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L^{∞} CONTINUATION PRINCIPLE FOR TWO-DIMENSIONAL COMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOWS*

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Abstract. We consider an initial boundary value problem of two-dimensional (2D) compressible nematic liquid crystal flows. Under a geometric condition for the initial orientation field, we show that the strong solution exists globally if the density is bounded from above. Our proof relies on elementary energy estimates and critical Sobolev inequalities of logarithmic type.

Keywords. compressible nematic liquid crystal flows; strong solutions; blow-up criterion.

AMS subject classifications. 76A15; 76N10; 35B65.

1. Introduction

Let $\Omega\subset\mathbb{R}^2$ be a bounded smooth domain, we investigate the motion of compressible nematic liquid crystal flows in Ω, which are governed by the following simplified version of the Ericksen-Leslie equations

$$\begin{cases}\rho_t+\operatorname{div}(\rho\mathbf{u})=0,\\ (\rho\mathbf{u})_t+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})-\mu\Delta\mathbf{u}-(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}+\nabla P=-\nabla\mathbf{d}\cdot\Delta\mathbf{d},\\ \mathbf{d}_t+(\mathbf{u}\cdot\nabla)\mathbf{d}=\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d},\\ |\mathbf{d}|=1.\end{cases}\tag{1.1}$$

The system (1.1) is supplemented with the initial condition

$$(\rho,\rho\mathbf{u},\mathbf{d})(x,0)=(\rho_0,\rho_0\mathbf{u}_0,\mathbf{d}_0)(x),\quad x\in\Omega,\tag{1.2}$$

and the boundary condition

$$\mathbf{u}=\mathbf{0},\quad \nabla\mathbf{d}\cdot\mathbf{n}=\mathbf{0},\quad x\in\partial\Omega.\tag{1.3}$$

Here $\rho,\mathbf{u},P=\rho^{\gamma}\ (\gamma>1),\mathbf{d}$ are the density, velocity, pressure, and the macroscopic average of the nematic liquid crystal orientation field, respectively. $\mathbf{n}$ is the unit outer normal vector of $\partial\Omega$. The constant viscosity coefficients μ and λ satisfy the physical restrictions

$$\mu>0,\quad \mu+\lambda\geq 0.\tag{1.4}$$

Liquid crystals can form and remain in an intermediate phase of matter between liquids and solids. When a solid melts, if the energy gain is enough to overcome the positional order but the shape of the molecules prevents the immediate collapse of orientational order, liquid crystals are formed. The nematic liquid crystals exhibit long range ordering in the sense that their rigid rod-like molecules arrange themselves with their long axes parallel to each other. Their molecules float around as in a liquid, but have the tendency to align along a preferred direction due to their orientation.

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The continuum theory of the nematic liquid crystals was first developed by Ericksen [2] and Leslie [14] during the period of 1958 through 1968. For more results on the simplified Ericksen-Leslie system modeling incompressible liquid crystal flows, refer to [5, 13, 15, 17–22] and references therein.

The study on the theory of well-posedness of solutions to the Cauchy problem and the initial boundary value problem (IBVP) for the compressible nematic liquid crystal flows has grown enormously in recent years. Based on adapting the standard three-level approximation scheme and the weak convergence arguments, Jiang et al. [11, 12] proved the global existence of weak solutions with finite energy for multi-dimensional compressible nematic liquid crystal flows. In [16], Li et al. established the global existence and uniqueness of classical solutions to the 3D Cauchy problem with smooth initial data which are of small energy but possibly large oscillations and vacuum, which is analogous to the result for compressible Navier-Stokes equations obtained by Huang et al. [8]. Later on, Wang [27] extended local strong solutions of the 2D Cauchy problem obtained in [24] to a global one provided that the smooth initial data are of small total energy. Nevertheless, many physical important and mathematical fundamental problems are still open due to the lack of smoothing mechanism and the strong nonlinearity.

Up to now, the regularity and uniqueness of weak solutions and the global well-posedness of the strong solutions to compressible nematic liquid crystal flows for general initial data are still open and challenge problems even in two dimensions. Therefore, it is important to study the mechanism of blow-up and structure of possible singularities of strong (or classical) solutions to the compressible nematic liquid crystal flows. For the Cauchy problem and IBVP of 3D compressible nematic liquid crystal flows, Huang et al. [9] obtained the following criterion

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)}+\|\nabla\mathbf{d}\|_{L^3(0,T;L^\infty)}\right)=\infty \tag{1.5}$$

under the assumption

$$7\mu>9\lambda. \tag{1.6}$$

Without the artificial restriction (1.6), the authors [10] showed that

$$\lim_{T\to T^*}\left(\|\mathfrak{D}(\mathbf{u})\|_{L^1(0,T;L^\infty)}+\|\nabla\mathbf{d}\|_{L^2(0,T;L^\infty)}\right)=\infty, \tag{1.7}$$

where $\mathfrak{D}(\mathbf{u})$ denotes the deformation tensor given by

$$\mathfrak{D}(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^{tr}).$$

Later on, Huang and Wang [7] established the following Serrin type criterion

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)}+\|\mathbf{u}\|_{L^{s_1}(0,T;L^{r_1})}+\|\nabla\mathbf{d}\|_{L^{s_2}(0,T;L^{r_2})}\right)=\infty \tag{1.8}$$

with s_i and r_i satisfying

$$\frac{2}{s_i}+\frac{3}{r_i}\le 1,\ s_i>1,\ 3<r_i\le\infty,\ i=1,2. \tag{1.9}$$

Recently, for the Cauchy problem of 2D compressible nematic liquid crystal flows, Wang [28] proved that

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)}+\|\nabla\mathbf{d}\|_{L^s(0,T;L^r)}\right)=\infty \tag{1.10}$$

with s and r satisfying

$$\frac{2}{s}+\frac{2}{r}\le 1,\ s>1,\ 2<r\le\infty. \tag{1.11}$$

Meanwhile, if the initial orientation $\mathbf{d_0}=(d_{01},d_{02})$ satisfies a geometric condition

$$d_{02}\ge\varepsilon_0 \tag{1.12}$$

for some positive constant ε_0, Liu and Wang [23] extended (1.10) to a refiner form

$$\lim_{T\to T^*}\|\rho\|_{L^\infty(0,T;L^\infty)}=\infty. \tag{1.13}$$

Our goal in this paper is to give a blow-up criterion of strong solutions to the problem (1.1)–(1.3) in terms of the density only.

Before stating our main result, we first explain the notations and conventions used throughout this paper. For $r>0$, set

$$\int\cdot dx:=\int_\Omega\cdot dx,\quad \dot{\mathbf{u}}:=\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}.$$

For $1\le p\le\infty$ and integer $k\ge 0$, the standard Sobolev spaces are denoted by:

$$L^p=L^p(\Omega),\ W^{k,p}=W^{k,p}(\Omega),\ H^k=H^{k,2}(\Omega).$$

Now we define precisely what we mean by strong solutions to the problem (1.1)–(1.3).

Definition 1.1 (Strong solutions). *$(\rho\ge 0,\mathbf{u},\mathbf{d})$ is called a strong solution to (1.1)–(1.3) in $\Omega\times(0,T)$ if for some $q>2$,*

$$\begin{cases}\rho\in C([0,T];W^{1,q}),\ \rho_t\in C([0,T];L^q),\\ \nabla\mathbf{u}\in L^\infty(0,T;H^1)\cap L^2(0,T;W^{1,q}),\\ \sqrt{\rho}\mathbf{u},\ \sqrt{\rho}\dot{\mathbf{u}}\in L^\infty(0,T;L^2),\\ \nabla\mathbf{d}\in C([0,T];H^2)\cap L^2(0,T;H^3),\\ \mathbf{d}_t\in C([0,T];H^1)\cap L^2(0,T;H^2),\end{cases}$$

and $(\rho,\mathbf{u},\mathbf{d})$ satisfies both (1.1) almost everywhere in $\Omega\times(0,T)$ and (1.2) almost everywhere in Ω.

