COMMUN. MATH. SCI.
Vol. 19, No. 1, pp. 175–192

ON DISSIPATIVE SOLUTIONS TO A SIMPLIFIED HYPERBOLIC ERICKSEN-LESLIE SYSTEM OF LIQUID CRYSTALS*

FENG CHENG†, NING JIANG‡, AND YI-LONG LUO§

Abstract. We study dissipative solutions to a 3D simplified hyperbolic Ericksen-Leslie system for liquid crystals with Ginzburg-Landau approximation. First, we establish a weak-strong stability principle, which leads to a suitable notion of dissipative solutions to the hyperbolic Ericksen-Leslie system. Then, we introduce a regularized system to approximate the original system, for which we can prove the existence of global-in-time weak solutions. Finally, we prove that there is at least one dissipative solution for this simplified hyperbolic Ericksen-Leslie system.

Keywords. Ericksen-Leslie system; dissipative solution; weak strong uniqueness.

AMS subject classifications. 35L51; 76A15; 82D15; 82D25.

1. Introduction

We consider the following three dimensional simplified hyperbolic Ericksen-Leslie liquid crystal model with Ginzburg-Landau approximation:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta u + \nabla p = -\operatorname{div}(\nabla d\odot\nabla d), \\ \operatorname{div} u = 0, \\ \partial_t\omega + u\cdot\nabla\omega + \omega = \Delta d - f(d), \\ \omega = \partial_t d + u\cdot\nabla d, \end{cases} \tag{1.1}$$

where $u(t,x)=(u_1,u_2,u_3)(t,x)$ is the velocity field, $d(t,x)=(d_1,d_2,d_3)(t,x)$ is the direction field of the liquid molecules, $p(t,x)$ is the scalar pressure function, $\mu>0$ is the viscosity constant, $(\nabla d\odot\nabla d)$ is a matrix with the ij-th entry $(\nabla d\odot\nabla d)_{ij}=\sum_{1\le k\le 3}\partial_i d_k\partial_j d_k$ and $f(d)$ is a penalization term to relax the constraint of $|d|=1$, which is vector-valued, smooth, function defined for all $d\in\mathbb{R}^3$. A usual example for $f(d)$ is the Ginzburg-Landau approximation $f(d)=\nabla\mathcal{F}(d)$ for

$$\mathcal{F}(d)=\frac{1}{4\kappa^2}(|d|^2-1)^2,$$

where κ is a parameter. The study of the asymptotic behavior as $\kappa\to 0$ is important and very challenging at the current state. From here and below we take $\kappa=1$ for simplicity of presentation.

The system (1.1) can be viewed as adding the second-order material derivative term $\partial_t\omega+u\cdot\nabla\omega$ to the simplified system that was proposed by Lin and Liu [22]. The second-order material derivative term is sometimes also called an inertial term. The presence of the inertial term provides a hyperbolic character compared with the model that was studied in [22] and is the main source of difficulties in the mathematical analysis. Jiang and Luo [13] studied the Ericksen-Leslie hyperbolic incompressible liquid crystal

*Received: March 08, 2020; Accepted (in revised form): August 12, 2020. Communicated by Chun Liu.

†Hubei Key Laboratory of Applied Mathematics, Faculty of Mathematics and Statistics, Hubei University, Wuhan 430062, P.R. China (fengcheng@hubu.edu.cn).

‡School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, P.R. China (njiang@whu.edu.cn).

§School of Mathematics, South China University of Technology, Guangzhou, 510641, P.R. China (luoylmath@scut.edu.cn).

model, in which the main difficulty arises from the inertial term and the additional presence of the unit-length constraint. Especially, if we set the Leslie's coefficients to be 0 and relax the unit-length constraint by the Ginzburg-Landau approximation in the Ericksen-Leslie hyperbolic liquid crystal model that was studied in [13], then it leads to the system (1.1) and for such reason we call it a simplified hyperbolic Ericksen-Leslie liquid crystal system.

For the sake of simplicity, we restrict ourselves to the periodic boundary conditions, for which the underlying spatial domain is identified as $\Omega=\mathbb{T}^3=[-\pi,\pi]^3$. We remark that the arguments in the subsequent section should be valid for bounded domain or whole space, and for bounded domain a suitable boundary condition for d is needed. Since there is a second-order derivative with respect to time variable, the system (1.1) is equipped with the following initial condition:

$$u(0,x)=u_0(x),\ d(0,x)=d_0,\ \omega(0,x)=\omega_0. \tag{1.2}$$

The hydrodynamic theory of incompressible liquid crystals was established by Ericksen [5–7] and Leslie [20, 21] in the 1960's. The original Ericksen-Leslie system was very complicated and there are many studies for its simplified model in many directions. The first mathematical analysis of a simplified Ericksen-Leslie model is due to Lin and Liu [22], in which they proved global existence of weak solution and local existence of strong solution. They proved partial regularity of weak solutions to the considered system in [23]. Subsequently, they generalize these results to a more realistic model in [24]. Since then there are many articles studying more complicated models, see [4, 9, 12, 28, 30, 31] and references therein. The above mentioned models are considered without the inertial term. Very recently, De Anna and Zarnescu [3] considered the inertial Qian-Sheng model of liquid crystals which couples a hyperbolic type equation involving a second-order material derivative with a forced incompressible Navier-Stokes equations. Jiang and Luo [13] proved the local existence of classical solution to the Ericksen-Leslie hyperbolic liquid crystal system with the inertial term. Moreover they also proved global existence of classical solution with small initial energy under a damping effect.

In this paper, we are concerned with the global existence of dissipative solution of the system (1.1)-(1.2). The notion of dissipative solutions was first introduced by P.-L. Lions in the context of the Euler equations [[26], Sec. 4.4] with ideals originating from the Boltzmann equation [27]. It is also applied in the context of incompressible viscous electro-magneto-hydrodynamics (see Arsénio and Saint-Raymond [2]), the two-component Camassa-Holm system [11], and equations of viscoelastic diffusion in polymers [29]. The term dissipative solutions is also used for solutions to the Navier-Stokes-Fourier system in [8], whereas this concept is a little different from the dissipative solutions in [26]. Recently, Kalousek [14] studied the dissipative solution to an incompressible visoelastic fluid. Subsequently, they also proved the existence of dissipative solutions to a magnetoviscoelastic system in [15]. Very recently, Lasarzik [17] studied the dissipative solutions to the parabolic Ericksen-Leslie system equipped with the Oseen-Frank energy in three dimensions. They also investigated the concept of measure-valued solutions to the considered system in [16, 18] and the relation between the measure-valued solution and the dissipative solution is revealed in [19]. Feireisl-Rocca-Schimperna-Zarnescu [10] proved a global existence of dissipative solution for the invisicd version of the Qian-Sheng model, in which they introduced the defect measures to define the new notion of dissipative solution.

In this paper we work in the spirit of P.-L. Lions on the dissipative solutions for Euler equation in [26]. The first difficulty is a suitable definition of dissipative solution on the Ericksen-Leslie system (1.1). Specifically, we first establish a novel weak-strong stable inequality both for u and d, based on which we can define dissipative solutions for the system (1.1). Then we introduce an approximation system to the system (1.1)-(1.2), for which we can prove a global existence of weak solution. Here the regularization is not on the velocity field u as P.-L. Lions' approach to Euler system, but on the material derivative ω. Finally, we approximate the dissipative solution by the weak solution to the approximate system.

The rest of the paper is orgonaized as follows. In Section 2, we introduce the notations and some preliminaries. In Section 3, we establish the weak strong inequality. In Section 4, we introduce the approximate system and prove the global existence of weak solution for the approximate system. Finally, in Section 5, we prove the existence of dissipative solution to the hyperbolic liquid crystal system (1.1)-(1.2).

2. Preliminaries and Basic facts

2.1. Notations. In this section, we introduce some notations of function spaces which will be used later. The space $L^p(\Omega)$, $1\le p\le\infty$, represents the usual Lebesgue space of scalar functions as well as that of vector-valued functions with norm denoted by $\|\cdot\|_{L^p}$. When $p=2$, $L^2(\Omega)$ is a Hilbert space and we denote $\langle\cdot,\cdot\rangle$ the inner product for scalar functions as well as vector-valued functions. We define $W^{k,p}(\Omega)$ the usual Sobolev space with the norm $\|\cdot\|_{W^{k,p}}$. When $p=2$, we denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$. Given a Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0,T;X)$, $1\le p\le\infty$, the set of functions $f(t)$ defined on $(0,T)$ with values in X such that $\int_0^T\|f(t)\|_X^p dt<\infty$. In this paper, we use C to represent an absolute constant which may change from line to line.

2.2. Basic energy law. Then, we can formally derive the basic energy law for the system (1.1)-(1.2). Throughout this section, the smoothness of the solution (u,d) to (1.1)-(1.2) is assumed.

PROPOSITION 2.1. *Assume (u,d) is a smooth solution to the system* (1.1) *with initial conditions* (1.2)*, then the following energy identity holds:*

$$\frac{d}{dt}\left(\frac{1}{2}\|u\|_{L^2}^2+\frac{1}{2}\|\omega\|_{L^2}^2+\frac{1}{2}\|\nabla d\|_{L^2}^2+\int_\Omega \mathcal{F}(d)dx\right)+\mu\|\nabla u\|_{L^2}^2+\|\omega\|_{L^2}^2=0. \tag{2.1}$$

Proof. Since (u,d) is a classical solution to the system (1.1)-(1.2). Multiply the first Equation (1.1) by u and integrate over Ω to give

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2+\mu\|\nabla u\|_{L^2}^2+\langle u,(\nabla d)^T\Delta d\rangle=0, \tag{2.2}$$

where we used the fact $\operatorname{div}(\nabla d\odot\nabla d)=\frac{1}{2}\nabla(|\nabla d|^2)+\sum_{1\le k\le 3}\nabla d_k\Delta d_k$ and the fact $\operatorname{div}u=0$. Then, multiply the third equation of (1.1) by ω and integrate over Ω to give

$$\frac{d}{dt}\left(\frac{1}{2}\|\omega\|_{L^2}^2+\frac{1}{2}\|\nabla d\|_{L^2}^2+\int_\Omega \mathcal{F}(d)dx\right)+\|\omega\|_{L^2}^2-\langle u\cdot\nabla d,\Delta d\rangle=0, \tag{2.3}$$

where $\nabla\mathcal{F}(d)=f(d)$. Note that

$$-\langle u,(\nabla d)^T\Delta d\rangle+\langle u\cdot\nabla d,\Delta d\rangle=0.$$

Then, add (2.2) and (2.3) together to get

$$\frac{d}{dt}\left(\frac{1}{2}\|u\|_{L^2}^2+\frac{1}{2}\|\omega\|_{L^2}^2+\frac{1}{2}\|\nabla d\|_{L^2}^2+\int_\Omega \mathcal{F}(d)dx\right)+\mu\|\nabla u\|_{L^2}^2+\|\omega\|_{L^2}^2=0,$$

which finishes the proof. ∎

REMARK 2.1. As pointed out by [13], the inertial term will cause difficulties in the justification of suitable weak solution in the energy space from the basic energy law. It is not known whether or not there is global well-posedness for the system (1.1)-(1.2) in the energy space, and for such reason we are led to the notion of dissipative solution.

