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THE UNIQUE GLOBAL SOLVABILITY OF MULTI-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES-POISSON-KORTEWEG MODEL*

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Abstract. The present paper is dedicated to the study of the Cauchy problem for compressible Navier-Stokes-Poisson-Korteweg model in any dimension $d \geq 2$, which simultaneously involves the lower order potential term and the higher order capillarity term. The unique global solvability of the system is obtained when the initial data are close to a stable equilibrium state in a functional setting invariant by the scaling of the associated equations. In particular, one may construct the unique global solution for a class of large highly oscillating initial velocities in physical dimensions $d = 2,3$.

Keywords. unique global solvability; highly oscillating velocity; compressible Navier-Stokes-Poisson-Korteweg model; critical Besov spaces.

AMS subject classifications. 35M20; 35Q35.

1. Introduction and main results

In this paper, we consider the following multi-dimensional compressible Navier-Stokes-Poisson-Korteweg model in $\mathbb{R}^d (d \geq 2)$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho) Du) - \nabla(\lambda(\rho)\operatorname{div} u) + \nabla P(\rho) = -\rho \nabla \phi + \operatorname{div} \mathcal{K}, \\ -\Delta \phi = \rho - \bar{\rho}, \end{cases} \tag{1.1}$$

where $\rho = \rho(t,x)$, $u = u(t,x)$ and $\phi = \phi(t,x)$ are the unknown functions, representing the density, the velocity and the potential force, respectively. $P = P(\rho)$ is pressure satisfying $P'(\rho) > 0$ and for all $\rho > 0$. The coefficients $\lambda = \lambda(\rho)$ and $\mu = \mu(\rho)$ designate the bulk and shear viscosities, respectively, and are assumed to satisfy in the neighborhood of some reference constant density $\bar{\rho} > 0$ the conditions

$$\mu > 0 \quad \text{and} \quad \nu \triangleq \lambda + 2\mu > 0. \tag{1.2}$$

$D(u) \overset{\text{def}}{=} \frac{1}{2}(Du + {}^T Du)$ is the deformation tensor and the capillarity tensor is given by

$$\mathcal{K} \triangleq \rho \operatorname{div}(\kappa(\rho)\nabla\rho) \mathrm{I}_{\mathbb{R}^d} + \frac{1}{2}\big(\kappa(\rho) - \rho\kappa'(\rho)\big)|\nabla\rho|^2 \mathrm{I}_{\mathbb{R}^d} - \kappa(\rho)\nabla\rho \otimes \nabla\rho.$$

The density-dependent capillarity function κ is assumed to be positive. Note that for smooth enough density and κ, we have (see [3])

$$\operatorname{div}\mathcal{K} = \rho \nabla \Big(\kappa(\rho)\Delta\rho + \frac{1}{2}\kappa'(\rho)|\nabla\rho|^2\Big)\cdot \tag{1.3}$$

Here, we are concerned with the Cauchy problem of the system (1.1) in $\mathbb{R}_+ \times \mathbb{R}^d$ subject to the initial data

$$(\rho, u, \phi)|_{t=0} = (\rho_0, u_0, \phi_0) \tag{1.4}$$

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satisfying compatibility condition

$$-\Delta\phi_0=\rho_0-\bar{\rho}.$$

System (1.1) can be used to describe physical phenomena in plasmas and semiconductors, see the pioneering work by Dunn and Serrin [15] and also Anderson et al. [1] and Cahn and Hilliard [5]. We would like to point out that System (1.1) includes several important models as special cases. When $\mathcal{K}=0$, (1.1) reduces to the compressible Navier-Stokes-Poisson model. As for classical solutions, Li-Matsumura-Zhang [22] proved the global existence and time decay estimates in the three-dimensional case under the assumption that data are close to the constant equilibrium state. In 2009, Hao-Li [16] proved the global existence and uniqueness of strong solutions in the framework of hybrid Besov spaces in three and higher dimensions. Recently, still in dimension $d\geq 3$, by adapting the works by Charve-Danchin [6] and Chen-Miao-Zhang [11], Zheng [30] removed the extra-assumption on the velocity from [16] and extended the global existence result to the L^p critical framework. In 2017, Chikami and Danchin [12] further improved the known result in [30] and established the unique global solvability and time decay estimates in any dimension $d\geq 2$ for small perturbations of a linearly stable constant state. It is a remarkable fact that the results mentioned above were obtained under the condition of constant viscosity coefficients. When there is no potential force ϕ, (1.1) becomes the compressible Navier-Stokes-Korteweg model, which attracted the attention of many researchers during the recent decades. Hattori and Li [20, 21] established the local existence of smooth solutions with large initial data and global existence of smooth solutions around constant states of the compressible Navier-Stokes-Korteweg model for small initial data (ρ_0,u_0) in Sobolev spaces $H^s(\mathbb{R}^d)\times H^{s-1}(\mathbb{R}^d)$ with $s\geq \frac{d}{2}+4$ and $d=2,3$, respectively. Recently, researchers in [27, 28] proved the global existence and decay rate of strong solutions for small initial data in some Sobolev spaces which have lower regularity than that of [21] in three dimensional case. In 2016, Li and Yong [23] investigated the zero Mach number limit for the three-dimensional model in the regime of smooth solutions. In L^2-critical Besov spaces, Danchin and Desjardins [14] and Haspot [18, 19] obtained the global well-posedness of strong solutions close to a stable equilibrium state. In 2018, Charve et al. [7] established the global existence, Gevrey analytic and algebraic time-decay estimates of strong solutions when the initial data are close to a stable equilibrium state in L^p-critical framework. When one simultaneously considers the effects of the electrostatic potential and the capillarity, that is, System (1.1), Wang and Yang [29] studied the quasi-neutral limit of global weak solutions in the torus $\mathbb{T}^3$. Li and Yong [24] presented the local-in-time existence of smooth solutions and studied the quasi-neutral limit. Later, Li et al. [25] showed the global-in-time existence of smooth solutions with small initial data and discussed some limit analysis. Here, it should be pointed out that the functional spaces with high Sobolev regularity is not *the lowest index* in the sense of the scaling invariant of the associated System (1.1) and the dimension of space is only limited to $d=3$.

A natural question follows then, that is, whether the global well-posedness in *the lowest index* functional spaces can be shown for the Cauchy problem (1.1)-(1.4). The main motivation of this paper is to give a positive answer to this question and establish the global solvability of strong solutions when the initial data are close to a stable equilibrium state in more general critical Besov spaces related to the L^p spaces for any dimension $d\geq 2$. At this stage, let us recall that, by definition, critical spaces for System (1.1) are norm invariant for all $l>0$ by the scaling transformations $T_l:\big(\rho(t,x),u(t,x),\phi(t,x)\big)\to\big(\rho(l^2t,lx),lu(l^2t,lx),l^2\phi(l^2t,lx)\big)$, in accordance with the fact that (ρ,u,ϕ) is a solution to System (1.1) if and only if so does $T_l(\rho,u,\phi)$, corresponding to the dilated initial data $(\rho_0(lx),lu_0(lx),l^2\phi_0(lx))$, provided that the pressure P has been changed into l^2P. Due to the similarity of the compressible Navier-Stokes-Poisson-Korteweg model to the compressible Navier-Stokes equations, we can apply some ideas developed in proving the existence of solutions to the compressible Navier-Stokes equations to deal with System (1.1). We refer the readers to [6,11,17]. However, it is non-trivial to apply directly the ideas from [6,11,17] to System (1.1) which simultaneously involves the lower order electrostatic potential term $\nabla(-\Delta)^{-1}\rho$ and the higher order capillarity term $\nabla\Delta\rho$, which makes it rather difficult to get the desired global *a priori* estimates. Now, let us explain some of the main difficulties and techniques involved in the process. In fact, System (1.1) is a hyperbolic-parabolic system with a non-local term $\nabla(-\Delta)^{-1}\rho$ arising from the lower order electrostatic potential ϕ. The symbol of this non-local term is singular in the low frequencies of the Fourier transform $\mathcal{G}_1(x,t)$ of Green's matrix $\mathcal{G}_1(x,t)$ (see Proposition 4.1), which plays a bad role in our analysis. In order to overcome the difficulty, we introduce a new unknown $a=\Lambda^{-1}c$ which transfers the system (4.1) into (4.8) without the pseudo-differential operator of order -1 in the linear part of the momentum equation. Here, for the new System (4.8), we do not directly study the hyperbolic-parabolic linear system with convection terms as in [16] but present an explicit derivation of the Fourier transform of Green's matrix $\mathcal{G}(x,t)$ corresponding to the linearized system by the Fourier transform and then perform its spectral analysis. In particular, we exhibit that $\widehat{\mathcal{G}}(\xi,t)$ behaves like the heat kernel in the low frequencies. Based on the important property, we further exploit the smoothing effects of (a,u) in the low frequencies, which naturally implies the same property of $(\rho-1,u)$. In contrast, in the high frequencies, we notice that the non-local term $\nabla(-\Delta)^{-1}\rho$ can be treated as the harmless perturbation term. Therefore, we only focus on the hyperbolic-parabolic system with the higher order capillarity term $\nabla\Delta\rho$. As in Haspot [17] for the standard compressible barotropic Navier-Stokes equations, we introduce some suitable effective velocity field $\omega=\mathcal{Q}u+\nu^{-1}(-\Delta)^{-1}\nabla c$ (named viscous effective flux in Hoff's work [10]) and observe a suitable linear combination of w and ∇c satisfying a heat equation involving some harmless lower-order terms. Taking advantage of the smoothing effects from the heat equation, we then fully show the parabolic properties for the density and velocity in the high frequencies. In fact, the important feature stems from the presence of the Korteweg tensor and enables us to apply fixed-point argument, which is different from the barotropic compressible Navier-Stokes equations. With these analysis tools in hand, employing contraction mapping principle, we eventually obtain the unique global solvability of strong solutions to the Cauchy problem (1.1)-(1.4). Finally, let us emphasize that the result allows us to construct global strong solutions for some highly oscillating initial velocity data.

