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TWO INEQUALITIES FOR CONVEX EQUIPOTENTIAL SURFACES*

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Abstract. We establish two geometric inequalities, respectively, for harmonic functions in exterior Dirichlet problems, and for Green's functions in interior Dirichlet problems, where the boundary surfaces are smooth and convex. Both inequalities involve integrals over the mean curvature and the Gaussian curvature on an equipotential surface, and the normal derivative of the harmonic potential thereupon. These inequalities generalize a geometric conservation law for equipotential curves in dimension two, and offer solutions to two free boundary problems in three-dimensional electrostatics.

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1. Introduction

Consider a three-dimensional exterior Dirichlet problem ("3-exD" below), where a non-constant harmonic function $U(\boldsymbol r),\boldsymbol r\in\Omega\subset\mathbb R^3$ solves a Laplace equation

$$\nabla^2 U(\boldsymbol r)=0,\quad \boldsymbol r\in\Omega \tag{1.1}$$

in an unbounded domain Ω, whose boundary $\partial\Omega$ is a smooth and connected surface, on which $U(\boldsymbol r)$ remains constant. The flux condition

$$-\oint_{\partial\Omega}\boldsymbol n\cdot\nabla U(\boldsymbol r)\,\mathrm dS=\Phi>0 \tag{1.2}$$

(with $\boldsymbol n$ being the outward unit normal on $\partial\Omega$, and $\mathrm dS$ the surface element) is equivalent to the following asymptotic behavior:

$$U(\boldsymbol r)\sim\frac{\Phi}{4\pi|\boldsymbol r|},\quad |\boldsymbol r|\to+\infty. \tag{1.3}$$

If $\mathbf 0\notin\Omega\cup\partial\Omega$, then one can define the Green's function $G(\boldsymbol r)=G_D^{\partial\Omega}(\mathbf 0,\boldsymbol r)$ in three-dimensional interior Dirichlet problem ("3-inD" below) as the solution to

$$\begin{cases}\nabla^2G(\boldsymbol r)=0, & \boldsymbol r\in\mathbb R^3\smallsetminus(\Omega\cup\partial\Omega\cup\{\mathbf 0\}),\\ G(\boldsymbol r)=0, & \boldsymbol r\in\partial\Omega,\\ -\lim\limits_{\varepsilon\to0^+}\oint_{|\boldsymbol r|=\varepsilon}\boldsymbol n\cdot\nabla G(\boldsymbol r)\,\mathrm dS=1. \end{cases} \tag{1.4}$$

According to the maximum principle for harmonic functions, we have $U(\boldsymbol r)>0,\boldsymbol r\in\Omega\cup\partial\Omega$ in 3-exD and $G(\boldsymbol r)>0,\boldsymbol r\in\mathbb R^3\smallsetminus(\Omega\cup\partial\Omega\cup\{\mathbf 0\})$ in 3-inD. In what follows, we write Σ_φ for the equipotential surface on which the harmonic function [either $U(\boldsymbol r)$ in 3-exD or $G(\boldsymbol r)$ in 3-inD] equals a given non-negative φ.

In classical physics, the 3-exD (resp. 3-inD) problem occurs in electrostatic equilibrium of an isolated metallic conductor (resp. a point charge enclosed in a metallic cavity), where our harmonic function of interest is the electrostatic potential, and

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$E(\boldsymbol{r})=|\nabla U(\boldsymbol{r})|$ (resp. $E(\boldsymbol{r})=|\nabla G(\boldsymbol{r})|$) is the magnitude of the electrostatic field, also known as "field intensity". If the boundary surface $\partial\Omega$ is smooth and convex (with non-negative Gaussian curvature $K(\boldsymbol{r})\geq 0,\boldsymbol{r}\in\partial\Omega$), then we have $E(\boldsymbol{r})\neq 0$ in both 3-exD and 3-inD problems [14, Proposition 3.2], and all the equipotential surfaces (excluding the boundary) are smooth and strictly convex (with positive Gaussian curvature $K(\boldsymbol{r})>0,\boldsymbol{r}\in\Sigma_\varphi\neq\partial\Omega$) [11, Theorem 1.1].

A quantitative understanding of the interplay between geometry (shape of an equipotential surface $\partial\Omega$) and physics (the distribution of field intensity $|\nabla U(\boldsymbol{r})|,\boldsymbol{r}\in\partial\Omega$) has practical consequences, ranging from the design of lightning-rods [18] to the self-assembly of metallic nanoparticles [12]. The "common knowledge" that strongest field accompanies greatest curvature is mathematically unfounded [18, Figure 5]. Therefore, instead of following *electricians' folklore* about pointwise causal relationship between curvature and field intensity, it is more sensible to study statistical correlations between geometric and physical quantities, in a non-local manner.

In this work, we focus on 3-exD and 3-inD problems with smooth and convex boundaries ("3-exDc" and "3-inDc" hereafter), and investigate integrals on equipotential surfaces Σ_φ with bounded mean curvature[1] $H(\boldsymbol{r})\leq 0,\boldsymbol{r}\in\Sigma_\varphi$, Gaussian curvature $K(\boldsymbol{r})\geq 0,\boldsymbol{r}\in\Sigma_\varphi$, and non-vanishing field intensity $E(\boldsymbol{r})\neq 0,\boldsymbol{r}\in\Sigma_\varphi$. (For convenience, we shall also use the term "electrostatic problems" to cover both 3-exDc and 3-inDc.) We will construct inequalities for surface integrals involving $H(\boldsymbol{r})$, $K(\boldsymbol{r})$ and $E(\boldsymbol{r})$, thereby presenting *a priori* bounds for statistical averages of field intensity fluctuation $|\boldsymbol{n}\times\nabla\log E(\boldsymbol{r})|^2$ through statistical averages of curvature fluctuation $H^2(\boldsymbol{r})-K(\boldsymbol{r})$.

After laying out the geometric settings in Section 2, we will prove our main result (Theorem 1.1) and its consequence (Corollary 1.1) in Section 3.

