral forms introduced by Chen et al. [3,4] to study the symmetry of positive solutions. Such a method basically consists of two steps. In the first step, we compare the values of $u(x)$ and $u_\lambda(x)$ on the domain Σ_λ. In fact, it will be verified that for sufficiently negative $\lambda<0$,

$$u(x)\geqslant u_\lambda(x), \qquad \forall\; x\in\Sigma_\lambda\setminus\{0\}. \tag{2.6}$$

In the second step, we will check that the plane $x_3=z(x)=\lambda$ can be continuously moved along x_3 direction from near negative infinity to the above so long as (2.6) holds. In the end, such a plane will stop at some limiting position denoted by $x_3=\lambda_0$.

Step 1. Set

$$\Sigma_\lambda^u \triangleq \left\{ x\in\Sigma_\lambda\setminus\{0\} \;\middle|\; u(x)<u_\lambda(x)\right\}.$$

Given $z^2(x^\lambda-y)\geqslant z^2(x-y)$, we obtain for $\forall\; x,y\in\Sigma_\lambda$,

$$\begin{aligned} u_\lambda(x)-u(x) &\leqslant \int_{\Sigma_\lambda^u}\frac{r^2(x-y)}{[r^2(x-y)+z^2(x-y)]^{\frac{3}{2}}}(u_\lambda^q(y)-u^q(y))\,dy \\ &\leqslant C(q)\int_{\Sigma_\lambda^u}\frac{r^2(x-y)u_\lambda^{q-1}(y)}{[r^2(x-y)+z^2(x-y)]^{\frac{3}{2}}}(u_\lambda(y)-u(y))\,dy. \end{aligned}$$

By Lemma 2.1 and Hölder's inequality, we get

$$\|u_\lambda-u\|_{L^t(\Sigma_\lambda^u)}\leqslant \|u_\lambda^{q-1}\|_{L^{\frac{3}{2}}(\Sigma_\lambda)}\;\|u_\lambda-u\|_{L^t(\Sigma_\lambda^u)}. \tag{2.7}$$

Noting that $u\in L^{[3(q-1)]/2}(\mathbb{R}^3)$, we can choose $N>0$ to be large enough such that for any $\lambda\leqslant -N<0$,

$$\|u_\lambda^{q-1}\|_{L^{\frac{3}{2}}(\Sigma_\lambda)}\leqslant\frac{1}{2},$$

which, together with (2.7), implies that

$$\|u_\lambda-u\|_{L^t(\Sigma_\lambda^u)}=0.$$

Thus, Σ_λ^u must be a zero-measure set.

Step 2. Keep moving the plane $x_3=\lambda$ to the limiting position $x_3=\lambda_0$ as long as (2.6) holds.

Let

$$\lambda_0 = \sup\left\{\lambda \in \mathbb{R} \;\middle|\; u(x) \geqslant u_\mu(x), \;\; \forall\, x \in \Sigma_\mu, \;\; \mu \leqslant \lambda\right\}. \tag{2.8}$$

Next, by a contradiction, we show that

$$\lambda_0 < \infty.$$

If $\lambda_0 = \infty$, by (2.5), then we have for $\forall\, x,y \in \Sigma_\mu$,

$$u(x) - u_\mu(x) \leqslant q \int_{\Sigma_\mu} \frac{r^2(x-y)u^{q-1}(y)[u(y)-u_\mu(y)]}{[r^2(x-y)+z^2(x-y)]^{\frac{3}{2}}}\, dy.$$

On the other hand, when $\mu > 0$ is sufficiently large, by the same estimate (2.7), we obtain

$$\|u_\mu^{q-1}\|_{L^{\frac{3}{2}}(\Sigma_\mu)} \leqslant \frac{1}{2},$$

and

$$m\left\{x \in \Sigma_\mu \setminus \{0\} \;\middle|\; u(x) > u_\mu(x)\right\} = 0. \tag{2.9}$$

Obviously, this is a paradox with (2.8). Now, we show that

$$u(x) \equiv u_{\lambda_0}(x), \qquad \forall\, x \in \Sigma_{\lambda_0}. \tag{2.10}$$

If (2.10) does not hold, by (2.8), then for $\forall\, x \in \Sigma_{\lambda_0} \setminus \{0\}$,

$$u(x) \geqslant u_{\lambda_0}(x), \;\; \text{but} \quad u(x) \not\equiv u_{\lambda_0}(x). \tag{2.11}$$

In this case, the plane $x_3 = \lambda_0$ can be moved further to the above. Precisely, there exists an $\epsilon > 0$ such that

$$u(x) \geqslant u_\lambda(x), \; x \in \; \Sigma_\lambda \setminus \{0\}, \quad \lambda \; \in \; [\lambda_0, \lambda_0 + \epsilon).$$

Indeed, by (2.5), we know that $u(x) > u_\lambda(x)$ in the interior of Σ_{λ_0}. Let

$$\widetilde{\Sigma}^u_{\lambda_0} = \left\{x \in \Sigma_{\lambda_0} \;\middle|\; u(x) \leqslant u_{\lambda_0}(x) \;\right\}.$$

Based on the above analyses, it is easy to verify that the measure $m(\widetilde{\Sigma}^u_{\lambda_0}) = 0$ and $\lim\limits_{\lambda \to \lambda_0^+} \Sigma^u_\lambda \subset \Sigma_{\lambda_0}$. This together with $u \in L^t(\mathbb{R}^n)$ ensures that one can choose ϵ to be small enough such that for any $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$,

$$\|u_\lambda - u\|_{L^t(\Sigma^u_\lambda)} \leqslant \frac{1}{2}\|u_\lambda - u\|_{L^t(\Sigma^u_\lambda)},$$

which implies that $\Sigma^u_\lambda = \{x \in \Sigma_\lambda \; | u(x) < u_\lambda(x)\}$ must be a zero-measure set. This clearly contradicts with the definition of λ_0.