3. Weak strong stability inequality

In this section, we first establish the weak strong stable inequality. Then, we are able to provide a suitable definition of dissipative solution for the system (1.1)-(1.2).

Analogous to [26], we will show how to modulate the basic energy and establish a weak-strong stability equality.

PROPOSITION 3.1. *Let (u,d) be a smooth solution to the system (1.1)-(1.2). Further, consider test functions $(\bar u,\bar d)\in C^\infty([0,\infty)\times\Omega)$ such that $\operatorname{div}\bar u=0$ and denote $\bar u(0,x)=\bar u_0,\bar d(0,x)=\bar d_0$ and*

$$A(\bar u,\bar d)=\begin{pmatrix}A_1(\bar u,\bar d)\\ A_2(\bar u,\bar d)\end{pmatrix}=\begin{pmatrix}-\partial_t\bar u-\mathbb{P}(\bar u\cdot\nabla\bar u)+\mu\Delta\bar u-\mathbb{P}\operatorname{div}(\nabla\bar d\odot\nabla\bar d)\\ -\partial_t\bar\omega-\bar u\cdot\nabla\bar\omega-\bar\omega+\Delta\bar d-f(\bar d)\end{pmatrix}, \tag{3.1}$$

where $\mathbb{P}$ is the projection onto periodic divergence-free vector fields and $\bar\omega=\partial_t\bar d+\bar u\cdot\nabla\bar d$.

We define the growth rate by

$$\begin{aligned}\lambda(t)=&2C\|\nabla\bar u\|_{L^\infty}+\|\nabla\bar\omega\|_{L^\infty}+\|\Delta\bar d\|_{L^\infty}+\frac{6C}{\mu}\|\nabla\bar d\|^2_{L^\infty}+2C\|\bar\omega\|_{L^3}\\ &+C(1+\|\nabla u\|^2_{L^2}+\|\nabla\bar u\|^2_{L^2}+\|\nabla d\|^2_{L^2}+\|\nabla\bar d\|^2_{L^2}+\|\nabla\bar d\|^2_{L^6}+\||\bar d|^2\|_{L^3}),\end{aligned} \tag{3.2}$$

where C is some constant produced by Sobolev embedding and Young's inequality. Then, one has the following stability inequality

$$\begin{aligned}&\delta\mathcal{E}(t)+\frac14\int_0^t\delta\mathcal{D}(s)e^{\int_s^t\lambda(\sigma)d\sigma}ds\\ \le&\delta\mathcal{E}(0)e^{\int_0^t\lambda(s)ds}+\int_0^t\left[\int_\Omega A(\bar u,\bar d)\cdot\begin{pmatrix}u-\bar u\\ \omega-\bar\omega\end{pmatrix}dx\right](s)e^{\int_s^t\lambda(\sigma)d\sigma}ds,\end{aligned} \tag{3.3}$$

where the modulated energy $\delta\mathcal{E}(t)$ and energy dissipation $\delta\mathcal{D}(t)$ are given by

$$\begin{aligned}&\delta\mathcal{E}(t)=\frac12\|(u-\bar u)(t)\|^2_{L^2}+\frac12\|\nabla(d-\bar d)(t)\|^2_{L^2}+\frac12\|(\omega-\bar\omega)(t)\|^2_{L^2}+\frac14\||d-\bar d|^2\|^2_{L^2},\\ &\delta\mathcal{D}(t)=\mu\|\nabla(u-\bar u)(t)\|^2_{L^2}+\|(\omega-\bar\omega)(t)\|^2_{L^2}.\end{aligned} \tag{3.4}$$

REMARK 3.1. The function $\lambda(t)$ of the growth rate not only depends on $\bar u,\bar d$ but also on u,d, this is due to the Ginzburg-Landau approximation term $f(d)$. However, from the basic energy law, one can easily derive that $\|u\|_{L^2(0,T;H^1)}$ and $\|d\|_{L^\infty(0,T;H^1)}$ are bounded. It then follows that, for all $T>0$, $\lambda(t)\in L^1([0,T])$.

Proof. Since (u,d) is a smooth solution to (1.1)-(1.2) and noting that $\bar u$ is divergence-free, we can rewrite the first Equation (1.1) as follows

$$\begin{aligned}&\partial_t(u-\bar u)+u\cdot\nabla(u-\bar u)+(u-\bar u)\cdot\nabla\bar u-\mu\Delta(u-\bar u)+\nabla\pi\\ &\qquad+\operatorname{div}(\nabla d\odot\nabla(d-\bar d))+\operatorname{div}(\nabla(d-\bar d)\odot\nabla\bar d)=A_1(\bar u,\bar d),\end{aligned} \tag{3.5}$$

where $A_1(\bar{u},\bar{d}) = -\partial_t\bar{u} - \mathbb{P}(\bar{u}\cdot\nabla\bar{u}) + \mu\Delta\bar{u} - \mathbb{P}\mathrm{div}(\nabla\bar{d}\odot\nabla\bar{d})$ and π is some scalar function. Denoting $\bar{\omega} = \partial_t\bar{d} + \bar{u}\cdot\nabla\bar{d}$, we can rewrite the third Equation (1.1) as follows

$$\partial_t(\omega-\bar{\omega}) + u\cdot\nabla(\omega-\bar{\omega}) + (u-\bar{u})\cdot\nabla\bar{\omega} + (\omega-\bar{\omega}) - \Delta(d-\bar{d}) + f(d) - f(\bar{d}) = A_2(\bar{u},\bar{d}), \tag{3.6}$$

where $A_2(\bar{u},\bar{d}) = -\partial_t\bar{\omega} - \bar{u}\cdot\nabla\bar{\omega} - \bar{\omega} + \Delta\bar{d} - f(\bar{d})$. Then, multiply (3.5) by $u-\bar{u}$ and integrate over Ω to give

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u-\bar{u})(t,\cdot)\|_{L^2}^2 + \langle (u-\bar{u})\cdot\nabla\bar{u}, u-\bar{u}\rangle + \mu\|\nabla(u-\bar{u})\|_{L^2}^2 \\
&\quad + \langle \mathrm{div}(\nabla d\odot\nabla(d-\bar{d})), u-\bar{u}\rangle + \langle \mathrm{div}(\nabla(d-\bar{d})\odot\nabla\bar{d}), u-\bar{u}\rangle \\
&= \int_\Omega A_1(\bar{u},\bar{d})\cdot(u-\bar{u})dx.
\end{aligned} \tag{3.7}$$

In a similar manner, multiply (3.6) by $\omega-\bar{\omega}$ and integrate over Ω to give

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\omega-\bar{\omega})(t,\cdot)\|_{L^2}^2 + \langle (u-\bar{u})\cdot\nabla\bar{\omega}, \omega-\bar{\omega}\rangle + \|\omega-\bar{\omega}\|_{L^2}^2 - \langle \Delta(d-\bar{d}), \omega-\bar{\omega}\rangle \\
&+ \langle f(d)-f(\bar{d}), \omega-\bar{\omega}\rangle = \int_\Omega A_2(\bar{u},\bar{d})\cdot(\omega-\bar{\omega})dx.
\end{aligned} \tag{3.8}$$

Notice that $\bar{\omega} = \partial_t\bar{d} + \bar{u}\cdot\nabla\bar{d}$, then

$$\omega-\bar{\omega} = \partial_t(d-\bar{d}) + u\cdot\nabla(d-\bar{d}) + (u-\bar{u})\cdot\nabla\bar{d}.$$

Therefore, the term $-\langle\Delta(d-\bar{d}),\omega-\bar{\omega}\rangle$ in (3.8) can be represented as follows

$$\begin{aligned}
&-\langle \Delta(d-\bar{d}), \omega-\bar{\omega}\rangle \\
=&\frac{1}{2}\frac{d}{dt}\|\nabla(d-\bar{d})\|_{L^2}^2 - \langle \Delta(d-\bar{d}), u\cdot\nabla(d-\bar{d})\rangle - \langle \Delta(d-\bar{d}), (u-\bar{u})\cdot\nabla\bar{d}\rangle.
\end{aligned} \tag{3.9}$$

Moreover, we can split the second term on the right-hand side of (3.9) as

$$\langle \Delta(d-\bar{d}), u\cdot\nabla(d-\bar{d})\rangle = \langle \Delta(d-\bar{d}), (u-\bar{u})\cdot\nabla(d-\bar{d})\rangle + \langle \Delta(d-\bar{d}), \bar{u}\cdot\nabla(d-\bar{d})\rangle. \tag{3.10}$$

Then, substituting (3.9) and (3.10) into (3.8), adding (3.7) and (3.8) together, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u-\bar{u}\|_{L^2}^2 + \|\omega-\bar{\omega}\|_{L^2}^2 + \|\nabla(d-\bar{d})\|_{L^2}^2\Big) + \mu\|\nabla(u-\bar{u})\|_{L^2}^2 + \|\omega-\bar{\omega}\|_{L^2}^2 \\
=&\underbrace{-\langle (u-\bar{u})\cdot\nabla\bar{u}, u-\bar{u}\rangle - \langle (u-\bar{u})\cdot\nabla\bar{\omega}, \omega-\bar{\omega}\rangle}_{I_1} \\
&\underbrace{-\langle \mathrm{div}(\nabla d\odot\nabla(d-\bar{d})), u-\bar{u}\rangle + \langle \Delta(d-\bar{d}), (u-\bar{u})\cdot\nabla(d-\bar{d})\rangle}_{I_2} \\
&\underbrace{-\langle \mathrm{div}(\nabla(d-\bar{d})\odot\nabla\bar{d}), u-\bar{u}\rangle}_{I_3} \\
&+\underbrace{\overbrace{\langle \Delta(d-\bar{d}), \bar{u}\cdot\nabla(d-\bar{d})\rangle}^{I_{41}} + \overbrace{\langle \Delta(d-\bar{d}), (u-\bar{u})\cdot\nabla\bar{d}\rangle}^{I_{42}}}_{I_4} - \underbrace{\langle f(d)-f(\bar{d}), \omega-\bar{\omega}\rangle}_{I_5} \\
&+\int_\Omega \Big[A_1(\bar{u},\bar{d})(u-\bar{u}) + A_2(\bar{u},\bar{d})(\omega-\bar{\omega})\Big]dx.
\end{aligned} \tag{3.11}$$

The next step is to estimate the terms on the right-hand side of (3.11) that are nonlinear in (u,d,ω) and to absorb the resulting expressions with the modulated energy $\delta\mathcal{E}(t)$ and the modulated energy dissipation $\delta\mathcal{D}(t)$ by suitable uses of Sobolev inequalities and Grönwall's lemma.

