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LOWER BOUNDS OF BLOW UP SOLUTIONS IN $\dot{H}^1_P(\mathbb{R}^3)$ OF THE NAVIER-STOKES EQUATIONS AND THE QUASI-GEOSTROPHIC EQUATION*

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Abstract. In this paper, we derive some new lower bounds of possible blow up solutions in $\dot{H}^1_p(\mathbb{R}^3)$ with $3/2<p<\infty$ to the 3D Navier-Stokes equations, which provides a new proof of the corresponding recent results involving blow up rates in $\dot{H}^s$ with $1\le s<5/2$ in [A. Cheskidov and M. Zaya, J. Math. Phys., 57:023101, 2016; J.C. Cortissoz and J.A. Montero, J. Math. Fluid Mech., 20:1–5, 2018; D.S. McCormick, E.J. Olson, J.C. Robinson, J.L. Rodrigo, A. Vidal-López, and Y. Zhou, SIAM J. Math. Anal., 48:2119–2132, 2016; J.C. Robinson, W. Sadowski, and R.P. Silva, 53:115618, 2012]. We apply this to study the upper box dimension of the set of singular times of weak solutions. In addition, blow up rates of solutions to the 2D supercritical surface quasi-geostrophic equation in $\dot{H}^1_p(\mathbb{R}^2)$ are established.

Keywords. Navier-Stokes equations; blow up; regularity.

AMS subject classifications. 76D03; 76D05; 35B33; 35Q35.

1. Introduction

The incompressible Newton fluid in the whole three-dimensional space is governed by the Navier-Stokes system

$$\begin{cases} u_t-\Delta u+u\cdot\nabla u+\nabla\Pi=0, \\ \operatorname{div} u=0, \\ u|_{t=0}=u_0, \end{cases} \tag{1.1}$$

where the unknown vector $u=u(x,t)$ describes the flow velocity field and the scalar function Π represents the pressure. The initial datum u_0 is given and satisfies the divergence-free condition.

In a fundamental work, Leray [14] proved that, if a smooth solution u of (1.1) has a singularity at time T, then

$$\|\nabla u\|_{L^2(\mathbb{R}^3)}\ge\frac{C}{(T-t)^{\frac14}}. \tag{1.2}$$

In what follows, we always suppose that T is the blow up time of a smooth solution to (1.1). In [14], Leray also mentioned the following result without proof, for $3<p<\infty$,

$$\|u\|_{L^p(\mathbb{R}^3)}\ge\frac{C}{(T-t)^{\frac{p-3}{2p}}}. \tag{1.3}$$

It is worth remarking that the proof of (1.3) is due to Giga in [9] and an alternative (elementary) proof was presented by Robinson and Sadowski in [20]. Using the bound

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(1.3), Giga [9] further showed that $(2p)^{-1}(2/p+3/q-1)$-dimensional Hausdorff measure of possible time-singularity of u is zero if u is in $L^p(0,T;L^q(\mathbb{R}^3))$ with $p\geq 1$ and $q>3$. More results concerning the Hausdorff measure of possible time-singularity can be found in [10, 13, 15].

On the one hand, by the Sobolev inequality, one derives from (1.3) that, for $1/2<s<3/2$,

$$\|u\|_{\dot{H}^s(\mathbb{R}^3)}\geq \frac{C}{(T-t)^{\frac{2s-1}{4}}}. \tag{1.4}$$

In the last decade, many authors studied the lower bounds of potential blow-up solutions in homogeneous Sobolev spaces $\dot{H}^s$ with $s\geq 3/2$ (see e.g. [1, 3, 5, 6, 16, 17] and references therein). The subtle case $s=3/2$ in (1.4) was independently solved by Cheskidov and Zaya [3], by Cortissoz and Montero [6] and by McCormick, Olson, Robinson, Rodrigo, Vidal-López, and Zhou [16]. The case $3/2<s<5/2$ in (1.4) was proved by Robinson, Sadowski, and Silva in [17]. Note that lower bounds of potential blow-up solutions (1.2), (1.3) and (1.4) with $1/2<s<5/2$ are optimal under the natural scaling of (1.1). Partial results involving (1.4) with $s\geq 5/2$ can be found in [1, 5, 16, 17].

