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A SPIN-WAVE SOLUTION TO THE LANDAU-LIFSHITZ-GILBERT EQUATION*

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Abstract. Magnetic materials possess the intrinsic spin order, whose disturbance leads to spin waves. From the mathematical perspective, a spin wave is known as a traveling wave, which is often seen in wave and transport equations. The dynamics of intrinsic spin order is modeled by the Landau-Lifshitz-Gilbert equation, a nonlinear parabolic system of equations with a pointwise length constraint. In this paper, a spin wave for this equation is obtained based on the assumption that the spin wave maintains its periodicity in space when propagating at a varying velocity. In the absence of magnetic field, an explicit form of spin wave is provided. When a magnetic field is applied, the spin wave does not have such an explicit form but its stability is justified rigorously. Moreover, an approximate explicit solution is constructed with approximation error depending quadratically on the strength of magnetic field and being uniform in time.

Keywords. Landau-Lifshitz-Gilbert equation; spin wave; asympotic analysis.

AMS subject classifications. 2010 34E10; 35B40; 35C07; 35C20; 35K55.

1. Introduction

Magnetic materials are commonly used for recording and data storage due to bistable structures of their intrinsic spin order. A spin wave is the disturbance of spin order and is usually excited using magnetic fields. This offers unique properties such as charge-less propagation and high group velocities, which are important for signal transformations and magnetic logic applications [2,8,9,12,15].

The dynamics of intrinsic spin order, also known as magnetization, is modeled by the Landau-Lifshitz-Gilbert (LLG) equation [4,11] in the dimensionless form

$$\mathbf{m}_t = -\mathbf{m}\times\mathbf{h} - \alpha\mathbf{m}\times(\mathbf{m}\times\mathbf{h}), \tag{1.1}$$

where the magnetization $\mathbf{m}=(m_1,m_2,m_3)^T$ is a 3D vector with unit length and $\alpha>0$ is the Gilbert damping parameter. The effective field $\mathbf{h}$ includes the exchange term, the anisotropy term with easy axis along the x-axis and the external field applied along the x-axis

$$\mathbf{h} = \Delta\mathbf{m} + qm_1\mathbf{e}_1 + h_{\text{ext}}\mathbf{e}_1. \tag{1.2}$$

Here q is the anisotropy constant and h_{ext} is the strength of the external field. The stray field term is neglected for simplicity since recent experiments have focused on magnetic thin films [2,12,15] and the stray field can be simplified [5].

There exist several attempts to find explicit solutions of (1.1)-(1.2) under different assumptions [1,3,6,14]. These results provide a rich understanding of magnetization dynamics. In particular, the Walker's solution [14] predicts that a domain wall moves at a constant velocity proportional to h_{ext}, which was verified in several experiments [7].

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However, the Walker's solution fails to interpret the experimental result when h_{ext} is larger than some critical value. Therefore, the Walker's solution is only valid under small external field. This refers to as the Walker's ansatz.

In this paper, we construct a spin wave to (1.1)-(1.2) based on the following assumption:

ASSUMPTION 1.1. *A spin-wave solution to* (1.1)-(1.2) *takes the form of*

$$\boldsymbol{m}(\boldsymbol{x},t)=\begin{pmatrix}\cos\theta(t)\\ \sin\theta(t)\cos(\boldsymbol{w}_0\cdot\boldsymbol{x}+\varphi(t))\\ \sin\theta(t)\sin(\boldsymbol{w}_0\cdot\boldsymbol{x}+\varphi(t))\end{pmatrix}. \tag{1.3}$$

Equation (1.3) describes a spin-wave structure in spherical coordinates since $|\mathbf{m}(\mathbf{x},t)|=1$, $\forall x,t$. It has fixed periodicity in space and varying propagation velocity. θ_0, φ_0, and $\mathbf{w}_0$ are specified by initial conditions, and we assume $0\le\theta_0\le\pi$ and $\varphi_0=0$ without loss of generality. We provide the motivation of (1.3) based on the method of characteristic lines in Section 2. Although it does not have a wall structure, one can show that it validates the Walker's ansatz under some conditions.

Throughout the paper, we denote $\mathbf{m}(\theta,\varphi)$ the spin-wave solution to (1.1)-(1.2) and $\mathbf{m}^*(\theta^*,\varphi^*)$ the spin-wave solution to (1.1)-(1.2) when $h_{\text{ext}}=0$, respectively. We then have the following explicit construction of $\mathbf{m}^*(\theta^*,\varphi^*)$:

THEOREM 1.1. *When $h_{ext}=0$,* (1.1)-(1.2) *admits a spin-wave solution of the form*

$$\boldsymbol{m}^*=\boldsymbol{m}^*(\theta^*,\varphi^*)=\begin{pmatrix}\cos\theta^*(t)\\ \sin\theta^*(t)\cos(\boldsymbol{w}_0\cdot\boldsymbol{x}+\varphi^*(t))\\ \sin\theta^*(t)\sin(\boldsymbol{w}_0\cdot\boldsymbol{x}+\varphi^*(t))\end{pmatrix},$$

and (i) if initially $0<\theta_0<\pi$ and $\theta_0\neq\frac{\pi}{2}$,

$$\tan\theta^*=C_1e^{-\alpha(|\boldsymbol{w}_0|^2+q)t},\quad \varphi^*=\frac{1}{\alpha}\ln\left|\cot\frac{\theta^*}{2}\right|+\frac{C_2}{\alpha} \tag{1.4}$$

with $C_1=\tan\theta_0$ and $C_2=-\frac{1}{2}\ln\left(\frac{1+\cos\theta_0}{1-\cos\theta_0}\right)$;