Our main result reads as follows:

Theorem 1.1. *Let the initial data $(\rho_0\ge 0,\mathbf{u}_0,\mathbf{d}_0)$ satisfy for any given number $q>2$,*

$$\rho_0\in W^{1,q},\ \mathbf{u}_0\in H_0^1\cap H^2,\ \nabla\mathbf{d}_0\in H^2,\ \nabla\mathbf{d}_0\cdot\mathbf{n}|_{\partial\Omega}=\mathbf{0},\ |\mathbf{d}_0|=1, \tag{1.14}$$

and the compatibility conditions

$$-\mu\Delta\mathbf{u}_0-(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}_0+\nabla\rho_0^\gamma+\nabla\mathbf{d}_0\cdot\Delta\mathbf{d}_0=\sqrt{\rho_0}\mathbf{g} \tag{1.15}$$

for some $\mathbf{g}\in L^2(\Omega)$. Moreover, we require the initial orientation $\mathbf{d_0}=(d_{01},d_{02})$ satisfies a geometric condition

$$d_{02}\ge\underline{d_{02}}, \tag{1.16}$$

where $\underline{d_{02}}$ *is defined as in Lemma 2.2. Let* $(\rho,\mathbf{u},\mathbf{d})$ *be a strong solution to the problem* (1.1)–(1.3). *If* $T^* < \infty$ *is the maximal time of existence for that solution, then we have*

$$\lim_{T\to T^*}\|\rho\|_{L^\infty(0,T;L^\infty)} = \infty. \tag{1.17}$$

Remark 1.1. Under the conditions of Theorem 1.1, the local existence of the strong solution can be obtained by the similar arguments as in [9]. Thus, the maximal time T^* is well-defined. Moreover, it is worth mentioning that the condition (1.16) is not needed to establish the local existence of strong solutions.

Remark 1.2. The conclusion in Theorem 1.1 reveals that the concentration of the density must be responsible for the loss of the regularity in finite time for two-dimensional compressible nematic liquid crystal flows.

We now make some comments on the analysis of this paper. We mainly make use of continuation arguments to prove Theorem 1.1. That is, suppose that (1.17) were false, i.e.,

$$\lim_{T\to T^*}\|\rho\|_{L^\infty(0,T;L^\infty)} \le M_0 < \infty.$$

We want to show that

$$\sup_{0\le t\le T^*}\left(\|\rho\|_{W^{1,q}} + \|\nabla\mathbf{u}\|_{H^1} + \|\nabla\mathbf{d}\|_{H^2}\right) \le C < +\infty.$$

We first prove (see Lemma 3.2) that a control of upper bound of the density implies a control on the $L_t^\infty L_x^2$-norm of $\nabla\mathbf{u}$ and $\nabla^2\mathbf{d}$. To this end, the key ingredient of the analysis is a logarithmic Sobolev inequality (see Lemma 2.5). The inequality implies the uniform estimates of $\|\mathbf{u}\|_{L^2(0,T;L^\infty)}$ and $\|\nabla\mathbf{d}\|_{L^2(0,T;L^\infty)}$ due to the a priori estimates of $\|\mathbf{u}\|_{L^2(0,T;H^1)}$ and $\|\nabla\mathbf{d}\|_{L^2(0,T;H^1)}$ from the energy estimates (3.2). Then we obtain the key a priori estimates on $L^\infty(0,T;L^q)$-norm of the density gradient by solving a logarithmic Grönwall inequality based on a Brézis-Waigner type inequality (see Lemma 2.6) and the a priori estimates we have just derived.

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

We begin with the following Gagliardo-Nirenberg inequality (see [3, Theorem 10.1, p. 27]).

Lemma 2.1 (Gagliardo-Nirenberg). *Let* $\Omega\subset\mathbb{R}^2$ *be a bounded smooth domain. Assume that* $1\le q,r\le\infty$, *and* j,m *are arbitrary integers satisfying* $0\le j<m$. *If* $v\in W^{m,r}(\Omega)\cap L^q(\Omega)$, *then we have*

$$\|D^j v\|_{L^p} \le C\|v\|_{L^q}^{1-a}\|v\|_{W^{m,r}}^a,$$

where

$$-j+\frac{2}{p} = (1-a)\frac{2}{q} + a\left(-m+\frac{2}{r}\right),$$

and

$$a \in \begin{cases} [\frac{j}{m},1), & \text{if } m-j-\frac{2}{r} \text{ is a nonnegative integer,} \\ [\frac{j}{m},1], & \text{otherwise.} \end{cases}$$

The constant C depends only on $m,j,q,r,a,$ and Ω.

Next, the following useful result was deduced in [4], which will help us to get the estimate of $\|\nabla^2\mathbf{d}\|_{L^2(0,T;L^2)}$ in terms of the basic energy inequality. We sketch it here for completeness.

LEMMA 2.2. *For the initial direction field $\mathbf{d}_0=(d_{01},d_{02}):\Omega\to\mathbb{S}^1$, assume that d_{02} satisfies the condition*

$$d_{02}\geq \underline{d_{02}}\geq 0, \tag{2.1}$$

where $\underline{d_{02}}$ is constant and is defined as

$$\begin{cases} \underline{d_{02}} > \sqrt{1-\frac{1}{A_1A_2}}, & \text{if } A_1A_2\geq 1, \\ \underline{d_{02}}=0, & \text{if } A_1A_2<1. \end{cases} \tag{2.2}$$

Here A_1 and A_2 are the best constants of the elliptic estimate

$$\|\nabla^2 f\|_{L^2}^2\leq A_1(\|\Delta f\|_{L^2}^2+\|f\|_{H^1}^2), \tag{2.3}$$

and Gagliardo-Nirenberg inequality

$$\|\nabla f\|_{L^4}^4\leq A_2(\|\nabla^2 f\|_{L^2}^2\|f\|_{L^\infty}^2+\|f\|_{L^\infty}^4). \tag{2.4}$$

Then we have

$$\|\nabla^2\mathbf{d}\|_{L^2}^2\leq C\left(\|\mathbf{d}-\mathbf{e}_2\|_{L^\infty}^4+\|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}\|_{L^2}^2+\|\mathbf{d}\|_{H^1}^2\right) \tag{2.5}$$

with $\mathbf{e}_2=(0,1)$.

Proof. Applying the maximum principle to the equation of d_2 (i.e., the 2-th component of $\mathbf{d}$) together with the geometric condition (2.1) yields that for any $t>0$ (see [12]),

$$\inf_{x\in\Omega} d_2(x,t)\geq \inf_{x\in\Omega} d_{02}(x)\geq \underline{d_{02}}\geq 0.$$

Due to $|\mathbf{d}|=1$, we have

$$\|\mathbf{d}-\mathbf{e}_2\|_{L^\infty}^2\leq 1-\underline{d_{02}}^2. \tag{2.6}$$