The first term I_1 is estimated as follows

$$\begin{aligned} I_1 &= -\langle (u-\bar{u})\cdot\nabla\bar{u}, u-\bar{u}\rangle - \langle (u-\bar{u})\cdot\nabla\bar{\omega}, \omega-\bar{\omega}\rangle \\ &\le \|\nabla\bar{u}\|_{L^\infty}\|u-\bar{u}\|_{L^2}^2 + \|\nabla\bar{\omega}\|_{L^\infty}\|u-\bar{u}\|_{L^2}\|\omega-\bar{\omega}\|_{L^2} \\ &\le (2\|\nabla\bar{u}\|_{L^\infty} + \|\nabla\bar{\omega}\|_{L^\infty})\delta\mathcal{E}(t). \end{aligned} \tag{3.12}$$

To estimate the second term I_2, we first rewrite the first expression $\langle \mathrm{div}(\nabla d\odot\nabla(d-\bar{d})), u-\bar{u}\rangle$ as follows

$$\begin{aligned} \langle \mathrm{div}(\nabla d\odot\nabla(d-\bar{d})), u-\bar{u}\rangle &= \langle \mathrm{div}(\nabla(d-\bar{d})\odot\nabla(d-\bar{d})), u-\bar{u}\rangle \\ &\quad + \langle \mathrm{div}(\nabla\bar{d}\odot\nabla(d-\bar{d})), u-\bar{u}\rangle. \end{aligned} \tag{3.13}$$

Notice that $u,\bar{u}$ are divergence-free, so is $u-\bar{u}$. Thus we have

$$\begin{aligned} &\langle \mathrm{div}(\nabla(d-\bar{d})\odot\nabla(d-\bar{d})), u-\bar{u}\rangle \\ =&\sum_{i,j,k}\int_\Omega \partial_j(\partial_i(d_k-\bar{d}_k)\partial_j(d_k-\bar{d}_k))(u_i-\bar{u}_i)dx \\ =&\int_\Omega (u_i-\bar{u}_i)\partial_i(\frac{1}{2}|\nabla(d-\bar{d})|^2)dx + \langle \Delta(d-\bar{d}), (u-\bar{u})\cdot\nabla(d-\bar{d})\rangle \\ =&\langle \Delta(d-\bar{d}), (u-\bar{u})\cdot\nabla(d-\bar{d})\rangle. \end{aligned} \tag{3.14}$$

From (3.13) and (3.14), I_2 can be rewritten as

$$I_2 = -\langle \mathrm{div}(\nabla\bar{d}\odot\nabla(d-\bar{d})), u-\bar{u}\rangle.$$

Integrating by parts and using Young's inequality, we have

$$\begin{aligned} |I_2| &\le \|\nabla\bar{d}\|_{L^\infty}\|\nabla(d-\bar{d})\|_{L^2}\|\nabla(u-\bar{u})\|_{L^2} \\ &\le \frac{C}{\mu}\|\nabla\bar{d}\|_{L^\infty}^2\|\nabla(d-\bar{d})\|_{L^2}^2 + \frac{\mu}{4}\|\nabla(u-\bar{u})\|_{L^2}^2 \\ &\le \frac{2C}{\mu}\|\nabla\bar{d}\|_{L^\infty}^2\delta\mathcal{E}(t) + \frac{\mu}{4}\|\nabla(u-\bar{u})\|_{L^2}^2, \end{aligned} \tag{3.15}$$

in which the latter term can be absorbed by the modulated dissipation $\delta\mathcal{D}(t)$. In a similar manner, for I_3, we have

$$\begin{aligned} |I_3| &\le \|\nabla\bar{d}\|_{L^\infty}\|\nabla(d-\bar{d})\|_{L^2}\|\nabla(u-\bar{u})\|_{L^2} \\ &\le \frac{C}{\mu}\|\nabla\bar{d}\|_{L^\infty}^2\|\nabla(d-\bar{d})\|_{L^2}^2 + \frac{\mu}{4}\|\nabla(u-\bar{u})\|_{L^2}^2 \\ &\le \frac{2C}{\mu}\|\nabla\bar{d}\|_{L^\infty}^2\delta\mathcal{E}(t) + \frac{\mu}{4}\|\nabla(u-\bar{u})\|_{L^2}^2. \end{aligned} \tag{3.16}$$

To estimate the fourth term I_4, we integrate by parts for the first term to get

$$\begin{aligned} I_{41} &= \langle \Delta(d-\bar{d}), \bar{u}\cdot\nabla(d-\bar{d})\rangle \\ &= \int_\Omega \sum_{i,j,k}\partial_k^2(d_j-\bar{d}_j)\bar{u}_i\partial_i(d_j-\bar{d}_j)dx \\ &= -\int_\Omega \sum_{i,j,k}\partial_k(d_j-\bar{d}_j)\partial_k\bar{u}_i\partial_i(d_j-\bar{d}_j)dx, \end{aligned}$$

where we used the fact that $\operatorname{div}\bar{u}=0$. Then

$$
\begin{aligned}
|I_{41}| &\le C\|\nabla\bar{u}\|_{L^\infty}\|\nabla(d-\bar{d})\|_{L^2}^2\\
&\le 2C\|\nabla\bar{u}\|_{L^\infty}\delta\mathcal{E}(t).
\end{aligned}
\tag{3.17}
$$

For the second term, by integration by parts we can write I_{42} as

$$
\begin{aligned}
I_{42} &= \langle\Delta(d-\bar{d}),(u-\bar{u})\cdot\nabla\bar{d}\rangle\\
&= \int_\Omega\sum_{i,j,k}(u_i-\bar{u}_i)\partial_i\bar{d}_j\partial_k^2(d_j-\bar{d}_j)dx\\
&= -\int_\Omega\sum_{i,j,k}\partial_k(u_i-\bar{u}_i)\partial_i\bar{d}_j\partial_k(d_j-\bar{d}_j)dx-\int_\Omega\sum_{i,j,k}(u-u_i)\partial_{ik}\bar{d}_j\partial_k(d_j-\bar{d}_j)dx.
\end{aligned}
$$

By Hölder inequality and Young's inequality, we have

$$
\begin{aligned}
|I_{42}| &\le \|\nabla\bar{d}\|_{L^\infty}\|\nabla(u-\bar{u})\|_{L^2}\|\nabla(d-\bar{d})\|_{L^2}+\|\Delta\bar{d}\|_{L^\infty}\|u-\bar{u}\|_{L^2}\|\nabla(d-\bar{d})\|_{L^2}\\
&\le \frac{C}{\mu}\|\nabla\bar{d}\|_{L^\infty}^2\|\nabla(d-\bar{d})\|_{L^2}^2+\frac{\mu}{4}\|\nabla(u-\bar{u})\|_{L^2}^2+\frac{\|\Delta\bar{d}\|_{L^\infty}}{2}(\|u-\bar{u}\|_{L^2}^2+\|\nabla(d-\bar{d})\|_{L^2}^2)\\
&\le \left(\frac{2C}{\mu}\|\nabla\bar{d}\|_{L^\infty}^2+\|\Delta\bar{d}\|_{L^\infty}\right)\delta\mathcal{E}(t)+\frac{\mu}{4}\|\nabla(u-\bar{u})\|_{L^2}^2.
\end{aligned}
\tag{3.18}
$$

Now it rests to estimate I_5. First, recall that $f(d)=(|d|^2-1)d$ and $\mathcal{F}(d)=\frac{1}{4}(|d|^2-1)^2$, we then have

$$
\begin{aligned}
I_5 &= \langle f(d)-f(\bar{d}),\omega-\bar{\omega}\rangle\\
&= \langle|d|^2d-|\bar{d}|^2\bar{d},\omega-\bar{\omega}\rangle-\langle d-\bar{d},\omega-\bar{\omega}\rangle\\
&= \langle|d-\bar{d}|^2(d-\bar{d})+(|d|^2\bar{d}-|\bar{d}|^2d)+2(d\cdot\bar{d})(d-\bar{d}),\omega-\bar{\omega}\rangle-\langle d-\bar{d},\omega-\bar{\omega}\rangle\\
&= \langle|d-\bar{d}|^2(d-\bar{d}),\omega-\bar{\omega}\rangle+\langle(|d|^2\bar{d}-|\bar{d}|^2d),\omega-\bar{\omega}\rangle+\langle 2(d\cdot\bar{d})(d-\bar{d}),\omega-\bar{\omega}\rangle-\langle d-\bar{d},\omega-\bar{\omega}\rangle\\
&:= I_{51}+I_{52}+I_{53}+I_{54}.
\end{aligned}
\tag{3.19}
$$

For the first term I_{51} on the right-hand side of (3.19), we have

$$
\begin{aligned}
I_{51} &= \langle|d-\bar{d}|^2(d-\bar{d}),\omega-\bar{\omega}\rangle\\
&= \langle|d-\bar{d}|^2(d-\bar{d}),\partial_t(d-\bar{d})+u\cdot\nabla(d-\bar{d})+(u-\bar{u})\cdot\nabla\bar{d}\rangle\\
&= \frac{d}{dt}\frac{1}{4}\||d-\bar{d}|^2\|_{L^2}^2+\langle|d-\bar{d}|^2(d-\bar{d}),(u-\bar{u})\cdot\nabla\bar{d}\rangle.
\end{aligned}
\tag{3.20}
$$

The latter term on the right of (3.20) is estimated as follows

$$
\begin{aligned}
|\langle|d-\bar{d}|^2(d-\bar{d}),(u-\bar{u})\cdot\nabla\bar{d}\rangle| &\le \||d-\bar{d}|^2\|_{L^2}\|d-\bar{d}\|_{L^6}\|u-\bar{u}\|_{L^6}\|\nabla\bar{d}\|_{L^6}\\
&\le C\||d-\bar{d}|^2\|_{L^2}\|\nabla(d-\bar{d})\|_{L^2}\|\nabla(u-\bar{u})\|_{L^2}\|\nabla\bar{d}\|_{L^6}\\
&\le C(\|\nabla u\|_{L^2}^2+\|\nabla\bar{u}\|_{L^2}^2+\|\nabla\bar{d}\|_{L^6}^2)\||d-\bar{d}|^2\|_{L^2}\|\nabla(d-\bar{d})\|_{L^2}\\
&\le C(\|\nabla u\|_{L^2}^2+\|\nabla\bar{u}\|_{L^2}^2+\|\nabla\bar{d}\|_{L^6}^2)\delta\mathcal{E}(t),
\end{aligned}
\tag{3.21}
$$

where C only depends on the spatial dimension. For the second term on the right-hand side of (3.19), we have

$$
I_{52}=\langle|d|^2\bar{d}-|\bar{d}|^2d,\omega-\bar{\omega}\rangle
$$

$$\begin{aligned}
&= \langle (|d|^2 - |\bar{d}|^2)\bar{d} + |\bar{d}|^2(\bar{d}-d), \omega - \bar{\omega}\rangle \\
&= \langle [(d-\bar{d})\cdot(d+\bar{d})]\bar{d}, \omega-\bar{\omega}\rangle + \langle |\bar{d}|^2(\bar{d}-d), \omega-\bar{\omega}\rangle \\
&:= I_{52}^1 + I_{52}^2. \qquad (3.22)
\end{aligned}$$