Now we state our main results as follows:

THEOREM 1.1. *Let $d\geq 2$, $p\in[2,\min(4,2d/(d-2))]$ with, additionally, $p\neq 4$ if $d=2$, and denote $c_0:=\rho_0-\bar{\rho}$. There exists a small enough constant η such that if $c_0^h\in\dot{B}^{\frac{d}{p}}_{p,1}$ and $u_0^h\in\dot{B}^{\frac{d}{p}-1}_{p,1}$ with besides $c_0^\ell\in\dot{B}^{\frac{d}{2}-2}_{2,1}$ and $u_0^\ell\in\dot{B}^{\frac{d}{2}-1}_{2,1}$ satisfy*

$$X_{p,0}\overset{def}{=}\|c_0\|^\ell_{\dot{B}^{\frac{d}{2}-2}_{2,1}}+\|u_0\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|c_0\|^h_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|u_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\leq\eta, \tag{1.5}$$

then (1.1)-(1.4) has a unique global-in-time solution (ρ,u) in the space X_p defined by

$$c^\ell\in\mathcal{C}(\mathbb{R}_+;\dot{B}^{\frac{d}{2}-2}_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{2}}_{2,1}),\quad u^\ell\in\mathcal{C}(\mathbb{R}_+;\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{2}+1}_{2,1}),$$

$$c^h \in \mathcal{C}(\mathbb{R}_+;\dot{B}^{\frac{d}{p}}_{p,1})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{p}+2}_{p,1}), \quad u^h \in \mathcal{C}(\mathbb{R}_+;\dot{B}^{\frac{d}{p}-1}_{p,1})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{p}+1}_{p,1}). \tag{1.6}$$

Remark 1.2. Compared with [24,25], we establish the global well-posedness of strong solutions in the so-called critical Besov spaces in any dimension $d\geq 2$ and the dimension of space is more extensive and is not limited to $d=3$.

Remark 1.3. In Theorem 1.1, the regularity index for the high frequency part of u_0 may be negative. Especially, this allows us to obtain the global well-posedness of the system (1.1) for the highly oscillating initial velocity u_0. For example, let

$$u_0(x)=\sin\left(\frac{x_1}{\varepsilon}\right)\phi(x), \quad \phi(x)\in\mathcal{S}(\mathbb{R}^d).$$

Thus for any $\varepsilon>0$

$$\|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\leq C\varepsilon^{1-\frac{d}{p}} \quad \text{for} \quad p>d.$$

Hence such data with small enough ϵ generate global unique solutions in dimension $d=2,3$.

The rest of the paper unfolds as follows. In the next section, we recall some basic facts about Littlewood-Paley decomposition, Besov spaces and some useful lemmas. Then, in Section 3, to make it more convenient to study, we reformulate the original system (1.1)-(1.4). Section 4 is devoted to the proof of the global well-posedness for initial data near equilibrium in critical Besov spaces.

Notations. We assume C be a positive generic constant throughout this paper that may vary at different places and denote $A\leq CB$ by $A\lesssim B$. We shall also need the notations

$$z^\ell \stackrel{\text{def}}{=} \sum_{j\leq k_0}\dot{\Delta}_j z \quad \text{and} \quad z^h \stackrel{\text{def}}{=} z-z^\ell, \quad \text{for some } j_0.$$

$$\|z\|^\ell_{\dot{B}^s_{p,1}} \stackrel{\text{def}}{=} \sum_{j\leq k_0} 2^{js}\|\dot{\Delta}_j z\|_{L^p} \quad \text{and} \quad \|z\|^h_{\dot{B}^s_{p,1}} \stackrel{\text{def}}{=} \sum_{j\geq k_0} 2^{js}\|\dot{\Delta}_j z\|_{L^p}, \quad \text{for some } j_0.$$

Noting the small overlap between low and high frequencies, we have

$$\|z^\ell\|_{\dot{B}^s_{p,1}}\lesssim\|z\|^\ell_{\dot{B}^s_{p,1}} \quad \text{and} \quad \|z^h\|_{\dot{B}^s_{p,1}}\lesssim\|z\|^h_{\dot{B}^s_{p,1}}.$$

2. Littlewood-Paley theory and some useful lemmas

Let us introduce the Littlewood-Paley decomposition. Choose a radial function $\varphi\in\mathcal{S}(\mathbb{R}^d)$ supported in $\mathcal{C}=\{\xi\in\mathbb{R}^d, \frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$ such that

$$\sum_{q\in\mathbb{Z}}\varphi(2^{-q}\xi)=1 \quad \text{for all}\, \xi\neq 0.$$

The homogeneous frequency localization operators $\dot{\Delta}_q$ and $\dot{S}_q$ are defined by

$$\dot{\Delta}_q f=\varphi(2^{-q}D)f, \quad \dot{S}_q f=\sum_{k\leq q-1}\dot{\Delta}_k f \quad \text{for} \quad q\in\mathbb{Z}.$$

With our choice of φ, one can easily verify that

$$\begin{aligned}&\dot{\Delta}_q\dot{\Delta}_k f=0 \quad \text{if} \quad |q-k|\geq 2 \quad \text{and}\\ &\dot{\Delta}_q(\dot{S}_{k-1}f\dot{\Delta}_k f)=0 \quad \text{if} \quad |q-k|\geq 5.\end{aligned}$$

We denote the space $\mathcal{Z}'(\mathbb{R}^d)$ by the dual space of $\mathcal{Z}(\mathbb{R}^d)=\{f\in\mathcal{S}(\mathbb{R}^d); D^\alpha\hat{f}(0)=0;\forall\alpha\in\mathbb{N}^d \text{multi-index}\}$, it also can be identified by the quotient space of $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$ with the polynomial space $\mathcal{P}$. The formal equality

$$f=\sum_{q\in\mathbb{Z}}\dot{\Delta}_q f$$

holds true for $f\in\mathcal{Z}'(\mathbb{R}^d)$ and is called the homogeneous Littlewood-Paley decomposition.

The basic tool of the paradifferential calculus is Bony's decomposition [4]. Formally, the product of two tempered distributions fu and g may be decomposed into

$$fg=T_f g+R(f,g)+T_g f$$

with

$$T_f g \overset{\text{def}}{=} \sum_q \dot{S}_{q-1}f\,\dot{\Delta}_q g \quad \text{and} \quad R(f,g)\overset{\text{def}}{=}\sum_q\sum_{|q'-q|\leq 1}\dot{\Delta}_q f\,\dot{\Delta}_{q'}g.$$

The usual product is continuous in many Besov spaces. The following proposition, the proof of which may be found in [26] Section 4.4 (see in particular inequality (28) page 174), will be very useful.

PROPOSITION 2.1. *For all $1\leq r,p,p_1,p_2\leq+\infty$, there exists a positive universal constant such that*

$$\begin{aligned}&\|fg\|_{\dot{B}^s_{p,r}}\lesssim\|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,r}}+\|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}}, \quad \textit{if} \quad s>0;\\ &\|fg\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,r}}\lesssim\|f\|_{\dot{B}^{s_1}_{p,r}}\|g\|_{\dot{B}^{s_2}_{p,\infty}}, \quad \textit{if} \quad s_1,s_2<\frac{d}{p}, \quad \textit{and} \quad s_1+s_2>0;\\ &\|fg\|_{\dot{B}^s_{p,r}}\lesssim\|f\|_{\dot{B}^s_{p,r}}\|g\|_{\dot{B}^{\frac{d}{p}}_{p,\infty}\cap L^\infty}, \quad \textit{if} \quad |s|<\frac{d}{p};\\ &\|fg\|_{\dot{B}^s_{2,1}}\lesssim\|f\|_{\dot{B}^{d/2}_{2,1}}\|g\|_{\dot{B}^s_{2,1}}, \quad \textit{if} \quad s\in(-d/2,d/2].\end{aligned}$$

The following Bernstein's inequalities will be frequently used.

LEMMA 2.2 ([8]). *Let $1\leq p_1\leq p_2\leq+\infty$. Assume that $f\in L^{p_1}(\mathbb{R}^d)$, then for any $\gamma\in(\mathbb{N}\cup\{0\})^d$, there exist constants C_1, C_2 independent of f, q such that*

$$\begin{aligned}&\text{supp}\hat{f}\subseteq\{|\xi|\leq A_0 2^q\}\Rightarrow\|\partial^\gamma f\|_{p_2}\leq C_1 2^{q|\gamma|+qd(\frac{1}{p_1}-\frac{1}{p_2})}\|f\|_{p_1},\\ &\text{supp}\hat{f}\subseteq\{A_1 2^q\leq|\xi|\leq A_2 2^q\}\Rightarrow\|f\|_{p_1}\leq C_2 2^{-q|\gamma|}\sup_{|\beta|=|\gamma|}\|\partial^\beta f\|_{p_1}.\end{aligned}$$

Let us recall the definition of homogeneous Besov spaces (see [2, 13]).