Substituting (2.1) and (2.2) into (2.6), we get

$$A_1A_2\|\mathbf{d}-\mathbf{e}_2\|_{L^\infty}^2<1. \tag{2.7}$$

Recalling that $|\mathbf{d}|=1$, and thus $\Delta(|\mathbf{d}|^2)=\Delta(\mathbf{d}\cdot\mathbf{d})=0$, which gives $\mathbf{d}\cdot\Delta\mathbf{d}=-|\nabla\mathbf{d}|^2$. Hence we have

$$\|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}\|_{L^2}^2=\int(\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d})\cdot(\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d})dx=\int|\Delta\mathbf{d}|^2dx-\int|\nabla\mathbf{d}|^4dx,$$

that is

$$\|\Delta \mathbf{d}\|_{L^2}^2 = \|\nabla \mathbf{d}\|_{L^4}^4 + \|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\|_{L^2}^2,$$

which combined with (2.3) and (2.4) leads to

$$\begin{aligned}\|\nabla^2 \mathbf{d}\|_{L^2}^2 &\le A_1 \left(\|\Delta \mathbf{d}\|_{L^2}^2 + \|\mathbf{d}\|_{H^1}^2\right)\\ &= A_1 \left(\|\nabla \mathbf{d}\|_{L^4}^4 + \|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\|_{L^2}^2 + \|\mathbf{d}\|_{H^1}^2\right)\\ &\le A_1 A_2 \|\mathbf{d} - \mathbf{e}_2\|_{L^\infty}^2 \|\nabla^2 \mathbf{d}\|_{L^2}^2\\ &\quad + A_1 \left(A_2 \|\mathbf{d} - \mathbf{e}_2\|_{L^\infty}^4 + \|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\|_{L^2}^2 + \|\mathbf{d}\|_{H^1}^2\right).\end{aligned}$$

This along with (2.2) and (2.7) yields the desired (2.5). □

Next, we give some regularity results for the following Lamé system with Dirichlet boundary condition, the proof can be found in [25, Proposition 2.1].

$$\begin{cases}\mu \Delta \mathbf{U} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{U} = \mathbf{F}, & x \in \Omega,\\ \mathbf{U} = \mathbf{0}, & x \in \partial\Omega.\end{cases} \tag{2.8}$$

Lemma 2.3. *Let $q \ge 2$ and $\mathbf{U}$ be a weak solution of (2.8). There exists a constant C depending only on q, μ, λ, and Ω such that the following estimates hold:*

- *If $\mathbf{F} \in L^q(\Omega)$, then*

$$\|\mathbf{U}\|_{W^{2,q}} \le C \|\mathbf{F}\|_{L^q};$$

- *If $\mathbf{F} \in W^{-1,q}(\Omega)$ (i.e., $\mathbf{F} = \operatorname{div} \mathbf{f}$ with $\mathbf{f} = (f_{ij})_{3\times 3}, f_{ij} \in L^q(\Omega)$), then*

$$\|\mathbf{U}\|_{W^{1,q}} \le C \|\mathbf{f}\|_{L^q};$$

- *If $\mathbf{F} \in W^{-1,q}(\Omega)$ (i.e., $\mathbf{F} = \operatorname{div} \mathbf{f}$ with $\mathbf{f} = (f_{ij})_{3\times 3}, f_{ij} \in L^\infty(\Omega)$), then*

$$[\nabla \mathbf{U}]_{BMO} \le C \|\mathbf{f}\|_{L^\infty}.$$

Here $BMO(\Omega)$ stands for the John-Nirenberg's space of bounded mean oscillation whose norm is defined by

$$\|f\|_{BMO} = \|f\|_{L^2} + [f]_{BMO}$$

with

$$[f]_{BMO} = \sup_{x \in \Omega, r \in (0,d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r}(x)| dy,$$

and

$$f_{\Omega_r}(x) = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y) dy,$$

where $\Omega_r(x) = B_r(x) \cap \Omega$, $B_r(x)$ is the ball with center x and radius r, and d is the diameter of Ω. $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$.

Next, motivated by [26], we decompose the velocity field into two parts in order to overcome the difficulty caused by the boundary, namely $\mathbf{u}=\mathbf{v}+\mathbf{w}$, where $\mathbf{v}$ is the solution to the Lamé system

$$\begin{cases}\mu\Delta\mathbf{v}+(\mu+\lambda)\nabla\operatorname{div}\mathbf{v}=\nabla P, & x\in\Omega,\\ \mathbf{v}=\mathbf{0}, & x\in\partial\Omega,\end{cases}\tag{2.9}$$

and $\mathbf{w}$ satisfies the following boundary value problem

$$\begin{cases}\mu\Delta\mathbf{w}+(\mu+\lambda)\nabla\operatorname{div}\mathbf{w}=\rho\dot{\mathbf{u}}+\nabla\mathbf{d}\cdot\Delta\mathbf{d}, & x\in\Omega,\\ \mathbf{w}=\mathbf{0}, & x\in\partial\Omega.\end{cases}\tag{2.10}$$

By virtue of Lemma 2.3, one has the following key estimates for $\mathbf{v}$ and $\mathbf{w}$.

LEMMA 2.4. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be a solution of* (2.9) *and* (2.10) *respectively. Then for any* $p\geq 2$, *there is a constant* $C>0$ *depending only on* p,μ,λ, *and* Ω *such that*

$$\|\mathbf{v}\|_{W^{1,p}}\leq C\|P\|_{L^p},\tag{2.11}$$

and

$$\|\mathbf{w}\|_{W^{2,p}}\leq C\|\rho\dot{\mathbf{u}}+\nabla\mathbf{d}\cdot\Delta\mathbf{d}\|_{L^p}.\tag{2.12}$$

Next, we state a critical Sobolev inequality of logarithmic type, which is originally due to Brézis-Wainger [1]. The reader can refer to [6, Section 2] for the proof.

LEMMA 2.5. *Assume* Ω *is a bounded smooth domain in* $\mathbb{R}^2$ *and* $f\in L^2(s,t;W^{1,q}(\Omega))$ *with some* $q>2$ *and* $0\leq s<t\leq\infty$, *then there is a constant* $C>0$ *depending only on* q *and* Ω *such that*

$$\|f\|_{L^2(s,t;L^\infty)}^2\leq C\left(1+\|f\|_{L^2(s,t;H^1)}^2\log(e+\|f\|_{L^2(s,t;W^{1,q})})\right).\tag{2.13}$$

Finally, the following variant of the Brézis-Waigner inequality (see [26, Lemma 2.3]) also plays a crucial role in the later proof.

LEMMA 2.6. *Assume* Ω *is a bounded smooth domain in* $\mathbb{R}^2$ *and* $f\in W^{1,q}(\Omega)$ *with some* $q>2$, *then there is a constant* $C>0$ *depending only on* q *and* Ω *such that*

$$\|f\|_{L^\infty}\leq C(1+\|f\|_{BMO}\log(e+\|f\|_{W^{1,q}})).$$

3. Proof of Theorem 1.1

Let $(\rho,\mathbf{u},\mathbf{d})$ be a strong solution described in Theorem 1.1. Suppose that (1.17) were false, that is, there exists a constant $M_0>0$ such that

$$\lim_{T\to T^*}\|\rho\|_{L^\infty(0,T;L^\infty)}\leq M_0<\infty.\tag{3.1}$$

Next, we have the following standard estimate.

LEMMA 3.1. *Under the condition* (3.1), *it holds that for any* $T\in[0,T^*)$,

$$\sup_{0\leq t\leq T}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^2}^2\right)+\int_0^T\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla^2\mathbf{d}\|_{L^2}^2\right)dt\leq C,\tag{3.2}$$

where and in what follows, C stands for generic positive constant depending only on M_0,λ,μ,T^*, and the initial data.