Hence, we obtain the symmetry and monotonicity of positive solutions of the integral Equation (1.1). The proof of the first part of Theorem 1.1 is completed.

2.3. Integrability. In this subsection, we will study the regularity of positive solutions $u(x)$ of the integral Equation (1.1). To be precise, we will use the regularity lifting lemma [2] to show that $u\in L^s(\mathbb{R}^3)$ with $s\in(3,\infty)$. Later on, we also prove $u\in L^\infty(\mathbb{R}^3)$.

For $A>0$, we set

$$u_A(x)=\begin{cases} u(x), & \text{if } u(x)\geqslant A \text{ or } |x|\geqslant A,\\ 0, & \text{otherwise.}\end{cases}$$

For $f\in L^s(\mathbb{R}^3)$ $(s>3)$, we define the operators $\mathscr{T}_A(f)$ and $\mathcal{F}(x)$:

$$\mathscr{T}_A(f)(x)\triangleq\int_{\mathbb{R}^3}\frac{r^2(x-y)\ u_A^{q-1}(y)\ f(y)dy}{[r^2(x-y)+z^2(x-y)]^{\frac{3}{2}}}$$

and

$$\mathcal{F}(x)\triangleq\int_{\mathbb{R}^3}\frac{r^2(x-y)\ (u(y)-u_A(y))^q dy}{[r^2(x-y)+z^2(x-y)]^{\frac{3}{2}}}.$$

Clearly, by Lemma 2.1 and Hölder's inequality, we have that for $s\in(3,\infty)$,

$$\|\mathcal{F}(x)\|_{L^s(\mathbb{R}^3)}\leqslant C\|(u-u_A)^q\|_{L^{\frac{3s}{3+2s}}(\mathbb{R}^3)}$$

and

$$\|\mathscr{T}_A(f)\|_{L^s(\mathbb{R}^3)}\leqslant C\|u_A^{q-1}f\|_{L^{\frac{3s}{3+2s}}(\mathbb{R}^3)}\leqslant C\|u_A\|^{q-1}_{L^{\frac{3(q-1)}{2}}(\mathbb{R}^3)}\|f\|_{L^s(\mathbb{R}^3)}.$$

Noting that $u\in L^{[3(q-1)]/2}(\mathbb{R}^3)$, we can choose A to be large enough such that

$$C\|u_A\|^{q-1}_{\frac{3(q-1)}{2}}\leqslant\frac{1}{2}.$$

This, together with $\mathscr{T}_A$ being a linear operator, implies that $\mathscr{T}_A$ is a contraction map from $L^t(\mathbb{R}^3)$ into itself. On the other hand, $u(x)$ is a solution of the following integral equation

$$f(x)=\mathscr{T}_A(f)(x)+\mathcal{F}(x). \tag{2.12}$$

Hence, taking $\mathcal{X}=L^{[3(q-1)]/2}(\mathbb{R}^3)$, $\mathcal{Y}=L^s(\mathbb{R}^3)$ in the regularity lifting lemma (see [2,10] for the details), we obtain

$$u(x)\in\mathcal{Z}=L^{[3(q-1)]/2}(\mathbb{R}^3)\cap L^s(\mathbb{R}^3),\qquad\forall\ s\in(3,\infty). \tag{2.13}$$

Next, we prove $u(x)\in L^\infty(\mathbb{R}^3)$. By (2.12), it suffices to obtain the boundedness of $\mathscr{T}_A(u)(x)$ and $\mathcal{F}(x)$ in $\mathbb{R}^3$. Indeed, if $|x|\leqslant 2A$, we have

$$\begin{aligned}|\mathcal{F}(x)|&\leqslant A^q\int_{|y|\leqslant A}\frac{r^2(x-y)}{[r^2(x-y)+z^2(x-y)]^{\frac{3}{2}}}dy\\&\leqslant A^q\int_{|x-y|\leqslant 3A}\frac{1}{|x-y|}dy=CA^{q+2}.\end{aligned} \tag{2.14}$$

At the same time, it is easy to check that for $|x|\geqslant 2A$,

$$|\mathcal{F}(x)|\leqslant \int_{|y|\leqslant A}\frac{(u-u_A^q)(y)}{|x-y|}\,dy=\frac{1}{A}\int_{|y|\leqslant A}(u-u_A^q)(y)\,dy. \tag{2.15}$$