Theorem 1.1 (Geometric inequalities on convex equipotential surfaces). *For every level set Σ in 3-exDc, we have the following inequality (strict unless $\partial\Omega$ is a sphere):*

$$\oint_\Sigma \frac{4[H^2(\boldsymbol{r})-K(\boldsymbol{r})]-|\boldsymbol{n}\times\nabla\log|\nabla U(\boldsymbol{r})||^2}{|\nabla U(\boldsymbol{r})|}\,\mathrm{d}S\geq 0. \tag{1.5}$$

For every level set Σ in 3-inDc, we have following inequality (strict unless $\partial\Omega$ is a sphere centered at the origin):

$$\oint_\Sigma \frac{4[H^2(\boldsymbol{r})-K(\boldsymbol{r})]-|\boldsymbol{n}\times\nabla\log|\nabla G(\boldsymbol{r})||^2}{|\nabla G(\boldsymbol{r})|}\,\mathrm{d}S\leq 0. \tag{1.6}$$

Corollary 1.1 (Spherical solutions to two free boundary value problems). *If there is a spherical equipotential surface in 3-exDc, then the boundary $\partial\Omega$ must be a sphere. If there is an equipotential surface in 3-inDc on which $|\nabla G(\boldsymbol{r})|$ remains constant, then $\partial\Omega$ must be a sphere centered at the origin.*

Two-dimensional analogs of electrostatic problems can be regarded as the situations of three-dimensional cylindrical surfaces with translational invariance along the z-axis. For the two-dimensional cross-section of such cylindrical surfaces, the curvature of an equipotential curve becomes $\kappa=-2H$, while the Gaussian curvature vanishes identically $K\equiv 0$. Therefore, the surface integrals appearing in Theorem 1.1 are reminiscent of the

[1] By choosing an outward unit normal vector, we are adopting a sign convention where the unit sphere has mean curvature $H=-1$.

following integrals on equipotential curves [22, (1.14)]:

$$\oint_{\Sigma}\frac{[\kappa(\boldsymbol r)]^2-|\boldsymbol n\times\nabla\log|\nabla U(\boldsymbol r)||^2}{|\nabla U(\boldsymbol r)|}\mathrm{d}s\quad\text{and}\quad\oint_{\Sigma}\frac{[\kappa(\boldsymbol r)]^2-|\boldsymbol n\times\nabla\log|\nabla G(\boldsymbol r)||^2}{|\nabla G(\boldsymbol r)|}\mathrm{d}s \tag{1.7}$$

for 2-exD and 2-inD, respectively. In our previous work [22, §2.2 and §3], we have shown that both integrals are constants (independent of φ) when the boundary $\partial\Omega$ is a smooth Jordan curve. Our proof in Section 3 will reveal a unified mechanism underlying the geometric inequalities in Theorem 1.1 and the geometric conservation laws in (1.7).

Theorem 1.1 unveils a subtle constraint between the fluctuations of curvatures and field intensity on a single equipotential surface. Its toy application (Corollary 1.1), by contrast, contains less surprising statements (cf. stronger results for the Green's functions in [16, Theorem III.2]). To conclude this article, we will strengthen the first half of Corollary 1.1 in $\mathbb{R}^d$ $(d\geq 2)$, and sharpen its second half in $\mathbb{R}^2$.

2. Geometric preparations

In this section, we set up a geometric framework for electrostatic problems (Section 2.1), and prepare some differential formulae (Section 2.2) that will be useful later. Unavoidably, we will recover some standard identities in classical differential geometry [8,15,20], as well as reproduce part of the modern investigations of level sets for Green's functions on manifolds [4–6,14]. Nevertheless, we choose to include our derivations here, for the sake of consistency and accessibility. Indeed, the availability of certain vector calculus identities in the flat Euclidean space $\mathbb{R}^3$ does make our computations more straightforward than generic cases on intrinsically curved Riemannian manifolds.

2.1. Curvilinear coordinates and Laplacian decomposition.

Akin to our previous work [22, Section 2.1], we set up a curvilinear coordinate system $\boldsymbol r(\varphi,u,v)\equiv\boldsymbol r(u^0,u^1,u^2)$ that is compatible with equipotential surfaces in $\mathbb{R}^3$. In this coordinate system, $\varphi\equiv u^0$ coincides with the value of the harmonic potential [$U(\boldsymbol r)$ in 3-exD, $G(\boldsymbol r)$ in 3-inD], and a pair of points on distinct equipotential surfaces share the same $(u,v)\equiv(u^1,u^2)$ coordinates if and only if they are joined by an integral curve of $\nabla\varphi$. Thus, a family of equipotential surfaces Σ_φ evolve according to the following equation

$$\frac{\partial\boldsymbol r(\varphi,u,v)}{\partial\varphi}=-\frac{\boldsymbol n}{E(\boldsymbol r)},\quad \boldsymbol r\in\Sigma_\varphi, \tag{2.1}$$

which conserves the total surface flux

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}\boldsymbol n\cdot\nabla\varphi\,\mathrm{d}S=0. \tag{2.2}$$

This conservation is expected from the Gauß law of electrostatics, which is part of the Maxwell equations for classical electrodynamics [10, §1.4, §1.7]. Hereafter, we will refer to (2.1) as the Gauß–Maxwell flow.

On each equipotential surface, we define the components of the *covariant metric tensor* (g_{ij}) as $g_{ij}:=\partial_i\boldsymbol r\cdot\partial_j\boldsymbol r$, where ∂_i is short-hand for $\partial/\partial u^i$. The *contravariant metric tensor* (g^{ij}) is the matrix inverse of (g_{ij}). The line element on each equipotential surface is given by $\mathrm{d}s^2=g_{ij}\,\mathrm{d}u^i\,\mathrm{d}u^j$, where the Einstein summation convention is applied hereinafter, and a Latin index takes values in $\{1,2\}$.

On each equipotential surface, we have the Gauss formula [8, §4.3]: $\partial_i\partial_j\boldsymbol r=\Gamma_{ij}^k\partial_k\boldsymbol r+b_{ij}\boldsymbol n$, for *connection coefficients* $\Gamma_{ij}^k:=g^{k\ell}\partial_i\partial_j\boldsymbol r\cdot\partial_\ell\boldsymbol r=\frac12 g^{k\ell}(\partial_i g_{\ell j}+\partial_j g_{i\ell}-\partial_\ell g_{ij})$ [8, §5.7] and the *coefficients of second fundamental form* $b_{ij}:=\partial_i\partial_j\boldsymbol r\cdot\boldsymbol n$. The components of the

Weingarten transform $\hat{W}=(b_i^j)$ is defined by $b_i^j:=g^{jk}b_{ki}$ and appears in the Weingarten formula: $\partial_i\boldsymbol{n}=-b_i^j\partial_j\boldsymbol{r}$, that is, $\mathrm{d}\boldsymbol{n}=-\hat{W}\,\mathrm{d}\boldsymbol{r}$ for infinitesimal changes tangent to the equipotential surface. The *mean curvature* is half the trace of the Weingarten transform: $H:=\frac{1}{2}\operatorname{Tr}(\hat{W})=\frac{1}{2}(b_1^1+b_2^2)=\frac{1}{2}g^{ij}b_{ij}$; while the *Gaussian curvature* is the determinant of the Weingarten transform: $K:=\det(\hat{W})=b_1^1b_2^2-b_2^1b_1^2$.