DEFINITION 2.3. *Let $s\in\mathbb{R}$, $1\leq p,r\leq+\infty$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by*

$$\dot{B}^s_{p,r}=\Big\{f\in\mathcal{Z}'(\mathbb{R}^d):\|f\|_{\dot{B}^s_{p,r}}<+\infty\Big\},$$

where

$$\|f\|_{\dot B^s_{p,r}} \stackrel{def}{=} \Big\| 2^{qs}\|\dot\Delta_q f(t)\|_p \Big\|_{\ell^r}.$$

REMARK 2.4. Some properties about the Besov spaces are as follows

- Derivation:

$$\|f\|_{\dot B^s_{2,1}} \approx \|\nabla f\|_{\dot B^{s-1}_{2,1}};$$

- Algebraic properties: for $s>0$, $\dot B^s_{2,1}\cap L^\infty$ is an algebra;
- Interpolation: for $s_1,s_2\in\mathbb{R}$ and $\theta\in[0,1]$, we have

$$\|f\|_{\dot B^{\theta s_1+(1-\theta)s_2}_{2,1}} \le \|f\|^{\theta}_{\dot B^{s_1}_{2,1}} \|f\|^{(1-\theta)}_{\dot B^{s_2}_{2,1}}.$$

DEFINITION 2.5. *Let $s\in\mathbb{R}$, $1\le p,\rho,r\le+\infty$. The homogeneous space-time Besov space $L^\rho_T(\dot B^s_{p,r})$ is defined by*

$$L^\rho_T(\dot B^s_{p,r}) = \Big\{ f\in\mathbb{R}_+\times\mathcal{Z}'(\mathbb{R}^d) : \|f\|_{L^\rho_T(\dot B^s_{p,r})} < +\infty \Big\},$$

where

$$\|f\|_{L^\rho_T(\dot B^s_{p,r})} \stackrel{def}{=} \Big\| \big\| 2^{qs}\|\dot\Delta_q f\|_{L^p} \big\|_{\ell^r} \Big\|_{L^\rho_T}.$$

We next introduce the Besov-Chemin-Lerner space $\widetilde L^q_T(\dot B^s_{p,r})$ which is initiated in [9].

DEFINITION 2.6. *Let $s\in\mathbb{R}$, $1\le p,q,r\le+\infty$, $0<T\le+\infty$. The space $\widetilde L^q_T(\dot B^s_{p,r})$ is defined by*

$$\widetilde L^q_T(\dot B^s_{p,r}) = \Big\{ f\in\mathbb{R}_+\times\mathcal{Z}'(\mathbb{R}^d) : \|f\|_{\widetilde L^q_T(\dot B^s_{p,r})} < +\infty \Big\},$$

where

$$\|f\|_{\widetilde L^q_T(\dot B^s_{p,r})} \stackrel{def}{=} \Big\| 2^{qs}\|\dot\Delta_q f(t)\|_{L^q(0,T;L^p)} \Big\|_{\ell^r}.$$

Obviously, $\widetilde L^1_T(\dot B^s_{p,1}) = L^1_T(\dot B^s_{p,1})$. By a direct application of Minkowski's inequality, we have the following relations between these spaces

$$L^\rho_T(\dot B^s_{p,r}) \hookrightarrow \widetilde L^\rho_T(\dot B^s_{p,r}), \text{if} \quad r\ge\rho,$$

$$\widetilde L^\rho_T(\dot B^s_{p,r}) \hookrightarrow L^\rho_T(\dot B^s_{p,r}), \text{if} \quad \rho\ge r.$$

For the composition of functions, we have the following estimates.

PROPOSITION 2.7 ([13]). *Let $s>0$, $1\le p\le\infty$ and $u\in\dot B^s_{p,1}\cap L^\infty$. If $F\in W^{[s]+2,\infty}_{loc}(\mathbb{R}^d)$ with $F(0)=0$, then $F(u)\in\dot B^s_{p,1}$. Moreover, there exists a function of one variable C_0 depending only on s and F, and such that*

$$\|F(u)\|_{\dot B^s_{p,1}} \le C_0(\|u\|_{L^\infty})\|u\|_{\dot B^s_{p,1}}.$$

PROPOSITION 2.8 ([2, 7]). *Let* $\sigma\in\mathbb{R},(p,r)\in[1,\infty]^2$ *and* $1\le\rho_2\le\rho_1\le\infty$. *Let* u *satisfy*

$$\begin{cases}\partial_t u-\mu\Delta u=f,\\ u\,|_{t=0}=u_0.\end{cases}\tag{2.1}$$

For all $T>0$, *(i) if* $\mu>0$, *we have*

$$\mu^{\frac{1}{\rho_1}}\|u\|_{\widetilde{L}_T^{\rho_1}(\dot{B}_{p,r}^{\sigma+\frac{2}{\rho_1}})}\lesssim\|u_0\|_{\dot{B}^{\sigma}_{p,r}}+\mu^{\frac{1}{\rho_2}-1}\|f\|_{\widetilde{L}_T^{\rho_2}(\dot{B}_{p,r}^{\sigma-2+\frac{2}{\rho_2}})}.\tag{2.2}$$

(ii) if $\mu\in\mathbb{C}$ *and* $\mathrm{Re}\mu>0$, *we have*

$$(\mathrm{Re}\mu)^{\frac{1}{\rho_1}}\|u\|_{\widetilde{L}_T^{\rho_1}(\dot{B}_{p,r}^{\sigma+\frac{2}{\rho_1}})}\lesssim\|u_0\|_{\dot{B}^{\sigma}_{p,r}}+(\mathrm{Re}\mu)^{\frac{1}{\rho_2}-1}\|f\|_{\widetilde{L}_T^{\rho_2}(\dot{B}_{p,r}^{\sigma-2+\frac{2}{\rho_2}})}.\tag{2.3}$$

3. Reformulation of the original system (1.1)-(1.4)

To make it more convenient to study, we reformulate the original system (1.1)-(1.4) into a different form. Without loss of generality, we will assume that $\bar{\rho}=1$, $P'(1)=1$, and denote that $c=\rho-1$. Then, in terms of the new variables (c,u), the system (1.1)-(1.4) rewrites

$$\begin{cases}\partial_t c+\mathrm{div}u=f,\\ \partial_t u-\mathcal{A}u+\nabla c+\nabla(-\Delta)^{-1}c-\kappa\nabla\Delta c=g,\\ (c,u)|_{t=0}=(c_0,u_0),\end{cases}\tag{3.1}$$

where

$$\begin{aligned}&f=f(c,u)=-\mathrm{div}(cu),\\ &g=g(c,u)=-u\cdot\nabla u-L_1(c)\mathcal{A}u+L_2(c)\nabla c+L_3(c)\Big(\mathrm{div}\big(2\widetilde{\mu}(c)D(u)\big)+\nabla\big(\widetilde{\lambda}(c))\mathrm{div}u\big)\Big)\\ &\qquad\qquad+\nabla\Big(\tilde{\kappa}(c)\Delta c+\frac{1}{2}\nabla\tilde{\kappa}(c)\cdot\nabla c\Big)\end{aligned}$$

with

$$\mathcal{A}u\stackrel{\mathrm{def}}{=}\mu\Delta u+(\lambda+\mu)\nabla\mathrm{div}u,\qquad L_1(c)\stackrel{\mathrm{def}}{=}\frac{c}{1+c},\qquad L_2(c)\stackrel{\mathrm{def}}{=}\frac{p'(1+c)}{1+c}-1,$$

$$L_3(c)\stackrel{\mathrm{def}}{=}\frac{1}{c+1},\qquad \lambda\stackrel{\mathrm{def}}{=}\lambda(1),\qquad \mu\stackrel{\mathrm{def}}{=}\mu(1),$$

$$\widetilde{\mu}(c)\stackrel{\mathrm{def}}{=}\mu(1+c)-\mu(1),\qquad \widetilde{\lambda}(c)\stackrel{\mathrm{def}}{=}\lambda(1+c)-\lambda(1),\qquad \kappa\stackrel{\mathrm{def}}{=}\kappa(1),$$

$$\tilde{\kappa}(c)\stackrel{\mathrm{def}}{=}\kappa(1+c)-\kappa(1).$$

For $s\in\mathbb{R}$, we denote $\Lambda^s h=\mathcal{F}^{-1}(|\xi|^s\widehat{h})$. Let us decompose u into $u=\mathcal{P}u+\mathcal{Q}u$, where $\mathcal{P}$ and $\mathcal{Q}$ are the projectors onto divergence-free and potential vector-fields, respectively (hence $\mathcal{P}=\mathrm{Id}+\nabla(-\Delta)^{-1}\mathrm{div}$). Set $v:=\Lambda^{-1}\mathrm{div}u=\Lambda^{-1}\mathrm{div}\mathcal{Q}u$ with $\Lambda=(-\Delta)^{\frac{1}{2}}$. Thus, $(c,v,\mathcal{P}u)$ satisfies

$$\begin{cases}\partial_t c+\Lambda v=f,\\ \partial_t v-\nu\Delta v-\Lambda c-\Lambda^{-1}c-\kappa\Lambda^3 c=g_1,\\ \partial_t\mathcal{P}u-\mu\Delta\mathcal{P}u=\mathcal{P}g,\end{cases}\tag{3.2}$$

where $g_1=-\Lambda^{-1}\mathrm{div}g$.