On the other hand, it follows from the Sobolev inequality and (1.3) that, for $3/2<p<3$,

$$\|\nabla u\|_{L^p(\mathbb{R}^3)}\geq \frac{C}{(T-t)^{\frac{2p-3}{2p}}}. \tag{1.5}$$

Considering recent works on blow up rates mentioned above, a natural question arises whether the cases $p\geq 3$ in (1.5) are valid. This paper is devoted to this problem. Our first result is stated as follows.

THEOREM 1.1. *Suppose that a classical solution u of the Navier-Stokes equations loses its regularity at time T. Then, for $3/2<p<\infty$,*

$$\|\nabla u\|_{L^p(\mathbb{R}^3)}\geq \frac{C}{(T-t)^{\frac{2p-3}{2p}}}. \tag{1.6}$$

REMARK 1.1. Theorem 1.1 is a generalization of Leary's classic result (1.2).

REMARK 1.2. Notice that, for $1\leq s<5/2$, the Sobolev inequality holds that

$$\|\nabla u\|_{L^p(\mathbb{R}^3)}\leq C\|u\|_{\dot{H}^s}, \quad \text{with} \quad p=\frac{6}{5-2s}.$$

Therefore, Theorem 1.1 unifies recent corresponding results of (1.4) with $1\leq s<5/2$ in [3, 6, 16, 17]. As a byproduct, we present the fourth proof of (1.4) with $s=3/2$.

REMARK 1.3. Using again the Sobolev inequality $\|\nabla u\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^3)}\leq C\|\nabla^2 u\|_{L^p(\mathbb{R}^3)}$ with $1<p<3$, we see that

$$\|\nabla^2 u\|_{L^p(\mathbb{R}^3)}\geq \frac{C}{(T-t)^{\frac{3p-3}{2p}}}.$$

It is unknown whether the case $p\geq 3$ holds or not in the latter inequality. More generally, it is an open problem to show, for $k>3$ and $p\neq 2$,

$$\|u\|_{\dot{H}^k_p(\mathbb{R}^3)}\geq \frac{C}{(T-t)^{\frac{(k+1)p-3}{2p}}}.$$

Unlike the $\dot{H}^s$ norm that was treated by energy method, the $\dot{H}^s_p$ norm with $p\neq 2$ requires addition estimate of the pressure. It seems to be difficult to obtain (1.6) with $p\geq 3$ via the equations of velocity (1.1). We refer the readers to [13] for (1.6) with $2\leq p<3$. To overcome this difficulty, we observe that there exist two absolute constants C_1 and C_2 such that

$$\|\nabla u\|_{L^p(\mathbb{R}^3)}\leq C_1\|\mathrm{curl}\, u\|_{L^p(\mathbb{R}^3)}\leq C_2\|\nabla u\|_{L^p(\mathbb{R}^3)}, \quad \text{with } 1<p<\infty. \tag{1.7}$$

Indeed, combining the Biot-Savart law and the Riesz transform $\mathcal{R}=(\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3)$ with $\mathcal{R}_i=\frac{\partial_i}{(-\Delta)^{1/2}}$, one can derive that $\partial_i u=\mathcal{R}_i(\mathcal{R}\times \mathrm{curl}\, u)$. This together with the Calderón-Zygmund estimate implies the first inequality in (1.7). On the other hand, the second inequality in (1.7) follows from the definition of vorticity. Therefore, it is enough to utilize the vorticity Equations (2.1) to derive (1.6).

The second goal of this paper is to apply the lower bound of blow up to study the upper box dimension of the set of singular times of weak solutions to (1.1), under the condition $u\in L^p(0,T;L^q(\mathbb{R}^3))$ or $\nabla u\in L^p(0,T;L^q(\mathbb{R}^3))$. The definition of the upper box dimension is presented in Section 2. It was shown that the upper box dimension of the set of singular times of weak solutions is at most $1/2$ by Robinson and Sadowski in [18]. Subsequently, in [19], Robinson and Sadowski proved that if a suitable weak solution u is in $u\in L^p(0,T;L^q(\mathbb{R}^3))$, then the upper box dimension of the set of space-time singularities is no greater than $\max\{p,q\}(2/p+3/q-1)$. Note that u is regular if u satisfies a Ladyzhenskaya-Prodi-Serrin-type condition $u\in L^p(0,T;L^q(\mathbb{R}^3))$ with $2/p+3/q=1$ and hence the set of singular times is empty (see e.g. [8] and references therein). Inspired by Robinson and Sadowski's work [19], we investigate the upper box dimension of the set of time singularities of Leray-Hopf weak solutions to the Navier-Stokes Equations (1.1) when $u\in L^p(0,T;L^q(\mathbb{R}^3))$ or $\nabla u\in L^p(0,T;L^q(\mathbb{R}^3))$. With blow up rates (1.6) in hand, we follow the path of [18] to discuss the upper box dimension of the set of singular times when the gradient of velocity ∇u is in $L^p(0,T;L^q(\mathbb{R}^3))$. Before formulating the results, we denote the set of singular times of Leray-Hopf weak solutions by