Being compatible with the Gauß–Maxwell flow equation in (2.1), we have $g_{00}:=\partial_0\boldsymbol{r}\cdot\partial_0\boldsymbol{r}=E^{-2}=1/g^{00}$ and $g_{0i}=g^{0i}:=\partial_0\boldsymbol{r}\cdot\partial_i\boldsymbol{r}=0$. In this way, the Euclidean line element $\mathrm{d}\underline{s}^2=\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2$ can be reformulated as

$$\mathrm{d}\underline{s}^2=g_{\mu\nu}\,\mathrm{d}u^\mu\,\mathrm{d}u^\nu=\frac{\mathrm{d}\varphi^2}{E^2}+g_{ij}\,\mathrm{d}u^i\,\mathrm{d}u^j, \tag{2.3}$$

where a Greek index takes values in $\{0,1,2\}$. One may extend the definition of connection coefficients as $\partial_\mu\partial_\nu\boldsymbol{r}=\Gamma_{\mu\nu}^\lambda\partial_\lambda\boldsymbol{r}$, where the newly-arisen connection coefficients will be computed in the following proposition.

PROPOSITION 2.1 (Connection coefficients). *We have the following computations for connection coefficients involving the index* 0*:*

$$\Gamma_{ij}^0=-Eb_{ij},\quad \Gamma_{j0}^0=-\frac{1}{E}\frac{\partial E}{\partial u^j},\quad \Gamma_{j0}^k=\frac{b_j^k}{E}; \tag{2.4}$$

$$\Gamma_{ij}^0=-\frac{E^2}{2}\frac{\partial g_{ij}}{\partial\varphi},\quad \Gamma_{00}^j=-\frac{1}{2}g^{jm}\frac{\partial g_{00}}{\partial u^m}=\frac{1}{E^3}g^{jm}\frac{\partial E}{\partial u^m}; \tag{2.5}$$

$$\Gamma_{00}^0=\frac{1}{2}g^{00}\partial_0 g_{00}=-\frac{\partial}{\partial\varphi}\log E=\Gamma_{m0}^m. \tag{2.6}$$

Proof. To prove the three identities in (2.4), it would suffice to compare the equation $\partial_\mu\partial_\nu\boldsymbol{r}=\Gamma_{\mu\nu}^\lambda\partial_\lambda\boldsymbol{r}$ with the Gauß and Weingarten formulae:

$$\frac{\partial^2\boldsymbol{r}}{\partial u^i\partial u^j}=\Gamma_{ij}^k\frac{\partial\boldsymbol{r}}{\partial u^k}-Eb_{ij}\frac{\partial\boldsymbol{r}}{\partial\varphi},\qquad \frac{\partial}{\partial u^j}\left(E\frac{\partial\boldsymbol{r}}{\partial\varphi}\right)=b_j^k\frac{\partial\boldsymbol{r}}{\partial u^k}. \tag{2.7}$$

The two identities in (2.5) follow from the Christoffel formula $\Gamma_{\mu\nu}^\lambda=\frac{1}{2}g^{\lambda\eta}(\partial_\mu g_{\eta\nu}+\partial_\nu g_{\mu\eta}-\partial_\eta g_{\mu\nu})$.

Before deducing (2.6), we compare the two expressions of Γ_{ij}^0 in (2.4) and (2.5) and write down

$$\frac{\partial g_{ij}}{\partial\varphi}=\frac{2}{E}b_{ij},\quad \frac{\partial\log\det(g_{ij})}{\partial\varphi}=g^{ij}\frac{\partial g_{ij}}{\partial\varphi}=\frac{2}{E}g^{ij}b_{ij}=\frac{4H}{E}. \tag{2.8}$$

On the other hand, the Laplace equation $\nabla^2\varphi(\boldsymbol{r})=-\nabla\cdot\boldsymbol{E}(\boldsymbol{r})=0$ implies zero divergence of $\boldsymbol{E}$-field, i.e. $\partial_0\log(E\sqrt{g})=0$, (hereafter $g=\det(g_{ij})$), thus (2.8) gives rise to

$$2H+\frac{\partial E}{\partial\varphi}=0,\qquad \text{i.e.}\quad \boldsymbol{n}\cdot\nabla\log E=2H. \tag{2.9}$$

It is easy to recast (2.9) into the harmonic coordinate condition $\Gamma^0:=g^{ij}\Gamma_{ij}^0+g^{00}\Gamma_{00}^0=0$, which leads to (2.6). □

REMARK 2.1. Using the identity $\partial_0 g_{ij}=2b_{ij}/E$, we can also readily deduce $\partial_0 g^{ij}=-2g^{ik}b_k^j/E$.

The three-dimensional Laplace operator $\Delta=\partial_x^2+\partial_y^2+\partial_z^2$ can be presented in curvilinear coordinates as

$$\Delta=g^{\mu\nu}(\partial_\mu\partial_\nu-\Gamma_{\mu\nu}^\lambda\partial_\lambda)=\frac{1}{\sqrt{\det(g_{\mu\nu})}}\partial_\lambda\left(g^{\lambda\eta}\sqrt{\det(g_{\mu\nu})}\partial_\eta\right). \tag{2.10}$$

Here, $\det(g_{\mu\nu})=g/E^2$ for $g=\det(g_{ij})$. Similarly, one can define the *Laplace operator on equipotential surface* Σ as

$$\Delta_\Sigma=g^{ij}(\partial_i\partial_j-\Gamma_{ij}^k\partial_k)=\frac{1}{\sqrt{g}}\partial_k\left(g^{k\ell}\sqrt{g}\partial_\ell\right). \tag{2.11}$$

PROPOSITION 2.2 (Decomposition of Laplacian). *The Laplace operator* Δ *can be rewritten as*

$$\Delta=\Delta_\Sigma+E^2\frac{\partial^2}{\partial\varphi^2}-\frac{1}{E}g^{jm}\frac{\partial E}{\partial u^m}\frac{\partial}{\partial u^j}. \tag{2.12}$$

Proof. By definition, we have

$$\Delta=\Delta_\Sigma-g^{ij}\Gamma_{ij}^0\frac{\partial}{\partial\varphi}+g^{00}\left(\frac{\partial^2}{\partial\varphi^2}-\Gamma_{00}^j\frac{\partial}{\partial u^j}-\Gamma_{00}^0\frac{\partial}{\partial\varphi}\right). \tag{2.13}$$

With the substitution of $g^{00}=E^2$ and the expressions for $\Gamma_{ij}^0,\Gamma_{00}^j,\Gamma_{00}^0$ from Proposition 2.1, we obtain the claimed result in (2.12). ∎

COROLLARY 2.1 (Geometric description of $\nabla\log E$). *We have*

$$\nabla\log E(\boldsymbol{r})=k(\boldsymbol{r})\boldsymbol{N}(\boldsymbol{r})+2H(\boldsymbol{r})\boldsymbol{n}(\boldsymbol{r}). \tag{2.14}$$