4. Global well-posedness for initial data near equilibrium

In this section, we will prove global existence and uniqueness of strong solutions to the system (3.1) in L^p-type critical regularity framework.

4.1. Global *a priori* estimates.

The subsection is devoted to exploiting important global *a priori* estimates for the system (3.1). It is divided into two steps as follows.

Step 1: Low frequencies. From (3.2), we find that the interaction between the velocity and the density only involves the compressible part of the velocity, namely v. The incompressible part $\mathcal{P}u$ satisfies a mere heat equation. The coupling system including c and v in (3.2) reads

$$\begin{cases} \partial_t c + \Lambda v = f, \\ \partial_t v - \nu\Delta v - \Lambda c - \Lambda^{-1} c - \kappa\Lambda^3 c = g_1, \\ (c,v)|_{t=0} = (c_0, v_0). \end{cases} \tag{4.1}$$

To better study properties of (c,v) in the low frequencies, we make some analysis for Green's matrix $\mathcal{G}_1(x,t)$ of the following linearized system without outer forces, namely

$$\begin{cases} \partial_t c + \Lambda v = 0, \\ \partial_t v - \nu\Delta v - \Lambda c - \Lambda^{-1} c - \kappa\Lambda^3 c = 0, \\ (c,v)|_{t=0} = (c_0, v_0). \end{cases} \tag{4.2}$$

Proposition 4.1. *Let $\mathcal{G}_1$ be the Green matrix of the system (4.2). Then we have the following explicit expression for $\widehat{\mathcal{G}_1}$:*

$$\widehat{\mathcal{G}_1}(\xi,t) = \begin{bmatrix} \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} & -\left(\frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right)|\xi| \\ -\left(\frac{e^{\lambda_- t} - e^{\lambda_+ t}}{\lambda_+ - \lambda_-}\right)(|\xi| + |\xi|^{-1} + \kappa|\xi|^3) & \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \end{bmatrix}, \tag{4.3}$$

where

$$\lambda_\pm = -\frac{1}{2}\nu|\xi|^2 \pm \frac{1}{2}\sqrt{(\nu^2 - 4\kappa)|\xi|^4 - 4(1+|\xi|^2)}.$$

Proof. Taking Fourier transforms to the linearized system (4.2) yields that

$$\begin{cases} \hat{c}_t + |\xi|\hat{v} = 0, \\ \hat{v}_t + \nu|\xi|^2\hat{v} - (|\xi| + |\xi|^{-1} + \kappa|\xi|^3)\hat{c} = 0. \end{cases} \tag{4.4}$$

Differentiating with respect to the time variable t in the second equation of (4.4) gives

$$\hat{v}_{tt} + \nu|\xi|^2\hat{v}_t - (|\xi| + |\xi|^{-1} + \kappa|\xi|^3)\hat{c}_t = 0.$$

Combining it with the first equation of (4.4), we get

$$\begin{cases} \hat{v}_{tt} + \nu|\xi|^2\hat{v}_t + (1 + |\xi|^2 + \kappa|\xi|^4)\hat{v} = 0, \\ \hat{v}(\xi,0) = \hat{v}_0(\xi), \quad \hat{v}_t(\xi,0) = -\nu|\xi|^2\hat{v}_0(\xi) + (|\xi| + |\xi|^{-1} + \kappa|\xi|^3)\hat{c}_0(\xi). \end{cases} \tag{4.5}$$

It is easy to check that $\lambda_\pm$ are two roots of the corresponding indicial equation of (4.5). Thus, we may assume that the solution of (4.5) has the form

$$\hat{v}(\xi,t)=A(\xi)e^{\lambda_-(\xi)t}+B(\xi)e^{\lambda_+(\xi)t}.$$

Using the initial conditions, we obtain

$$A=\frac{(\lambda_+ +\nu|\xi|^2)\hat{v}_0-(|\xi|+|\xi|^{-1}+\kappa|\xi|^3)\hat{c}_0}{\lambda_+-\lambda_-},$$

$$B=\frac{-(\lambda_- +\nu|\xi|^2)\hat{v}_0+(|\xi|+|\xi|^{-1}+\kappa|\xi|^3)\hat{c}_0}{\lambda_+-\lambda_-},$$

which imply

$$\hat{v}(\xi,t)=-\left(\frac{e^{\lambda_-t}-e^{\lambda_+t}}{\lambda_+-\lambda_-}\right)(|\xi|+|\xi|^{-1}+\kappa|\xi|^3)\hat{a}_0(\xi)+\left(\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}\right)\hat{v}_0(\xi). \tag{4.6}$$

This determines $\widehat{\mathcal{G}_1^{21}}$ and $\widehat{\mathcal{G}_2^{22}}$.

On the other hand, from the first equation of (4.4), we have

$$\hat{c}(\xi,t)=\hat{c}(\xi,0)-|\xi|\int_0^t\hat{v}(\xi,\tau)d\tau.$$

Plugging (4.6) into the above equality and using the following relations

$$\lambda_\pm+\nu|\xi|^2=-\lambda_\mp,\quad \lambda_-\lambda_+=1+|\xi|^2+\kappa|\xi|^4,$$

we finally get

$$\hat{a}(\xi,t)=\left(\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}\right)\hat{a}_0(\xi)-\left(\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}\right)|\xi|\hat{v}_0(\xi), \tag{4.7}$$

which determines $\widehat{\mathcal{G}_1^{11}}$ and $\widehat{\mathcal{G}_1^{12}}$. $\square$

From (4.3), we observe that $|\xi|+|\xi|^{-1}+\kappa|\xi|^3\sim|\xi|^{-1}$ when $|\xi|\to 0$ in $\widehat{\mathcal{G}_1^{21}}$, which is different from the Fourier transform of Green's matrix for compressible Navier-Stokes equations. Obviously, the term $|\xi|^{-1}$ in $\widehat{\mathcal{G}_1^{21}}$ of $\widehat{\mathcal{G}_1}(\xi,t)$ is singular and causes some difficulty in low frequencies. Thus, it is impossible to obtain decay estimates of $\widehat{\mathcal{G}_1}(\xi,t)$ like the heat kernel. In fact, the term $|\xi|^{-1}$ comes from the symbol of the nonlocal term $\Lambda^{-1}c$ in the system (4.2). To overcome the difficulty, we notice that $\Lambda^{-1}c$ should have the same regularity as $\nu\Delta v$ in low frequencies, which induces us to introduce a new unknown $a=\Lambda^{-1}c$. Thus, the system (4.1) is equivalent to the following form

$$\begin{cases}\partial_t a+v=f_1,\\ \partial_t v-\nu\Delta v-a-\Lambda^2 a-\kappa\Lambda^4 a=g_1,\\ (a,v)|_{t=0}=(\Lambda^{-1}c_0,v_0)\end{cases} \tag{4.8}$$

with $f_1=\Lambda^{-1}f$.

Similar to Proposition 4.1, we also show an explicit derivation of the Fourier transform of Green's matrix $\mathcal{G}(x,t)$ corresponding to the linearized system (4.8) without outer forces.

PROPOSITION 4.2. *Let $\mathcal{G}$ be the Green matrix of the system (4.8). Then we have the following explicit expression for $\widehat{\mathcal{G}}$:*

$$\widehat{\mathcal{G}}(\xi,t)=\begin{bmatrix} \frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-} & -\left(\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}\right) \\ -\left(\frac{e^{\lambda_-t}-e^{\lambda_+t}}{\lambda_+-\lambda_-}\right)(1+|\xi|^2+\kappa|\xi|^4) & \frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-} \end{bmatrix}, \tag{4.9}$$

where

$$\lambda_\pm=-\frac{1}{2}\nu|\xi|^2\pm\frac{1}{2}\sqrt{(\nu^2-4\kappa)|\xi|^4-4(1+|\xi|^2)}. \tag{4.10}$$

Based on Proposition 4.2, we exhibit that $\widehat{\mathcal{G}}(\xi,t)$ behaves like the heat kernel in the low frequencies.

LEMMA 4.3. *Let $\mathcal{G}$ be the Green matrix of Lemma 4.2. Given $R>0$, there is a positive number ϑ such that for $|\xi|<R$*

$$|\widehat{\mathcal{G}}(\xi,t)|\le Ce^{-\vartheta|\xi|^2t}, \tag{4.11}$$

where $C=C(R)$.

Proof. Here, we will prove it in the following two cases.

Case 1. For $\nu^2\le 4\kappa$, in this case, $\lambda_\pm$ are complex numbers for any fixed ξ. We denote $b=\frac{1}{2}\sqrt{4(1+|\xi|^2)-(\nu^2-4\kappa)|\xi|^4}$, thus $b>0$ and $\lambda_\pm=-\frac{1}{2}\nu|\xi|^2\pm bi$. Employing Euler's formula, we have

$$\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-}=\frac{\sin(bt)}{b}e^{-\frac{1}{2}\nu|\xi|^2t},$$

$$\frac{\lambda_+e^{\lambda_-t}-\lambda_-e^{\lambda_+t}}{\lambda_+-\lambda_-}=\left[\cos(bt)+\frac{1}{2}\nu\frac{\sin(bt)}{b}|\xi|^2\right]e^{-\frac{1}{2}\nu|\xi|^2t},$$

$$\frac{\lambda_+e^{\lambda_+t}-\lambda_-e^{\lambda_-t}}{\lambda_+-\lambda_-}=\left[\cos(bt)-\frac{1}{2}\nu\frac{\sin(bt)}{b}|\xi|^2\right]e^{-\frac{1}{2}\nu|\xi|^2t}.$$

For $|\xi|<R$, noticing that $b=O(1)$, one can easily find that

$$|\widehat{\mathcal{G}}(\xi,t)|\le Ce^{-\frac{1}{2}\nu|\xi|^2t},$$

where $C=C(R)$.