Proof. Multiplying $(1.1)_2$ and $(1.1)_3$ by $\mathbf{u}$ and $-(\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d})$ respectively, then adding the two resulting equations together, and integrating over Ω, we obtain after integrating by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left(\rho|\mathbf{u}|^2+|\nabla\mathbf{d}|^2\right)dx+\int\left[\mu|\nabla\mathbf{u}|^2+(\lambda+\mu)(\operatorname{div}\mathbf{u})^2+|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}|^2\right]dx\\&=\int P\operatorname{div}\mathbf{u}dx.\end{aligned}\tag{3.3}$$

Due to $P=\rho^\gamma$, it follows from $(1.1)_1$ that P satisfies

$$P_t+\operatorname{div}(P\mathbf{u})+(\gamma-1)P\operatorname{div}\mathbf{u}=0.\tag{3.4}$$

Integrating (3.4) over Ω and then adding the resulting equality to (3.3) give rise to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left(\rho|\mathbf{u}|^2+|\nabla\mathbf{d}|^2+\frac{2P}{\gamma-1}\right)dx\\&\quad+\int\left[\mu|\nabla\mathbf{u}|^2+(\lambda+\mu)(\operatorname{div}\mathbf{u})^2+|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}|^2\right]dx=0.\end{aligned}\tag{3.5}$$

Thus, integrating (3.5) with respect to t leads to

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^2}^2\right)+\int_0^T\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}\|_{L^2}^2\right)dt\le C.\tag{3.6}$$

This along with (2.5) gives rise to

$$\int_0^T\|\nabla^2\mathbf{d}\|_{L^2}^2dt\le C.\tag{3.7}$$

So the desired (3.2) follows from (3.6) and (3.7). □

LEMMA 3.2. *Under the condition* (3.1), *it holds that for any $T\in[0,T^*)$,*

$$\sup_{0\le t\le T}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla^2\mathbf{d}\|_{L^2}^2\right)+\int_0^T\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{d}_t\|_{L^2}^2+\|\nabla^3\mathbf{d}\|_{L^2}^2\right)dt\le C.\tag{3.8}$$

Proof.

(1) Multiplying $(1.1)_2$ by $\mathbf{u}_t$ and integrating the resulting equation over Ω give rise to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\left(\mu|\nabla\mathbf{u}|^2+(\mu+\lambda)(\operatorname{div}\mathbf{u})^2\right)dx+\int\rho|\dot{\mathbf{u}}|^2dx\\&=\int\rho\dot{\mathbf{u}}\cdot\mathbf{u}\cdot\nabla\mathbf{u}dx+\int P\operatorname{div}\mathbf{u}_tdx-\int\mathbf{u}_t\cdot\operatorname{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d})dx+\frac{1}{2}\int\mathbf{u}_t\cdot\nabla|\nabla\mathbf{d}|^2dx\\&=:K_1+K_2+K_3+K_4.\end{aligned}\tag{3.9}$$

It follows from Cauchy-Schwarz inequality and (3.1) that

$$K_1\le\frac{1}{4}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+C\||\mathbf{u}||\nabla\mathbf{u}|\|_{L^2}^2.\tag{3.10}$$

To bound K_2, we decompose $\mathbf{u}$ into $\mathbf{u}=\mathbf{v}+\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ satisfy (2.9) and (2.10), respectively. Then we have

$$
\begin{aligned}
K_2 =& \frac{d}{dt}\left(\int P \operatorname{div}\mathbf{u} dx\right) - \int P_t \operatorname{div}\mathbf{u} dx \\
=& \frac{d}{dt}\left(\int P \operatorname{div}\mathbf{u} dx\right) - \int P_t \operatorname{div}\mathbf{v} dx - \int P_t \operatorname{div}\mathbf{w} dx \\
=& \frac{d}{dt}\left(\int P \operatorname{div}\mathbf{u} dx\right) + \int \nabla P_t \cdot \mathbf{v} dx + \int \operatorname{div}(P\mathbf{u}) \operatorname{div}\mathbf{w} dx \\
&+ (\gamma-1)\int P \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{w} dx \\
=& \frac{d}{dt}\left(\int P \operatorname{div}\mathbf{u} dx\right) + \int (\mu\Delta\mathbf{v} + (\mu+\lambda)\nabla\mathbf{v})_t \cdot \mathbf{v} dx - \int P\mathbf{u}\cdot\nabla \operatorname{div}\mathbf{w} dx \\
&+ (\gamma-1)\int P \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{w} dx \\
=& \frac{1}{2}\frac{d}{dt}\int \left(2P\operatorname{div}\mathbf{u} - \mu|\nabla\mathbf{v}|^2 - (\mu+\lambda)(\operatorname{div}\mathbf{v})^2\right) dx - \int P\mathbf{u}\cdot\nabla\operatorname{div}\mathbf{w} dx \\
&+ (\gamma-1)\int P \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{w} dx
\end{aligned} \tag{3.11}
$$

due to (3.4). By Hölder's inequality, Sobolev's inequality, Lemma 2.4, and (3.1), we derive that

$$
\begin{aligned}
&\left|\int P\mathbf{u}\cdot\nabla\operatorname{div}\mathbf{w} dx\right| + \left|(\gamma-1)\int P\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{w} dx\right| \\
\le& \|P\|_{L^3}\|\mathbf{u}\|_{L^6}\|\nabla\operatorname{div}\mathbf{w}\|_{L^2} + (\gamma-1)\|P\|_{L^\infty}\|\operatorname{div}\mathbf{u}\|_{L^2}\|\operatorname{div}\mathbf{w}\|_{L^2} \\
\le& \varepsilon\|\nabla^2\mathbf{w}\|_{L^2}^2 + C\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{w}\|_{L^2}^2\right) \\
\le& C\varepsilon\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{v}\|_{L^2}^2 + \||\nabla\mathbf{d}||\Delta\mathbf{d}|\|_{L^2}^2\right),
\end{aligned}
$$

which together with (3.11) and choosing ε suitably small yield

$$
\begin{aligned}
K_2 \le& \frac{1}{2}\frac{d}{dt}\int \left(2P\operatorname{div}\mathbf{u} - \mu|\nabla\mathbf{v}|^2 - (\mu+\lambda)(\operatorname{div}\mathbf{v})^2\right) dx \\
&+ \frac{1}{4}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{v}\|_{L^2}^2 + \||\nabla\mathbf{d}||\Delta\mathbf{d}|\|_{L^2}^2\right).
\end{aligned} \tag{3.12}
$$

Integration by parts and Cauchy-Schwarz inequality imply that

$$
\begin{aligned}
K_3 =& \int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}_t dx \\
=& \frac{d}{dt}\int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u} dx - \int (\nabla\mathbf{d}_t\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u} dx - \int (\nabla\mathbf{d}\odot\nabla\mathbf{d}_t)\cdot\nabla\mathbf{u} dx \\
\le& \frac{d}{dt}\int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u} dx + \delta\|\nabla\mathbf{d}_t\|_{L^2}^2 + C(\delta)\||\nabla\mathbf{d}||\nabla\mathbf{u}|\|_{L^2}^2.
\end{aligned} \tag{3.13}
$$

Similarly, one has

$$
K_4 \le \frac{1}{2}\frac{d}{dt}\int |\nabla\mathbf{d}|^2 \operatorname{div}\mathbf{u} dx + \delta\|\nabla\mathbf{d}_t\|_{L^2}^2 + C(\delta)\||\nabla\mathbf{d}||\nabla\mathbf{u}|\|_{L^2}^2. \tag{3.14}
$$

Putting (3.10), (3.12), (3.13), and (3.14) into (3.9), we get

$$B'(t)+\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2\le C\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{v}\|_{L^2}^2+\||\nabla\mathbf{d}||\Delta\mathbf{d}|\|_{L^2}^2\right)\\ +4\delta\|\nabla\mathbf{d}_t\|_{L^2}^2+C(\delta)\||\nabla\mathbf{d}||\nabla\mathbf{u}|\|_{L^2}^2, \tag{3.15}$$

where

$$B(t)\triangleq\frac{\mu}{2}\|\nabla\mathbf{u}\|_{L^2}^2+\frac{\lambda+\mu}{2}\|\operatorname{div}\mathbf{u}\|_{L^2}^2+\frac{\mu}{2}\|\nabla\mathbf{v}\|_{L^2}^2+\frac{\lambda+\mu}{2}\|\operatorname{div}\mathbf{v}\|_{L^2}^2\\ -\int P\operatorname{div}\mathbf{u}dx+\frac{1}{2}\int|\nabla\mathbf{d}|^2\operatorname{div}\mathbf{u}dx-\int(\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}dx. \tag{3.16}$$