$$\mathcal{IR}_1=\{t:\|\nabla u\|_{L^p(\mathbb{R}^3)}\}=\infty\}. \tag{1.8}$$

and

$$\mathcal{IR}_2=\{t:\|u(t)\|_{L^p(\mathbb{R}^3)}=\infty\}. \tag{1.9}$$

With blow up rates (1.6) in hand, following the path of [18], we prove the following theorems for the upper box dimension of the set of singular times.

THEOREM 1.2. *The upper box dimension of the set of singular times $\mathcal{IR}_1$ is at most $\frac{p}{2}(2/p+3/q-2)$ if ∇u is in $L^p(0,T;L^q(\mathbb{R}^3))$ with $p\geq 2$.*

THEOREM 1.3. *The upper box dimension of the set of singular times $\mathcal{IR}_2$ is less than or equal to $\frac{p}{2}(2/p+3/q-1)$ if u belongs to $L^p(0,T;L^q(\mathbb{R}^3))$ with $q>3$.*

REMARK 1.4. Recall that each Leray-Hopf weak solution u satisfies that $\nabla u\in L^2(0,T;L^2(\mathbb{R}^3))$ and $u\in L^2(0,T;L^6(\mathbb{R}^3))$. Thus Theorem 1.2 and Theorem 1.3 generalize Robinson and Sadowski's corresponding results in [19].

REMARK 1.5. Note that the Hausdorff dimension is less than or equal to the upper box dimension (see e.g. [7]). Hence, there is no comparability between Theorem 1.3 and Giga's time singularities result aforementioned in [9] .

Next, we are concerned with the 2D supercritical surface quasi-geostrophic equation below, for $\alpha\in(0,1)$,

$$\begin{cases}\theta_t+\Lambda^\alpha\theta+v\cdot\nabla\theta=0,\\ v=(-\mathcal{R}_2\theta,\mathcal{R}_1\theta),\\ \theta|_{t=0}=\theta_0,\end{cases}\tag{1.10}$$

where the unknown scalar function $\theta(x,t):\mathbb{R}^2\to\mathbb{R}$ stands for the temperature, $\mathcal{R}_1$ and $\mathcal{R}_2$ are the usual Riesz transforms in $\mathbb{R}^2$, and v is the velocity field which is given by the Riesz transforms of θ.

The surface quasi-geostrophic equation arises in geophysical fluids and can be seen as a model of the 3D Navier-Stokes system (see [4] and references therein). Very recently, this equation has attracted a lot of attention (see, e.g. [2, 4, 11, 22–26] and references therein). The local-in-time solutions with large initial datum and global solutions with small data have been constructed in [2, 11, 22–25]. In particular, the lower bound of θ in $\dot{H}^s$ obtained in [11] is that, for $2-\alpha<s<2-\alpha/2$,

$$\|\theta\|_{\dot{H}^s(\mathbb{R}^2)}\geq\frac{C}{(T-t)^{\frac{\alpha+s-2}{\alpha}}}.\tag{1.11}$$

This is optimal in the sense of the natural scaling of Equation (1.10), namely, if θ solves system (1.10), then $\theta_\lambda=\lambda^{\alpha-1}\theta(\lambda x,\lambda^\alpha t)$ is also a solution of (1.10) for any $\lambda\in\mathbb{R}^+$. As in [4], applying the operator $\nabla^\perp=(-\partial_{x_2},\partial_{x_1})$ to (1.10), we have

$$\Theta_t+\Lambda^\alpha\Theta+v\cdot\nabla\Theta-\Theta\cdot\nabla^\perp v=0\tag{1.12}$$

where $\Theta=\nabla^\perp\theta$. The nonlinear term $\Theta\cdot\nabla^\perp v$ in (1.12) is similar to the vortex-stretching term $w\cdot\nabla u$ in vorticity Equations (2.1) of the Navier-Stokes equations. Now, we state the result of supercritical surface quasi-geostrophic equation.