Case 2. For $\nu^2>4\kappa$, in this case, if we define $h=\nu-\sqrt{\nu^2-4\kappa}$, then $h>0$ and

$$\mathrm{Re}(\lambda_\pm)\le-\frac{h}{2}|\xi|^2,\quad\forall\xi\in\mathbb{R}^d. \tag{4.12}$$

On the other hand, for $|\xi|<R$, since $|\lambda_\pm|, |\lambda_+-\lambda_-|=O(1)$, we derive from the expression (4.9) and (4.12) that

$$|\widehat{\mathcal{G}}(\xi,t)|\le Ce^{-\frac{1}{2}h|\xi|^2t},$$

where $C=C(R)$. The proof of Lemma 4.3 is complete. □

We have the following smoothing effects of Green's matrix $\mathcal{G}$ in the low frequencies.

LEMMA 4.4. *Let $\mathcal{C}$ be a ring centered at 0 in $\mathbb{R}^d$. Then there exist positive constants R_0, C, c such that, if* $\operatorname{supp}\hat{u}\subset\lambda\mathcal{C}$, *$\lambda\le R_0$, then we have*

$$\|\mathcal{G}*u\|_{L^2}\le Ce^{-c\lambda^2t}\|u\|_{L^2}. \tag{4.13}$$

Proof. Thanks to the Plancherel theorem, we get

$$\|\mathcal{G}*u\|_{L^2}=\|\widehat{\mathcal{G}}(\xi)\hat{u}(\xi)\|_{L^2}\le C\|e^{-\vartheta|\xi|^2t}\hat{u}(\xi)\|_2\le Ce^{-c\lambda^2t}\|u\|_2,$$

where we have used (4.11) and the support property of $\hat{u}(\xi)$. □

The following Lemma states some optimal *a priori* estimates for the solution to the system (4.8), and exhibits the smoothing properties of a and v in the low frequencies, assuming that f_1 and g_1 are given.

LEMMA 4.5. *Let (a,v) be a solution of the system (4.8). Let m_0 be any integer number. There exists a positive constant C depending only on ν and m_0, such that the following inequality holds for all $t\ge0$ and $1\le r\le\infty$*

$$\|(a,v)\|^{\ell}_{\widetilde{L}^r_t(\dot{B}^{s+\frac{2}{r}}_{2,1})}\le C\Big(\|(a_0,v_0)\|^{\ell}_{\dot{B}^s_{2,1}}+\|(f_1,g_1)\|^{\ell}_{L^1_t(\dot{B}^s_{2,1})}\Big). \tag{4.14}$$

Proof. In terms of Green's matrix and Duhamel's principle, the solution of (4.8) can be expressed as

$$\begin{pmatrix} a\\ v\end{pmatrix}=\mathcal{G}(x,t)*\begin{pmatrix} a_0\\ v_0\end{pmatrix}+\int_0^t\mathcal{G}(x,t-\tau)*\begin{pmatrix} f_1\\ g_1\end{pmatrix}d\tau. \tag{4.15}$$

Applying homogeneous frequency localization operators $\dot{\Delta}_j$ on both sides of (4.15), we get

$$\begin{pmatrix} \dot{\Delta}_j a\\ \dot{\Delta}_j v\end{pmatrix}=\mathcal{G}(x,t)*\begin{pmatrix} \dot{\Delta}_j a_0\\ \dot{\Delta}_j v_0\end{pmatrix}+\int_0^t\mathcal{G}(x,t-\tau)*\begin{pmatrix} \dot{\Delta}_j f_1\\ \dot{\Delta}_j g_1\end{pmatrix}d\tau.$$

From Lemma 4.4 and Young's inequality, we infer that

$$\begin{aligned}\|\dot{\Delta}_j a(t)\|_{L^2}+\|\dot{\Delta}_j v(t)\|_{L^2}\le{}& Ce^{-c2^{2j}t}(\|\dot{\Delta}_j a_0\|_{L^2}+\|\dot{\Delta}_j v_0\|_{L^2})\\ &+C\int_0^t e^{-c2^{2j}(t-\tau)}(\|\dot{\Delta}_j f_1(\tau)\|_{L^2}+\|\dot{\Delta}_j g_1(\tau)\|_{L^2})d\tau.\end{aligned}$$

Taking L^r norm with respect to t gives

$$\|\dot{\Delta}_j a\|_{L^r_tL^2}+\|\dot{\Delta}_j v\|_{L^r_tL^2}\le C2^{-\frac{2j}{r}}(\|\dot{\Delta}_j a_0\|_{L^2}+\|\dot{\Delta}_j v_0\|_{L^2}+\|\dot{\Delta}_j f_1\|_{L^1_tL^2}+\|\dot{\Delta}_j g_1\|_{L^1_tL^2}). \tag{4.16}$$

Multiplying 2^{js} on both sides of (4.16), and then summing for $j\le m_0$, we get (4.14). The proof of Lemma 4.5 is complete. □

Taking advantage of Proposition 2.8 for the following heat equation

$$\partial_t\mathcal{P}u-\mu\Delta\mathcal{P}u=\mathcal{P}g,$$

we have

$$\|\mathcal{P}u\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|\mathcal{P}u\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\le C\Big(\|\mathcal{P}u_0\|^{\ell}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\mathcal{P}g\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\Big). \tag{4.17}$$

Combining with (4.14) and (4.17) yields

$$\|(a,u)\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|(a,u)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\le C\Big(\|(a_0,u_0)\|^{\ell}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|(\Lambda^{-1}f,g_1)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\Big).$$

Noticing that $a=\Lambda^{-1}c$, the above inequality is actually equivalent to

$$\begin{aligned}&\|c\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-2}_{2,1})}+\|c\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\|u\|^{\ell}_{\widetilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|u\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\ \le& C\Big(\|c_0\|^{\ell}_{\dot{B}^{\frac{d}{2}-2}_{2,1}}+\|u_0\|^{\ell}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|f\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-2}_{2,1})}+\|g\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\Big).\end{aligned} \tag{4.18}$$

Next, we bound the terms $\|f\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-2}_{2,1})}$ and $\|g\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}$ as follows. For $\|f\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-2}_{2,1})}$, it suffices to bound $\|cu\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}$. Employing Bony's decomposition, we have

$$(cu)^{\ell}=(T_cu)^{\ell}+\big(R(c,u)\big)^{\ell}+(T_uc)^{\ell}.$$

Recall that $T:\dot{B}^{\frac{d}{p}-1}_{p,1}\times\dot{B}^{\frac{d}{p}}_{p,1}\to\dot{B}^{\frac{d}{2}-1}_{2,1}$ for $2\le p\le\min\big(4,\frac{2d}{d-2}\big)$. Hence, we have

$$\begin{aligned}&\|(T_uc)^{\ell}\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\le C\|u\|_{L^{\infty}_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|c\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\ \le& C\big(\|u\|^{\ell}_{L^{\infty}_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|u\|^{h}_{L^{\infty}_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\big)\big(\|c\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|c\|^{h}_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big)\\ \le& C\big(\|u\|^{\ell}_{L^{\infty}_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|u\|^{h}_{L^{\infty}_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\big)\big(\|c\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\|c\|^{h}_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\big)\\ \le& C\|(c,u)\|^2_{X_p(t)},\end{aligned}$$

$$\begin{aligned}&\|(T_cu)^{\ell}\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\le C\|c\|_{L^2_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|u\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\ \le& C\big(\|c\|^{\ell}_{L^2_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|c\|^{h}_{L^2_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\big)\big(\|u\|^{\ell}_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|u\|^{h}_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big)\\ \le& C\big(\|c\|^{\ell}_{L^2_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|c\|^{h}_{L^2_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\big)\big(\|u\|^{\ell}_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\|u\|^{h}_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big)\\ \le& C\|(c,u)\|^2_{X_p(t)},\end{aligned}$$

where we have used the following interpolation inequalities,

$$\|c\|^{\ell}_{\tilde{L}^2_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\le\Big(\|c\|^{\ell}_{\tilde{L}^{\infty}_t(\dot{B}^{\frac{d}{2}-2}_{2,1})}\Big)^{\frac12}\Big(\|c\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}\Big)^{\frac12},$$

$$\|c\|^h_{\tilde{L}^2_t(\dot{B}^{\frac{d}{p}+1}_{p,1})} \leq \Big(\|c\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big)^{\frac{1}{2}} \Big(\|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\Big)^{\frac{1}{2}},$$

$$\|u\|^\ell_{\tilde{L}^2_t(\dot{B}^{\frac{d}{2}}_{2,1})} \leq \Big(\|u\|^\ell_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\Big)^{\frac{1}{2}} \Big(\|u\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\Big)^{\frac{1}{2}}$$

and

$$\|u\|^h_{\tilde{L}^2_t(\dot{B}^{\frac{d}{p}}_{p,1})} \leq \Big(\|u\|^h_{\tilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\Big)^{\frac{1}{2}} \Big(\|u\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\Big)^{\frac{1}{2}}.$$