(ii) if initially $\theta_0=0$, $\frac{\pi}{2}$ or π, then θ and φ are independent of time, and m^* takes the value of

$$\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \begin{pmatrix}0\\ \cos(\boldsymbol{w}_0\cdot\boldsymbol{x})\\ \sin(\boldsymbol{w}_0\cdot\boldsymbol{x})\end{pmatrix},\quad \begin{pmatrix}-1\\0\\0\end{pmatrix}, \tag{1.5}$$

respectively.

Furthermore, it holds that

$$\lim_{\alpha\to0^+}\boldsymbol{m}^*=\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\cos\left(\boldsymbol{w}_0\cdot\boldsymbol{x}+\cos\theta_0\left(|\boldsymbol{w}_0|^2+q\right)t\right)\\ \sin\theta_0\sin\left(\boldsymbol{w}_0\cdot\boldsymbol{x}+\cos\theta_0\left(|\boldsymbol{w}_0|^2+q\right)t\right)\end{pmatrix}, \tag{1.6}$$

where the right-hand side of (1.6) *is the solution to* (1.1)-(1.2) *when $h_{ext}=0$ and $\alpha=0$.*

The proof of Theorem 1.1 is given in Section 3.

When $h_{\text{ext}} \neq 0$, such an explicit construction is not available, though the global existence of spin-wave solutions can be proved. Instead, we prove the stability of the spin-wave solution. Furthermore, we construct an explicit approximation using the asymptotic expansion and give some uniform-in-time error estimates of this approximation.

THEOREM 1.2. *When $0 \le \theta_0 < \frac{\pi}{2}$ for arbitrary $h_{ext} > 0$, or $\frac{\pi}{2} < \theta_0 \le \pi$ for $0 < h_{ext} < -(|\boldsymbol{w}_0|^2 + q)\cos\theta_0$, the spin-wave solution $\boldsymbol{m}(\theta,\varphi)$ to (1.1)-(1.2) satisfies*

$$0 \le \theta^*(t) - \theta(t) \le \alpha C_1 C_3 t e^{-\alpha(|\boldsymbol{w}_0|^2+q)t} h_{ext}, \tag{1.7}$$

$$t h_{ext} \le \varphi(t) - \varphi^*(t) \le \frac{1}{2} C_1^2 |C_3| h_{ext} + t h_{ext}, \tag{1.8}$$

where $C_1 = \tan\theta_0$ and $C_3 = \frac{1}{\cos\theta_0}$.

When $\theta_0 = \pi/2$, the spin-wave solution satisfies

$$0 \le \theta^*(t) - \theta(t) \le \frac{1}{|\boldsymbol{w}_0|^2 + q}(e^{\alpha(|\boldsymbol{w}_0|^2+q)t} - 1) h_{ext}, \tag{1.9}$$

$$t h_{ext} \le \varphi(t) - \varphi^*(t) \le \frac{1}{\alpha(|\boldsymbol{w}_0|^2 + q)}(e^{\alpha(|\boldsymbol{w}_0|^2+q)t} - 1) h_{ext}, \tag{1.10}$$

where $\theta^*(t) = \frac{\pi}{2}$, and $\varphi^* = 0$.

Moreover, for the approximate solution

$$\begin{cases} \hat{\theta} = \theta^* + (|\boldsymbol{w}_0|^2 + q)^{-1}(\sin\theta^* - (\alpha\varphi^* + C_3)\sin\theta^*\cos\theta^*) h_{ext}, \\ \hat{\varphi} = \varphi^* + \alpha^{-1}(|\boldsymbol{w}_0|^2 + q)^{-1}(\alpha\varphi^*\cos\theta^* + C_3\cos\theta^* - 1) h_{ext}, \end{cases}$$

it holds that, under the assumption $0 \le \theta_0 < \frac{\pi}{2}$ for arbitrary $h_{ext} > 0$, or $\frac{\pi}{2} \le \theta_0 \le \pi$ for $0 < h_{ext} < -(|\boldsymbol{w}_0|^2 + q)\cos\theta_0$,

$$|\hat{\theta} - \theta| \le C t e^{-\alpha(|\boldsymbol{w}_0|^2+q)t} h_{ext}^2, \tag{1.11}$$

$$|\hat{\varphi} - \varphi| \le C h_{ext}^2, \tag{1.12}$$

where C is a constant independent of t and h_{ext}.

Estimates (1.7)-(1.10) imply that the spin-wave solution to (1.1)-(1.2) is stable under magnetic field, and estimates (1.11)-(1.12) show that the approximate solution is second-order accurate in terms of magnetic field and the error is uniform in time. The proof of Theorem 1.2 is given in Section 4.

2. Motivation of Assumption 1.1

In this section, we shall elaborate the origin of Assumption 1.1, which is inspired by the method of characteristics in simple situations.