Here, $k(\boldsymbol{r})$ *is the curvature (inverse of the radius of curvature) of the electric field line (*$\boldsymbol{E}$*-line) that passes* $\boldsymbol{r}$*, and* $H(\boldsymbol{r})$ *is the mean curvature of the equipotential surface that passes* $\boldsymbol{r}$*, with* $\boldsymbol{N}$ *and* $\boldsymbol{n}$ *being the respective unit normal vectors for the* $\boldsymbol{E}$*-line and equipotential surface.*

Proof. As we already have the normal derivative $\boldsymbol{n}\cdot\nabla\log E=2H$ in (2.9), it is sufficient to show that the tangential gradient $[\boldsymbol{n}\times\nabla\log E(\boldsymbol{r})]\times\boldsymbol{n}=g^{jm}(\partial_m\log E)\partial_j\boldsymbol{r}$ is equal to $k\boldsymbol{N}$. To fulfill this task, we compute

$$\begin{aligned}\boldsymbol{0}&=\Delta\boldsymbol{r}=\Delta_\Sigma\boldsymbol{r}+E^2\frac{\partial^2\boldsymbol{r}}{\partial\varphi^2}-\frac{1}{E}g^{jm}\frac{\partial E}{\partial u^m}\frac{\partial\boldsymbol{r}}{\partial u^j}\\&=2H\boldsymbol{n}-E^2\frac{\partial(\boldsymbol{n}/E)}{\partial\varphi}-g^{jm}\frac{\partial\log E}{\partial u^m}\frac{\partial\boldsymbol{r}}{\partial u^j}=-E\frac{\partial\boldsymbol{n}}{\partial\varphi}-g^{jm}\frac{\partial\log E}{\partial u^m}\frac{\partial\boldsymbol{r}}{\partial u^j},\end{aligned} \tag{2.15}$$

where the definition for the curvature of a curve $k\boldsymbol{N}=-E\partial_0\boldsymbol{n}$ can be substituted in the last step. ∎

REMARK 2.2. The result in (2.14) is well known in physics, as the tangential and normal components of $\nabla\log E$ can be easily derived from elementary vector analysis [10, p. 591] and the Gauß theorem of electrostatic field [10, p. 52, Problem 1.11], respectively. We have rederived (2.14) in our curvilinear coordinate system as a double check of the computations involving the connection coefficients and the Laplacian.

Later on, we will often use the notation $Df:=g^{ij}\partial_i f\partial_j\boldsymbol{r}$ for the tangential gradient of a smooth function f. This allows us to abbreviate (2.15) as $\partial_\varphi\boldsymbol{n}=D(1/E)$ for the Gauß–Maxwell flow. It follows immediately from (2.15) that

$$\partial_\varphi\boldsymbol{E}=\partial_\varphi(E\boldsymbol{n})=ED(1/E)-2H\boldsymbol{n}=-\nabla\log E. \tag{2.16}$$

It is also easy to verify, for the Gauß–Maxwell flow, that the following commutation relation holds:

$$\partial_\varphi(Df)-D(\partial_\varphi f)=-\frac{\hat{W}Df}{E}-\boldsymbol{n}(Df)\cdot\left(D\frac{1}{E}\right):=-\frac{g^{ik}b_k^j\partial_i f\partial_j\boldsymbol{r}}{E}-\boldsymbol{n}g^{ij}\partial_i f\partial_j\frac{1}{E}. \tag{2.17}$$

In particular, (2.15) and the commutation relation above would entail

$$\begin{aligned}\Delta\boldsymbol{n}-\Delta_\Sigma\boldsymbol{n}&=E^2\partial_\varphi[D(1/E)]+\hat{W}D\log E\\&=2DH+2(\hat{W}-2H)D\log E-\boldsymbol{n}|D\log E|^2,\end{aligned} \tag{2.18}$$

a formula that will be used later in Section 2.2.

2.2. Evolution of mean and Gaussian curvatures on equipotential surfaces. Since we will be interested in tracking down the changes of curvatures across different equipotential surfaces, it is sensible to derive formulae for the derivatives of curvatures with respect to the potential φ.

PROPOSITION 2.3 (Evolution of the second fundamental form). *We have the following identities*

$$\frac{\partial b_{ij}}{\partial\varphi}=(b_j^k b_{ki}-\partial_i\partial_j+\Gamma_{ij}^k\partial_k)\frac{1}{E} \tag{2.19}$$

and

$$2\frac{\partial H}{\partial\varphi}=-\Delta_\Sigma\frac{1}{E}-\frac{4H^2-2K}{E}. \tag{2.20}$$

Proof. From the identity $\partial_0(\partial_i\partial_j\boldsymbol{r})=\partial_i(\partial_0\partial_j\boldsymbol{r})$, we may deduce $\partial_0\Gamma_{ij}^0+\Gamma_{ij}^\nu\Gamma_{\nu 0}^0=\partial_i\Gamma_{j0}^0+\Gamma_{j0}^\nu\Gamma_{\nu i}^0$. This results in (2.19), upon substitution of the connection coefficients. Combining $2H=g^{ij}b_{ij}$ and $\partial_0 g^{ij}=-2g^{ik}b_k^j/E$ with (2.19), we obtain

$$2\frac{\partial H}{\partial\varphi}=-\Delta_\Sigma\frac{1}{E}-\frac{b_j^k b_k^j}{E},\quad\text{where } b_j^k b_k^j=\operatorname{Tr}(\hat{W}^2)=4H^2-2K.$$

This verifies (2.20). □

PROPOSITION 2.4 (Evolution of Gaussian curvature). *We have the following formula*

$$\frac{\partial}{\partial\varphi}(K\sqrt{g})=-\partial_i\left(\beta^{ij}\sqrt{g}\partial_j\frac{1}{E}\right)\Longleftrightarrow\frac{\partial}{\partial\varphi}\frac{K}{E}=-\frac{1}{E\sqrt{g}}\partial_i\left(\beta^{ij}\sqrt{g}\partial_j\frac{1}{E}\right) \tag{2.21}$$

for $\beta^{ij}:=2Hg^{ij}-g^{ik}b_k^j=\mathfrak{b}^{ij}/g$, *where* $(\mathfrak{b}^{ij})=\begin{pmatrix}b_{22}&-b_{12}\\-b_{12}&b_{11}\end{pmatrix}$ *is the adjugate matrix of* (b_{ij}).