Thus, we obtain that $\mathcal{F}(x)\in L^\infty(\mathbb{R}^3)$. We now show that $\mathscr{T}_A(u)(x)\in L^\infty(\mathbb{R}^3)$. Firstly, we decompose it into two parts

$$\begin{aligned}\mathscr{T}_A(u)(x)&=\int_{\mathbb{R}^3}\frac{r^2(x-y)\;u_A^q(y)\,dy}{[r^2(x-y)+z^2(x-y)]^{\frac{3}{2}}}\\&=\left(\int_{|y|\leqslant A}+\int_{|y|\geqslant A}\right)\frac{r^2(x-y)\;u_A^q(y)\,dy}{[r^2(x-y)+z^2(x-y)]^{\frac{3}{2}}}\\&\triangleq \mathscr{T}_A(u)_1(x)+\mathscr{T}_A(u)_2(x).\end{aligned} \tag{2.16}$$

Since $q>3$, it is easy to check that $2q>3(q-1)/2>3$. Hence, by Hölder's inequality, we have for $|x|\leqslant 2A$,

$$\begin{aligned}\mathscr{T}_A(u)_1(x)&\leqslant \int_{|y|\leqslant A}\frac{u_A^q(y)}{|x-y|}\,dy\\&\leqslant \left(\int_{|x-y|\leqslant 3A}|x-y|^{-2}\,dy\right)^{\frac{1}{2}}\left(\int_{|y|\leqslant A}u_A^{2q}(y)\,dy\right)^{\frac{1}{2}}\\&\leqslant C(A)\|u\|^q_{L^{2q}(\mathbb{R}^3)}.\end{aligned} \tag{2.17}$$

On the other hand, we have for $|x|\geqslant 2A$,

$$\begin{aligned}\mathscr{T}_A(u)_1(x)&\leqslant \int_{|y|\leqslant A}\frac{u_A^q(y)}{|x-y|}\,dy\leqslant \frac{1}{A}\int_{|y|\leqslant A}u_A^q(y)\,dy\\&\leqslant \frac{1}{A}\left(\int_{|y|\leqslant A}\chi_{|y|\leqslant A}(y)\,dy\right)^{\frac{1}{2}}\left(\int_{|y|\leqslant A}u_A^{2q}(y)\,dy\right)^{\frac{1}{2}}.\end{aligned} \tag{2.18}$$

Next, we estimate $\mathscr{T}_A(u)_2(x)$, which is decomposed into two parts as follows.

$$\begin{aligned}\mathscr{T}_A(u)_2(x)&\leqslant \int_{|y|\geqslant A}\frac{u_A^q(y)}{|x-y|}\,dy\\&\leqslant \left(\int_{\{|y|\geqslant A\}\setminus B_A(x)}+\int_{B_A(x)}\right)\frac{u_A^q(y)}{|x-y|}\,dy\\&\triangleq \mathscr{T}_A(u)_{2,1}(x)+\mathscr{T}_A(u)_{2,2}(x).\end{aligned} \tag{2.19}$$

Similar to (2.17), we easily obtain that $\mathscr{T}_A(u)_{2,2}(x)\in L^\infty(\mathbb{R}^3)$. Now, we consider $\mathscr{T}_A(u)_{2,1}(x)$. Taking $\tau=3+1/(2q)>3$, we have

$$2\times(3q+\frac{1}{2})\geqslant 3(q-1)(2+\frac{1}{2q})\ \text{ and }\ q\tau'=q\times\frac{3+1/(2q)}{2+1/(2q)}\geqslant\frac{3(q-1)}{2}.$$

Thus, by Hölder's inequality, we have

$$\mathscr{T}_A(u)_{2,1}(x)\leqslant \int_{|x-y|\geqslant A}\frac{u_A^q(y)}{|x-y|}\,dy\leqslant \left(\int_{|x-y|\geqslant A}\frac{1}{|x-y|^\tau}\,dy\right)^{\frac{1}{\tau}}\times\left(\int_{|x-y|\geqslant A}u_A^{q\tau'}(y)\,dy\right)^{\frac{1}{\tau'}}.$$

This, together with (2.13), completes the proof of the second part of Theorem 1.1.

3. The proof of Theorem 1.2

This section is devoted to showing Theorem 1.2, which will be divided into two steps. In step 1, under the additional condition that $u(x)$ is a radial positive solution of the integral equation (1.1), we will obtain the sharp decay rate of $u(x)$ in x_1ox_2-plane. Precisely, we will prove the follow optimal horizontal asymptotic behaviors

$$\lim_{\substack{r(x)\to\infty\\ z(x)=0}} r(x)u(x)=\|u\|^q_{L^q(\mathbb{R}^3)}. \tag{3.1}$$

In step 2, we will derive the decay rate of $u(x)$ on the axis x_3 with $x_1=x_2=0$. Namely, we will show that

$$\limsup_{|x_3|\to\infty}|x_3|^3u(x)=C_1\int_{\mathbb{R}^3} r^2(y)u^q(y)\,dy. \tag{3.2}$$

Comparing (3.1) with (3.2), this implies that the integral Equation (1.1) does not admit a radially symmetric positive solution. Firstly, we consider the horizontal asymptotic behaviors of $u(x)$. The results can be stated as follows.