Then

$$\begin{aligned}
|I_{52}^1| &= |\langle [(d-\bar{d})\cdot(d+\bar{d})]\bar{d}, \omega-\bar{\omega}\rangle| \\
&\le \langle |d-\bar{d}||d+\bar{d}||\bar{d}|, |\omega-\bar{\omega}|\rangle \\
&\le \|d-\bar{d}\|_{L^6}\|d+\bar{d}\|_{L^6}\|\bar{d}\|_{L^6}\|\omega-\bar{\omega}\|_{L^2} \\
&\le C\|\nabla(d-\bar{d})\|_{L^2}\|\nabla(d+\bar{d})\|_{L^2}\|\bar{d}\|_{L^6}\|\omega-\bar{\omega}\|_{L^2} \\
&\le C(\|\nabla d\|_{L^2}^2 + \|\nabla\bar{d}\|_{L^2}^2 + \|\bar{d}\|_{L^6}^2)\|\nabla(d-\bar{d})\|_{L^2}\|\omega-\bar{\omega}\|_{L^2} \\
&\le C(\|\nabla d\|_{L^2}^2 + \|\nabla\bar{d}\|_{L^2}^2 + \|\bar{d}\|_{L^6}^2)\delta\mathcal{E}(t), \qquad (3.23)
\end{aligned}$$

where C is the Sobolev embedding constant. For I_{52}^2, we have

$$\begin{aligned}
|I_{52}^2| &= |\langle |\bar{d}|^2(\bar{d}-d), \omega-\bar{\omega}\rangle| \\
&\le \||\bar{d}|^2\|_{L^3}\|d-\bar{d}\|_{L^6}\|\omega-\bar{\omega}\|_{L^2} \\
&\le C\||\bar{d}|^2\|_{L^3}\|\nabla(d-\bar{d})\|_{L^2}\|\omega-\bar{\omega}\|_{L^2} \\
&\le C\||\bar{d}|^2\|_{L^3}\delta\mathcal{E}(t). \qquad (3.24)
\end{aligned}$$

For I_{53}, it follows that

$$\begin{aligned}
|I_{53}| &= 2|\langle (d\cdot\bar{d})(d-\bar{d}), \omega-\bar{\omega}\rangle| \\
&\le 2\|d\|_{L^6}\|\bar{d}\|_{L^6}\|d-\bar{d}\|_{L^6}\|\omega-\bar{\omega}\|_{L^2} \\
&\le C\|\nabla d\|_{L^2}\|\bar{d}\|_{L^6}\|\nabla(d-\bar{d})\|_{L^2}\|\omega-\bar{\omega}\|_{L^2} \\
&\le C(\|\nabla d\|_{L^2}^2 + \|\bar{d}\|_{L^6}^2)\|\nabla(d-\bar{d})\|_{L^2}\|\omega-\bar{\omega}\|_{L^2} \\
&\le C(\|\nabla d\|_{L^2}^2 + \|\bar{d}\|_{L^6}^2)\delta\mathcal{E}(t), \qquad (3.25)
\end{aligned}$$

where C is the Sobolev embedding constant. Finally, it rests to estimate I_{54}. By Poincaré's inequality, it follows that

$$\begin{aligned}
|I_{54}| &= |\langle d-\bar{d}, \omega-\bar{\omega}\rangle| \\
&\le \|d-\bar{d}\|_{L^2}\|\omega-\bar{\omega}\|_{L^2} \\
&\le C\|\nabla(d-\bar{d})\|_{L^2}\|\omega-\bar{\omega}\|_{L^2} \\
&\le C\delta\mathcal{E}(t), \qquad (3.26)
\end{aligned}$$

where C is the Sobolev embedding constant.

Then, substituting the estimates (3.12), (3.15), (3.16), (3.17), (3.18), (3.19) and (3.20)-(3.26) into (3.11), we have

$$\frac{d}{dt}\delta\mathcal{E}(t) + \frac{1}{4}\delta\mathcal{D}(t) \le \lambda(t)\delta\mathcal{E}(t) + \int_\Omega A\cdot\begin{pmatrix} u-\bar{u} \\ \omega-\bar{\omega}\end{pmatrix} dx,$$

which concludes the proof of the Proposition with the application of Grönwall's lemma.
∎

By analogy with Lions' dissipative solution to the incompressible Euler system [[26], Section 4.4], we then provide the definition of dissipative solution for the system (1.1)-(1.2).

Definition 3.1. *For any $T>0$, we say that*

$$(u,\nabla d)\in L^\infty(0,T;L^2(\Omega))\cap C(0,T;w-L^2(\Omega),$$

such that $\operatorname{div} u=0$, *in the sense of distributions, is a dissipative solution of the hyperbolic liquid crystal system* (1.1)-(1.2), *if* $u(0)=u_0, d(0)=d_0, \omega(0)=\omega_0$ *satisfy* $\operatorname{div} u_0=0$ *and the stability inequality* (3.3) *holds for any test functions* $(\bar u,\bar d,\bar\omega)\in C_c^\infty([0,\infty)\times\Omega)$ *such that* $\operatorname{div}\bar u=0$.

Following the idea of Lions [26], we explain how the stability (3.3) ensures the so-called weak-strong uniqueness for the hyperbolic liquid crystal system (1.1)-(1.2). If the test function $(\bar u,\bar d)$ satisfying $\operatorname{div}\bar u=0$ is the strong solution to the system (1.1)-(1.2) with the same initial data, then one has

$$\delta\mathcal{E}(0)=0, \quad A(\bar u,\bar d)=\vec 0.$$

It then follows that

$$\delta\mathcal{E}(t)\equiv 0,$$

which indicates $u\equiv\bar u, d\equiv\bar d$.

Remark 3.2. Actually, we will see in the last section that even though (u,d) is the limit of some weak solution to the regularized system of (1.1)-(1.2) the norm $\|u\|_{L^2(H^1)}$ and $\|d\|_{L^\infty(H^1)}$ are also bounded, which makes the Definition 3.1 reasonable.

4. Weak solution for the approximate equation

As previously mentioned, dissipative solutions define actual solutions in the sense that they coincide with the unique strong solution when the latter exists. To prove such solutions exist, we need to find a sequence of approximate solutions to the hyperbolic liquid crystal system (1.1)-(1.2). To this end, we introduce the following regularized system

$$\partial_t u^\varepsilon+u^\varepsilon\cdot\nabla u^\varepsilon+\nabla p^\varepsilon-\mu\Delta u^\varepsilon=-\operatorname{div}(\nabla d^\varepsilon\odot\nabla d^\varepsilon), \tag{4.1}$$

$$\operatorname{div} u^\varepsilon=0, \tag{4.2}$$

$$\partial_t\omega^\varepsilon+u^\varepsilon\cdot\nabla\omega^\varepsilon+\omega^\varepsilon=\Delta d^\varepsilon-f(d^\varepsilon), \tag{4.3}$$

$$\omega^\varepsilon=\partial_t d^\varepsilon+u^\varepsilon\cdot\nabla d^\varepsilon-\varepsilon\Delta d^\varepsilon, \tag{4.4}$$

Formally, let $\varepsilon=0$ in the above system, it then reduces to the hyperbolic liquid crystal system (1.1)-(1.2). It is supplmented with the following initial data

$$u^\varepsilon|_{t=0}=u_0, \quad d^\varepsilon|_{t=0}=d_0, \quad \omega^\varepsilon+\varepsilon\Delta d^\varepsilon|_{t=0}=\omega_0, \tag{4.5}$$

where of course $\operatorname{div} u_0=0$. The domain considered here is also the torus $\Omega=\mathbb{T}^3$, thus the initial data are assumed to satisfy the periodic boundary conditions. We are concerned with the global well-posedness of the regularized system (4.1)-(4.5).

For the regularized system (4.1)-(4.5), we aim to prove the global-in-time weak solution. First, we shall state the weak formulation of the Cauchy problem for the system (4.1)-(4.5).

Definition 4.1 (Weak solution). *Let* $u_0\in L^2(\Omega), \nabla d_0\in L^2(\Omega), \omega_0\in L^2(\Omega)$ *satisfying* $\operatorname{div} u_0=0$ *in the sense of distributions. Then, for any* $T>0$, $(u^\varepsilon,d^\varepsilon)$ *is called a weak solution to* (4.1)-(4.5) *if*

$$u^\varepsilon\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega));$$
$$d^\varepsilon\in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega));$$
$$\omega^\varepsilon\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;L^2(\Omega))$$

and for any scalar test function $\phi \in C^\infty(\Omega)$*, vector test function* $\varphi(t,x) \in C_0^\infty([0,T]\times\Omega)$ *satisfying* $\operatorname{div}\varphi(t,x)=0$ *and vector test function* $\psi(t,x)\in C_0^\infty([0,T]\times\Omega)$*, the following weak formulation*

$$
\begin{aligned}
&\int_\Omega u^\varepsilon \cdot \nabla\phi dx = 0,\\
&-\langle u^\varepsilon, \partial_t\varphi\rangle - \int_\Omega (u^\varepsilon\otimes u^\varepsilon):\nabla\varphi dx + \mu\int_\Omega \nabla u^\varepsilon\cdot\nabla\varphi dx = \int_\Omega (\nabla d^\varepsilon \odot \nabla d^\varepsilon):\nabla\varphi dx,\\
&-\langle \omega^\varepsilon, \partial_t\psi\rangle - \int_\Omega (u^\varepsilon\otimes\omega^\varepsilon):\nabla\psi dx + \int_\Omega \omega^\varepsilon\cdot\psi dx + \int_\Omega \nabla d^\varepsilon\cdot\nabla\psi dx + \int_\Omega f(d^\varepsilon)\cdot\psi dx = 0
\end{aligned}
$$

holds for almost every $t\in[0,T]$.

The following theorem asserts the existence of weak solution for the approximate Equation (4.1)-(4.5).

THEOREM 4.1. *Let* $u_0\in L^2(\Omega), \nabla d_0\in L^2(\Omega), \omega_0\in L^2(\Omega)$ *satisfying* $\operatorname{div}u_0=0$ *in the weak sense. Then for each* $0<\varepsilon\le\varepsilon_0$*, for any* $T>0$*, there exists a weak solution* $(u^\varepsilon,d^\varepsilon)$ *to the system* (4.1)-(4.5) *satisfying*

$$
\begin{aligned}
&u^\varepsilon \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega));\\
&d^\varepsilon \in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega));\\
&\omega^\varepsilon \in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;L^2(\Omega)),
\end{aligned}
$$

and additionally satisfying, for a.e. $t\in(0,T)$*, the energy inequality*

$$
\mathcal{E}^\varepsilon(t) + \int_0^t \mathcal{D}^\varepsilon(s) e^{C\varepsilon(t-s)} ds \le \mathcal{E}^\varepsilon(0)e^{C\varepsilon t}, \tag{4.6}
$$

where C *is some constant depending only on the dimension and the energy* $\mathcal{E}^\varepsilon(t)$ *and the energy dissipation* $\mathcal{D}^\varepsilon(t)$ *are given as*

$$
\begin{aligned}
&\mathcal{E}^\varepsilon(t) = \frac{1}{2}\|u^\varepsilon(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla d^\varepsilon(t)\|_{L^2}^2 + \frac{1}{2}\|\omega^\varepsilon(t)\|_{L^2}^2 + \int_\Omega \mathcal{F}(d^\varepsilon(t))dx,\\
&\mathcal{D}^\varepsilon(t) = \mu\|\nabla u^\varepsilon(t)\|_{L^2}^2 + \varepsilon\|\Delta d^\varepsilon(t)\|_{L^2}^2 + \|\omega^\varepsilon(t)\|_{L^2}^2.
\end{aligned}
$$

Proof. The proof of Theorem 4.1 consists of the following three steps.