Proof. Using Jacobi's formula for the derivative of a determinant, we may verify that

$$\frac{\partial}{\partial\varphi}(K\sqrt{g})=\frac{\partial}{\partial\varphi}\left(\frac{\det(b_{ij})}{\sqrt{g}}\right)=\frac{\mathfrak{b}^{ij}}{\sqrt{g}}\frac{\partial b_{ij}}{\partial\varphi}-\frac{2HK\sqrt{g}}{E}$$

$$=\frac{\mathfrak{b}^{ij}}{\sqrt{g}}(b^k_j b_{ki}-\partial_i\partial_j+\Gamma^m_{ij}\partial_m)\frac{1}{E}-\frac{2HK\sqrt{g}}{E}$$
$$=\beta^{ij}\sqrt{g}(-\partial_i\partial_j+\Gamma^m_{ij}\partial_m)\frac{1}{E},\tag{2.22}$$

where we have quoted (2.19) in the penultimate step, before using the relation $\mathfrak{b}^{ij}b^k_j b_{ki}/g=K\delta^j_k b^k_j=2HK$ in the last step. Then, we note that the Codazzi–Mainardi equation $\partial_k b_{ij}-\partial_j b_{ik}+\Gamma^\ell_{ij}b_{\ell k}-\Gamma^\ell_{ik}b_{\ell j}=0$ and the vanishing covariant derivatives of the metric $(g^{ik})_{;\ell}:=\partial_\ell g^{ik}+g^{im}\Gamma^k_{m\ell}+g^{km}\Gamma^i_{m\ell}=0$ allow us to compute $\partial_i b^k_j+\Gamma^k_{i\ell}b^\ell_j-\Gamma^\ell_{ij}b^k_\ell=:b^k_{j;i}=b^k_{i;j}$ and $(\beta^{ik})_{;k}=b^\ell_{\ell;k}g^{ik}-g^{ij}b^k_{j;k}=b^\ell_{\ell;k}g^{ik}-g^{ij}b^k_{k;j}=0$, thereby leading to

$$-\partial_i\left(\beta^{ij}\sqrt{g}\partial_j f\right)=\beta^{ij}\sqrt{g}(-\partial_i\partial_j+\Gamma^m_{ij}\partial_m)f,\tag{2.23}$$

for every smooth function f. Combining the results in (2.22) and (2.23), we arrive at the claimed formula in (2.21). □

REMARK 2.3. When the field intensity E is non-vanishing on an entire equipotential surface Σ_φ, we may double-check the reasonability of (2.21) by the following computation

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}K\,\mathrm{d}S=\oint_{\Sigma_\varphi}\frac{1}{\sqrt{g}}\frac{\partial}{\partial\varphi}(K\sqrt{g})\,\mathrm{d}S=-\oint_{\Sigma_\varphi}\frac{1}{\sqrt{g}}\partial_i\left(\beta^{ij}\sqrt{g}\partial_j\frac{1}{E}\right)\mathrm{d}S=0.\tag{2.24}$$

On the other hand, we know from the Gauß–Bonnet theorem that $\oint_{\Sigma_\varphi}K\,\mathrm{d}S=2\pi\chi(\Sigma_\varphi)$, where $\chi(\Sigma_\varphi)$ is the Euler–Poincaré characteristic that determines the topology of Σ_φ. The result in (2.24) is thus expected from the non-critical $\boldsymbol{E}$-lines that establish diffeomorphisms among all the equipotential surfaces in a neighborhood of Σ_φ.

COROLLARY 2.2 (Weatherburn formula [20, p. 231]). *The following identity holds on every smooth surface*

$$\Delta_\Sigma\boldsymbol{n}=(2K-4H^2)\boldsymbol{n}-2(\boldsymbol{n}\times\nabla H)\times\boldsymbol{n}.\tag{2.25}$$

Proof. Applying (2.23) to the three Euclidean components of $\boldsymbol{r}=x\boldsymbol{e}_x+y\boldsymbol{e}_y+z\boldsymbol{e}_z$, we can quickly recover the following formula of Minkowski [15]:

$$\frac{1}{\sqrt{g}}\partial_i\left(\beta^{ij}\sqrt{g}\partial_j\boldsymbol{r}\right)=2K\boldsymbol{n},\tag{2.26}$$

with the computation

$$\frac{1}{\sqrt{g}}\partial_i\left(\beta^{ij}\sqrt{g}\partial_j\boldsymbol{r}\right)=\beta^{ij}(\partial_i\partial_j-\Gamma^m_{ij}\partial_m)\boldsymbol{r}=\frac{\mathfrak{b}^{ij}b_{ij}\boldsymbol{n}}{g}=\frac{2\det(b_{ij})}{g}\boldsymbol{n}=2K\boldsymbol{n}.\tag{2.27}$$

This in turn allows us to verify the Weatherburn formula via

$$\Delta_\Sigma\boldsymbol{n}-2K\boldsymbol{n}=\frac{1}{\sqrt{g}}\partial_i\left(g^{ij}\sqrt{g}\partial_j\boldsymbol{n}-\beta^{ij}\sqrt{g}\partial_j\boldsymbol{r}\right)$$
$$=-\frac{1}{\sqrt{g}}\partial_i(2Hg^{ij}\sqrt{g}\partial_j\boldsymbol{r})=-4H^2\boldsymbol{n}-2DH,\tag{2.28}$$

where we have exploited $g^{ij}\sqrt{g}\partial_j\boldsymbol{n}=-g^{ik}b^j_k\sqrt{g}\partial_j\boldsymbol{r}$, $\beta^{ij}:=2Hg^{ij}-g^{ik}b^j_k$ and a familiar relation $\Delta_\Sigma\boldsymbol{r}=\frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\partial_j\boldsymbol{r})=2H\boldsymbol{n}$. □

Combining (2.18) and (2.25), we immediately arrive at the following representation of $\Delta\boldsymbol{n}:=\boldsymbol{e}_x\Delta(\boldsymbol{n}\cdot\boldsymbol{e}_x)+\boldsymbol{e}_y\Delta(\boldsymbol{n}\cdot\boldsymbol{e}_y)+\boldsymbol{e}_z\Delta(\boldsymbol{n}\cdot\boldsymbol{e}_z)$:

$$\Delta\boldsymbol{n}=2(\hat{W}-2H)D\log E-\boldsymbol{n}(|D\log E|^2+4H^2-2K), \tag{2.29}$$

a result that will be used later in Corollary 3.1.

3. Main result and applications

Like previous studies of level sets for harmonic functions [4–6, 14, 22], we will build a monotonicity result (Section 3.1) on positive-definite quadratic forms, before subsequently applying it to the proof of Theorem 1.1 and Corollary 1.1 in Section 3.2. We will finally devote Section 3.3 to some generalizations of Corollary 1.1.

3.1. Monotonicity of an integral on equipotential surface.

To prepare for the proof of Theorem 1.1, we compute two more quantities: $\Delta\log E$ and $\Delta\frac{\boldsymbol{n}}{E}$. Both these quantities vanish in two-dimensional electrostatic problems, as one can easily check by complex analytic techniques.

Proposition 3.1 (Laplacian representation of Gaussian curvature). *There is a geometric identity*

$$\Delta\log E+2K=0, \tag{3.1}$$

which is a special case of [21, Proposition 1.4].