THEOREM 1.4. *Suppose that a classical solution θ of the quasi-geostrophic equation loses regularity at time T. Then, for $2/\alpha<p<\infty$,*

$$\|\nabla\theta\|_{L^p(\mathbb{R}^2)}\geq\frac{C}{(T-t)^{\frac{\alpha p-2}{\alpha p}}}.\tag{1.13}$$

REMARK 1.6. The Sobolev embedding guarantees that Theorem 1.4 recovers (1.11).

REMARK 1.7. The blow up rates of higher derivatives of solutions to Equation (1.10) are unknown except (1.11) and (1.13).

2. Proof of theorems

2.1. Preliminaries. We begin with the notations. For $q\in[1,\infty]$, $L^q(0,T;X)$ stands for the set of measurable functions on the interval $(0,T)$ with values in X and $\|f(t,\cdot)\|_X$ belongs to $L^q(0,T)$. The Fourier transform $\hat{f}$ of a tempered distribution $f(x)$ is defined as $\hat{f}(\xi)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}f(x)e^{-i\xi\cdot x}\,dx$. We denote the square root of the Laplacian $(-\Delta)^{1/2}$ by Λ. The Riesz transforms $\mathcal{R}_j$ are defined by $\widehat{\mathcal{R}_j f}=-\frac{i\xi_j}{|\xi|}\hat{f}(\xi)$, where $\hat{f}(\xi)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}f(x)e^{-i\xi\cdot x}\,dx$. $\Lambda^s f$ is defined via $\widehat{\Lambda^s f}(\xi)=|\xi|^{2\alpha}\hat{f}(\xi)$. The homogenous Sobolev norm $\|\cdot\|_{\dot{H}^s_p(\mathbb{R}^3)}$ is defined as $\|f\|_{\dot{H}^s_p(\mathbb{R}^3)}=\|\Lambda^s f\|_{L^p(\mathbb{R}^3)}$. We denote $\dot{H}^s_2$ by $\dot{H}^s$ for simplicity. Throughout this paper, C is an absolute constant which may be different from line to line unless otherwise stated.

Next, we recall the following well-known ODE lemma which is helpful in the study of blow up rates.

Lemma 2.1 ([16, 17]). *Assume that there exist two positive constants C and γ such that*

$$\frac{dX(t)}{dt} \leq CX^{1+\gamma}(t),$$

and $X(t) \to \infty$ as $t \to T$. Then, for $X(t) > 0$, there holds

$$X(t) \geq \Big[\frac{1}{\gamma C(T-t)}\Big]^{\frac{1}{\gamma}}.$$

Hausdorff dimension and box dimension are widely used fractal dimensions. The usual definition of upper box-counting dimension and related material can be found in [7]. Here, we will use its equivalent definition showed in [21].

Definition 2.1 ([21]). *The (upper) box-counting dimension of a set X is defined as*

$$d_{box}(X) = \limsup_{\epsilon \to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon},$$

where $N(X,\epsilon)$ denotes the maximal number of disjoint balls of radius ϵ centred at points of X.

2.2. Blow up rates of the Navier-Stokes equations.

Proof. **(Proof of Theorem 1.1.)** We work with the vorcity equation. Denote $\omega = \operatorname{curl} u$. Taking the curl of the first equation of (1.1), we get

$$\omega_t - \Delta\omega + u\cdot\nabla\omega = \omega\cdot\nabla u. \tag{2.1}$$

Multiplying (2.1) by $\omega|\omega|^{p-2}$, using the incompressible condition, we arrive at

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|\omega|^p + \int_{\mathbb{R}^3}|\nabla\omega|^2|\omega|^{p-2}dx + \frac{4(p-2)}{p^2}\int_{\mathbb{R}^3}|\nabla|\omega|^{\frac{p}{2}}|^2dx = \int_{\mathbb{R}^3}\omega\cdot\nabla u\omega|\omega|^{p-2}dx. \tag{2.2}$$

The Calderón-Zygmund estimate ensures that, for $0 < p < \infty$,

$$\|\nabla u\|_{L^{p+1}(\mathbb{R}^3)} \leq C\|\omega\|_{L^{p+1}(\mathbb{R}^3)}. \tag{2.3}$$