Lemma 3.1. *Set $x=(x_1,x_2,0)\in\mathbb{R}^3$. If u is a positive radial solution of the integral Equation (1.1), then*

$$\lim_{\substack{r(x)\to\infty\\ z(x)=0}} r(x)u(x)=\lim_{\substack{r(x)\to\infty\\ z(x)=0}}\int_{\mathbb{R}^3}\frac{r(x)r^2(x-y)}{(r^2(x-y)+y_3^2))^{\frac{3}{2}}}u^q(y)\,dy=\int_{\mathbb{R}^3}u^q(y)\,dy. \tag{3.3}$$

Proof. We denote

$$\mathbb{D}_1\triangleq\left\{y\in\mathbb{R}^3\ \middle|r(y)\leqslant R \text{ and } |z(y)|\leqslant R\right\}\quad\text{and}\quad \mathbb{D}_2\triangleq\left\{y\in\mathbb{R}^3\middle|\ |y-x|\leqslant\frac{|x|}{2}\right\}.$$

To obtain (3.3), we rewrite

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{r(x)r^2(x-y)}{(r^2(x-y)+y_3^2)^{\frac{3}{2}}}u^q(y)dy&=\int_{\mathbb{D}_1}\frac{r(x)r^2(x-y)}{(r^2(x-y)+y_3^2)^{\frac{3}{2}}}u^q(y)\,dy\\&+\int_{\mathbb{R}^3\setminus(\mathbb{D}_1\cup\mathbb{D}_2)}\frac{r(x)r^2(x-y)}{(r^2(x-y)+y_3^2)^{\frac{3}{2}}}u^q(y)\,dy+\int_{\mathbb{D}_2}\frac{r(x)r^2(x-y)}{(r^2(x-y)+y_3^2)^{\frac{3}{2}}}u^q(y)\,dy\\&\triangleq\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3.\end{aligned} \tag{3.4}$$

Obviously, when $y\in\mathbb{R}^3\setminus(\mathbb{D}_1\cup\mathbb{D}_2)$ and $x=(x_1,x_2,0)\neq 0$, we have $|y-x|\geqslant|x|/2\geqslant r(x)/2$ and

$$0<\frac{r(x)r^2(x-y)}{(r^2(x-y)+y_3^2)^{\frac{3}{2}}}\leqslant\frac{r(x)}{(r^2(x-y)+y_3^2)^{\frac{1}{2}}}\leqslant 2.$$

Therefore, together with the dominated convergence theorem, we gain

$$\mathcal{A}_2=\int_{\mathbb{R}^3\setminus(\mathbb{D}_1\cup\mathbb{D}_2)}\frac{r(x)r^2(x-y)}{(r^2(x-y)+y_3^2))^{\frac{3}{2}}}u^q(y)\,dy\leqslant 2\int_{\mathbb{R}^3\setminus\mathbb{D}_1}u^q(y)\,dy\to 0,\quad\text{as}\quad R\to\infty. \tag{3.5}$$

For $r(x)\geqslant 2R$ and $y\in\mathbb{D}_1$, it is easy to check that

$$0\leqslant\frac{r(x)r^2(x-y)}{(r^2(x-y)+y_3^2))^{\frac{3}{2}}}\leqslant 32\quad\text{and}\quad\lim_{r(x)\to\infty}\frac{r(x)r^2(x-y)}{(r^2(x-y)+y_3^2))^{\frac{3}{2}}}=1.$$

Therefore, together with the dominated convergence theorem and Theorem 1.1, we obtain

$$\begin{aligned}&\lim_{R\to\infty}\lim_{\substack{r(x)\to\infty\\ z(x)=0}}\mathcal{A}_1\\ &=\lim_{R\to\infty}\lim_{\substack{r(x)\to\infty\\ z(x)=0}}\int_{|y|\leqslant R}\frac{r(x)r^2(x-y)}{(r^2(x-y)+y_3^2))^{\frac{3}{2}}}u^q(y)\,dy=\int_{\mathbb{R}^3}u^q(y)\,dy.\end{aligned}\tag{3.6}$$

Next, we estimate $\mathcal{A}_3$. Since $u(x)$ is a positive radial solution of the integral Equation (1.1), by Theorem 1.1, we deduce that

$$u^s(\tfrac{|x|}{2})|x|^3\leqslant C_1\int_{\frac{|x|}{4}\leqslant|y|\leqslant\frac{|x|}{2}}u^s(y)\,dy\leqslant C,\qquad\forall\ s\in(3,\ \infty).$$

Note that

$$\begin{aligned}\mathcal{A}_3&=\int_{|y-x|\leqslant\frac{|x|}{2}}\frac{r(x)r^2(x-y)}{(r^2(x-y)+y_3^2)^{\frac{3}{2}}}u^q(y)\,dy\leqslant|x|\ u^q(\frac{|x|}{2})\int_{|y-x|\leqslant\frac{|x|}{2}}\frac{1}{|y-x|}dy\\ &=C|x|^3u^q(\frac{|x|}{2})\leqslant C[|x|^3u^s(\frac{|x|}{2})]^{\frac{q}{s}}\ |x|^{3-\frac{3q}{s}}.\end{aligned}$$

Therefore, taking $s=q-\varepsilon$ (ε *small enough*), we have

$$\mathcal{A}_3\to 0,\quad\text{as}\quad|x|\to\infty.$$

This together with (3.4)–(3.6) implies that (3.3) holds. Hence, the proof of Lemma 3.1 is completed. ∎

Secondly, we discuss the vertical asymptotic behaviors of $u(x)$ and give the following lemma.