Step I (A priόri estimate). In this step, we assume $(u^\varepsilon,d^\varepsilon)$ is a classical solution to (4.1)-(4.5). Multiply the first Equation (4.1) by u^ε and integrate over Ω to give

$$
\frac{1}{2}\frac{d}{dt}\|u^\varepsilon\|_{L^2}^2 + \mu\|\nabla u^\varepsilon\|_{L^2}^2 + \langle(\nabla d^\varepsilon)^T\cdot\Delta d^\varepsilon, u^\varepsilon\rangle = 0. \tag{4.7}
$$

Then, multiply the third Equation (4.3) by ω^ε and integrate over Ω to get

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\omega^\varepsilon\|_{L^2}^2 + \frac{1}{2}\|\nabla d^\varepsilon\|_{L^2}^2 + \int_\Omega \mathcal{F}(d^\varepsilon)dx\right) + \|\omega^\varepsilon\|_{L^2}^2\\
&\qquad + \varepsilon\|\Delta d^\varepsilon\|_{L^2}^2 - \langle u^\varepsilon\cdot\nabla d^\varepsilon, \Delta d^\varepsilon\rangle = \langle f(d^\varepsilon), \varepsilon\Delta d^\varepsilon\rangle.
\end{aligned} \tag{4.8}
$$

Notice that $\langle(\nabla d^\varepsilon)^T\cdot\Delta d^\varepsilon, u^\varepsilon\rangle = \langle u^\varepsilon\cdot\nabla d^\varepsilon, \Delta d^\varepsilon\rangle$, and add (4.7) and (4.8) together to obtain

$$
\frac{d}{dt}\left(\frac{1}{2}\|u^\varepsilon\|_{L^2}^2 + \frac{1}{2}\|\omega^\varepsilon\|_{L^2}^2 + \frac{1}{2}\|\nabla d^\varepsilon\|_{L^2}^2 + \int_\Omega \mathcal{F}(d^\varepsilon)dx\right)
$$

$$+\|\omega^\varepsilon\|_{L^2}^2+\mu\|\nabla u^\varepsilon\|_{L^2}^2+\varepsilon\|\Delta d^\varepsilon\|_{L^2}^2-\langle f(d^\varepsilon),\varepsilon\Delta d^\varepsilon\rangle=0. \quad (4.9)$$

Notice that $f(d^\varepsilon)=(|d^\varepsilon|^2-1)d^\varepsilon$, and if we denote $d_\varepsilon=(d_1^\varepsilon,d_2^\varepsilon,d_3^\varepsilon)$, it follows by integration by parts

$$\begin{aligned}\varepsilon\langle f(d^\varepsilon),-\Delta d^\varepsilon\rangle&=\varepsilon\int_\Omega\sum_{i,k=1}^3\big(\sum_{j=1}^3(d_j^\varepsilon)^2-1\big)d_k^\varepsilon(-\partial_{x_i}^2d_k^\varepsilon)dx\\&=\varepsilon\int_\Omega\sum_{i,k=1}^3\big(\sum_{j=1}^3(d_j^\varepsilon)^2-1\big)(\partial_{x_i}d_k^\varepsilon)^2dx+2\varepsilon\int_\Omega\sum_{i,j,k=1}^3 d_j^\varepsilon\partial_{x_i}d_j^\varepsilon d_k^\varepsilon\partial_{x_i}d_k^\varepsilon dx\\&=\varepsilon\int_\Omega\sum_{i,k=1}^3\big(\sum_{j=1}^3(d_j^\varepsilon)^2-1\big)(\partial_{x_i}d_k^\varepsilon)^2dx+2\varepsilon\int_\Omega\sum_{i=1}^n\big(\sum_{j=1}^n d_j^\varepsilon\partial_{x_i}d_j^\varepsilon\big)^2dx, \quad (4.10)\end{aligned}$$

The second term on the right-hand side of (4.10) is nonegative. For the first term, we have

$$\begin{aligned}\int_\Omega\sum_{i,k=1}^3\big(\sum_{j=1}^3(d_j^\varepsilon)^2-1\big)(\partial_{x_i}d_k^\varepsilon)^2dx=&\int_{\{x||d^\varepsilon|\geq1\}}\sum_{i,k=1}^3(|d^\varepsilon|^2-1)(\partial_{x_i}d_k^\varepsilon)^2dx\\&+\int_{\{x||d_\varepsilon|<1\}}\sum_{i,k=1}^3(|d^\varepsilon|^2-1)(\partial_{x_i}d_k^\varepsilon)^2dx, \quad (4.11)\end{aligned}$$

where the first term on the right-hand side of (4.11) is also nonegative. For the second term, we directly have

$$\begin{aligned}\left|\int_{\{x||d^\varepsilon|<1\}}\sum_{i,k=1}^3(|d^\varepsilon|^2-1)(\partial_{x_i}d_k^\varepsilon)^2dx\right|\leq&\int_{\{x||d^\varepsilon|<1\}}\sum_{i,k=1}^3\big||d^\varepsilon|^2-1\big|(\partial_{x_i}d_k^\varepsilon)^2dx\\\leq& C\int_\Omega\sum_{i,k}(\partial_{x_i}d_k^\varepsilon)^2dx=C\|\nabla d^\varepsilon\|_{L^2}^2, \quad (4.12)\end{aligned}$$

where C is some constant depending only on the dimension. Then, substitute (4.10)-(4.12) into (4.9) to give

$$\frac{d}{dt}\mathcal{E}^\varepsilon(t)+\mathcal{D}^\varepsilon(t)\leq C\varepsilon\mathcal{E}^\varepsilon(t). \quad (4.13)$$

By a Groñwall's inquality, we have, for any $t>0$

$$\mathcal{E}^\varepsilon(t)+\int_0^t\mathcal{D}^\varepsilon(s)e^{C\varepsilon(t-s)}ds\leq\mathcal{E}^\varepsilon(0)e^{C\varepsilon t}, \quad (4.14)$$

which concludes the a priori estimate (4.6).

Step II (Approximate solution). In this step, we follow the standard Faedo-Galerkin method to construct the approximate solution to the system (4.1)-(4.5). Note that the Equation (4.3) is a hyperbolic equation of d, thus we can not apply the approximation procedure of [22] to construct approximate solutions. Indeed we construct the approximate solution both for u and d by Galerkin method here.

To start with the Galerkin approximation procedure, we state some known facts about the Stokes operator. Denote $\mathbf{H}=\{v\in L^2(\Omega)|\operatorname{div}v=0\}$. Let $\{e_i\}_{i=1}^\infty$ be the orthonormal basis of $\mathbf{H}$ satisfying

$$Ae_i=\lambda_ie_i,\quad i=1,2,\ldots \quad (4.15)$$

where A is the Stokes operator and λ_i satisfies $0<\lambda_1\le\lambda_2\le\ldots$. Let $\{\phi_j\}_{j=1}^{\infty}$ be the orthonormal basis of $L^2(\Omega)$, for instance ϕ_j can be chosen as the eigenvectors of $-\Delta$ in $H^1(\Omega)$ which satisfies $-\Delta\phi_j=\sigma_j\phi_j,\ j=1,2,\ldots$ and $0<\sigma_1\le\sigma_2\le\sigma_3\le\ldots$.

For fixed $N\in\mathbb{N}^+$, denote $\mathbb{P}_N$ the projection from $\mathbf{H}$ onto the space spanned by $\{e_1,e_2,\ldots,e_N\}$ and $\mathcal{P}_N$ the projection from $L^2(\Omega)$ onto the space spanned by $\{\phi_1,\phi_2,\ldots,\phi_N\}$. We construct approximate solutions $(u^{\varepsilon,N},d^{\varepsilon,N})$ in the following form

$$u^{\varepsilon,N}=\sum_{i=1}^{N}g_N^i(t)e_i,\quad d^{\varepsilon,N}=\sum_{j=1}^{N}h_N^j(t)\phi_j$$

to satisfy the following approximate equation

$$\partial_t u^{\varepsilon,N}-\mu\Delta u^{\varepsilon,N}=\mathbb{P}_N\mathbb{P}(-u^{\varepsilon,N}\cdot\nabla u^{\varepsilon,N}-\operatorname{div}(\nabla d^{\varepsilon,N}\odot\nabla d^{\varepsilon,N})),\tag{4.16}$$

$$\partial_t\mathcal{P}_N\omega^{\varepsilon,N}+\mathcal{P}_N(u^{\varepsilon,N}\cdot\nabla(\mathcal{P}_N\omega^{\varepsilon,N})+\omega^{\varepsilon,N})-\Delta d^{\varepsilon,N}+f(d^{\varepsilon,N})=0,\tag{4.17}$$

$$\omega^{\varepsilon,N}=\partial_t d^{\varepsilon,N}+(u^{\varepsilon,N}\cdot\nabla d^{\varepsilon,N})-\varepsilon\Delta d^{\varepsilon,N},$$

$$u^{\varepsilon,N}(0)=\mathbb{P}_N u_0,\ d^{\varepsilon,N}(0)=\mathcal{P}_N d_0,\ \partial_t d^{\varepsilon,N}(0)+\mathcal{P}_N(u_0\cdot\nabla d_0)=\mathcal{P}_N\omega_0,\tag{4.18}$$

where $\mathbb{P}$ is the Leray projection. To solve the approximate system (4.16)-(4.18), we first note that

$$\begin{aligned}\mathcal{P}_N\omega_\varepsilon^N&=\sum_{k=1}^{N}\langle\omega_\varepsilon^N,\phi_k\rangle\phi_k\\&=\sum_{k=1}^{N}\frac{d}{dt}h_N^k(t)\phi_k+\sum_{k=1}^{N}\langle u_\varepsilon^N\cdot\nabla d_\varepsilon^N,\phi_k\rangle\phi_k+\varepsilon\sum_{k=1}^{N}\sigma_k h_N^k(t)\phi_k\\&=\sum_{k=1}^{N}\frac{d}{dt}h_N^k(t)\phi_k+\sum_{k=1}^{N}\sum_{i,j=1}^{N}\alpha_{ijk}g_N^i(t)h_N^k(t)\phi_k+\varepsilon\sum_{k=1}^{N}\sigma_k h_N^k(t)\phi_k,\end{aligned}$$

where $\alpha_{ijk}=\langle e_i\cdot\nabla\phi_j,\phi_k\rangle$.