Proof. We first employ (2.20) to compute

$$-\frac{\partial^2}{\partial\varphi^2}\log E=\frac{\partial}{\partial\varphi}\left(\frac{2H}{E}\right)=\frac{4H^2}{E^2}+\frac{2}{E}\frac{\partial H}{\partial\varphi}=-\frac{1}{E}\Delta_\Sigma\frac{1}{E}+\frac{2K}{E^2}. \tag{3.2}$$

Meanwhile, we may use the definition of Laplacian Δ in the curvilinear coordinate system to evaluate

$$\Delta\log E=\frac{E}{\sqrt{g}}\partial_\mu\left(\frac{\sqrt{g}}{E}g^{\mu\nu}\partial_\nu\log E\right)=-\frac{E}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\frac{1}{E}\right)=-E\Delta_\Sigma\frac{1}{E}+E^2\frac{\partial^2}{\partial\varphi^2}\log E. \tag{3.3}$$

Combining (3.2) with (3.3), we arrive at the claimed identity. ∎

Corollary 3.1 (A geometric representation of $\Delta\frac{\boldsymbol{n}}{E}$). *We have the following formula:*

$$\Delta\frac{\boldsymbol{n}}{E}=4\left(\beta^{ij}\partial_i\frac{1}{E}\partial_j\boldsymbol{r}+\frac{K\boldsymbol{n}}{E}\right). \tag{3.4}$$

Proof. Combining our formula for $\Delta\boldsymbol{n}$ in (2.29) with the identity $\Delta\log E+2K=0$, and noting that $(\nabla\log E\cdot\nabla)\boldsymbol{n}=2H(\boldsymbol{n}\cdot\nabla)\boldsymbol{n}+(D\log E\cdot\nabla)\boldsymbol{n}=2HD\log E-\hat{W}D\log E$, we can compute

$$\begin{aligned}\Delta\frac{\boldsymbol{n}}{E}&=\frac{\Delta\boldsymbol{n}}{E}+\boldsymbol{n}\Delta\frac{1}{E}+2g^{\mu\nu}\partial_\mu\frac{1}{E}\partial_\nu\boldsymbol{n}\\&=2\beta^{ij}\partial_i\frac{1}{E}\partial_j\boldsymbol{r}-\frac{(4H^2-2K+|D\log E|^2)\boldsymbol{n}}{E}-\boldsymbol{n}\nabla\cdot\left(\frac{1}{E}\nabla\log E\right)-\frac{2}{E}(\nabla\log E\cdot\nabla)\boldsymbol{n}\\&=2\beta^{ij}\partial_i\frac{1}{E}\partial_j\boldsymbol{r}+\frac{4K\boldsymbol{n}}{E}-\frac{2}{E}(2H-\hat{W})D\log E=4\left(\beta^{ij}\partial_i\frac{1}{E}\partial_j\boldsymbol{r}+\frac{K\boldsymbol{n}}{E}\right),\end{aligned} \tag{3.5}$$

as claimed. ∎

Corollary 3.2 (Evolution of a surface integral). *We have the following derivative formula:*

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}\left(H^2-K-\frac{|D\log E|^2}{4}\right)\frac{\mathrm{d}S}{E}=-\frac{3}{2}\oint_{\Sigma_\varphi}\beta^{ij}\partial_i\frac{1}{E}\partial_j\frac{1}{E}\mathrm{d}S. \tag{3.6}$$

Proof. One can verify (3.6) by a brute-force computation, using the derivatives of H, K, E and g^{ij} studied in Section 2. Here, we will build our proof on Proposition 3.1 and Corollary 3.1, to highlight the mechanism shared by the derivative formula (3.6) for three-dimensional electrostatics and its two-dimensional counterpart [22, (2.25)].

Specializing the vector Green identity [7, p. 156]

$$\begin{aligned}&\int_{\mathfrak{D}}[\boldsymbol{Q}\cdot\Delta\boldsymbol{F}-\boldsymbol{F}\cdot\Delta\boldsymbol{Q}]\mathrm{d}^3\boldsymbol{r}\\&=\oint_{\partial\mathfrak{D}}[(\boldsymbol{\nu}\times\boldsymbol{Q})\cdot(\nabla\times\boldsymbol{F})+(\boldsymbol{\nu}\cdot\boldsymbol{Q})(\nabla\cdot\boldsymbol{F})-(\boldsymbol{\nu}\times\boldsymbol{F})\cdot(\nabla\times\boldsymbol{Q})-(\boldsymbol{\nu}\cdot\boldsymbol{F})(\nabla\cdot\boldsymbol{Q})]\mathrm{d}S\end{aligned} \tag{3.7}$$

to $\boldsymbol{Q}(\boldsymbol{r})=\nabla\log E(\boldsymbol{r})$ and $\boldsymbol{F}(\boldsymbol{r})=\boldsymbol{n}(\boldsymbol{r})/E(\boldsymbol{r})$, we may put down

$$\begin{aligned}&\int_{\mathfrak{D}}\left[\nabla\log E(\boldsymbol{r})\cdot\Delta\frac{\boldsymbol{n}(\boldsymbol{r})}{E(\boldsymbol{r})}-\frac{\boldsymbol{n}(\boldsymbol{r})}{E(\boldsymbol{r})}\cdot\Delta\nabla\log E(\boldsymbol{r})\right]\mathrm{d}^3\boldsymbol{r}\\&=\oint_{\partial\mathfrak{D}}\left\{[\boldsymbol{\nu}\times\nabla\log E(\boldsymbol{r})]\cdot\left[\nabla\times\frac{\boldsymbol{n}(\boldsymbol{r})}{E(\boldsymbol{r})}\right]+[\boldsymbol{\nu}\cdot\nabla\log E(\boldsymbol{r})]\left[\nabla\cdot\frac{\boldsymbol{n}(\boldsymbol{r})}{E(\boldsymbol{r})}\right]\right.\\&\quad\left.-\left[\boldsymbol{\nu}\cdot\frac{\boldsymbol{n}(\boldsymbol{r})}{E(\boldsymbol{r})}\right]\Delta\log E(\boldsymbol{r})\right\}\mathrm{d}S,\end{aligned} \tag{3.8}$$

where $\boldsymbol{\nu}$ is the outward normal vector with respect to the domain boundary $\partial\mathfrak{D}$.