Firstly, when $\alpha = q = h_{\text{ext}} = 0$, one can check that

$$\mathbf{m}(x,t) = \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \cos\left(\frac{c}{\cos\theta_0}(x+ct)\right) \\ \sin\theta_0 \sin\left(\frac{c}{\cos\theta_0}(x+ct)\right) \end{pmatrix} \tag{2.1}$$

solves $\mathbf{m}_t = -\mathbf{m} \times \mathbf{m}_{xx}$. Equation (2.1) is derived by the method of characteristic lines; see Chapter 2 of [6] for details. It provides a spin-wave solution with the traveling speed c.

Secondly, a generalization of the above method yields a spin wave to $\mathbf{m}_t=-\mathbf{m}\times\Delta\mathbf{m}$ in the 3D case as

$$\mathbf{m}(\mathbf{x},t)=\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\cos\frac{v}{\cos\theta_0}\\ \sin\theta_0\sin\frac{v}{\cos\theta_0}\end{pmatrix}, \tag{2.2}$$

where $v=\mathbf{c}\cdot\mathbf{x}+(\mathbf{c}\cdot\mathbf{c})t$ with $\mathbf{c}=(c_1,c_2,c_3)^T$ a given velocity field. In fact, (2.2) can be rewritten as

$$\mathbf{m}(\mathbf{x},t)=\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\cos(\mathbf{w}_0\cdot\mathbf{x}+\varphi(t))\\ \sin\theta_0\sin(\mathbf{w}_0\cdot\mathbf{x}+\varphi(t))\end{pmatrix}, \tag{2.3}$$

where $\mathbf{w}_0=\mathbf{c}/\cos\theta_0$ and $\varphi(t)=\left(|\mathbf{c}|^2/\cos\theta_0\right)t$.

Finally, we generalize the spin-wave profile in (2.3) to take Gilbert damping and other terms in (1.2) into account. This leads to Assumption 1.1 where θ and φ are functions of t.

Substituting the ansatz (1.3) into (1.1)-(1.2), one can get the following ordinary differential equations for θ and φ

$$\begin{cases}\theta_t=-\alpha(|\mathbf{w}_0|^2+q)\sin\theta\cos\theta-\alpha h_{\text{ext}}\sin\theta\\ \varphi_t=(|\mathbf{w}_0|^2+q)\cos\theta+h_{\text{ext}}\end{cases} \tag{2.4}$$

with initial conditions $\theta(0)=\theta_0$ and $\varphi(0)=0$.

3. Explicit solution without magnetic field

In this section, we will give a detailed derivation of Theorem 1.1 in the absence of magnetic field, i.e., $h_{\text{ext}}=0$.

Proof. **(Proof of Theorem 1.1.)** When $h_{\text{ext}}=0$, (2.4) reduces to

$$\begin{cases}\theta_t=-\alpha(|\mathbf{w}_0|^2+q)\sin\theta\cos\theta,\\ \varphi_t=(|\mathbf{w}_0|^2+q)\cos\theta.\end{cases} \tag{3.1}$$

It is easy to check that

$$\theta_1(t)=0,\quad \theta_2(t)=\frac{\pi}{2},\quad \theta_3(t)=\pi$$

are three special solutions to the first equation of (3.1). Hence, by the existence and uniqueness theorem for ODEs, for every $\theta_0\in[0,\pi]$, (3.1) admits one unique global solution (θ^*,φ^*). In fact, solving the first equation in (3.1) by separation of variables, one has

$$\alpha(|\mathbf{w}_0|^2+q)t=\ln\cot\theta^*+\ln C_1,$$

where $C_1=\tan\theta_0$. Equivalently,

$$\tan\theta^*=C_1e^{-\alpha(|\mathbf{w}_0|^2+q)t}. \tag{3.2}$$

As for φ, it follows from (3.1) that

$$\frac{\mathrm{d}\varphi^*}{\mathrm{d}\theta^*}=\frac{\mathrm{d}\varphi^*}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}\theta^*}=-\frac{1}{\alpha\sin\theta^*}.$$

Hence,

$$\alpha\varphi^* = \frac{1}{2}\ln\left(\frac{1+\cos\theta^*}{1-\cos\theta^*}\right) + C_2 = \ln\left|\cot\left(\frac{\theta^*}{2}\right)\right| + C_2, \tag{3.3}$$

where $C_2 = -\frac{1}{2}\ln\left(\frac{1+\cos\theta_0}{1-\cos\theta_0}\right)$.

Next, consider the limit of $\mathbf{m}^*$ when α goes to 0^+. When $\theta_0 \neq \frac{\pi}{2}$, for every fixed $t>0$, it follows from (3.2) that

$$\lim_{\alpha\to 0^+} \tan\theta^*(t) = \tan\theta_0,$$

which implies

$$\lim_{\alpha\to 0^+} \theta^*(t) = \theta_0. \tag{3.4}$$

On the other hand, by L'Hospital's rule,

$$\begin{aligned}\lim_{\alpha\to 0^+} \varphi^*(t) &= \lim_{\alpha\to 0^+} \tfrac{d}{d\alpha}\left[\ln\left|\cot\tfrac{\theta^*}{2}\right| + C_2\right] \\ &= \lim_{\alpha\to 0^+} \tfrac{-1}{\sin\theta^*(t)}\tfrac{d\theta^*}{d\alpha}.\end{aligned}$$

It follows from (3.2) that

$$\frac{d\theta^*}{d\alpha} = -\sin\theta^*\cos\theta^*(|\mathbf{w}_0|^2 + q)t.$$

Hence,

$$\lim_{\alpha\to 0^+} \varphi^*(t) = \lim_{\alpha\to 0^+} \cos\theta^*(|\mathbf{w}_0|^2 + q)t = \cos\theta_0(|\mathbf{w}_0|^2 + q)t. \tag{3.5}$$

When $\theta_0 = \frac{\pi}{2}$, it is obvious that both (3.4) and (3.5) hold.