To solve the Cauchy problem for the system (4.16)-(4.18), we need to transform the system (4.16)-(4.18) into an ordinary system of equations. To this end, take the L^2-inner product of the first Equation (4.16) with e_i for each $1\le i\le N$ and the second Equation (4.17) with ϕ_j for each $1\le j\le N$ to obtain

$$\frac{d}{dt}g_N^i(t)+\mu\lambda_i g_N^i(t)+\sum_{j,k=1}^{N}b_{jki}g_N^j(t)g_N^k(t)+\sum_{j,k=1}^{N}c_{jki}h_N^j(t)h_N^k(t)=0,\ 1\le i\le N,\tag{4.19}$$

$$\begin{aligned}&\frac{d^2}{dt^2}h_N^j(t)+\frac{d}{dt}\left(\sum_{k,\ell=1}^{N}\alpha_{k\ell j}g_N^k(t)h_N^\ell(t)\right)+\varepsilon\sigma_j\frac{d}{dt}h_N^j(t)+\sum_{k,\ell=1}^{N}\alpha_{k\ell j}g_N^k(t)\frac{d}{dt}h_N^\ell(t)\\&+\sum_{k,\ell,i,m=1}^{N}\alpha_{k\ell j}g_N^k(t)\alpha_{im\ell}g_N^i(t)h_N^m(t)+\varepsilon\sum_{k,\ell=1}^{N}\sigma_\ell\alpha_{k\ell j}g_N^k(t)h_N^\ell(t)+\frac{d}{dt}h_N^j(t)\\&+\sum_{k,\ell=1}^{N}\alpha_{k\ell j}g_N^k(t)h_N^\ell(t)+\varepsilon\sigma_j h_N^j(t)\\&+(\sigma_j-1)h_N^j(t)+\sum_{k,\ell,m=1}^{N}\beta_{k\ell mj}h_N^k(t)h_N^\ell(t)h_N^m(t)=0,\ 1\le j\le N,\end{aligned}\tag{4.20}$$

$$g_N^i(0)=\langle u_0,e_i\rangle,\quad h_N^j(0)=\langle d_0,\phi_j\rangle,\ 1\le i,j\le N,$$
$$\frac{d}{dt}h_N^j(0)+\sum_{k,\ell=1}^{N}\alpha_{k\ell j}g_N^k(0)h_N^\ell(0)=\langle\omega_0,\phi_j\rangle, \tag{4.21}$$

where the coefficients are determined by

$$b_{jki}=\langle e_j\cdot\nabla e_k,e_i\rangle,\quad c_{jki}=\langle \mathrm{div}(\nabla\phi_j\odot\nabla\phi_k),e_i\rangle$$
$$\alpha_{k\ell j}=\langle e_k\cdot\nabla\phi_\ell,\phi_j\rangle,\quad \beta_{k\ell mj}=\langle(\phi_k\cdot\phi_\ell)\phi_m,\phi_j\rangle.$$

Since the ODE system is continuous and locally Lipschitz, according to the standard existing theory of ODEs, there exists a unique sequence of C^1 functions $\{g_N^1(t),g_N^2(t),\dots,g_N^N(t)\}$ and a sequence of C^2 functions $\{h_N^1(t),h_N^2(t),\dots,h_N^N(t)\}$, satisfying (4.19)-(4.21), and solving (4.19)-(4.20) on $[0,T_N]$. Since the ODE system (4.19)-(4.21) is equivalent to the PDE system (4.16)-(4.18), thus it follows that the system (4.16)-(4.18) has a local solution $(u^{\varepsilon,N},d^{\varepsilon,N})$ on $[0,T_N]$ for fixed $N\in\mathbb{N}^+$.

Then we will show that for all $N\in\mathbb{N}^+$ the local solutions $(u^{\varepsilon,N},d^{\varepsilon,N})$ of the system (4.16)-(4.18) can be extended to $[0,T]$ for arbitrary $T>0$. To this end, we perform the energy estimates on the system (4.16)-(4.18). Multiply the first Equation (4.16) by $u^{\varepsilon,N}$ and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\|u^{\varepsilon,N}\|_{L^2}^2+\mu\|\nabla u^{\varepsilon,N}\|_{L^2}^2+\langle\mathrm{div}(\nabla d^{\varepsilon,N}\odot\nabla d^{\varepsilon,N}),u^{\varepsilon,N}\rangle=0, \tag{4.22}$$

where we used the fact that $\mathbb{P}$ and $\mathbb{P}_N$ are symmetric. Then, multiply the second Equation (4.17) by $\mathcal{P}_N\omega^{\varepsilon,N}$ and integrate over Ω to give

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{P}_N\omega^{\varepsilon,N}\|_{L^2}^2+\|\mathcal{P}_N\omega^{\varepsilon,N}\|_{L^2}^2+\frac{1}{2}\frac{d}{dt}\|\nabla d^{\varepsilon,N}\|_{L^2}^2-\langle\Delta d^{\varepsilon,N},u^{\varepsilon,N}\cdot\nabla d^{\varepsilon,N}\rangle$$
$$+\varepsilon\|\Delta d^{\varepsilon,N}\|_{L^2}^2+\langle f(d^{\varepsilon,N}),\omega^{\varepsilon,N}\rangle=0. \tag{4.23}$$

Sum up (4.22) with (4.23), and recall (4.10)-(4.12) to obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|u^{\varepsilon,N}\|_{L^2}^2+\frac{1}{2}\|\mathcal{P}_N\omega^{\varepsilon,N}\|_{L^2}^2+\frac{1}{2}\|\nabla d^{\varepsilon,N}\|_{L^2}^2+\int_\Omega\mathcal{F}(d^{\varepsilon,N})dx\right)$$
$$+\mu\|\nabla u^{\varepsilon,N}\|_{L^2}^2+\|\mathcal{P}_N\omega^{\varepsilon,N}\|_{L^2}^2+\varepsilon\|\Delta d^{\varepsilon,N}\|_{L^2}^2\le C\varepsilon\|\nabla d^{\varepsilon,N}\|_{L^2}^2,$$

where C is a constant which is independent of N. By a Grönwall's inequality, we derive for any $t>0$, for any $N\in\mathbb{N}^+$, for each $\varepsilon>0$,

$$\mathcal{E}_N^\varepsilon(t)+\int_0^t\mathcal{D}_N^\varepsilon(s)e^{C\varepsilon(t-s)}ds\le\mathcal{E}_N^\varepsilon(0)e^{C\varepsilon t}, \tag{4.24}$$

where

$$\mathcal{E}_N^\varepsilon(t)=\frac{1}{2}\|u^{\varepsilon,N}(t)\|_{L^2}^2+\frac{1}{2}\|\mathcal{P}_N\omega^{\varepsilon,N}(t)\|_{L^2}^2+\frac{1}{2}\|\nabla d^{\varepsilon,N}(t)\|_{L^2}^2+\int_\Omega\mathcal{F}(d^{\varepsilon,N})(t)dx,$$
$$\mathcal{D}_N^\varepsilon(t)=\mu\|\nabla u^{\varepsilon,N}(t)\|_{L^2}^2+\|\mathcal{P}_N\omega^{\varepsilon,N}(t)\|_{L^2}^2+\varepsilon\|\Delta d^{\varepsilon,N}(t)\|_{L^2}^2.$$

It can be easily seen that $\mathcal{E}_N^\varepsilon(0)\le\widetilde{\mathcal{E}^\varepsilon}(0)$, where

$$\widetilde{\mathcal{E}^\varepsilon}(0)=\frac{1}{2}\|u(0)\|_{L^2}^2+\frac{1}{2}\|\omega(0)\|_{L^2}^2+\frac{1}{2}\|\nabla d(0)\|_{L^2}^2+\int_\Omega\frac{\left(|d_0|^2+1\right)^2}{4}dx.$$

That is to say that the right-hand side of the energy estimate (4.24) has an upper bound for any fixed T and the bound is independent of N, which ensures that the local solution $(u^{\varepsilon,N}, d^{\varepsilon,N})$ on $[0,T_N]$ can be extended to $[0,T]$ for any $T>0$.

Step III (Convergence) In this step, we shall take limit in the approximate Equation (4.16)-(4.18) as $N\to\infty$. First, according to the energy estimate (4.24) and for every $T>0$ fixed, we have

$$\begin{aligned}
&\sup_{0\le t\le T}\|u^{\varepsilon,N}(t)\|_{L^2}^2+\mu\int_0^T\|\nabla u^{\varepsilon,N}(t)\|_{L^2}^2dt\le C_{T,\varepsilon};\\
&\sup_{0\le t\le T}\|\nabla d^{\varepsilon,N}(t)\|_{L^2}^2+\varepsilon\int_0^T\|\Delta d^{\varepsilon,N}(t)\|_{L^2}^2dt\le C_{T,\varepsilon};\\
&\sup_{0\le t\le T}\|\omega^{\varepsilon,N}(t)\|_{L^2}^2+\int_0^T\|\omega^{\varepsilon,N}(t)\|_{L^2}^2dt\le C_{T,\varepsilon},
\end{aligned}$$

where $C_{T,\varepsilon}=\widetilde{\mathcal{E}^\varepsilon}(0)e^{C\varepsilon T}$ is independent of N. That is to say,

$$\begin{aligned}
&\{u^{\varepsilon,N}\}\text{ is uniformly bounded in }L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega));\\
&\{d^{\varepsilon,N}\}\text{ is uniformly bounded in }L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega));\\
&\{\mathcal{P}_N\omega^{\varepsilon,N}\}\text{ is uniformly bounded in }L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;L^2(\Omega)).
\end{aligned}\tag{4.25}$$

From the Gagliado-Nirenberg inequality,

$$\|u^{\varepsilon,N}\|_{L^4}\le C\|\nabla u^{\varepsilon,N}\|_{L^2}^{\frac34}\|u^{\varepsilon,N}\|_{L^2}^{\frac14},$$

and

$$\|\nabla d^{\varepsilon,N}\|_{L^4}\le C\|\Delta d^{\varepsilon,N}\|_{L^2}^{\frac34}\|\nabla d^{\varepsilon,N}\|_{L^2}^{\frac14},$$

we have

$$\{\partial_t u^{\varepsilon,N}\}\text{ is uniformly bounded in }L^{\frac43}(0,T;H^{-1}(\Omega)).$$

Then, by Aubin-Lions Lemma (see e.g. [1, 25]), we infer that there exists u^ε and a subsequence (still labeled) $u^{\varepsilon,N}$ such that

$$u^{\varepsilon,N}\to u^\varepsilon\ \text{ in }L^2(0,T;L^2(\Omega)),\text{ as }N\to\infty.$$

Noting that $\mathcal{P}_N\omega^{\varepsilon,N}=\partial_t d^{\varepsilon,N}+\mathcal{P}_N(u^{\varepsilon,N}\cdot\nabla d^{\varepsilon,N})-\varepsilon\Delta d^{\varepsilon,N}$, we directly have

$$\partial_t d^{\varepsilon,N}=\mathcal{P}_N\omega^{\varepsilon,N}+\varepsilon\Delta d^{\varepsilon,N}-\mathcal{P}_N(u^{\varepsilon,N}\cdot\nabla d^{\varepsilon,N}).$$

From (4.25), we have

$$\{\partial_t d^{\varepsilon,N}\}\text{ is uniformly bounded in }L^{\frac43}([0,T];L^2(\Omega)),$$

also by Aubin-Lions lemma, there exists d_ε such that

$$d^{\varepsilon,N}\to d^\varepsilon\text{ in }L^2(0,T;H^1(\Omega)),\text{ as }N\to\infty,$$

Denoting $\omega^\varepsilon:=\partial_t d^\varepsilon+u^\varepsilon\cdot\nabla d^\varepsilon-\varepsilon\Delta d^\varepsilon$, it follows that

$$\omega^{\varepsilon,N}\rightharpoonup\omega^\varepsilon\text{ in }L^2([0,T];L^2(\Omega)),\text{ as }N\to\infty.$$

In fact, we can deduce that

$$\{\partial_t \omega^{\varepsilon,N}\} \text{ is uniformly bounded in } L^2(0,T;W^{-1,\frac{3}{2}}(\Omega))+L^2(0,T;L^2(\Omega)).$$

Then we are able to take limits in the approximate Equations (4.16)-(4.18) and the limit $(u^\varepsilon, d^\varepsilon)$ is one of the weak solutions satisfying Definition 4.1, which concludes the proof of Theorem 4.1. □

REMARK 4.1. Whether or not the weak solution in Theorem 4.1 is unique is unknown and is beyond the main concern of the present paper, which is analogous to the similar issue for 3D incompressible Navier-Stokes equations. Actually, the so-called weak-strong uniqueness considered in the present paper does not need to ensure the uniqueness of the weak solution for the approximate system.