LEMMA 3.2. *Set $x=(0,0,x_3)\in\mathbb{R}^3$ $(x_3\neq 0)$. If $u\in L^t(\mathbb{R}^3)$ with $t=[3(q-1)]/2>1$ is a positive solution of the integral Equation (1.1) and*

$$\int_{\mathbb{R}^3}r^2(y)u^q(y)\,dy<\infty,\tag{3.7}$$

then we have

$$\begin{aligned}&\overline{\lim_{|x_3|\to\infty}}|x|^3u(x)=\limsup_{|x_3|\to\infty}|x|^3u(x)\\ &=\overline{\lim_{|x_3|\to\infty}}\int_{\mathbb{R}^3}\frac{r^2(y)|x_3|^3u^q(y)}{(r^2(y)+(x_3-y_3)^2)^{\frac{3}{2}}}u^q(y)\,dy=C_1\int_{\mathbb{R}^3}r^2(y)u^q(y)\,dy,\end{aligned}\tag{3.8}$$

where $C_1\in(\frac{1}{8},8)$.

Proof. To obtain (3.8), for $x=(0,0,x_3)\neq 0$, we rewrite

$$\begin{aligned}|x_3|^3u(0,0,x_3)&=\int_{\mathbb{R}^3}\frac{r^2(y)\ |x_3|^3\ u^q(y)\,dy}{[r^2(y)+(x_3-y_3)^2]^{\frac{3}{2}}}\\ &=\left(\int_{\mathbb{G}_1}+\int_{\mathbb{G}_2}+\int_{\mathbb{R}^3\setminus(\mathbb{G}_1\cup\mathbb{G}_2)}\right)\frac{r^2(y)\ |x_3|^3\ u^q(y)\,dy}{[r^2(y)+(x_3-y_3)^2]^{\frac{3}{2}}}\end{aligned}$$

$$\triangleq \mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3,$$

where

$$\mathbb{G}_1 \triangleq \left\{ y\in\mathbb{R}^3 \;\middle|\; r(y)\leqslant \frac{|x_3|}{2} \quad \text{and} \quad |z(y)|\leqslant \frac{|x_3|}{2} \right\}$$

and

$$\mathbb{G}_2 \triangleq \left\{ y\in\mathbb{R}^3 \;\middle|\; |r(y)-|x_3||\leqslant \frac{|x_3|}{2} \quad \text{and} \quad |z(y-x)|\leqslant \frac{|z(x)|}{2}=\frac{|x_3|}{2} \right\}.$$

Noting that $y\in\mathbb{G}_1$ and $|x_3|\neq 0$, by the dominated convergence theorem and (3.7), we have

$$\frac{1}{8}\leqslant \frac{|x_3|^3}{[r^2(y)+(x_3-y_3)^2]^{\frac{3}{2}}}\leqslant 8$$

and

$$\begin{aligned}
\overline{\lim_{|x_3|\to\infty}}\mathcal{B}_1 &= \overline{\lim_{|x_3|\to\infty}}\int_{\mathbb{R}^3}\frac{r^2(y)\;|x_3|^3\;u^q(y)\,dy}{[r^2(y)+(x_3-y_3)^2]^{\frac{3}{2}}}\\
&= C_1\int_{\mathbb{R}^3} r^2(y)u^q(y)\,dy, \quad C_1\in(\tfrac{1}{8},8).
\end{aligned}\tag{3.9}$$

Next, we turn to estimate $\mathcal{B}_2$. For $y\in\mathbb{G}_2$, it is easy to check that

$$\frac{|x_3|}{2}\leqslant r(y)\leqslant \frac{3|x_3|}{2}, \quad \frac{|x_3|}{2}\leqslant |z(y)|\leqslant \frac{3|x_3|}{2},$$

and by (3.7)

$$\begin{aligned}
\mathcal{B}_2 &= \int_{\mathbb{G}_2}\frac{r^2(y)\;|x_3|^3\;u^q(y)\,dy}{[r^2(y)+(x_3-y_3)^2]^{\frac{3}{2}}}\\
&\leqslant 8\int_{|y|\geqslant\frac{|x_3|}{2}} r^2(y)\;u^q(y)\,dy\to 0, \quad \text{as} \quad |x_3|\to\infty.
\end{aligned}\tag{3.10}$$