(2) Applying ∇ on $(1.1)_3$, we have

$$\nabla\mathbf{d}_t-\Delta\nabla\mathbf{d}=-\nabla(\mathbf{u}\cdot\nabla\mathbf{d})+\nabla(|\nabla\mathbf{d}|^2\mathbf{d}), \tag{3.17}$$

from which one deduces that

$$\begin{aligned}&\frac{d}{dt}\int|\Delta\mathbf{d}|^2dx+\int\left(|\nabla\mathbf{d}_t|^2+|\nabla\Delta\mathbf{d}|^2\right)dx\\ =&\int|\nabla\mathbf{d}_t-\nabla\Delta\mathbf{d}|^2dx\\ =&\int|-\nabla(\mathbf{u}\cdot\nabla\mathbf{d})+\nabla(|\nabla\mathbf{d}|^2\mathbf{d})|^2dx\\ \le&C\int\left(|\nabla\mathbf{d}|^6+|\nabla\mathbf{d}|^2|\nabla^2\mathbf{d}|^2+|\nabla\mathbf{u}|^2|\nabla\mathbf{d}|^2+|\mathbf{u}|^2|\nabla^2\mathbf{d}|^2\right)dx.\end{aligned} \tag{3.18}$$

Multiplying (3.17) by $4|\nabla\mathbf{d}|^2\nabla\mathbf{d}$ and integrating by parts yield

$$\begin{aligned}&\frac{d}{dt}\int|\nabla\mathbf{d}|^4dx+4\int\left(|\nabla\mathbf{d}|^2|\nabla^2\mathbf{d}|^2+2|\nabla\mathbf{d}|^2|\nabla(\nabla\mathbf{d})|^2\right)dx\\ =&2\int_{\partial\Omega}|\nabla\mathbf{d}|^2\nabla|\nabla\mathbf{d}|^2\cdot\mathbf{n}d\sigma+4\int|\nabla\mathbf{d}|^2\nabla\mathbf{d}(-\nabla(\mathbf{u}\cdot\nabla\mathbf{d})+\nabla(|\nabla\mathbf{d}|^2\mathbf{d}))dx\\ \le&C(\varepsilon)\int\left(|\nabla\mathbf{d}|^2|\nabla^2\mathbf{d}|^2+|\nabla\mathbf{d}|^2|\nabla\mathbf{u}|^2+|\nabla^2\mathbf{d}|^2|\mathbf{u}|^2+|\nabla\mathbf{d}|^6+|\nabla\mathbf{d}|^4\right)dx\\ &+\varepsilon\int|\nabla^3\mathbf{d}|^2dx,\end{aligned}$$

where we have used Cauchy-Schwarz inequality and the following fact

$$\begin{aligned}&\left|2\int_{\partial\Omega}|\nabla\mathbf{d}|^2\nabla|\nabla\mathbf{d}|^2\cdot\mathbf{n}d\sigma\right|\\ \le&4\int_{\partial\Omega}|\nabla\mathbf{d}|^3|\nabla^2\mathbf{d}|d\sigma\\ \le&C\||\nabla\mathbf{d}|^3|\nabla^2\mathbf{d}|\|_{W^{1,1}}\\ \le&C\int\left(|\nabla\mathbf{d}|^3|\nabla^2\mathbf{d}|+|\nabla\mathbf{d}|^2|\nabla^2\mathbf{d}|^2+|\nabla\mathbf{d}|^3|\nabla^3\mathbf{d}|\right)dx\\ \le&C(\varepsilon)\int\left(|\nabla\mathbf{d}|^2|\nabla^2\mathbf{d}|^2+|\nabla\mathbf{d}|^6+|\nabla\mathbf{d}|^4\right)dx+\varepsilon\int|\nabla^3\mathbf{d}|^2dx.\end{aligned}$$

Thus, we derive

$$\begin{aligned}
&\frac{d}{dt}\int|\nabla\mathbf{d}|^4dx+\int|\nabla\mathbf{d}|^2|\nabla^2\mathbf{d}|^2dx\\
\leq&C\int\left(|\nabla\mathbf{d}|^2|\nabla^2\mathbf{d}|^2+|\nabla\mathbf{d}|^2|\nabla\mathbf{u}|^2+|\nabla^2\mathbf{d}|^2|\mathbf{u}|^2+|\nabla\mathbf{d}|^6+|\nabla\mathbf{d}|^4\right)dx+\varepsilon\int|\nabla^3\mathbf{d}|^2dx,
\end{aligned}$$

which along with (3.18) and choosing ε suitably small leads to

$$\begin{aligned}
&\frac{d}{dt}\left(\|\Delta\mathbf{d}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^4}^4\right)+\left(\|\nabla\mathbf{d}_t\|_{L^2}^2+\|\nabla^3\mathbf{d}\|_{L^2}^2\right)\\
\leq&C\int\left(|\nabla\mathbf{d}|^6+|\nabla\mathbf{d}|^4+|\nabla\mathbf{d}|^2|\nabla^2\mathbf{d}|^2+|\nabla\mathbf{d}|^2|\nabla\mathbf{u}|^2+|\nabla^2\mathbf{d}|^2|\mathbf{u}|^2\right)dx. \qquad (3.19)
\end{aligned}$$

(3) Choosing C_1 suitably large such that

$$\begin{aligned}
\frac{\mu}{4}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{v}\|_{L^2}^2\right)+\|\nabla\mathbf{d}\|_{L^4}^4-C\leq&B(t)+C_1\|\nabla\mathbf{d}\|_{L^4}^4\\
\leq&C\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{v}\|_{L^2}^2\right)+C\|\nabla\mathbf{d}\|_{L^4}^4+C. \qquad (3.20)
\end{aligned}$$

Adding (3.19) multiplied by C_1 to (3.15) and choosing δ small enough yield

$$\begin{aligned}
&\frac{d}{dt}\left(B(t)+C_1\|\Delta\mathbf{d}\|_{L^2}^2+C_1\|\nabla\mathbf{d}\|_{L^4}^4\right)+\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{d}_t\|_{L^2}^2\\
&\quad+\|\nabla^3\mathbf{d}\|_{L^2}^2+\||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2}^2\\
\leq&C\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{v}\|_{L^2}^2+\||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2}^2+\||\nabla\mathbf{d}||\nabla\mathbf{u}|\|_{L^2}^2\right)\\
&\quad+C\left(\||\mathbf{u}||\nabla^2\mathbf{d}|\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^4}^4+\|\nabla\mathbf{d}\|_{L^6}^6\right)\\
\leq&C\left(1+\|\nabla\mathbf{d}\|_{L^\infty}^2+\|\mathbf{u}\|_{L^\infty}^2\right)\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{v}\|_{L^2}^2+\|\nabla^2\mathbf{d}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^4}^4\right). \qquad (3.21)
\end{aligned}$$

Let

$$\begin{aligned}
\Phi(t)\triangleq e+&\sup_{0\leq\tau\leq t}\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{v}\|_{L^2}^2+\|\nabla^2\mathbf{d}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^4}^4\right)\\
&+\int_0^t\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{d}_\tau\|_{L^2}^2+\|\nabla^3\mathbf{d}\|_{L^2}^2+\||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2}^2\right)d\tau. \qquad (3.22)
\end{aligned}$$

Then by virtue of (3.21), (3.20), and Grönwall's inequality, we obtain that for every $0\leq s\leq T<T^*$,

$$\Phi(T)\leq C\Phi(s)\exp\left\{C\int_s^T\left(\|\mathbf{u}\|_{L^\infty}^2+\|\nabla\mathbf{d}\|_{L^\infty}^2\right)d\tau\right\}. \qquad (3.23)$$