Substituting (3.4) and (3.5) into (1.3), we have

$$\lim_{\alpha\to 0^+} \mathbf{m}^* = \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0\cos\left(\mathbf{w}_0\cdot\mathbf{x} + \cos\theta_0\left(|\mathbf{w}_0|^2 + q\right)t\right) \\ \sin\theta_0\sin\left(\mathbf{w}_0\cdot\mathbf{x} + \cos\theta_0\left(|\mathbf{w}_0|^2 + q\right)t\right) \end{pmatrix}, \tag{3.6}$$

where the right-hand side is exactly the solution to (1.1)-(1.2) when $h_{\text{ext}} = 0$ and $\alpha = 0$. This completes the proof. ∎

It is easy to check that when $h_{\text{ext}} = 0$ and $\alpha = 0$, the spin wave propagates at a constant velocity (see the right-hand side of (3.6)). The increment of velocity field is $q\cos\theta_0\frac{\mathbf{w}_0}{|\mathbf{w}_0|^2}$ with magnitude $\frac{|q\cos\theta_0|}{|\mathbf{w}_0|}$, due to the magnetic anisotropy.

In [10], the authors used the stereographic projection and observed that the effect of Gilbert damping was only a rescaling of time by a complex constant. However, this was latter found to be valid only for a single spin in a constant magnetic field [13]. Our derivation provides an explicit characterization of magnetization dynamics in the presence of Gilbert damping as follows.

REMARK 3.1. The formula (1.4) reveals that, as $t\to+\infty$, the spin wave converges to $(1,0,0)^T$ (the easy-axis direction) and $(-1,0,0)^T$ (the easy-axis direction) exponentially fast if $0 \le \theta_0 < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta_0 \le \pi$, respectively. The rate of convergence is proportional to the damping parameter α. When $\theta_0 = \frac{\pi}{2}$, Gilbert damping does not have any influence on magnetization dynamics, which is shown by (1.5). In addition, when $\alpha\to 0^+$, the spin wave recovers its profile without damping; see (1.6).

4. Stability and approximate solution with magnetic field

In this section, we will discuss the case when the magnetic field is applied. The stability of spin wave (1.3) with respect to h_{ext} is proved. Although it is difficult to get the explicit solution of (2.4), an approximate solution can be obtained using the asymptotic expansion. Errors of approximate solution are proved to be second order in terms of h_{ext} and uniform in time.

4.1. Stability under the magnetic field

Proof. **(Proof of** (1.7)-(1.10) **in Theorem 1.2.)** It is easy to check that

$$\theta_1(t)=0, \quad \theta_2(t)=\pi$$

are two special solutions of (2.4). The existence and uniqueness theorem for ODEs imply that (2.4) admits one unique global solution (θ,φ).

Recall that (θ,φ) and (θ^*,φ^*) are solutions of (2.4) and (3.1), respectively. Then

$$\begin{cases} \theta_t^* = -\alpha(|\mathbf{w}_0|^2+q)\sin\theta^*\cos\theta^*, \\ \theta_t = -\alpha(|\mathbf{w}_0|^2+q)\sin\theta\cos\theta - \alpha h_{\text{ext}}\sin\theta, \end{cases}$$

with $\theta^*(0)=\theta(0)=\theta_0$.

When $0\le\theta_0<\frac{\pi}{2}$, define $e=e(t)$ as

$$\theta(t-e(t))=\theta^*(t). \tag{4.1}$$

Obviously, $e(0)=0$. Taking the derivative of (4.1) with respect to t, one gets that

$$\begin{aligned} &(1-e')[\alpha(|\mathbf{w}_0|^2+q)\sin\theta(t-e)\cos\theta(t-e)+\alpha h_{\text{ext}}\sin\theta(t-e)] \\ =&\alpha(|\mathbf{w}_0|^2+q)\sin\theta^*(t)\cos\theta^*(t). \end{aligned}$$

This together with (4.1) implies

$$e'(t)=\frac{h_{\text{ext}}}{(|\mathbf{w}_0|^2+q)\cos\theta^*+h_{\text{ext}}}.$$

Since $\theta_2(t)=\frac{\pi}{2}$ is a special solution of the first equation of (3.1), it holds that $0\le\theta^*(t)\le\frac{\pi}{2}$ when $0\le\theta_0\le\frac{\pi}{2}$. Hence, θ^* is non-increasing and consequently,

$$e(t)=\int_0^t \frac{h_{ext}}{(|\mathbf{w}_0|^2+q)\cos\theta^*(s)+h_{ext}}ds \le \frac{h_{ext}}{(|\mathbf{w}_0|^2+q)\cos\theta_0+h_{\text{ext}}}t. \tag{4.2}$$