Finally, we estimate $\mathcal{B}_3$. Note that

$$\begin{aligned}
(\mathbb{G}_2)^C &= \left\{ y\in\mathbb{R}^3 \;\middle|\; |r(y)-|x_3||\geqslant \frac{|x_3|}{2} \quad \text{or} \quad |z(y-x)|\geqslant \frac{|z(x)|}{2} \right\}\\
&\subseteq \left\{ y\in\mathbb{R}^3 \;\middle|\; |r(y)-|x_3||\geqslant \frac{|x_3|}{2} \right\}\cup\left\{ y\in\mathbb{R}^3 \;\middle|\; |z(y-x)|\geqslant \frac{|z(x)|}{2} \right\}\\
&\triangleq \mathbb{G}_{2,1}^C\cup\mathbb{G}_{2,2}^C
\end{aligned}$$

and

$$\mathbb{R}^3\setminus(\mathbb{G}_1\cup\mathbb{G}_2)=\mathbb{R}^3\setminus\mathbb{G}_1\setminus\mathbb{G}_2=(\mathbb{G}_2)^C\setminus\mathbb{G}_1=(\mathbb{G}_{2,1}^C\setminus\mathbb{G}_1)\cup(\mathbb{G}_{2,2}^C\setminus\mathbb{G}_1).$$

Here we will discuss $\mathcal{B}_3$ in the following two cases:

Case 1. For $y\in\mathbb{G}_{2,1}^C=\{y\in\mathbb{R}^3 \mid |r(y)-|x_3||\geqslant |x_3|/2\}$, we have

$$y\in\left\{y\in\mathbb{R}^3 \middle|\; r(y)\geqslant\frac{3|x_3|}{2}\right\} \quad \text{or} \quad y\in\left\{y\in\mathbb{R}^3 \middle|\; r(y)\leqslant\frac{|x_3|}{2}\right\},$$

which implies that

$$\mathbb{G}_{2,1}^C \setminus \mathbb{G}_1 \subseteq (\mathbb{Q}_1 \cup \mathbb{Q}_2) \cap (\mathbb{Q}_3 \cup \mathbb{Q}_4),$$

where

$$\mathbb{Q}_1 = \left\{ y \in \mathbb{R}^3 \middle| \; r(y) > \frac{3|x_3|}{2} \right\}, \; \mathbb{Q}_2 = \left\{ y \in \mathbb{R}^3 \middle| \; r(y) \leqslant \frac{|x_3|}{2} \right\},$$
$$\mathbb{Q}_3 = \left\{ y \in \mathbb{R}^3 \middle| \; r(y) > \frac{|x_3|}{2} \right\}, \; \mathbb{Q}_4 = \left\{ y \in \mathbb{R}^3 \middle| \; |z(y)| > \frac{|x_3|}{2} \right\}.$$

Note that

$$\mathbb{Q}_2 \cap \mathbb{Q}_3 = \emptyset.$$

Therefore, it holds that

$$(\mathbb{Q}_1 \cup \mathbb{Q}_2) \cap (\mathbb{Q}_3 \cup \mathbb{Q}_4) = (\mathbb{Q}_1 \cap \mathbb{Q}_3) \cup (\mathbb{Q}_1 \cap \mathbb{Q}_4) \cup (\mathbb{Q}_2 \cap \mathbb{Q}_4).$$

With Fubini's theorem, it is easy to verify that

$$\begin{aligned}
&\int\int_{(\mathbb{Q}_1 \cap \mathbb{Q}_3) \cup (\mathbb{Q}_1 \cap \mathbb{Q}_4)} \frac{r^2(y)\ |x_3|^3\ u^q(y)}{[r^2(y) + (x_3 - y_3)^2]^{\frac{3}{2}}} \, dy_1 dy_2 dy_3 \\
\leqslant & \frac{16}{27} \int\int_{r(y) \geqslant \frac{3|x_3|}{2}} \frac{r^2(y)\ |x_3|^3\ u^q(y)}{|x_3|^3} \, dy_1 dy_2 dy_3 \\
\leqslant & \int_{r(y) \geqslant \frac{3|x_3|}{2}} \left(\int_{\mathbb{R}} r^2(y) u^q(y) \, dy_3 \right) dy_1 dy_2 \to 0, \quad |x_3| \to \infty.
\end{aligned}$$

For the case of $\mathbb{Q}_2 \cap \mathbb{Q}_4$, we rewrite

$$\begin{aligned}
\int\int_{\mathbb{Q}_2 \cap \mathbb{Q}_4} \frac{r^2(y)\ |x_3|^3\ u^q(y) \, dy}{[r^2(y) + (x_3 - y_3)^2]^{\frac{3}{2}}} &= \int_{r(y) \leqslant \frac{|x_3|}{2}} \int_{|z(y)| > \frac{|x_3|}{2}} \frac{r^2(y)\ |x_3|^3\ u^q(y) \, dy}{[r^2(y) + (x_3 - y_3)^2]^{\frac{3}{2}}} \\
&= \int_{r(y) \leqslant \frac{|x_3|}{2}} \left(\int_{|z(y)| > \frac{3|x_3|}{2}} + \int_{\frac{|x_3|}{2} < |z(y)| \leqslant \frac{3|x_3|}{2}} \right) \frac{r^2(y)\ |x_3|^3\ u^q(y) \, dy}{[r^2(y) + (x_3 - y_3)^2]^{\frac{3}{2}}} \\
&\triangleq \mathbb{I}_1 + \mathbb{I}_2.
\end{aligned}$$