For the remainder term, one can use that $R: \dot{B}^{\frac{d}{p}-1}_{p,1} \times \dot{B}^{\frac{d}{p}}_{p,1} \to \dot{B}^{\frac{d}{2}-1}_{2,1}$ for $2 \leq p \leq 4$, thus

$$\begin{aligned}
&\|(R(c,u))^\ell\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} \leq C\|u\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|c\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} \\
\leq & C\big(\|u\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|u\|^h_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\big)\big(\|c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big) \\
\leq & C\big(\|u\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|u\|^h_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\big)\big(\|c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\big) \\
\leq & C\|(c,u)\|^2_{X_p(t)}.
\end{aligned}$$

In order to bound $\|g\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}$, we set

$$\begin{aligned}
g = & -u\cdot\nabla u - L_1(c)\mathcal{A}u + L_2(c)\nabla c + L_3(c)\Big(\mathrm{div}\big(2\widetilde{\mu}(c)D(u)\big) + \nabla\big(\widetilde{\lambda}(c))\mathrm{div}u\big)\Big) \\
& + \nabla\Big(\tilde{\kappa}(c)\Delta c + \frac{1}{2}\nabla\tilde{\kappa}(c)\cdot\nabla c\Big) \\
\stackrel{\text{def}}{=} & g^1 + g^2 + g^3 + g^4 + g^5.
\end{aligned}$$

For g^1, we use Bony's decomposition with the summation convention over repeated indices,

$$u\cdot\nabla u^i = T_{\nabla u^i}\cdot u + R(u^i, \partial_j u^i) + T_u\cdot\nabla u^i, \quad \text{with} \quad i = 1,2,\cdots,d.$$

Similar to the bound of $\|cu\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}$, we have

$$\begin{aligned}
\|(T_{\nabla u^i}\cdot u + R(u^i,\partial_j u^i))^\ell\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} & \leq C\|T_{\nabla u^i}\cdot u + R(u^i,\partial_j u^i)\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\
& \leq C\|\nabla u\|_{L^2_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|u\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})} \\
& \leq C\|u\|^2_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})} \\
& \leq C\|(c,u)\|^2_{X_p(t)},
\end{aligned}$$

and

$$\begin{aligned}
\|(T_u\cdot\nabla u^i)^\ell\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} & \leq C\|T_u\cdot\nabla u^i\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\
& \leq C\|u\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|\nabla u\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|u\|_{L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}\|u\|_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}\\
&\le C\|(c,u)\|^2_{X_p(t)}.
\end{aligned}
$$

The second term $g^2=L_2(c)\nabla c$ satisfies for some smooth function L_2 vanishing at 0. For bounding $L_2(c)$, one cannot use directly Proposition 2.7 as it may happen that $\frac{d}{p}-1<0$. Using the Taylor expansion, we have

$$
L_2(c)=L_2'(0)c+c\tilde L_2(c)\quad \text{with}\quad \tilde L_2(0)=0.
$$

Combining Proposition 2.7 and product laws in Besov spaces, we get for $2\le p<d$,

$$
\|L_2(c)\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\le C\big(L_2'(0)+\|a\|_{\dot B^{\frac{d}{p}}_{p,1}}\big)\|a\|_{\dot B^{\frac{d}{p}-1}_{p,1}}.
$$

Further, from $T:\dot B^{\frac{d}{p}-1}_{p,1}\times \dot B^{\frac{d}{p}}_{p,1}\to \dot B^{\frac{d}{2}-1}_{2,1}$ for $2\le p\le \min\big(4,\frac{2d}{d-2}\big)$, $R:\dot B^{\frac{d}{p}-1}_{p,1}\times \dot B^{\frac{d}{p}}_{p,1}\to \dot B^{\frac{d}{2}-1}_{2,1}$ for $2\le p\le 4$, Bernstein's inequality and the following decomposition:

$$
\big(L_2(c)\nabla c\big)^\ell=\big(T_{\nabla c}L_2(c)+R(\nabla c,L_2(c))\big)^\ell+\big(T_{L_2(c)}\nabla c)\big)^\ell,
$$

we have

$$
\begin{aligned}
\|\big(R(\nabla c,L_2(c))\big)^\ell\|_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})}&\le C\|R(\nabla c,L_2(c))\|_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})}\\
&\le C\|\nabla c\|_{L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}\|L_2(c)\|_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})}\\
&\le C\|c\|_{L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})}\|c\|_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})}\\
&\le C\|(c,u)\|^2_{X_p(t)},
\end{aligned}
$$

$$
\begin{aligned}
\|\big(T_{\nabla c}L_2(c))\big)^\ell\|_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})}&\le C\|T_{\nabla c}L_2(c))\|_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})}\\
&\le C\|\nabla c\|_{L^2_t(\dot B^{\frac{d}{p}-1}_{p,1})}\|L_2(c)\|_{L^2_t(\dot B^{\frac{d}{p}}_{p,1})}\\
&\le C\Big(\big(\|c\|^\ell_{L^2_t(\dot B^{\frac{d}{p}}_{p,1})}\big)^2+\big(\|c\|^h_{L^2_t(\dot B^{\frac{d}{p}}_{p,1})}\big)^2\Big)\\
&\le C\Big(\big(\|c\|^\ell_{L^2_t(\dot B^{\frac{d}{2}}_{2,1})}\big)^2+\big(\|c\|^h_{L^2_t(\dot B^{\frac{d}{p}}_{p,1})}\big)^2\Big)\\
&\le C\Big(\big(\|c\|^\ell_{L^2_t(\dot B^{\frac{d}{2}-1}_{2,1})}\big)^2+\big(\|c\|^h_{L^2_t(\dot B^{\frac{d}{p}+1}_{p,1})}\big)^2\Big)\\
&\le C\|(c,u)\|^2_{X_p(t)},
\end{aligned}
$$

$$
\begin{aligned}
&\|\big(T_{L_2(c)}\nabla c\big)^\ell\|_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})}\\
&\le C\|L_2(c)\|_{L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}\|\nabla c\|_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})}\\
&\le C\big(1+\|c\|_{L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})}\big)\|c\|_{L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}\|c\|_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}\\
&\le C\big(1+\|c\|_{L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})}\big)\big(\|c\|^\ell_{L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|c\|^h_{L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}\big)\times\big(\|c\|^\ell_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}+\|c\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}\big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\big(1+\|c\|_{L_t^\infty(\dot B_{p,1}^{\frac{d}{p}})}\big)\big(\|c\|^\ell_{L_t^\infty(\dot B_{2,1}^{\frac{d}{2}-1})}+\|c\|^h_{L_t^\infty(\dot B_{p,1}^{\frac{d}{p}-1})}\big)\times\big(\|c\|^\ell_{L_t^1(\dot B_{2,1}^{\frac{d}{2}+1})}+\|c\|^h_{L_t^1(\dot B_{p,1}^{\frac{d}{p}+1})}\big)\\
&\leq C\big(1+\|c\|_{L_t^\infty(\dot B_{p,1}^{\frac{d}{p}})}\big)\big(\|c\|^\ell_{L_t^\infty(\dot B_{2,1}^{\frac{d}{2}-2})}+\|c\|^h_{L_t^\infty(\dot B_{p,1}^{\frac{d}{p}})}\big)\times\big(\|c\|^\ell_{L_t^1(\dot B_{2,1}^{\frac{d}{2}})}+\|c\|^h_{L_t^1(\dot B_{p,1}^{\frac{d}{p}+2})}\big)\\
&\leq C\|(c,u)\|^2_{X_p(t)}+C\|(c,u)\|^3_{X_p(t)}.
\end{aligned}
$$

To bound the third term $g^3=L_1(c)\mathcal{A}u$ with $L_1(c)=\frac{c}{1+c}$, we use as for g^2 and omit it.

For the term g^4: For the terms of nonconstant viscosity coefficients, it is only a matter of checking that g^4 satisfies quadratic estimates. We write that

$$
L_3(c)\mathrm{div}\big(2\widetilde{\mu}(c)D(u)\big)=2\frac{\widetilde{\mu}(c)}{1+c}\mathrm{div}\big(D(u)\big)+2\frac{\widetilde{\mu}'(c)}{1+c}D(u)\cdot\nabla c,
$$

$$
L_3(c)\nabla\big((\widetilde{\mu}(c)+\widetilde{\lambda}(c))\mathrm{div}u\big)=\frac{\widetilde{\mu}(c)+\widetilde{\lambda}(c)}{1+c}\nabla\mathrm{div}u+\frac{\widetilde{\mu}'(c)+\widetilde{\lambda}'(c)}{1+c}\mathrm{div}u\nabla c.
$$

Since $\frac{\widetilde{\mu}(c)}{1+c}$ and $\frac{\widetilde{\mu}(c)+\widetilde{\lambda}(c)}{1+c}$ are two smooth functions vanishing at 0, then the terms $\frac{\widetilde{\mu}(c)}{1+c}\mathrm{div}\big(D(u)\big)$ and $\frac{\widetilde{\mu}(c)+\widetilde{\lambda}(c)}{1+c}\nabla\mathrm{div}u$ may be handled exactly as g^3. On the other hand, we observe that the fact $\frac{\widetilde{\mu}'(c)}{1+c}\nabla c=\nabla(L_4(c))$ and $\frac{\widetilde{\lambda}'(c)}{1+c}\nabla c=\nabla(L_5(c))$, where $L_4(c)$ and $L_5(c)$ are two smooth functions vanishing at 0. Thus, we introduce the following decomposition:

$$
\big(D(u)\nabla(L_4(c))\big)^\ell=\big(T_{\nabla(L_4(c))}D(u)+R(D(u),\nabla(L_4(c)))\big)^\ell+\big(T_{D(u)}\nabla(L_4(c))\big)^\ell.
$$