5. Dissipative solution for the liquid crystal system

In this section, we shall prove that there exists at least one dissipative solution for the hyperbolic liquid crystal system (1.1)-(1.2).

THEOREM 5.1. *Let the initial data* $u_0 \in L^2(\Omega), d_0 \in H^1(\Omega), \omega_0 \in L^2(\Omega)$ *such that* $\operatorname{div} u_0 = 0$ *in the sense of distributions, then there exists a dissipative solution to the hyperbolic liquid crystals system* (1.1)-(1.2).

Proof. We consider $(u^\varepsilon, d^\varepsilon)$, a weak solution of the regularized system (4.1)-(4.5) with initial data (u_0, d_0, ω_0) satisfy the assumption. Consider test functions $(\bar{u}, \bar{d}) \in C^\infty([0,T]\times\Omega)$ which satisfy $\operatorname{div}\bar{u}=0$ and denote $\bar{\omega}=\partial_t \bar{d}+\bar{u}\cdot\nabla\bar{d}$, then rewrite the Equation (4.1), (4.2) as

$$\begin{aligned} \partial_t(u^\varepsilon-\bar{u})+u^\varepsilon\cdot\nabla(u^\varepsilon-\bar{u})+(u^\varepsilon-\bar{u})\cdot\nabla\bar{u}-\mu\Delta(u^\varepsilon-\bar{u})+\nabla\pi \\ +\operatorname{div}(\nabla d^\varepsilon\odot\nabla(d^\varepsilon-\bar{d}))+\operatorname{div}(\nabla(d^\varepsilon-\bar{d})\odot\nabla\bar{d})=A_1(\bar{u},\bar{d}), \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \partial_t(\omega^\varepsilon-\bar{\omega})+u^\varepsilon\cdot\nabla(\omega^\varepsilon-\bar{\omega})+(u^\varepsilon-\bar{u})\cdot\nabla\bar{\omega}+(\omega^\varepsilon-\bar{\omega})-\Delta(d^\varepsilon-\bar{d}) \\ +f(d^\varepsilon)-f(\bar{d})=A_2(\bar{u},\bar{d}), \end{aligned} \tag{5.2}$$

where π is some scalar function and $A_1(\bar{u},\bar{d}), A_2(\bar{u},\bar{d})$ are given as (3.1). Then, multiply (5.1) by $u^\varepsilon-\bar{u}$ and integrate over Ω to give

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^\varepsilon-\bar{u}\|_{L^2}^2+\mu\|\nabla(u^\varepsilon-\bar{u})\|_{L^2}^2+\langle(u^\varepsilon-\bar{u})\cdot\nabla\bar{u},u^\varepsilon-\bar{u}\rangle \\ &\qquad+\langle\operatorname{div}(\nabla d^\varepsilon\odot\nabla(d^\varepsilon-\bar{d})),u^\varepsilon-\bar{u}\rangle+\langle\operatorname{div}(\nabla(d^\varepsilon-\bar{d})\odot\nabla\bar{d}),u^\varepsilon-\bar{u}\rangle \\ &\qquad=\langle A_1(\bar{u},\bar{d}),u^\varepsilon-\bar{u}\rangle. \end{aligned} \tag{5.3}$$

Similarly, multiply (5.2) by $\omega^\varepsilon-\bar{\omega}$ and integrate over Ω to give

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\omega^\varepsilon-\bar{\omega}\|_{L^2}^2+\|\omega^\varepsilon-\bar{\omega}\|_{L^2}^2+\langle(u^\varepsilon-\bar{u})\cdot\nabla\bar{\omega},\omega^\varepsilon-\bar{\omega}\rangle-\langle\Delta(d^\varepsilon-\bar{d}),\omega^\varepsilon-\bar{\omega}\rangle \\ +\langle f(d^\varepsilon)-f(\bar{d}),\omega^\varepsilon-\bar{\omega}\rangle=\langle A_2(\bar{u},\bar{d}),\omega^\varepsilon-\bar{\omega}\rangle. \end{aligned} \tag{5.4}$$

Noting that $\omega^\varepsilon=\partial_t d^\varepsilon+u^\varepsilon\cdot\nabla d^\varepsilon-\varepsilon\Delta d^\varepsilon$ and $\bar{\omega}=\partial_t\bar{d}+\bar{u}\cdot\nabla\bar{d}$, then

$$\omega^\varepsilon-\bar{\omega}=\partial_t(d^\varepsilon-\bar{d})+u^\varepsilon\cdot\nabla(d^\varepsilon-\bar{d})+(u^\varepsilon-\bar{u})\cdot\nabla\bar{d}-\varepsilon\Delta d^\varepsilon. \tag{5.5}$$

Thus we have

$$
\begin{aligned}
-\langle \Delta(d^\varepsilon-\bar d),\omega^\varepsilon-\bar\omega\rangle =&\frac{1}{2}\frac{d}{dt}\|\nabla(d^\varepsilon-\bar d)\|_{L^2}^2-\langle \Delta(d^\varepsilon-\bar d),u^\varepsilon\cdot\nabla(d^\varepsilon-\bar d)\rangle\\
&-\langle \Delta(d^\varepsilon-\bar d),(u^\varepsilon-\bar u)\cdot\nabla\bar d\rangle+\varepsilon\langle \Delta(d^\varepsilon-\bar d),\Delta d^\varepsilon\rangle. \qquad (5.6)
\end{aligned}
$$

Recall that $f(d^\varepsilon)=(|d^\varepsilon|^2-1)d^\varepsilon$, then

$$
\begin{aligned}
\langle f(d^\varepsilon)-f(\bar d),\omega^\varepsilon-\bar\omega\rangle =&\langle f(d^\varepsilon)-f(\bar d),\partial_t d^\varepsilon+u^\varepsilon\cdot\nabla d^\varepsilon-\partial_t\bar d-\bar u\cdot\nabla\bar d-\varepsilon\Delta d^\varepsilon\rangle\\
=&\langle f(d^\varepsilon)-f(\bar d),\partial_t(d^\varepsilon-\bar d)+u\cdot\nabla(d^\varepsilon-\bar d)+(u^\varepsilon-\bar u)\cdot\nabla\bar d\rangle\\
&-\varepsilon\langle f(d^\varepsilon)-f(\bar d),\Delta d^\varepsilon\rangle. \qquad (5.7)
\end{aligned}
$$

Then, take summation of (5.3) and (5.4) and use (5.5)-(5.7) to obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac12\|u^\varepsilon-\bar u\|_{L^2}^2+\frac12\|\omega^\varepsilon-\bar\omega\|_{L^2}^2+\frac12\|\nabla(d^\varepsilon-\bar d)\|_{L^2}^2+\frac14\||d^\varepsilon-d|^2\|_{L^2}^2\right)\\
&\quad+\mu\|\nabla(u^\varepsilon-\bar u)\|_{L^2}^2+\|\omega^\varepsilon-\bar\omega\|_{L^2}^2+\varepsilon\|\Delta d^\varepsilon\|_{L^2}^2-\varepsilon\langle f(d^\varepsilon)-f(\bar d),\Delta d^\varepsilon\rangle\\
=&-\langle (u^\varepsilon-\bar u)\cdot\nabla\bar u,u^\varepsilon-\bar u\rangle-\langle (u^\varepsilon-\bar u)\cdot\nabla\bar\omega,\omega^\varepsilon-\bar\omega\rangle+\varepsilon\langle\Delta\bar d,\Delta d^\varepsilon\rangle+\langle f(d^\varepsilon),\bar\omega\rangle\\
&\quad-\langle \mathrm{div}(\nabla d^\varepsilon\odot\nabla(d^\varepsilon-\bar d)),u^\varepsilon-\bar u\rangle+\langle \Delta(d^\varepsilon-\bar d),(u^\varepsilon-\bar u)\cdot\nabla(d^\varepsilon-\bar d)\rangle\\
&\quad-\langle \mathrm{div}(\nabla(d^\varepsilon-\bar d)\odot\nabla\bar d),u^\varepsilon-\bar u\rangle+\langle \Delta(d^\varepsilon-\bar d),\bar u\cdot\nabla(d^\varepsilon-\bar d)\rangle\\
&\quad+\langle \Delta(d^\varepsilon-\bar d),(u^\varepsilon-\bar u)\cdot\nabla\bar d\rangle+\int_\Omega\left[A_1(\bar u,\bar d)(u^\varepsilon-\bar u)+A_2(\bar u,\bar d)(\omega^\varepsilon-\bar\omega)\right]dx.
\end{aligned}
$$

With the same computations as Proposition 3.1 by replacing u,d,ω by $u^\varepsilon,d^\varepsilon,\omega^\varepsilon$, we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\delta\mathcal{E}^\varepsilon(t)+\frac14\delta\mathcal{D}^\varepsilon(t)+\varepsilon\|\Delta d^\varepsilon\|_{L^2}^2-\varepsilon\langle f(d^\varepsilon),\Delta d^\varepsilon\rangle\\
\le&\lambda^\varepsilon(t)\delta\mathcal{E}^\varepsilon(t)+\varepsilon\langle\Delta\bar d-f(\bar d),\Delta d^\varepsilon\rangle+\int_\Omega A\cdot\begin{pmatrix}u^\varepsilon-\bar u\\ \omega^\varepsilon-\bar\omega\end{pmatrix}dx, \qquad (5.8)
\end{aligned}
$$

where the modulate energy $\delta\mathcal{E}^\varepsilon(t)$, the modulate energy dissipation $\delta\mathcal{D}^\varepsilon(t)$ and the growth rate $\lambda^\varepsilon(t)$ are defined by simply replacing (u,d,ω) by $(u^\varepsilon,d^\varepsilon,\omega^\varepsilon)$ in (3.4). Use Cauchy inequality to give

$$
|\varepsilon\langle\Delta\bar d-f(\bar d),\Delta d^\varepsilon\rangle|\le\frac{\varepsilon}{2}\left(\|\Delta\bar d-f(\bar d)\|_{L^2}^2+\|\Delta d^\varepsilon\|_{L^2}^2\right).
$$

Then, employ the estimate (4.10)-(4.12) to give

$$
\begin{aligned}
\frac{d}{dt}\delta\mathcal{E}^\varepsilon(t)+\frac14\delta\mathcal{D}^\varepsilon(t)+\varepsilon\|\Delta d^\varepsilon\|_{L^2}^2\le&\lambda^\varepsilon(t)\delta\mathcal{E}^\varepsilon(t)+\frac{\varepsilon}{2}\left(\|\Delta\bar d-f(\bar d)\|_{L^2}^2+\|\Delta d^\varepsilon\|_{L^2}^2\right)\\
&+C\varepsilon\|\nabla d^\varepsilon\|_{L^2}^2+\int_\Omega A\cdot\begin{pmatrix}u^\varepsilon-\bar u\\ \omega^\varepsilon-\bar\omega\end{pmatrix}dx, \qquad (5.9)
\end{aligned}
$$