Similarly, when $0\le\theta_0\le\frac{\pi}{2}$, $\theta(t)$ is non-increasing as well. Using the mean-value theorem and (2.4), one gets

$$\begin{aligned} 0\le\theta^*(t)-\theta(t)&=\theta(t-e)-\theta(t)=-\theta_t(t_\xi)e(t) \\ &\le\alpha\sin\theta(t_\xi)(|\mathbf{w}_0|^2+q+h_{\text{ext}})e(t), \end{aligned} \tag{4.3}$$

where $t-e\le t_\xi\le t$. Noting that $\theta(t_\xi)\le\theta(t-e)=\theta^*(t)$ and substituting (4.2) into (4.3), one has

$$\theta^*(t)-\theta(t)\le\alpha t\sin\theta^*(t)\frac{|\mathbf{w}_0|^2+q+h_{\text{ext}}}{(|\mathbf{w}_0|^2+q)\cos\theta_0+h_{\text{ext}}}h_{\text{ext}}. \tag{4.4}$$

Using the fact $\sin\theta^*(t)\le\tan\theta^*(t)$ and (3.2), (4.4) yields

$$0\le\theta^*(t)-\theta(t)\le\alpha\frac{\tan\theta_0}{\cos\theta_0}h_{\text{ext}}\cdot te^{-\alpha(|\mathbf{w}_0|^2+q)t}. \tag{4.5}$$

When $\frac{\pi}{2}<\theta_0\le\pi$, define $e=e(t)$ as

$$\theta(t+e(t))=\theta^*(t).$$

Taking the derivative, one gets

$$e'(t)=-\frac{h_{\text{ext}}}{(|\mathbf{w}_0|^2+q)\cos\theta^*+h_{\text{ext}}}.$$

If we take

$$h_{\text{ext}}<-(|\mathbf{w}_0|^2+q)\cos\theta_0, \tag{4.6}$$

then $\theta=\pi$ and $(|\mathbf{w}_0|^2+q)\cos\theta+h_{\text{ext}}=0$ are two steady solutions of (2.4), and it follows that

$$e(t)\le-\frac{h_{ext}}{(|\mathbf{w}_0|^2+q)\cos\theta_0+h_{ext}}t,$$

since $\theta^*(t)$ is non-decreasing.

On the other hand, it follows from (4.6) and (2.4) that $\theta(t)$ is increasing. Hence one has

$$\begin{aligned}0\le\theta^*(t)-\theta(t)&=\theta(t+e(t))-\theta(t)=\theta_t(t_\xi)e(t)\\&\le-\alpha t\sin\theta^*(t)\frac{|\mathbf{w}_0|^2+q-h_{\text{ext}}}{(|\mathbf{w}_0|^2+q)\cos\theta_0+h_{\text{ext}}}h_{\text{ext}},\end{aligned}$$

which leads to

$$0\le\theta^*(t)-\theta(t)\le\alpha\frac{\tan\theta_0}{\cos\theta_0}h_{\text{ext}}\cdot te^{-\alpha(|\mathbf{w}_0|^2+q)t}. \tag{4.7}$$

When $\theta_0=\pi/2$, it follows from (1.5) that $\theta^*=\pi/2$. We use first-order Taylor expansion on the right-hand side of (2.4) at point $\pi/2$ with respect to the variable θ, one can deduce

$$\begin{aligned}(\frac{\pi}{2}-\theta)_t=&\alpha(|\mathbf{w}_0|^2+q)\sin\frac{\pi}{2}\cos\frac{\pi}{2}+\alpha h_{\text{ext}}\sin\frac{\pi}{2}\\&+\alpha(|\mathbf{w}_0|^2+q)(1-2\sin^2\theta_\xi)(\frac{\pi}{2}-\theta)-\alpha h_{\text{ext}}\cos\theta_\xi(\frac{\pi}{2}-\theta),\end{aligned}$$

where $\theta\le\theta_\xi\le\pi/2$. Noting the fact $0\le\theta\le\pi/2$, it follows that

$$(\frac{\pi}{2}-\theta)_t\le\alpha(|\mathbf{w}_0|^2+q)(\frac{\pi}{2}-\theta)+\alpha h_{\text{ext}}.$$

Therefore Grönwall's inequality implies the stability

$$0\le\frac{\pi}{2}-\theta\le\frac{1}{|\mathbf{w}_0|^2+q}(e^{\alpha(|\mathbf{w}_0|^2+q)t}-1)h_{\text{ext}}. \tag{4.8}$$

As for φ and φ^*, we have

$$\begin{cases} \varphi_t^* = (|\mathbf{w}_0|^2+q)\cos\theta^* \\ \varphi_t = (|\mathbf{w}_0|^2+q)\cos\theta + h_{\text{ext}} \end{cases}$$

with $\varphi(0)=\varphi^*(0)=0$. Subtracting these two equations, one has

$$\varphi_t - \varphi_t^* = (|\mathbf{w}_0|^2+q)(\cos\theta - \cos\theta^*) + h_{\text{ext}}. \tag{4.9}$$

If $\theta_0 \neq \pi/2$, applying (4.5), (3.2), and (4.7), (4.9) leads to

$$\begin{aligned} h_{\text{ext}} \le \varphi_t - \varphi_t^* &\le (|\mathbf{w}_0|^2+q)(\theta^*-\theta)\sin\theta^* + h_{\text{ext}} \\ &\le \alpha t(|\mathbf{w}_0|^2+q)\frac{\tan^2\theta_0}{|\cos\theta_0|}e^{-2\alpha(|\mathbf{w}_0|^2+q)t}h_{\text{ext}} + h_{\text{ext}}. \end{aligned}$$