(4) From Lemma 2.5, we get

$$\begin{aligned}
&\|\mathbf{u}\|_{L^2(s,T;L^\infty)}^2+\|\nabla\mathbf{d}\|_{L^2(s,T;L^\infty)}^2\\
\leq&C\big[1+\left(\|\mathbf{u}\|_{L^2(s,T;H^1)}^2+\|\nabla\mathbf{d}\|_{L^2(s,T;H^1)}^2\right)\\
&\qquad\log\left(e+\|\mathbf{u}\|_{L^2(s,T;W^{1,3})}+\|\nabla\mathbf{d}\|_{L^2(s,T;W^{1,3})}\right)\big]\\
\leq&C_2\big[1+\left(\|\nabla\mathbf{u}\|_{L^2(s,T;L^2)}^2+\|\nabla^2\mathbf{d}\|_{L^2(s,T;L^2)}^2\right)\log(C\Phi(T))\big], \qquad (3.24)
\end{aligned}$$

where one has used the Poincaré inequality, (3.2), and the following facts

$$\begin{aligned}\|\mathbf{u}\|_{W^{1,3}}^2 &\le \|\mathbf{w}\|_{W^{1,3}}^2+\|\mathbf{v}\|_{W^{1,3}}^2\\ &\le C\|\mathbf{w}\|_{W^{2,2}}^2+C\|P\|_{L^3}^2\\ &\le C\left(1+\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+\||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2}^2\right),\end{aligned}$$

and

$$\|\nabla\mathbf{d}\|_{W^{1,3}}^2\le C\|\nabla\mathbf{d}\|_{W^{2,2}}^2\le C\left(1+\|\nabla^2\mathbf{d}\|_{L^2}^2+\|\nabla^3\mathbf{d}\|_{L^2}^2\right).$$

The combination (3.23) and (3.24) gives rise to

$$\Phi(T)\le C\Phi(s)(C\Phi(T))^{C_2\left(\|\nabla\mathbf{u}\|_{L^2(s,T;L^2)}^2+\|\nabla^2\mathbf{d}\|_{L^2(s,T;L^2)}^2\right)}. \tag{3.25}$$

Recalling (3.2), one can choose s close enough to T^* such that

$$\lim_{T\to T^{*-}} C_2\left(\|\nabla\mathbf{u}\|_{L^2(s,T;L^2)}^2+\|\nabla^2\mathbf{d}\|_{L^2(s,T;L^2)}^2\right)\le\frac{1}{2}.$$

Hence, for $s<T<T^*$, we have

$$\Phi(T)\le C\Phi^2(s)<\infty.$$

This completes the proof of Lemma 3.2. □

LEMMA 3.3. *Under the condition* (3.1), *then it holds that for any* $T\in[0,T^*)$,

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{d}_t\|_{L^2}^2\right)+\int_0^T\left(\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla^2\mathbf{d}_t\|_{L^2}^2\right)dt\le C. \tag{3.26}$$

Proof.

(1) Operating $\partial_t+\mathrm{div}(\mathbf{u}\cdot)$ to the j-th component of $(1.1)_2$ and multiplying the resulting equation by $\dot{u}^j$, one gets by some calculations that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|\dot{\mathbf{u}}|^2dx\\ &=\mu\int\dot{u}^j(\partial_t\Delta u^j+\mathrm{div}(\mathbf{u}\Delta u^j))dx+(\lambda+\mu)\int\dot{u}^j(\partial_t\partial_j(\mathrm{div}\,\mathbf{u})+\mathrm{div}(\mathbf{u}\partial_j(\mathrm{div}\,\mathbf{u})))dx\\ &\quad-\int\dot{u}^j(\partial_jP_t+\mathrm{div}(\mathbf{u}\partial_jP))dx-\int\dot{u}^j\left(\partial_t(\nabla\mathbf{d}\cdot\Delta d^j)+\mathrm{div}(\mathbf{u}\cdot\nabla\mathbf{d}\cdot\Delta d^j)\right)dx\\ &=:J_1+J_2+J_3+J_4.\end{aligned} \tag{3.27}$$

Integration by parts leads to

$$\begin{aligned}J_1&=-\mu\int(\partial_i\dot{u}^j\partial_t\partial_iu^j+\Delta u^j\mathbf{u}\cdot\nabla\dot{u}^j)dx\\ &=-\mu\int(|\nabla\dot{\mathbf{u}}|^2-\partial_i\dot{u}^ju^k\partial_k\partial_iu^j-\partial_i\dot{u}^j\partial_iu^k\partial_ku^j+\Delta u^j\mathbf{u}\cdot\nabla\dot{u}^j)dx\\ &=-\mu\int(|\nabla\dot{\mathbf{u}}|^2+\partial_i\dot{u}^j\partial_ku^k\partial_iu^j-\partial_i\dot{u}^j\partial_iu^k\partial_ku^j-\partial_iu^j\partial_iu^k\partial_k\dot{u}^j)dx\\ &\le-\frac{3\mu}{4}\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^4}^4.\end{aligned} \tag{3.28}$$

Similarly, one has

$$J_2 \le -\frac{\lambda+\mu}{2}\|\operatorname{div}\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^4}^4. \tag{3.29}$$

It follows from integration by parts, (3.4), (3.1), and (3.8) that

$$\begin{aligned} J_3 &= \int (\partial_j \dot{u}^j P_t + \partial_j P \mathbf{u}\cdot\nabla\dot{u}^j)dx \\ &= \int \partial_j\dot{u}^j P_t dx - \int P\partial_j(\mathbf{u}\cdot\nabla\dot{u}^j)dx \\ &= \int \partial_j\dot{u}^j\,(P_t + \operatorname{div}(P\mathbf{u}))\,dx - \int P\partial_j\mathbf{u}\cdot\nabla\dot{u}^j\,dx \\ &= (1-\gamma)\int \partial_j\dot{u}^j P\operatorname{div}\mathbf{u}dx - \int P\partial_j\mathbf{u}\cdot\nabla\dot{u}^j\,dx \\ &\le C\|P\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\dot{\mathbf{u}}\|_{L^2} \\ &\le \frac{\mu}{4}\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C. \end{aligned} \tag{3.30}$$

Integrating by parts and applying (3.2), (3.8), and Sobolev's inequality, we arrive at

$$\begin{aligned} |J_4| &\le C\int |\nabla\dot{\mathbf{u}}|\big(|\nabla\mathbf{d}||\nabla\mathbf{d}_t| + |\mathbf{u}||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\big)\,dx \\ &\le \frac{\mu}{8}\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{d}\|_{L^4}^2\|\nabla\mathbf{d}_t\|_{L^4}^2 + C\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{d}\|_{L^6}^2\|\nabla^2\mathbf{d}\|_{L^6}^2 \\ &\le \frac{\mu}{8}\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{d}\|_{H^1}^2\|\nabla\mathbf{d}_t\|_{L^4}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{d}\|_{H^1}^2\|\nabla^2\mathbf{d}\|_{H^1}^2 \\ &\le \frac{\mu}{8}\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{d}_t\|_{L^4}^2 + C\|\nabla^3\mathbf{d}\|_{L^2}^2 + C. \end{aligned} \tag{3.31}$$

Inserting (3.28)–(3.31) into (3.27) yields

$$\frac{d}{dt}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \mu\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 \le C\|\nabla\mathbf{u}\|_{L^4}^4 + C\|\nabla\mathbf{d}_t\|_{L^4}^2 + C\|\nabla^3\mathbf{d}\|_{L^2}^2 + C. \tag{3.32}$$