It can be checked directly that

$$\mathbb{I}_1 \to 0, \qquad \text{as } |x_3| \to \infty.$$

In fact, by Fubini's theorem, we have

$$\begin{aligned}
\mathbb{I}_1 \leqslant & 8 \int_{r(y) \leqslant \frac{|x_3|}{2}} \int_{|z(y)| > \frac{3|x_3|}{2}} \frac{r^2(y)\ |x_3|^3\ u^q(y)}{|x_3|^3} \, dy_1 dy_2 dy_3 \\
\leqslant & \int_{|y_3| > \frac{3|x_3|}{2}} \left(\int_{\mathbb{R}^2} r^2(y) u^q(y) \, dy_1 dy_2 \right) dy_3 \to 0, \quad \text{as } |x_3| \to \infty.
\end{aligned}$$

Next, we estimate the term $\mathbb{I}_2$. Firstly we recall an important a priori estimate of $u(x)$. By Theorem 1.1, we obtain that all positive solutions of the integral Equation (1.1), on the axis x_3, are symmetric and decreasing with respect to some point $x_3 = \lambda_0$. At the same time, the solution of (1.1) is invariant under the translation transformation. Without loss of generality, we can assume that the positive solution $u(x)$ is radially symmetric and decreasing about $x_3=0$. Therefore, on $y\in\{r(y)\leqslant|x_3|/2$ and $|x_3|/2<|z(y)|\leqslant(3|x_3|)/2\}$, we obtain

$$\begin{aligned}&\int_{\frac{|x_3|}{2}<|z(y)|\leqslant\frac{3|x_3|}{2}}\frac{1}{[r^2(y)+(x_3-y_3)^2]^{\frac{3}{2}}}dy_3\\ &\leqslant\int_{|y_3-x_3|\leqslant\frac{5|x_3|}{2}}\frac{d(y_3-x_3)}{[r^2(y)+(x_3-y_3)^2]^{\frac{3}{2}}}\leqslant\frac{C\ |x_3|}{r^2(y)\sqrt{r^2(y)+|x_3|^2}}=\frac{C}{r^2(y)}.\end{aligned}$$

From the above, by Theorem 1.1, we have

$$\begin{aligned}\mathbb{I}_2=&\int_{r(y)\leqslant\frac{|x_3|}{2}}\int_{\frac{|x_3|}{2}<|z(y)|\leqslant\frac{3|x_3|}{2}}\frac{r^2(y)\ |x_3|^3\ u^q(y)}{[r^2(y)+(x_3-y_3)^2]^{\frac{3}{2}}}dy_1dy_2dy_3\\ \leqslant&\int_{r(y)\leqslant\frac{|x_3|}{2}}r^2(y)\ |x_3|^3\ u^q(y_1,y_2,\tfrac{|x_3|}{2})\,dy_1dy_2\int_{|y_3-x_3|\leqslant\frac{5|x_3|}{2}}\frac{d(y_3-x_3)}{[r^2(y)+(x_3-y_3)^2]^{\frac{3}{2}}}\\ \leqslant& C\int_{r(y)\leqslant\frac{|x_3|}{2}}r^2(y)\ |x_3|^3\ u^q(y_1,y_2,\tfrac{|x_3|}{2})\frac{|x_3|}{r^2(y)\sqrt{r^2(y)+|x_3|^2}}dy_1dy_2.\end{aligned}$$

On the other hand, since $q>3$ and $u\in L^q(\mathbb{R}^3)$, it is easy to check that for all $|x_3|>1$,

$$\begin{aligned}&C|x_3|^3\int_{\mathbb{R}^2}u^q(y_1,y_2,\frac{x_3}{2})\,dy_1dy_2\\ &=\int_{\mathbb{R}^2}\int_{\frac{|x_3|^3}{4}<|z(y)|\leqslant\frac{|x_3|^3}{2}}u^q(y_1,y_2,\frac{x_3}{2})\,dy_1dy_2dy_3\\ &\leqslant\int_{\frac{|x_3|^3}{4}<|z(y)|}\int_{\mathbb{R}^2}u^q(y_1,y_2,y_3)\,dy_1dy_2dy_3\to0,\quad\text{as } |x_3|\to\infty.\end{aligned}$$

Thus, we get

$$\begin{aligned}\mathbb{I}_2\leqslant& C\int_{r(y)\leqslant\frac{|x_3|}{2}}|x_3|^3\ u^q(y_1,y_2,\tfrac{|x_3|}{2})\,dy_1dy_2\\ \leqslant& C|x_3|^3\int_{\mathbb{R}^2}u^q(y_1,y_2,\frac{x_3}{2})\,dy_1dy_2\to0\quad\text{as}\quad|x_3|\to\infty.\end{aligned}\tag{3.11}$$

Case 2. Note that

$$\begin{aligned}y\in&\mathbb{G}_{2,2}^C\setminus\mathbb{G}_1\\ \subseteq&\left\{\ y\in\mathbb{R}^3\ \middle|\ |\ r(y)\geqslant\frac{|x_3|}{2}\ \text{and}\ |z(y-x)|\geqslant\frac{|z(x)|}{2}\right\}\\ &\cup\left\{\ y\in\mathbb{R}^3\ \middle|\ |z(y-x)|\geqslant\frac{|z(x)|}{2}\ \text{and}\ |z(y)|\geqslant\frac{|x_3|}{2}\right\}\triangleq\mathbb{S}_1\cup\mathbb{S}_2.\end{aligned}$$