We first look at the integral over $\mathfrak{D}$ (which vanishes in the two-dimensional electrostatics where $\Delta\boldsymbol{Q}=\Delta\boldsymbol{F}=\boldsymbol{0}$). We can rewrite the integrand as

$$\begin{aligned}\nabla\log E\cdot\Delta\frac{\boldsymbol{n}}{E}-\frac{\boldsymbol{n}}{E}\cdot\Delta\nabla\log E=&4\beta^{ij}\partial_i\frac{1}{E}\partial_j\log E+\frac{8HK}{E}+\frac{2}{E}\boldsymbol{n}\cdot\nabla K\\=&4\beta^{ij}\partial_i\frac{1}{E}\partial_j\log E+\frac{8HK}{E}+\frac{2}{E}\boldsymbol{n}\cdot\nabla K\\&-8\boldsymbol{n}\cdot\nabla\frac{K}{E}+\frac{8}{\sqrt{g}}\partial_i\left(\beta^{ij}\sqrt{g}\partial_j\frac{1}{E}\right),\end{aligned} \tag{3.9}$$

after employing the relations in (2.21), (3.1) and (3.5).

We then turn our attention to the boundary contributions. If we pick the boundary $\partial\mathfrak{D}=\Sigma_{\varphi_1}\cup\Sigma_{\varphi_2}$ as the union of two equipotential surfaces Σ_{φ_1} and Σ_{φ_2}, with the latter surface enclosing the former, then $\boldsymbol{\nu}$ corresponds to $\boldsymbol{n}$ on Σ_{φ_2} and $-\boldsymbol{n}$ on Σ_{φ_1}. Meanwhile, it is straightforward to compute that

$$\nabla\times\frac{\boldsymbol{n}}{E}=-\nabla\times\frac{\nabla\varphi}{E^2}=-\nabla\frac{1}{E^2}\times\nabla\varphi=\frac{2\boldsymbol{n}\times\nabla\log E}{E}, \tag{3.10}$$

$$\nabla\cdot\frac{\boldsymbol{n}}{E}=-\nabla\cdot\frac{\nabla\varphi}{E^2}=-\nabla\frac{1}{E^2}\cdot\nabla\varphi=-\frac{2\boldsymbol{n}\cdot\nabla\log E}{E}=-\frac{4H}{E}. \tag{3.11}$$

Plugging the results from the last two paragraphs into the vector Green identity, we obtain

$$\begin{aligned}&2\oint_{\Sigma_{\varphi_2}}\frac{|\boldsymbol{n}\times\nabla\log E|^2-|\boldsymbol{n}\cdot\nabla\log E|^2+K}{E}\,\mathrm{d}S\\&\quad-2\oint_{\Sigma_{\varphi_1}}\frac{|\boldsymbol{n}\times\nabla\log E|^2-|\boldsymbol{n}\cdot\nabla\log E|^2+K}{E}\,\mathrm{d}S\\&=\int_{\varphi_2}^{\varphi_1}\mathrm{d}\varphi\left\{\oint_{\Sigma_\varphi}\left[4\beta^{ij}\partial_i\frac{1}{E}\partial_j\log E+\frac{12HK}{E}-6\boldsymbol{n}\cdot\nabla\frac{K}{E}+\frac{8}{\sqrt{g}}\partial_i\left(\beta^{ij}\sqrt{g}\partial_j\frac{1}{E}\right)\right]\frac{\mathrm{d}S}{E}\right\}\\&=\int_{\varphi_2}^{\varphi_1}\mathrm{d}\varphi\left\{-12\oint_{\Sigma_\varphi}\beta^{ij}\partial_i\frac{1}{E}\partial_j\frac{1}{E}\,\mathrm{d}S+6\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}\frac{K}{E}\,\mathrm{d}S\right\}.\end{aligned}\tag{3.12}$$

Here, in the last step, we have integrated by parts, and used the fact that $\partial_0\sqrt{g}=2H\sqrt{g}/E$ [see the second half of (2.8)]. Now, differentiating both sides of (3.12) with respect to φ_1, we arrive at

$$\begin{aligned}8\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}\left(H^2-K-\frac{|D\log E|^2}{4}\right)\frac{\mathrm{d}S}{E}&=-2\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}\frac{|\boldsymbol{n}\times\nabla\log E|^2-|\boldsymbol{n}\cdot\nabla\log E|^2+4K}{E}\,\mathrm{d}S\\&=-12\oint_{\Sigma_\varphi}\beta^{ij}\partial_i\frac{1}{E}\partial_j\frac{1}{E}\,\mathrm{d}S,\end{aligned}\tag{3.13}$$

as claimed in (3.6). ∎

When we are dealing with the convex boundary surfaces $\partial\Omega$ in Theorem 1.1, all the equipotential surfaces $\Sigma_\varphi\neq\partial\Omega$ in question are strictly convex [11, Theorem 1.1], on which $(\beta^{ij})=(\bar{b}^{ij}/g)$ is negative definite. Therefore, we have a monotonicity statement

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\oint_{\Sigma_\varphi}\left(H^2-K-\frac{|D\log E|^2}{4}\right)\frac{\mathrm{d}S}{E}\geq 0,\tag{3.14}$$

where the inequality is strict unless $E(\boldsymbol{r}),\boldsymbol{r}\in\Sigma_\varphi\neq\partial\Omega$ is a constant.

It is worth noting that the last inequality is the first instance where the strict convexity of equipotential surfaces has played an indispensable rôle in our derivations. All our previous theoretical developments are applicable to both convex and non-convex equipotential surfaces alike. Since Theorem 1.1 and Corollary 1.1 both require convex equipotential surfaces, a diligent reader may rework all our main results in this article using the support function of convex equipotential surfaces, as in Ma–Zhang [13].

3.2. Geometric inequalities and their applications. Our next task is to show that

$$\lim_{\varphi\to0}\oint_{\Sigma_\varphi}\left(H^2-K-\frac{|D\log E|^2}{4}\right)\frac{\mathrm{d}S}{E}=0\tag{3.15}$$

in 3-exD and

$$\lim_{\varphi\to+\infty}\oint_{\Sigma_\varphi}\left(H^2-K-\frac{|D\log E|^2}{4}\right)\frac{\mathrm{d}S}{E}=0\tag{3.16}$$

in 3-inD. Once this is done, we can deduce the two inequalities in Theorem 1.1 from (3.14).