Integrating the above equation over $[0,t]$ yields

$$th_{\text{ext}} \le \varphi(t) - \varphi^*(t) \le \frac{1}{2}\frac{\tan^2\theta_0}{|\cos\theta_0|}h_{\text{ext}} + th_{\text{ext}}. \tag{4.10}$$

If $\theta_0 = \pi/2$, applying (4.8), (4.9) leads to

$$\begin{aligned} h_{\text{ext}} \le \varphi_t - \varphi_t^* &\le (|\mathbf{w}_0|^2+q)(\theta^*-\theta)\sin\theta^* + h_{\text{ext}} \\ &\le e^{\alpha(|\mathbf{w}_0|^2+q)t}h_{\text{ext}}. \end{aligned}$$

Integrating the above equation over $[0,t]$ produces

$$th_{\text{ext}} \le \varphi(t) - \varphi^*(t) \le \frac{1}{\alpha(|\mathbf{w}_0|^2+q)}(e^{\alpha(|\mathbf{w}_0|^2+q)t} - 1)h_{\text{ext}}.$$

Proof of (1.7)-(1.10) is complete. □

Above stability results of θ and θ^* are illustrated in Figure 4.1. When $0 \le \theta_0 < \pi/2$, both θ and θ^* converge exponentially fast to 0, and the difference between them is controlled by h_{ext}. When $\theta_0 = \pi/2$, θ^* does not change, while θ decreases to 0. When $\pi/2 < \theta_0 \le \pi$, θ^* converges exponentially fast to π, but the behavior of θ depends on h_{ext}. Let $h_b = -(|\mathbf{w}_0|^2+q)\cos\theta_0 > 0$. θ converges to π if $h_{\text{ext}} < h_b$; θ does not change if $h_{\text{ext}} = h_b$; θ decreases to 0 if $h_{\text{ext}} > h_b$.

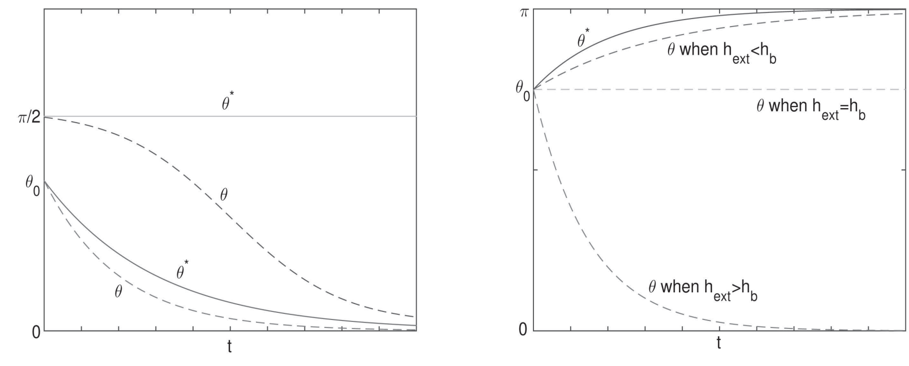
Equations (4.5) and (4.7) show that $\theta - \theta^*$ converges exponentially fast to 0, when $\theta_0 \neq \frac{\pi}{2}$. Therefore, in the long time, (4.9) shows that the increment of velocity field due to the magnetic field is $h_{\text{ext}}\frac{\mathbf{w}_0}{|\mathbf{w}_0|^2}$ with magnitude $\frac{|h_{\text{ext}}|}{|\mathbf{w}_0|}$. However, when $\frac{\pi}{2} < \theta_0 \le \pi$ and h_{ext} is large enough to break the condition (4.6), the above result does not hold any more. This validates the Walker's ansatz for the spin wave.

4.2. The approximate solution

Proof. **(Proof of** (1.11)-(1.12) **in Theorem 1.2.)** We use the method of asymptotic expansion to construct an approximate solution of (2.4) and prove error estimates. Recall that (θ,φ) and (θ^*,φ^*) are solutions of (2.4) and (3.1), respectively.

Under the assumption that h_{ext} is small, θ and φ admit the following formal expansions

$$\begin{cases} \theta(t) = \theta^0(t) + \theta^1(t)h_{\text{ext}} + \theta^2(t)h_{\text{ext}}^2 + \cdots, \\ \varphi(t) = \varphi^0(t) + \varphi^1(t)h_{\text{ext}} + \varphi^2(t)h_{\text{ext}}^2 + \cdots. \end{cases} \tag{4.11}$$



(a) $0 \leq \theta_0 \leq \pi/2$ (b) $\pi/2 < \theta_0 \leq \pi$

Fig. 4.1. *Stability diagram. (a) When $0 \leq \theta_0 < \pi/2$, θ and θ^* both converge exponentially fast to 0, and the difference between them is controlled by h_{ext}; When $\theta_0 = \pi/2$, θ^* does not change, while θ decreases to 0. (b) When $\pi/2 < \theta_0 \leq \pi$, θ^* converges exponentially fast to π, but the behavior of θ depends on h_{ext}. Let $0 < h_b = -(|\boldsymbol{w}_0|^2 + q)\cos\theta_0$. θ converges to π if $h_{ext} < h_b$; θ does not change if $h_{ext} = h_b$; θ decreases to 0 if $h_{ext} > h_b$.*