From the aforementioned properties of maps T and R, Proposition 2.7, Bernstein's inequality and interpolation inequalities, we have

$$
\begin{aligned}
&\|\big(T_{\nabla(L_4(c))}D(u)+R(D(u),\nabla(L_4(c)))\big)^\ell\|_{L_t^1(\dot B_{2,1}^{\frac{d}{2}-1})}\\
&\leq C\|T_{\nabla(L_4(c))}D(u)+R(D(u),\nabla(L_4(c)))\|_{L_t^1(\dot B_{2,1}^{\frac{d}{2}-1})}\\
&\leq C\|\nabla(L_4(c))\|_{L_t^\infty(\dot B_{p,1}^{\frac{d}{p}-1})}\|\nabla u\|_{L_t^1(\dot B_{p,1}^{\frac{d}{p}})}\\
&\leq C\|L_4(c)\|_{L_t^\infty(\dot B_{p,1}^{\frac{d}{p}})}\|u\|_{L_t^1(\dot B_{p,1}^{\frac{d}{p}+1})}\\
&\leq C\|c\|_{L_t^\infty(\dot B_{p,1}^{\frac{d}{p}})}\|u\|_{L_t^1(\dot B_{p,1}^{\frac{d}{p}+1})}\\
&\leq C\|(c,u)\|^2_{X_p(t)},
\end{aligned}
$$

$$
\begin{aligned}
\|\big(T_{D(u)}\nabla(L_4(c))\big)^\ell\|_{L_t^1(\dot B_{2,1}^{\frac{d}{2}-1})}&\leq C\|\big(T_{D(u)}\nabla(L_4(c))\big)\|^\ell_{L_t^1(\dot B_{2,1}^{\frac{d}{2}-2})}\\
&\leq C\|T_{D(u)}\nabla(L_4(c))\|_{L_t^1(\dot B_{2,1}^{\frac{d}{2}-2})}\\
&\leq C\|\nabla u\|_{L_t^2(\dot B_{p,1}^{\frac{d}{p}-1})}\|\nabla(L_4(c))\|_{L_t^2(\dot B_{p,1}^{\frac{d}{p}-1})}\\
&\leq C\|u\|_{L_t^2(\dot B_{p,1}^{\frac{d}{p}})}\|L_4(c)\|_{L_t^2(\dot B_{p,1}^{\frac{d}{p}})}\\
&\leq C\|u\|_{L_t^2(\dot B_{p,1}^{\frac{d}{p}})}\|c\|_{L_t^2(\dot B_{p,1}^{\frac{d}{p}})}\\
&\leq C\|(c,u)\|^2_{X_p(t)}.
\end{aligned}
$$

Similarly, we also obtain the estimate of the term $\nabla(L_5(c))\mathrm{div}u$.

Finally, we bound the last term g^5. Indeed,

$$g^5 \approx \nabla\Big(\tilde{\kappa}(c)\Delta c\Big) + \nabla\Big(\nabla\tilde{\kappa}(c)\cdot\nabla c\Big) \stackrel{\mathrm{def}}{=} g^{51} + g^{52}.$$

For g^{51}, using Bernstein's inequality we have

$$\|\nabla\big(\tilde{\kappa}(c)\Delta c\big)^\ell\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} \le C\|\tilde{\kappa}(c)\Delta c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}.$$

Now, we only focus on the estimate of $\|\tilde{\kappa}(c)\Delta c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}$ and further using Bony's decomposition, we have

$$\big(\tilde{\kappa}(c)\Delta c\big)^\ell = \big(T_{\tilde{\kappa}(c)}\Delta c + R(\Delta c, \tilde{\kappa}(c))\big)^\ell + \big(T_{\Delta c}\tilde{\kappa}(c)\big)^\ell.$$

Employing the properties of maps T and R, we deduce that

$$\begin{aligned}
\|\big(T_{\Delta c}\tilde{\kappa}(c)\big)^\ell\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} &\le C\|\Delta c\|_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|\tilde{\kappa}(c)\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\le C\|c\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\|c\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\le C\big(\|c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})} + \|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\big)\big(\|c\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|c\|^h_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big)\\
&\le C\big(\|c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\big)\big(\|c\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|c\|^h_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big)\\
&\le C\big(\|c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\big)\big(\|c\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{2}-2}_{2,1})} + \|c\|^h_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big)\\
&\le C\|(c,u)\|^2_{X_p(t)},
\end{aligned}$$

and

$$\begin{aligned}
&\|\big(T_{\tilde{\kappa}(c)}\Delta c + R(\Delta c, \tilde{\kappa}(c))\big)^\ell\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\\
\le{}& C\|\tilde{\kappa}(c)\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|\Delta c\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\
\le{}& C\big(1+\|c\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big)\|c\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|c\|_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\\
\le{}& C\big(1+\|c\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big)\big(\|c\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|c\|^h_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\big)\times\big(\|c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})} + \|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\big)\\
\le{}& C\big(1+\|c\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big)\big(\|c\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|c\|^h_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\big)\times\big(\|c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+2}_{2,1})} + \|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\big)\\
\le{}& C\big(1+\|c\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big)\big(\|c\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{2}-2}_{2,1})} + \|c\|^h_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\big)\times\big(\|c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\big)\\
\le{}& C\|(c,u)\|^2_{X_p(t)} + C\|(c,u)\|^3_{X_p(t)},
\end{aligned}$$

where the bound of $\tilde{\kappa}(c)$ is similar to the $L_2(c)$. For g^{52}, we also have similar estimates and omit it.

Putting all the previous estimates together, we have proved the following inequality:

$$\begin{aligned}
&\|c\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-2}_{2,1})} + \|c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})} + \|u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|u\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\
\le{}& CX_{p,0} + C\|(c,u)\|^2_{X_p(t)} + C\|(c,u)\|^3_{X_p(t)}.
\end{aligned} \tag{4.19}$$

Step 2: High frequencies. First, we bound $\mathcal{P}u$, we just use the fact that

$$\partial_t \mathcal{P}u - \mu\Delta\mathcal{P}u = \mathcal{P}g.$$

Hence, according to Proposition 2.8 (i) (restricted to high frequencies)

$$\begin{aligned}\|\mathcal{P}u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \mu\|\mathcal{P}u\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})} &\le C\big(\|\mathcal{P}u_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|\mathcal{P}g\|^h_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\big)\\ &\le C\big(\|u_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|g\|^h_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\big).\end{aligned} \tag{4.20}$$

Next, we consider the following system including c and $\mathcal{Q}u$,

$$\begin{cases}\partial_t c + \operatorname{div}\mathcal{Q}u = f,\\ \partial_t\mathcal{Q}u - \nu\Delta\mathcal{Q}u + \nabla c - \kappa\Delta\nabla c = \mathcal{Q}g - (-\Delta)^{-1}\nabla c.\end{cases} \tag{4.21}$$

As in [17], we introduce a new auxiliary function $\omega \overset{\text{def}}{=} \mathcal{Q}u + \nu^{-1}(-\Delta)^{-1}\nabla c$, which satisfies that

$$\begin{cases}\partial_t\nabla c + \nu^{-1}\nabla c + \Delta\omega = \nabla f,\\ \partial_t\omega - \nu\Delta\omega - \kappa\Delta\nabla c = \mathcal{Q}g - (1+\nu^{-2})(-\Delta)^{-1}\nabla c + \nu^{-1}\omega + \nu^{-1}(-\Delta)^{-1}\nabla f.\end{cases} \tag{4.22}$$

We now consider suitable linear combinations of w and ∇c in order to exploit the property of parabolic behavior in the high frequencies similar to the heat equation. Indeed, for all $\beta \in \mathbf{C}$ with β to be determined later, we have

$$\begin{aligned}&\partial_t(\omega + \beta\nu\nabla c) - (1-\beta)\nu\Delta\omega - \kappa\Delta\nabla c + \beta\nabla c\\ =&\mathcal{Q}g - (1+\nu^{-2})(-\Delta)^{-1}\nabla c + \nu^{-1}\omega + \nu^{-1}(-\Delta)^{-1}\nabla f + \nu\nabla f.\end{aligned} \tag{4.23}$$

Therefore, if we set

$$\chi \triangleq \omega + \beta\nu\nabla c \quad \text{with}\,\beta \text{ satisfying } \beta = \frac{\kappa}{\nu^2(1-\beta)},$$

then

$$\begin{aligned}&\partial_t\chi - (1-\beta)\nu\Delta\chi\\ =& -\beta\nabla c + \mathcal{Q}g - (1+\nu^{-2})(-\Delta)^{-1}\nabla c + \nu^{-1}\omega + \nu^{-1}(-\Delta)^{-1}\nabla f + \nu\nabla f.\end{aligned} \tag{4.24}$$

Here, a possible value of β is

$$\beta = \frac{1}{2} + \frac{\sqrt{\nu^2 - 4\kappa}}{2\nu} \quad \text{such that } \ 1-\beta = \frac{1}{2}\big(1 - \sqrt{1 - \frac{4\kappa}{\nu^2}}\big).$$