Integrating (5.9) from 0 to t, we finally get

$$
\begin{aligned}
\delta\mathcal{E}^\varepsilon(t)+\int_0^t\left(\frac14\delta\mathcal{D}^\varepsilon+\frac{\varepsilon}{2}\|\Delta d^\varepsilon\|_{L^2}^2\right)ds\le&\,\delta\mathcal{E}^\varepsilon(0)+\int_0^t\left(\frac{\varepsilon}{2}\|\Delta\bar d-f(\bar d)\|_{L^2}^2+C\varepsilon\|\nabla d^\varepsilon\|_{L^2}^2\right)ds\\
&+\int_0^t\left[\lambda^\varepsilon(s)\delta\mathcal{E}^\varepsilon(s)+\left[\int_\Omega A\cdot\begin{pmatrix}u^\varepsilon-\bar u\\ \omega^\varepsilon-\bar\omega\end{pmatrix}dx\right]\right]ds.
\end{aligned}
$$

Then an application of Grönwall's lemma yields

$$\begin{aligned}\delta\mathcal{E}^\varepsilon(t)+\frac{1}{4}\int_0^t \delta\mathcal{D}^\varepsilon(t)e^{\int_s^t\lambda^\varepsilon(\sigma)d\sigma}ds\le \delta\mathcal{E}^\varepsilon(0)e^{\int_0^t\lambda(s)ds}\\ +\int_0^t\left[\frac{\varepsilon}{2}\|\Delta\bar{d}-f(\bar{d})\|_{L^2}^2+C\varepsilon\|\nabla d^\varepsilon\|_{L^2}^2+\left[\int_\Omega A\cdot\begin{pmatrix}u^\varepsilon-\bar{u}\\ \omega^\varepsilon-\bar{\omega}\end{pmatrix}dx\right]\right](s)e^{\int_s^t\lambda^\varepsilon(\sigma)d\sigma}ds.\end{aligned} \tag{5.10}$$

Then, we need to pass to the limit in the above stability inequality. Note that for all $\varepsilon\ge 0$ and for every fixed $T>0$, we have $\{u^\varepsilon\}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$. Then we may assume that for a subsequence of $\{u^\varepsilon\}$ (still labeled as $\{u^\varepsilon\}$),

$$u^\varepsilon \overset{*}{\rightharpoonup} u \ \text{ in } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)).$$

And notice that $\{\nabla d^\varepsilon\}$ is uniformly bounded in $L^\infty(0,T;L^2)$, then we infer that

$$\nabla d^\varepsilon \overset{*}{\rightharpoonup} \nabla d \ \text{ in } L^\infty(0,T;L^2(\Omega)).$$

Furthermore, we observe that

$$\partial_t u^\varepsilon = -\mathbb{P}(\nabla\cdot(u^\varepsilon\otimes u^\varepsilon)+\mathrm{div}(\nabla d^\varepsilon\odot\nabla d^\varepsilon))+\mu\Delta u^\varepsilon,$$

and thus $\{\partial_t u^\varepsilon\}$ is bounded in $L^2(0,T;H^{-1}(\Omega))+L^\infty(0,T;W^{-(1+\eta),1}(\Omega))$ for all $\eta>0$, it is possible to show (see [[26], Appendix C]) that u^ε converges to $u\in C([0,T];w-L^2(\Omega))$ weakly in L^2 uniformly in $t\in[0,T]$ for all $T>0$.

Similarly, we observe that $\omega^\varepsilon=\partial_t d^\varepsilon+u^\varepsilon\cdot\nabla d^\varepsilon-\varepsilon\Delta d^\varepsilon$, then $\partial_t d^\varepsilon=\omega^\varepsilon+\varepsilon\Delta d^\varepsilon-u^\varepsilon\cdot\nabla d^\varepsilon$. It follows that $\{\partial_t\nabla d^\varepsilon\}$ is uniformly bounded in $L^2(0,T;H^{-1}(\Omega))+L^\infty(0,T;W^{-(1+\eta),1}(\Omega))$. Then, we infer that ∇d^ε converges to $\nabla d\in C([0,T];w-L^2(\Omega))$ weakly in $L^2(\Omega)$ uniformly in $t\in[0,T]$ for all $T>0$.

In fact we recall that $\{u^\varepsilon\}$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$, then we actually infer that u^ε converges to u strongly in $L^2(0,T;L^2(\Omega))$ by the classical Aubin and Lions Lemma. Thus, denote $\omega=\partial_t d+u\cdot\nabla d$, it follows that $\omega^\varepsilon=\partial_t d^\varepsilon+u^\varepsilon\cdot\nabla d^\varepsilon-\varepsilon\Delta d^\varepsilon$ converges to $\omega=\partial_t d+u\cdot\nabla d$ weakly in $L^2([0,T];L^2(\Omega))$ as $\varepsilon\to 0$. Therefore, by the weak lower semi-continuity of the norms, we obtain that, for every $t>0$,

$$\delta\mathcal{E}(t)+\frac{1}{4}\int_0^t\delta\mathcal{D}(s)e^{\int_s^t\lambda(\sigma)d\sigma}ds\le\liminf_{\varepsilon\to0}\delta\mathcal{E}^\varepsilon(t)+\frac{1}{4}\int_0^t\delta\mathcal{D}^\varepsilon(s)e^{\int_s^t\lambda^\varepsilon(\sigma)d\sigma}ds.$$

Hence, let $\varepsilon\to0$ in (5.10) and use the fact $\int_0^t\left(\frac{\varepsilon}{2}\|\Delta\bar{d}-f(\bar{d})\|_{L^2}^2+C\varepsilon\|\nabla d^\varepsilon\|_{L^2}^2\right)ds\to0$, then the stability inequality (3.3) holds, which concludes the proof. □

Acknowledgements. The first author wishes to thank Dr. Shaojun Tang for providing many useful suggestions. The second author N. J. was supported by a grant from the National Natural Science Foundation of China under contract Nos. 11971360 and 11731008.

REFERENCES

[1] J.-P. Aubin, *Un théoreme de compacité*, C.R. Acad. Sci. Paris, 256:5042–5044, 1963.

[2] D. Arsénio and L. Saint-Raymond, *From the Vlasov–Maxwell–Boltzmann system to incompressible viscous electro-magneto-hydrodynamics*, arXiv e-print, 2016.

[3] F. De Anna and A. Zarnescu, *Global well-posedness and twist-wave solutions for the inertial Qian-Sheng model of liquid crystals*, J. Diff. Eqs., 264:1080–1118, 2018.

[4] C. Cavaterra, E. Rocca, and H. Wu, *Global weak solution and blow-up criterion of the general Ericksen–Leslie system for nematic liquid crystal flows*, J. Diff. Eqs., 255:24–57, 2013.

[5] J.L. Ericksen, *Conservation laws for liquid crystals*, Trans. Soc. Rheology, 5:23–34, 1961.

[6] J.L. Ericksen, *Continuum theory of nematic liquid crystals*, Res. Mechanica, 21:381–392, 1987.

[7] J.L. Ericksen, *Liquid crystals with variable degree of orientation*, Arch. Ration. Mech. Anal., 113:97–120, 1990.

[8] E. Feireisl, *Relative entropies, dissipative solutions, and singular limits of complete fluid systems*, in E. Feireisl, A. Bressan, F. Ancona, P. Marcati and A. Marson (eds.), Hyperbolic Problems: Theory, Numerics, Applications, AIMS on Applied Mathematics, AIMS, Springfield, USA, 8:11–28, 2014.

[9] E. Feireisl, E. Rocca, and G. Schimperna, *On a non-isothermal model for nematic liquid crystals*, Nonlinearity, 24:243–257, 2011.

[10] E. Feireisl, E. Rocca, G. Schimperna, and A. Zarnescu, *On a hyperbolic system arising in liquid crystals modeling*, J. Hyperbolic Diff. Eqs., 15:15–35, 2018.

[11] K. Grunert, H. Holden, and X. Raynaud, *Global dissipative solutions of the two-component Camassa-Holm system for initial data with nonvanishing asymptotics*, Nonlinear Anal. Real World Appl., 17:203–244, 2014.

[12] M.-C. Hong and Z. Xin, *Global existence of solutions of the liquid crystal fow for the Oseen-Frank model in $\mathbb{R}^2$*, Adv. Math., 231:1364–1400, 2012.

[13] N. Jiang and Y.-L. Luo, *On well-posedness of Ericksen-Leslie's hyperbolic incompressible liquid crystal model*, SIAM J. Math. Anal., 51(1):403–434, 2019.

[14] M. Kalousek, *On dissipative solutions to a system arising in viscoelasticity*, J. Math. Fluid Mech., 21:56, 2019.

[15] M. Kalousek and A. Schlomerkemper, *Dissipative solutions to a system for the flow of magneto-viscoelastic materials*, J. Diff. Eqs. 271:1023–1057, 2020

[16] R. Lasarzik, *Weak-strong uniqueness for measure-valued solutions to the Ericksen-Leslie model equipped with the Oseen-Frank free energy*, J. Math. Anal. Appl., 470(1):36–90, 2019.

[17] R. Lasarzik, *Dissipative solution to the Ericksen-Leslie system equipped with the Oseen-Frank energy*, Z. Angew. Math. Phys., 70(1):1–39, 2019.

[18] R. Lasarzik, *Measure-valued solutions to the Ericksen-Leslie model equipped with the Oseen-Frank energy*, Nonlinear Anal., 179:146–183, 2019.

[19] R. Lasarzik, *Approximation and optimal control of dissipative solutions to the Ericksen-Leslie system*, Numer. Funct. Anal. Optim., 40(15):1721–1767, 2019.

[20] F. Leslie, *Theory of flow phenomena in liquid crystals*, Adv. Liq. Cryst., 4:1–81, 1979.

[21] F. Leslie, *Some constitutive equations for liquid crystals*, Arch. Ration. Mech. Anal., 28:265–283, 1968.

[22] F.H. Lin and C. Liu, *Nonparabolic dissipative systems modeling the flow of liquid crystals*, Comm. Pure Appl. Math., 48:501–537, 1995.

[23] F.-H. Lin and C. Liu, *Partial regularity of the dynamic system modeling the flow of liquid crystals*, Discrete Contin. Dyn. Syst., 2:1-22, 1996.

[24] F.-H. Lin and C. Liu, *Existence of solutions for the Ericksen-Leslie system*, Arch. Ration. Mech. Anal., 154:135-156, 2000.

[25] J.-L. Lions, *Équations différentielles opérationnelles et problemes aux limites*, Die Grundlehren der mathematischen Wissenschaften 111, Springer-Verlag, Berlin, 1961.

[26] P.-L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. I. Incompressible Models*, Oxford Lecture Series in Mathematics and its Applications 3, The Clarendon Press Oxford University Press, New York, 1996.

[27] P.-L. Lions, *Compactness in Boltzmann's equation via Fourier integral operators and applications. I, II.*, J. Math. Kyoto Univ., 34:391-427, 429-461, 1994.

[28] J. Li, E.S. Titi, and Z. Xin, *On the uniqueness of weak solutions to the Ericksen-Leslie liquid crystal model in $\mathbb{R}^2$*, Math. Models Meth. Appl. Sci., 26:803-822, 2016.

[29] D.A. Vorotnikov, *Dissipative solutions for equations of viscoelastic diffusion in polymers*, J. Math. Anal. Appl., 339:876–888, 2008.

[30] W. Wang, P. Zhang, and Z. Zhang, *Well-posedness of the Ericksen-Leslie system*, Arch. Ration. Mech. Anal., 210:837–855, 2013.

[31] M. Wang and W. Wang, *Global existence of weak solution for the 2-D Ericksen-Leslie system*, Calc. Var. Part. Diff. Eqs., 51:915–962, 2014.