Substituting (4.11) into (2.4), for zero-order terms, one has

$$\begin{cases} \theta_t^0 = -\alpha(|\mathbf{w}_0|^2 + q)\sin\theta^0\cos\theta^0 \\ \varphi_t^0 = (|\mathbf{w}_0|^2 + q)\cos\theta^0 \end{cases} \tag{4.12}$$

with $\theta^0(0) = \theta_0$ and $\varphi^0(0) = 0$. Obviously,

$$\begin{cases} \theta^0(t) = \theta^*(t), \\ \varphi^0(t) = \varphi^*(t). \end{cases} \tag{4.13}$$

As for the first-order terms, one has

$$\begin{cases} \theta_t^1 = -\alpha(|\mathbf{w}_0|^2 + q)\theta^1\cos 2\theta^0 - \alpha\sin\theta^0 \\ \varphi_t^1 = -(|\mathbf{w}_0|^2 + q)\theta^1\sin\theta^0 + 1 \end{cases} \tag{4.14}$$

with $\theta^1(0) = \varphi^1(0) = 0$.

By the method of constant variation, one can assume

$$\theta^1 = \frac{C(t)}{C_1 e^{-\alpha(|\mathbf{w}_0|^2+q)t} + C_1^{-1} e^{\alpha(|\mathbf{w}_0|^2+q)t}}.$$

It follows that

$$C(t) = \int -\alpha(\tan\theta^* + \tan^{-1}\theta^*)\sin\theta^* \mathrm{d}t,$$

and by using (3.1)

$$\int -\alpha\tan\theta^*\sin\theta^*\mathrm{d}t = \int (|\mathbf{w}_0|^2 + q)^{-1}\frac{\sin\theta^*}{\cos^2\theta^*}\mathrm{d}\theta^*$$

$$=(|\mathbf{w}_0|^2+q)^{-1}\frac{1}{\cos\theta^*}+C,$$

$$\int -\alpha\tan^{-1}\theta^*\sin\theta^*\mathrm{d}t=-\alpha(|\mathbf{w}_0|^2+q)^{-1}\varphi^*+C.$$

Therefore we obtain $C(t)=(|\mathbf{w}_0|^2+q)^{-1}(\frac{1}{\cos\theta^*}-\alpha\varphi^*-C_3)$ with $C_3=\cos^{-1}\theta_0$, and

$$\theta^1=(|\mathbf{w}_0|^2+q)^{-1}(\sin\theta^*-(\alpha\varphi^*+C_3)\sin\theta^*\cos\theta^*). \tag{4.15}$$

As for φ^1, substituting (4.15) into the second equation of (4.14), one has

$$\varphi^1=t-\int \sin^2\theta^*-(\alpha\varphi^*+C_3)\sin^2\theta^*\cos\theta^*\mathrm{d}t,$$

and

$$\int(\alpha\varphi^*+C_3)\sin^2\theta^*\cos\theta^*\mathrm{d}t=-\frac{1}{\alpha(|\mathbf{w}_0|^2+q)}\int(\alpha\varphi^*+C_3)\sin\theta^*\mathrm{d}\theta^*$$
$$=\frac{1}{\alpha(|\mathbf{w}_0|^2+q)}(\alpha\varphi^*+C_3)\cos\theta^*-\int\cos^2\theta^*\mathrm{d}t.$$

It follows that

$$\varphi^1=\frac{1}{\alpha(|\mathbf{w}_0|^2+q)}(\alpha\varphi^*\cos\theta^*+C_3\cos\theta^*-1). \tag{4.16}$$

Hence, we get an approximate solution

$$\begin{cases}\hat{\theta}=\theta^0+\theta^1h_{ext}=\theta^*+(|\mathbf{w}_0|^2+q)^{-1}(\sin\theta^*-(\alpha\varphi^*+C_3)\sin\theta^*\cos\theta^*)h_{\mathrm{ext}},\\ \hat{\varphi}=\varphi^0+\varphi^1h_{ext}=\varphi^*+\alpha^{-1}(|\mathbf{w}_0|^2+q)^{-1}(\alpha\varphi^*\cos\theta^*+C_3\cos\theta^*-1)h_{\mathrm{ext}}.\end{cases}$$

Next, let us give error estimates for the above approximate solution. It follows from (4.12) and (4.14) that

$$\theta_t^0+\theta_t^1h_{\mathrm{ext}}=-\alpha(|\mathbf{w}_0|^2+q)\sin\theta^*\cos\theta^*$$
$$-\alpha(|\mathbf{w}_0|^2+q)h_{\mathrm{ext}}\theta^1\cos2\theta^*-\alpha h_{\mathrm{ext}}\sin\theta^*. \tag{4.17}$$