Therefore, by Fubini's theorem, we have

$$\begin{aligned}&\int\int_{\mathbb{S}_1}\frac{r^2(y)\ |x_3|^3\ u^q(y)}{[r^2(y)+(x_3-y_3)^2]^{\frac{3}{2}}}dy_1dy_2dy_3\\ &\leqslant8\int_{r(y)>\frac{|x_3|}{2}}\left(\int_{\mathbb{R}}r^2(y)\ u^q(y)\,dy_3\right)dy_1dy_2\to0\quad\text{as } |x_3|\to\infty,\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}&\int\int_{\mathbb{S}_1} \frac{r^2(y)\ |x_3|^3\ u^q(y)}{[r^2(y)+(x_3-y_3)^2]^{\frac{3}{2}}}dy_1dy_2dy_3\\ &\leqslant 8\int_{z(y)>\frac{|x_3|}{2}}\left(\int_{\mathbb{R}^2} r^2(y)\ u^q(y)\,dy_1dy_2\right)dy_3\to 0 \quad \text{as } |x_3|\to\infty. \end{aligned}\tag{3.13}$$

By (3.9)–(3.13), we obtain (3.8). Hence, the proof of Lemma 3.2 is completed. □

By Lemmas 3.1–3.2, we immediately obtain Theorem 1.2.

REFERENCES

[1] W.X. Chen and C.M. Li, *An integral system and the Lane-Emden conjecture*, Discrete Contin. Dyn. Syst., 24(4):1167–1184, 2009. 1
[2] W.X. Chen and C.M. Li, *Methods on Nonlinear Elliptic Equations*, AIMS Series on Differential Equations & Dynamical Systems, American Institute of Mathematical Sciences (AIMS), Springfield, MO, 4, 2010. 2.3, 2.3
[3] W.X. Chen, C.M. Li, and B. Ou, *Classification of solutions for a system of integral equations*, Comm. Part. Diff. Eqs., 30(1-3):59–65, 2005. 2.2
[4] W.X. Chen, C.M. Li, and B. Ou, *Classification of solutions for an integral equation*, Comm. Pure Appl. Math., 59(3):330–343, 2006. 1, 1.1, 2.2
[5] Z. Cheng and C.M. Li, *An extended discrete Hardy-Littlewood-Sobolev inequality*, Discrete Contin. Dyn. Syst., 34(5):1951–1959, 2014. 1
[6] P. Federbush, T. Luo, and J. Smoller, *Existence of magnetic compressible fluid stars*, Arch. Ration. Mech. Anal., 215(2):611–631, 2015. 1, 1
[7] F. Jiang and S. Jiang, *On the stabilizing effect of the magnetic fields in the magnetic Rayleigh-Taylor problem*, SIAM J. Math. Anal., 50(1):491–540, 2018. 1
[8] F. Jiang and S. Jiang, *Nonlinear stability and instability in the Rayleigh-Taylor problem of stratified compressible MHD fluids*, Calc. Var. Part. Diff. Eqs., 58(1):29, 2019. 1
[9] F. Jiang and S. Jiang, *On the dynamical stability and instability of Parker problem*, Phys. D, 391:17–51, 2019. 1
[10] C. Jin and C.M. Li, *Quantitative analysis of some system of integral equations*, Calc. Var. Part. Diff. Eqs., 26(4):447–457, 2006. 1, 2.3
[11] C. Jin and C.M. Li, *Symmetry of solutions to some systems of integral equations*, Proc. Amer. Math. Soc., 134(6):1661–1670, 2006. 1
[12] Y.T. Lei, *Asymptotic properties of positive solutions of the Hardy-Sobolev type equations*, J. Diff. Eqs., 254(4):1774–1799, 2013. 1
[13] Y.T. Lei and C.M. Li, *Sharp criteria of Liouville type for some nonlinear systems*, Discrete Contin. Dyn. Syst., 36(6):3277–3315, 2016. 1
[14] C.M. Li and J. Villavert, *An extension of the Hardy-Littlewood-Pólya inequality*, Acta Math. Sci. Ser. B (Engl. Ed.), 31(6):2285–2288, 2011. 1
[15] Y.Y. Li, *Remark on some conformally invariant integral equations: the method of moving spheres*, J. Eur. Math. Soc., 6(2):153–180, 2004. 1, 1.1
[16] E.H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. Math., 118(2):349–374, 1983. 1, 1, 2.1
[17] G.Z. Lu and J.Y. Zhu, *Symmetry and regularity of extremals of an integral equation related to the Hardy-Sobolev inequality*, Calc. Var. Part. Diff. Eqs., 42(3-4):563–577, 2011. 1
[18] J.K. Xu, H.X. Wu, and Z. Tan, *The non-existence results for a class of integral equation*, J. Diff. Eqs., 256(5):1873–1902, 2014. 1, 1.1
[19] X.W. Xu, *Uniqueness theorem for integral equations and its application*, J. Funct. Anal., 247(1):95–109, 2007. 1, 1.1