Obviously, $\mathrm{Re}(1-\beta)$ is positive for any values of κ and ν. Therefore, Proposition 2.8 (ii) and the fact that $\Delta^{-1}\Lambda$ is a homogeneous Fourier multiplier of degree -1 imply that

$$\begin{aligned}&\|\chi\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\chi\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\\ \le& C\Big(\|\chi_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|g\|^h_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|f\|^h_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} + \|\nabla c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\end{aligned}$$

$$
\begin{aligned}
&+\|\omega\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|(-\Delta)^{-1}\nabla c\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}\Big)\\
&\le C\Big(\|\chi_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}}+\|g\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|f\|^h_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})}+2^{-2k_0}\|c\|^h_{L^1_t(\dot B^{\frac{d}{p}+2}_{p,1})}\\
&\quad+2^{-2k_0}\|\omega\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}+2^{-4k_0}\|c\|^h_{L^1_t(\dot B^{\frac{d}{p}+2}_{p,1})}\Big).
\end{aligned}
\tag{4.25}
$$

Furthermore, in order to bound ω, from the definition of χ and the second equation in (4.22), we deduce that the following heat equation:

$$
\partial_t\omega-\frac{\beta\nu^2-\kappa}{\beta\nu}\Delta\omega=\frac{\kappa}{\beta\nu}\Delta\chi+\mathcal{Q}g-(1+\nu^{-2})(-\Delta)^{-1}\nabla c+\nu^{-1}\omega+\nu^{-1}(-\Delta)^{-1}\nabla f.
\tag{4.26}
$$

Noticing that

$$
\frac{\beta\nu^2-\kappa}{\beta\nu}=\frac{\kappa}{\nu(1-\beta)}.
$$

Since the real part of $(1-\beta)$ is positive, we have $\mathrm{Re}\big(\frac{\beta\nu^2-\kappa}{\beta\nu}\big)>0$. Thus, owing to the high frequency cut-off, from Proposition 2.8 (ii) we have

$$
\begin{aligned}
&\|\omega\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|\omega\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}\\
&\le C\Big(\|\omega_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}}+\|\chi\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}+\|g\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|f\|^h_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})}+\|\nabla c\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}\\
&\quad+\|\omega\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|(-\Delta)^{-1}\nabla c\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}\Big)\\
&\le C\Big(\|\omega_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}}+\|\chi\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}+\|g\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|f\|^h_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})}+2^{-2k_0}\|c\|^h_{L^1_t(\dot B^{\frac{d}{p}+2}_{p,1})}\\
&\quad+2^{-2k_0}\|\omega\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}+2^{-4k_0}\|c\|^h_{L^1_t(\dot B^{\frac{d}{p}+2}_{p,1})}\Big),
\end{aligned}
$$

choosing k_0 large enough yields

$$
\begin{aligned}
&\|\omega\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|\omega\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}\\
&\le C\Big(\|\omega_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}}+\|\chi\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}+\|g\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|f\|^h_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})}+\|\nabla c\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}\\
&\quad+\|\omega\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|(-\Delta)^{-1}\nabla c\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}\Big)\\
&\le C\Big(\|\omega_0\|^h_{\dot B^{\frac{d}{p}-1}_{p,1}}+\|\chi\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}+\|g\|^h_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|f\|^h_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})}+2^{-2k_0}\|c\|^h_{L^1_t(\dot B^{\frac{d}{p}+2}_{p,1})}\\
&\quad+2^{-4k_0}\|c\|^h_{L^1_t(\dot B^{\frac{d}{p}+2}_{p,1})}\Big).
\end{aligned}
\tag{4.27}
$$

Putting (4.27) into (4.25) and taking k_0 large enough, we conclude that

$$
\|\chi\|^h_{\widetilde L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}+\|\chi\|^h_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}
$$

$$\leq C\Big(\|\chi_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|g\|^h_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|f\|^h_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+2^{-2k_0}\|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}+2^{-4k_0}\|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\Big). \tag{4.28}$$

Further, using the fact $\nabla c=\frac{\chi-\omega}{\beta\nu}$, taking k_0 large enough, we obtain from (4.27) and (4.28)

$$\|(\nabla c,\omega)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|(\nabla c,\omega)\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\leq C\Big(\|\chi_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|f\|^h_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|g\|^h_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\Big). \tag{4.29}$$

Finally, keeping in mind that $u=\omega-(-\Delta)^{-1}\nabla c+\mathcal{P}u$, and employing (4.20) and (4.29), we deduce that

$$\begin{aligned}&\|(\nabla c,u)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|(\nabla c,u)\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\\ \leq& C\Big(\|(\nabla c_0,u_0)\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|f\|^h_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|g\|^h_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\Big).\end{aligned} \tag{4.30}$$

Employing Proposition 2.7, for the last two terms on the right-hand side of (4.30), we easily get for $1\leq p<2d$,

$$\begin{aligned}\|f\|^h_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\leq& C\Big(\|c\|_{L^2_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\|u\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|\nabla u\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|c\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big),\\ \|g\|^h_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\leq& C\Big(\|u\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|\nabla u\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|c\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|\nabla^2 u\|_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\\ &+\|c\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|\nabla c\|_{L^2_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|c\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|c\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\Big),\end{aligned}$$

which implies that

$$\|u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|u\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|c\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\leq CX_{p,0}+CX_p^2(t). \tag{4.31}$$

Combining with (4.19) and (4.31), we finally conclude the following global *a priori* estimates.

Lemma 4.6. *Let $T\geq 0$, $d\geq 2$, $p\in[2,\min(4,2d/(d-2))]$ with, additionally, $p\neq 4$ if $d=2$ and (c,u) be a solution to the system (3.1) on $[0,T]\times\mathbb{R}^d$, that belongs to the space X_p defined in (1.6), we have*

$$\|(c,u)\|_{X_p(t)}\leq C\Big(X_p(0)+\|(c,u)\|^2_{X_p(t)}+\|(c,u)\|^3_{X_p(t)}\Big),\quad \textit{for}\quad \forall t\in[0,T], \tag{4.32}$$

where

$$\begin{aligned}\|(c,u)\|_{X_p(t)}\stackrel{def}{=}&\|c\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-2}_{2,1})}+\|c\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\|u\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|u\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\ &+\|c\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|c\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}+\|u\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|u\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}.\end{aligned}$$

4.2. Global existence and uniqueness. In order to solve the system (3.1) by fixed-point theorem, we define the following map

$$\Phi:(c,u)\to(b,v) \tag{4.33}$$

with (b,v) the solution to

$$\begin{cases}\partial_t b+\mathrm{div} v=f(c,u),\\ \partial_t v-\mathcal{A}v+\nabla b+\nabla(-\Delta)^{-1}b-\kappa\nabla\Delta b=g(c,u),\\ (b,v)|_{t=0}=(c_0,u_0).\end{cases} \tag{4.34}$$

Obviously, to prove the existence part of the theorem, we just have to show that Φ is a contraction map in a ball of X_p. We define a ball $B(O,R)$ centered at the origin by

$$B(0,R)=\Big\{(c,u)\in X_p:\|(c,u)\|_{X_p(t)}\le R\Big\}. \tag{4.35}$$

Assuming $R\le 1$, from Lemma 4.6 we have

$$\begin{aligned}\|\Phi(c,u)\|_{X_p(t)}&\le C\Big(X_p(0)+\|(c,u)\|^2_{X_p(t)}+\|(c,u)\|^3_{X_p(t)}\Big)\\ &\le C\Big(\eta+R^2+R^3\Big)\\ &\le C\Big(\eta+2R^2\Big).\end{aligned} \tag{4.36}$$

Choosing (R,η) such that

$$R\le\min\{1,(4C)^{-1}\}\quad\text{and}\quad\eta\le 2R^2. \tag{4.37}$$

Thus, from (4.36), we finally deduce that

$$\Phi(B(0,R))\subseteq B(0,R).$$

In order to show Φ is a contraction map, one chooses two elements (c_1,u_1) and (c_2,u_2) in $B(0,R)$. According to (4.34), (4.18) and (4.30), we have

$$\begin{aligned}&\|\Phi(c_1,u_1)-\Phi(c_2,u_2)\|_{X_p(t)}\\ \le& C\Big(\|f(c_1,u_1)-f(c_2,u_2)\|^{\ell}_{L^1_t(\dot B^{\frac d2-2}_{2,1})}+\|g(c_1,u_1)-g(c_2,u_2)\|^{\ell}_{L^1_t(\dot B^{\frac d2-1}_{2,1})}\\ &+\|f(c_1,u_1)-f(c_2,u_2)\|^{h}_{L^1_t(\dot B^{\frac dp}_{p,1})}+\|g(c_1,u_1)-g(c_2,u_2)\|^{h}_{L^1_t(\dot B^{\frac dp-1}_{p,1})}\Big).\end{aligned} \tag{4.38}$$

Similar to the estimates (4.19) and (4.31), we get

$$\begin{aligned}&\|\Phi(c_1,u_1)-\Phi(c_2,u_2)\|_{X_p(t)}\\ \le& C\|(c_1-c_2,u_1-u_2)\|_{X_p(t)}\big(\|(c_1,u_1)\|_{X_p(t)}+\|(c_2,u_2)\|_{X_p(t)}\big).\end{aligned} \tag{4.39}$$

From (4.37) we finally deduce that

$$\|\Phi(c_1,u_1)-\Phi(c_2,u_2)\|_{X_p(t)}\le\frac12\|(c_1-c_2,u_1-u_2)\|_{X_p(t)}, \tag{4.40}$$

and the proof of the existence part of Theorem 1.1 is achieved. Moreover, the uniqueness part of Theorem 1.1 in $B(0,R)$ naturally follows.

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