Denote $\omega=\hat{\theta}-\theta$. Subtracting (4.17) from the first equation of (2.4) and using the mean value theorem, one has

$$\omega_t=\frac{1}{2}\alpha(|\mathbf{w}_0|^2+q)(\sin2\theta-\sin2\theta^*)-\alpha(|\mathbf{w}_0|^2+q)h_{\mathrm{ext}}\theta^1\cos2\theta^*+\alpha h_{\mathrm{ext}}(\sin\theta-\sin\theta^*)$$
$$=\alpha(|\mathbf{w}_0|^2+q)\cos2\theta_\xi(\theta-\theta^*)-\alpha(|\mathbf{w}_0|^2+q)h_{\mathrm{ext}}\theta^1\cos2\theta^*+\alpha h_{\mathrm{ext}}\cos\theta_\xi(\theta-\theta^*)$$

with $\theta\le\theta_\xi\le\theta^*$. Noticing that $\theta-\theta^*=-\omega+\theta^1h_{\mathrm{ext}}$ and using the mean value theorem again, one gets

$$\omega_t=-\alpha(|\mathbf{w}_0|^2+q)\cos2\theta_\xi\omega+\alpha(|\mathbf{w}_0|^2+q)h_{\mathrm{ext}}\theta^1(\cos2\theta_\xi-\cos2\theta^*)+\alpha h_{\mathrm{ext}}\cos\theta_\xi(\theta-\theta^*)$$
$$=-\alpha(|\mathbf{w}_0|^2+q)\cos2\theta_\xi\omega-2\alpha(|\mathbf{w}_0|^2+q)h_{\mathrm{ext}}\theta^1\sin2\theta_{\xi^*}(\theta_\xi-\theta^*)+\alpha h_{\mathrm{ext}}\cos\theta_\xi(\theta-\theta^*)$$
$$=a(t)\omega+b(t),$$

where $\theta_\xi\le\theta_{\xi^*}\le\theta^*$, $a(t)=-\alpha(|\mathbf{w}_0|^2+q)\cos2\theta_\xi$, and

$$b(t)=-2\alpha(|\mathbf{w}_0|^2+q)h_{\mathrm{ext}}\theta^1\sin2\theta_{\xi^*}(\theta_\xi-\theta^*)+\alpha h_{\mathrm{ext}}\cos\theta_\xi(\theta-\theta^*).$$

Using Grönwall's inequality, it follows that

$$\omega = \int_0^t e^{\int_s^t a(r)\mathrm{d}r} b(s)\mathrm{d}s. \tag{4.18}$$

On the other hand, it is easy to check that

$$\begin{aligned}\int_s^t a(r)\mathrm{d}r = &\int_s^t -\alpha(|\mathbf{w}_0|^2+q)(1-2\sin^2\theta_\xi)\mathrm{d}r \\ = &-\alpha(|\mathbf{w}_0|^2+q)(t-s-2\int_s^t \sin^2\theta_\xi \mathrm{d}r),\end{aligned} \tag{4.19}$$

and

$$|b(s)| \le Ch_{\text{ext}}(\theta^* - \theta) \le Ch_{\text{ext}}^2 e^{-\alpha(|\mathbf{w}_0|^2+q)s}. \tag{4.20}$$

From (4.7), one has

$$|\sin\theta^* - \sin\theta| \le |\theta^* - \theta| \le Cte^{-\alpha(|\mathbf{w}_0|^2+q)t},$$

and thus

$$\sin^2\theta \le Ct^2 e^{-2\alpha(|\mathbf{w}_0|^2+q)t} + \sin^2\theta^*.$$

It follows from (3.2) $\tan^2\theta^*$ is integrable over $(0,\infty)$ and so is $\sin^2\theta^*$ since $\sin^2\theta^* \le \tan^2\theta^*$. Noticing $\sin^2\theta_\xi \le \max\{\sin^2\theta^*, \sin^2\theta\}$, one can deduce that

$$\int_s^t \sin^2\theta_\xi \mathrm{d}r \le C. \tag{4.21}$$

Combining (4.18)-(4.21) we finally get

$$|\omega| \le \int_0^t e^{\int_s^t a(r)\mathrm{d}r}|b(s)|\mathrm{d}s \le Cte^{-\alpha(|\mathbf{w}_0|^2+q)t}h_{\text{ext}}^2. \tag{4.22}$$

As for $\hat{\varphi}$ and φ, one has

$$\begin{aligned}\hat{\varphi}_t - \varphi_t = &\varphi_t^* + \varphi_t^1 h_{\text{ext}} - \varphi_t \\ = &(|\mathbf{w}_0|^2+q)(\cos\theta^* - \cos\theta) - (|\mathbf{w}_0|^2+q)\theta^1\sin\theta^* h_{\text{ext}}.\end{aligned}$$

Using $\theta^1 h_{\text{ext}} = \omega + \theta - \theta^*$, one gets

$$\hat{\varphi}_t - \varphi_t = (|\mathbf{w}_0|^2+q)\sin\theta_\xi(\theta-\theta^*) - (|\mathbf{w}_0|^2+q)\sin\theta^*(\omega+\theta-\theta^*),$$

which leads to

$$\begin{aligned}|\hat{\varphi}_t - \varphi_t| \le &(|\mathbf{w}_0|^2+q)\cos\theta_{\xi^*}(\theta-\theta^*)^2 + (|\mathbf{w}_0|^2+q)\sin\theta^*|\omega| \\ \le &C(t+t^2)e^{-2\alpha(|\mathbf{w}_0|^2+q)t}h_{\text{ext}}^2,\end{aligned}$$

where $\theta \le \theta_\xi \le \theta_{\xi^*} \le \theta^*$. Integrating with respect to t over $[0,t]$ gives

$$|\hat{\varphi} - \varphi| \le Ch_{\text{ext}}^2. \tag{4.23}$$

This completes the proof of (1.11)-(1.12). □

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