By this and the Hölder inequality, we infer that

$$\begin{aligned}\int_{\mathbb{R}^3}\omega\cdot\nabla u\omega|\omega|^{p-2}dx &\leq \Big(\int_{\mathbb{R}^3}|\nabla u|^{p+1}dx\Big)^{\frac{1}{p+1}}\Big(\int_{\mathbb{R}^3}|\omega|^{p+1}dx\Big)^{1-\frac{1}{p+1}}\\ &\leq \Big(\int_{\mathbb{R}^3}|\omega|^{p+1}dx\Big).\end{aligned} \tag{2.4}$$

It follows from the Sobolev inequality that

$$\|\omega\|_{L^{3p}(\mathbb{R}^3)} = \||\omega|^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^6(\mathbb{R}^3)} \leq C\|\nabla|\omega|^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2(\mathbb{R}^3)}, \tag{2.5}$$

which together with the interpolation inequality leads to

$$\|\omega\|^{p+1}_{L^{p+1}(\mathbb{R}^3)} \leq \|\omega\|^{\frac{2p-1}{2}}_{L^p(\mathbb{R}^3)}\|\omega\|^{\frac{3}{2}}_{L^{3p}(\mathbb{R}^3)} \leq C\|\omega\|^{\frac{2p-1}{2}}_{L^p(\mathbb{R}^3)}\|\nabla|\omega|^{\frac{p}{2}}\|^{\frac{3}{p}}_{L^2(\mathbb{R}^3)}.$$

Plugging this into (2.4) and using the Young inequality, we obtain

$$\int_{\mathbb{R}^3} \omega\cdot\nabla u\omega|\omega|^{p-2}dx \leq C\|\omega\|_{L^p(\mathbb{R}^3)}^{\frac{p(2p-1)}{2p-3}} + \frac{4(p-2)}{p^2}\|\nabla|\omega|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^3)}^2.$$

We derive from the latter inequality and (2.2) that

$$\frac{d}{dt}\|\omega\|_{L^p(\mathbb{R}^3)}^p \leq C\|\omega\|_{L^p(\mathbb{R}^3)}^{\frac{p(2p-1)}{2p-3}} = C\|\omega\|_{L^p(\mathbb{R}^3)}^{p(1+\frac{2}{2p-3})}.$$

Applying Lemma 2.1 completes the proof. □

2.3. Proof of Theorem 1.2 and Theorem 1.3.

Proof. **(Proof of Theorem 1.2.)** As in [18], we proceed by contradiction. Thanks to the eventual regularity of weak solutions, we know that the set of irregular points $\mathcal{IR}_1$ is compact. We derive from the Definition 2.1 that if $\delta < d_{\text{box}}(\mathcal{IR}_1)$, then, there exists a sequence $\epsilon_j \to 0$ such that

$$N(\mathcal{IR}_1,\epsilon_j) > \epsilon_j^{-\delta},$$

where $N(\mathcal{IR}_1,\epsilon_j)$ represents the maximal number of disjoint balls centred on $t_j \in \mathcal{IR}_1$ of radius ϵ_j.

Consequently, we assume that $d_{\text{box}}(\mathcal{IR}_1) > 1-\frac{p(2q-3)}{2q}$, then we can choose a constant d such that $1-\frac{p(2q-3)}{2q} < d < d_{\text{box}}(\mathcal{IR}_1)$. Thus, there exists a decreasing sequence $\epsilon_j \to 0$ such that

$$N(\mathcal{IR}_1,\epsilon_j) > \epsilon_j^{-d}.$$

Let $\{t_i\}_{i=1}^{N(\mathcal{IR}_1,\epsilon_j)}$ be a collection of ϵ_j-separated points in $\mathcal{IR}_1$. For any $t_i \in \mathcal{IR}_1$, according to blow up rates of (1.6), we get

$$\int_{t_i-\epsilon_j}^{t_i+\epsilon_j} \|\nabla u\|_{L^q(\mathbb{R}^3)}^p dt \geq \int_{t_i-\epsilon_j}^{t_i} \frac{C}{(t_i-t)^{\frac{p(2q-3)}{2q}}} dt > \epsilon_j^{1-\frac{p(2q-3)}{2q}} C.$$

Combining the estimates above, we infer that

$$\int_0^T \|\nabla u\|_{L^q(\mathbb{R}^3)}^p dt \geq \sum_{i=1}^{N(\mathcal{IR}_1,\epsilon_j)} \int_{t_i-\epsilon_j}^{t_i+\epsilon_j} \|\nabla u\|_{L^q(\mathbb{R}^3)}^p dt > \epsilon_j^{1-\frac{p(2q-3)}{2q}-d} C.$$