(2) Differentiation (3.17) with respect to t leads to

$$\nabla\mathbf{d}_{tt} - \Delta\nabla\mathbf{d}_t = -\nabla(\mathbf{u}\cdot\nabla\mathbf{d})_t + \nabla(|\nabla\mathbf{d}|^2\mathbf{d})_t. \tag{3.33}$$

Multiplying (3.33) by $\nabla\mathbf{d}_t$ and integrating the resulting equation over Ω gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int|\nabla\mathbf{d}_t|^2dx + \int|\nabla^2\mathbf{d}_t|^2dx \\ &\le C\int|\nabla\mathbf{d}||\nabla\mathbf{u}_t||\nabla\mathbf{d}_t|dx + C\int|\mathbf{u}_t||\nabla^2\mathbf{d}_t||\nabla\mathbf{d}|dx + C\int|\nabla\mathbf{d}_t|^2|\nabla\mathbf{u}|dx \\ &\quad + C\int|\nabla\mathbf{d}|^2|\mathbf{d}_t||\nabla^2\mathbf{d}_t|dx + C\int|\nabla\mathbf{d}||\nabla\mathbf{d}_t||\nabla^2\mathbf{d}_t|dx =: \sum_{i=1}^5 S_i. \end{aligned} \tag{3.34}$$

By Hölder's, Young's, Sobolev's inequalities, (3.2), and (3.8), one gets

$$S_1 + S_2 \le C\int|\nabla\mathbf{d}||\nabla\dot{\mathbf{u}}||\nabla\mathbf{d}_t|dx + C\int|\dot{\mathbf{u}}||\nabla^2\mathbf{d}_t||\nabla\mathbf{d}|dx$$

$$
\begin{aligned}
&+C\int|\nabla^2\mathbf{d}||\mathbf{u}\cdot\nabla\mathbf{u}||\nabla\mathbf{d}_t|dx+C\int|\mathbf{u}\cdot\nabla\mathbf{u}||\nabla^2\mathbf{d}_t||\nabla\mathbf{d}|dx\\
\le& C\|\nabla\dot{\mathbf{u}}\|_{L^2}\|\nabla\mathbf{d}\|_{L^4}\|\nabla\mathbf{d}_t\|_{L^4}+C\|\nabla^2\mathbf{d}_t\|_{L^2}\|\dot{\mathbf{u}}\|_{L^4}\|\nabla\mathbf{d}\|_{L^4}\\
&+C\|\nabla^2\mathbf{d}\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^{12}}\|\nabla\mathbf{d}_t\|_{L^4}\\
&+C\|\nabla^2\mathbf{d}_t\|_{L^2}\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{d}\|_{L^6}\|\mathbf{u}\|_{L^{12}}\\
\le&\frac{\delta}{2}\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C(\delta)\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+C\|\nabla^3\mathbf{d}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^4}^2+C\|\nabla\mathbf{d}_t\|_{L^4}^2\\
\le&\delta\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C(\delta)\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+C\|\nabla^3\mathbf{d}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^4}^2+C\|\nabla\mathbf{d}_t\|_{L^2}^2;\\
S_3\le& C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{d}_t\|_{L^4}^2\le C\|\nabla\mathbf{d}_t\|_{L^2}\|\nabla\mathbf{d}_t\|_{H^1}\le\delta\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C\|\nabla\mathbf{d}_t\|_{L^2}^2;\\
S_5\le& C\|\nabla^2\mathbf{d}_t\|_{L^2}\|\nabla\mathbf{d}_t\|_{L^4}\|\nabla\mathbf{d}\|_{L^4}\\
\le&\frac{\delta}{2}\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C\|\nabla\mathbf{d}_t\|_{L^2}\|\nabla\mathbf{d}_t\|_{H^1}\|\nabla\mathbf{d}\|_{H^1}^2\\
\le&\delta\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C\|\nabla\mathbf{d}_t\|_{L^2}^2.
\end{aligned}
$$

To bound S_4, it follows from $(1.1)_3$, (3.2), and (3.8) that

$$
\begin{aligned}
\|\mathbf{d}_t\|_{L^2}=&\|-\mathbf{u}\cdot\nabla\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}+\Delta\mathbf{d}\|_{L^2}\\
\le& C\left(\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{d}\|_{L^3}+\|\nabla\mathbf{d}\|_{L^4}^2+\|\nabla^2\mathbf{d}\|_{L^2}\right)\\
\le& C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{d}\|_{H^1}+\|\nabla\mathbf{d}\|_{H^1}^2+C\\
\le& C.
\end{aligned}\tag{3.35}
$$

Then we deduce from Hölder's, Young's, Sobolev's inequalities, (3.2), (3.8), and (3.35) that

$$
\begin{aligned}
S_4\le& C\|\nabla^2\mathbf{d}_t\|_{L^2}\|\nabla\mathbf{d}\|_{L^8}^2\|\mathbf{d}_t\|_{L^4}\\
\le&\delta\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C\|\mathbf{d}_t\|_{H^1}^2\\
\le&\delta\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C\|\nabla\mathbf{d}_t\|_{L^2}^2+C.
\end{aligned}
$$

Substituting the above estimates on S_i $(i=1,\cdots,5)$ into (3.34), we obtain after choosing δ suitably small that

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\mathbf{d}_t\|_{L^2}^2+\|\nabla^2\mathbf{d}_t\|_{L^2}^2\\
\le& C\|\nabla\mathbf{d}_t\|_{L^2}^2+C_1\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+C\|\nabla^3\mathbf{d}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^4}^2+C.
\end{aligned}\tag{3.36}
$$

(3) Adding (3.32) multiplied by $\frac{C_1+1}{\mu}$ to (3.36), one has

$$
\begin{aligned}
&\frac{d}{dt}\left(\mu^{-1}(C_1+1)\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{d}_t\|_{L^2}^2\right)+\mu\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla^2\mathbf{d}_t\|_{L^2}^2\\
\le& C\|\nabla\mathbf{u}\|_{L^4}^4+C\|\nabla\mathbf{d}_t\|_{L^2}^2+C\|\nabla^3\mathbf{d}\|_{L^2}^2+C,
\end{aligned}
$$

which together with Gagliardo-Nirenberg inequality, Lemma 2.4, (3.1), and (3.8) implies that

$$
\begin{aligned}
&\frac{d}{dt}\left(\mu^{-1}(C_1+1)\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{d}_t\|_{L^2}^2\right)+\mu\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla^2\mathbf{d}_t\|_{L^2}^2\\
\le& C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{H^1}^2+C\|\nabla\mathbf{d}_t\|_{L^2}^2+C\|\nabla^3\mathbf{d}\|_{L^2}^2+C
\end{aligned}
$$

$$\le C\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+\|\nabla\mathbf{d}_t\|_{L^2}^2\right)+C\|\nabla^3\mathbf{d}\|_{L^2}^2+C.$$

This along with Grönwall's inequality and (3.8) leads to the desired (3.26). □

LEMMA 3.4. *Under the condition* (3.1)*, and let* $q>2$ *be as in Theorem 1.1, then it holds that for any* $T\in[0,T^*)$,

$$\sup_{0\le t\le T}(\|\rho\|_{W^{1,q}}+\|\nabla\mathbf{u}\|_{H^1}+\|\nabla\mathbf{d}\|_{H^2})\le C. \tag{3.37}$$

Proof.

(1) For $q>2$, it follows from the mass equation $(1.1)_1$ that $\nabla\rho$ satisfies

$$\begin{aligned}\frac{d}{dt}\|\nabla\rho\|_{L^q}&\le C(q)(1+\|\nabla\mathbf{u}\|_{L^\infty})\|\nabla\rho\|_{L^q}+C(q)\|\nabla^2\mathbf{u}\|_{L^q}\\&\le C(1+\|\nabla\mathbf{w}\|_{L^\infty}+\|\nabla\mathbf{v}\|_{L^\infty})\|\nabla\rho\|_{L^q}+C\left(\|\nabla^2\mathbf{w}\|_{L^q}+\|\nabla^2\mathbf{v}\|_{L^q}\right)\\&\le C(1+\|\nabla\mathbf{w}\|_{L^\infty}+\|\nabla\mathbf{v}\|_{L^\infty})\|\nabla\rho\|_{L^q}+C\|\nabla^2\mathbf{w}\|_{L^q}\end{aligned} \tag{3.38}$$

due to the following fact

$$\|\nabla^2\mathbf{v}\|_{L^q}\le C\|\nabla P\|_{L^q}\le C\|\nabla\rho\|_{L^q},$$

which follows from the standard L^q-estimate for the following elliptic system

$$\begin{cases}\mu\Delta\mathbf{v}+(\lambda+\mu)\nabla\operatorname{div}\mathbf{v}=\nabla P, & x\in\Omega,\\ \mathbf{v}=\mathbf{0},\ x\in\partial\Omega.\end{cases}$$

From Lemma 2.6 and (2.11), one gets

$$\begin{aligned}\|\nabla\mathbf{v}\|_{L^\infty}&\le C\left(1+\|\nabla\mathbf{v}\|_{BMO}\log(e+\|\nabla\mathbf{v}\|_{W^{1,q}})\right)\\&\le C\left(1+(\|P\|_{L^2}+\|P\|_{L^\infty})\log(e+\|\nabla\mathbf{v}\|_{W^{1,q}})\right)\\&\le C\left(1+\log(e+\|\nabla\rho\|_{L^q})\right).\end{aligned} \tag{3.39}$$

By virtue of Hölder's inequality, Sobolev's inequality, (3.2), and (3.8), one deduces that

$$\sup_{0\le t\le T}\||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^q}\le\left(\sup_{0\le t\le T}\|\nabla\mathbf{d}\|_{L^{q^2}}^{\frac{q(q-2)}{q^2-2}}\right)\cdot\left(\sup_{0\le t\le T}\|\nabla^2\mathbf{d}\|_{L^2}^{\frac{2(q-1)}{q^2-2}}\right)\le C, \tag{3.40}$$

which combined with Sobolev's embedding theorem, (2.12), and (3.1) yields

$$\begin{aligned}\|\nabla\mathbf{w}\|_{L^\infty}&\le\|\mathbf{w}\|_{W^{2,q}}\\&\le C\left\|\rho\dot{\mathbf{u}}+\nabla\mathbf{d}\cdot\Delta\mathbf{d}\right\|_{L^q}\\&\le C\|\dot{\mathbf{u}}\|_{L^q}+C\||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^q}\\&\le C\|\nabla\dot{\mathbf{u}}\|_{L^2}+C.\end{aligned} \tag{3.41}$$

Moreover, we have

$$\|\nabla^2\mathbf{w}\|_{L^q}\le\|\mathbf{w}\|_{W^{2,q}}\le C\|\nabla\dot{\mathbf{u}}\|_{L^2}+C. \tag{3.42}$$

Substituting (3.39)–(3.41) into (3.38), we derive that

$$\frac{d}{dt}\log(e+\|\nabla\rho\|_{L^q})\le C(1+\|\nabla\dot{\mathbf{u}}\|_{L^2})\log(e+\|\nabla\rho\|_{L^q}).$$

This along with Grönwall's inequality and (3.26) leads to

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^q}\le C. \tag{3.43}$$

(2) We infer from (2.11), (2.12), and (3.1) that

$$\begin{aligned}\|\nabla^2\mathbf{u}\|_{L^2}&\le\|\nabla^2\mathbf{v}\|_{L^2}+\|\nabla^2\mathbf{w}\|_{L^2}\\&\le C\|\nabla\rho\|_{L^2}+C\|\rho\dot{\mathbf{u}}+\nabla\mathbf{d}\cdot\Delta\mathbf{d}\|_{L^2}\\&\le C\|\nabla\rho\|_{L^2}+C\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}+C\||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2},\end{aligned}$$

which combined with (3.43), (3.26), (3.40), and Hölder's inequality implies that

$$\sup_{0\le t\le T}\|\nabla^2\mathbf{u}\|_{L^2}\le C. \tag{3.44}$$

(3) It follows from standard L^2-estimate of elliptic system (3.17), Hölder's inequality, and Sobolev's inequality that

$$\begin{aligned}\|\nabla^3\mathbf{d}\|_{L^2}&\le C\|\nabla\mathbf{d}_t\|_{L^2}+C\|\nabla(\mathbf{u}\cdot\nabla\mathbf{d})\|_{L^2}+C\|\nabla(|\nabla\mathbf{d}|^2\mathbf{d})\|_{L^2}\\&\le C\|\nabla\mathbf{d}_t\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{d}\|_{L^4}+C\|\mathbf{u}\|_{L^\infty}\|\nabla^2\mathbf{d}\|_{L^2}\\&\quad+C\|\nabla\mathbf{d}\|_{L^6}^3+C\||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2}\\&\le C\|\nabla\mathbf{d}_t\|_{L^2}+C\|\nabla\mathbf{u}\|_{H^1}\|\nabla\mathbf{d}\|_{H^1}+C\|\mathbf{u}\|_{H^2}\|\nabla^2\mathbf{d}\|_{L^2}\\&\quad+C\|\nabla\mathbf{d}\|_{H^1}^3+C\||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2},\end{aligned}$$

which together with (3.26), (3.2), (3.8), (3.44), and (3.40) gives rise to

$$\sup_{0\le t\le T}\|\nabla^3\mathbf{d}\|_{L^2}\le C. \tag{3.45}$$

Consequently, the desired (3.37) follows from (3.1), (3.43), (3.8), (3.44), (3.2), and (3.45). The proof of Lemma 3.4 is finished.

□

With Lemmas 3.1–3.4 at hand, we are now in a position to prove Theorem 1.1.

Proof. **(Proof of Theorem 1.1.)** We argue by contradiction. Suppose that (1.17) were false, that is, (3.1) holds. Note that the general constant C in Lemmas 3.1–3.4 is independent of $t<T^*$, that is, all the a priori estimates obtained in Lemmas 3.1–3.4 are uniformly bounded for any $t<T^*$. Hence, the functions

$$(\rho,\mathbf{u},\mathbf{d})(x,T^*):=\lim_{t\to T^*}(\rho,\mathbf{u},\mathbf{d})(x,t)$$

satisfies the initial condition (1.14) at $t=T^*$.

Furthermore, standard arguments yield that $\rho\dot{\mathbf{u}}\in C([0,T];L^2)$, which implies

$$\rho\dot{\mathbf{u}}(x,T^*)=\lim_{t\to T^*}\rho\dot{\mathbf{u}}\in L^2.$$

Hence,

$$-\mu\Delta\mathbf{u}-(\lambda+\mu)\nabla\mathrm{div}\mathbf{u}+\nabla P+\nabla\mathbf{d}\cdot\Delta\mathbf{d}|_{t=T^*}=\sqrt{\rho}(x,T^*)g(x)$$

with

$$g(x)\triangleq\begin{cases}\rho^{-1/2}(x,T^*)(\rho\dot{\mathbf{u}})(x,T^*), & \text{for } x\in\{x|\rho(x,T^*)>0\},\\ 0, & \text{for } x\in\{x|\rho(x,T^*)=0\},\end{cases}$$

satisfying $g\in L^2$ due to (3.37). Therefore, one can take $(\rho,\mathbf{u},\mathbf{d})(x,T^*)$ as the initial data and extend the local strong solution beyond T^*. This contradicts the assumption on T^*. Thus we finish the proof of Theorem 1.1. □

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