Since $d > 1-\frac{p(q-3)}{2q}$, we obtain a contradiction as $j\to\infty$, which completes the proof of Theorem 1.2. □

Proof. **(Proof of Theorem 1.3.)** A slight variant of the proof of Theorem 1.2 provides the proof of Theorem 1.3. We omit the details. □

2.4. Blow up rates of quasi-geostrophic equation.

Proof. **(Proof of Theorem 1.4.)** A lower bound on the fractional Laplacian (see [2, 12]) entails us to obtain that

$$\Theta|\Theta|^{p-2}\Lambda^\alpha\Theta \geq \frac{2}{p}|\Lambda^{\alpha/2}\Theta^{p/2}|^2.$$

Multiplying (1.12) by $\Theta|\Theta|^{p-2}$ and using the incompressible condition, we get

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\Theta|^p dx+\frac{2}{p}\int_{\mathbb{R}^2}|\Lambda^{\alpha/2}\Theta^{\frac{p}{2}}|^2 dx\leq\int_{\mathbb{R}^2}\left|\Theta\cdot\nabla^{\perp}v\Theta|\Theta|^{p-2}\right|dx. \tag{2.6}$$

The Calderón-Zygmund estimate ensures that, for $0<p<\infty$,

$$\|\nabla^{\perp}v\|_{L^{p+1}(\mathbb{R}^2)}\leq C\|\Theta\|_{L^{p+1}(\mathbb{R}^3)}. \tag{2.7}$$

By the Hölder inequality and (2.3), we infer that

$$\begin{aligned}\int_{\mathbb{R}^2}\Theta\cdot\nabla v\nabla^{\perp}\theta|\Theta|^{p-2}dx&\leq\Big(\int_{\mathbb{R}^2}|\nabla^{\perp}v|^{p+1}dx\Big)^{\frac{1}{p+1}}\Big(\int_{\mathbb{R}^2}|\Theta|^{p+1}dx\Big)^{1-\frac{1}{p+1}}\\&\leq\Big(\int_{\mathbb{R}^2}|\Theta|^{p+1}dx\Big).\end{aligned} \tag{2.8}$$

It follows from the Sobolev inequality that

$$\|\Theta\|_{L^{\frac{2p}{2-\alpha}}(\mathbb{R}^2)}=\||\Theta|^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^{\frac{4}{2-\alpha}}(\mathbb{R}^2)}\leq\|\Lambda^{\alpha/2}\Theta^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2(\mathbb{R}^2)}. \tag{2.9}$$

By the interpolation inequality and (2.9), we arrive at

$$\|\Theta\|_{L^{p+1}(\mathbb{R}^2)}\leq\|\Theta\|^{1-\frac{2}{\alpha(p+1)}}_{L^p(\mathbb{R}^2)}\|\Theta\|^{\frac{2}{\alpha(p+1)}}_{L^{\frac{2p}{2-\alpha}}(\mathbb{R}^2)}\leq\|\Theta\|^{1-\frac{2}{\alpha(p+1)}}_{L^p(\mathbb{R}^2)}\|\Lambda^{\frac{\alpha}{2}}\Theta^{\frac{p}{2}}\|^{\frac{4}{\alpha p(p+1)}}_{L^2(\mathbb{R}^2)}.$$

Plugging this into (2.8) and using the Young inequality, we get

$$\int_{\mathbb{R}^2}\Theta\cdot\nabla^{\perp}v\Theta|\Theta|^{p-2}dx\leq C\|\Theta\|^{\frac{p(\alpha p+\alpha-2)}{\alpha p-2}}_{L^p(\mathbb{R}^2)}+\frac{1}{p}\|\Lambda^{\frac{\alpha}{2}}\Theta^{\frac{p}{2}}\|^2_{L^2(\mathbb{R}^2)},$$

where we require $\frac{4}{\alpha p}<2$. We derive from the latter inequality and (2.6) that

$$\frac{d}{dt}\|\Theta\|^p_{L^p(\mathbb{R}^2)}\leq C\|\Theta\|^{\frac{p(\alpha p+\alpha-2)}{\alpha p-2}}_{L^p(\mathbb{R}^2)}=C\|\Theta\|^{p(1+\frac{\alpha}{\alpha p-2})}_{L^p(\mathbb{R}^2)}.$$

Applying Lemma 2.1 completes the proof. □

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