As we go to sufficiently large distances $|\boldsymbol{r}|$ in Ω, say, away from the circumsphere of $\mathbb{R}^3\setminus\Omega$, the spherical harmonic expansion of $U(\boldsymbol{r})$ converges uniformly and absolutely [10, §4.1]:

$$U(\boldsymbol{r})=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\frac{4\pi}{2\ell+1}}c_{\ell m}\frac{Y_{\ell m}(\theta,\phi)}{|\boldsymbol{r}|^{\ell+1}}, \tag{3.17}$$

where the spherical coordinates $\boldsymbol{r}=|\boldsymbol{r}|(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$ are employed, along with the spherical harmonic function $Y_{\ell m}(\theta,\phi)$ and the constants $c_{\ell m}$ [the multi-pole coefficients associated with the (ℓ,m)-modes]. The only significant contributors to our surface integral are the two leading ℓ-modes: $\ell=0,1$, as all the higher-order terms amount to infinitesimal corrections to our surface integral for equipotential surfaces at infinite distances. Without loss of generality, we may evaluate the left-hand side of (3.15) by investigating the dipole field

$$U(\boldsymbol{r})=\frac{c_{00}}{|\boldsymbol{r}|}+\frac{c_{10}\cos\theta}{|\boldsymbol{r}|^2},\quad c_{00}>0,c_{10}\neq 0, \tag{3.18}$$

which is rotationally symmetric about the z-axis. We parametrize the equipotential surface Σ_U with

$$\begin{cases} x=\dfrac{c_{00}+\sqrt{(c_{00})^2+4c_{10}U\cos\theta}}{2U}\sin\theta\cos\phi \\ y=\dfrac{c_{00}+\sqrt{(c_{00})^2+4c_{10}U\cos\theta}}{2U}\sin\theta\sin\phi \\ z=\dfrac{c_{00}+\sqrt{(c_{00})^2+4c_{10}U\cos\theta}}{2U}\cos\theta \end{cases}\quad (0\leq\theta\leq\pi,0\leq\phi\leq 2\pi) \tag{3.19}$$

so that the surface element is given by

$$\mathrm{d}S=\left[\frac{(c_{00})^2\sin\theta}{U^2}+\frac{c_{10}\sin 2\theta}{U}+O(U^0)\right]\mathrm{d}\theta\,\mathrm{d}\phi; \tag{3.20}$$

the two principal curvatures (eigenvalues of the Weingarten transform $\hat{W}$) read

$$k_\theta=\frac{U}{c_{00}}+\frac{(c_{10})^2(1-9\cos 2\theta)U^3}{4(c_{00})^5}+O(U^4),\quad k_\phi=\frac{U}{c_{00}}-\frac{(c_{10})^2(5+3\cos 2\theta)U^3}{4(c_{00})^5}+O(U^4), \tag{3.21}$$

leading us to

$$H^2-K=\frac{(k_\theta-k_\phi)^2}{4}=O(U^6); \tag{3.22}$$

the surface distribution of $E=E(\theta,\phi)$ satisfies

$$E=\frac{U^2}{c_{00}}-\frac{(c_{10})^2(3\cos 2\theta+1)U^4}{4(c_{00})^5}+O(U^5); \tag{3.23}$$

$$|\boldsymbol{n}\times\nabla\log E|^2=\frac{9(c_{10})^4\sin^2\theta\cos^2\theta}{(c_{00})^{10}}U^6+O(U^7). \tag{3.24}$$

Now it becomes clear that our integral on Σ_U has order $O(U^2)$ for the dipole field. Hence, the limit formula (3.15) is true.

So far, we have established

$$\oint_{\Sigma_\varphi}\left(H^2-K-\frac{|D\log E|^2}{4}\right)\frac{\mathrm{d}S}{E}=-\frac{3}{2}\int_0^{\varphi}\left(\oint_{\Sigma_U}\beta^{ij}\partial_i\frac{1}{E}\partial_j\frac{1}{E}\mathrm{d}S\right)\mathrm{d}U\geq 0 \tag{3.25}$$

for 3-exDc, where (β^{ij}) is negative definite on Σ_U for all $U\in(0,\varphi)$.

If the equality holds for a certain given Σ_φ, then we will have $D\log E(\boldsymbol{r})=0,\boldsymbol{r}\in\Sigma_U$ for all $U\in(0,\varphi)$, and also

$$\oint_{\Sigma_U}\frac{H^2-K}{E}\mathrm{d}S=\oint_{\Sigma_U}\left(H^2-K-\frac{|D\log E|^2}{4}\right)\frac{\mathrm{d}S}{E}=0 \tag{3.26}$$

for all $U\in(0,\varphi)$. This implies that at every point on the strictly convex surface Σ_U, the two eigenvalues of the Weingarten transform $\hat{W}$ are equal, so Σ_U must be a sphere [8, §5.2, Theorem 1b]. The condition $D\log E(\boldsymbol{r})=0,\boldsymbol{r}\in\Sigma_U$ also means that the spheres $\Sigma_U,U\in(0,\varphi)$ are all concentric. If the center of these spheres is $\boldsymbol{r}_0\in\mathbb{R}^3$, then we will have $U(\boldsymbol{r})=\frac{\Phi}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|}$ whenever $|\boldsymbol{r}-\boldsymbol{r}_0|>\frac{\Phi}{4\pi\varphi}$. By the unique continuation principle [3, 19], we know that $U(\boldsymbol{r})=\frac{\Phi}{4\pi|\boldsymbol{r}-\boldsymbol{r}_0|}$ holds for all $\boldsymbol{r}\in\Omega\cup\partial\Omega$. This proves that $\partial\Omega$ is spherical.

After establishing the first half of Theorem 1.1, we can move on to the 3-exDc case of Corollary 1.1. Suppose that we have a spherical equipotential surface Σ_φ on which

$$-\oint_{\Sigma_\varphi}\frac{|D\log E|^2}{4E}\mathrm{d}S=\oint_{\Sigma_\varphi}\left(H^2-K-\frac{|D\log E|^2}{4}\right)\frac{\mathrm{d}S}{E}\geq 0 \tag{3.27}$$

entails $D\log E(\boldsymbol{r})=0,\boldsymbol{r}\in\Sigma_\varphi$, so equality holds in (3.25). We are then reduced to the situations in the last paragraph, whereupon a spherical $\partial\Omega$ becomes inevitable.

It is much easier to prove the limit formula (3.16) for 3-inDc, because

$$\frac{\mathrm{d}S}{E}=O(|\boldsymbol{r}|^4\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi),\quad H^2-K=O\left(\frac{1}{|\boldsymbol{r}|^2}\right),\quad |D\log E|=O\left(\frac{1}{|\boldsymbol{r}|}\right), \tag{3.28}$$

as $|\boldsymbol{r}|\to 0$. This quickly leads us to

$$\oint_{\Sigma_\varphi}\left(H^2-K-\frac{|D\log E|^2}{4}\right)\frac{\mathrm{d}S}{E}=\frac{3}{2}\int_{\varphi}^{+\infty}\left(\oint_{\Sigma_G}\beta^{ij}\partial_i\frac{1}{E}\partial_j\frac{1}{E}\mathrm{d}S\right)\mathrm{d}G\leq 0 \tag{3.29}$$

for 3-inDc, where (β^{ij}) is negative definite on Σ_G for all $G\in(\varphi,+\infty)$.

If the equality holds for a certain given Σ_φ, then we will have $D\log E(\boldsymbol{r})=0,\boldsymbol{r}\in\Sigma_G$ for all $G\in(\varphi,+\infty)$, and also

$$\oint_{\Sigma_G}\frac{H^2-K}{E}\mathrm{d}S=\oint_{\Sigma_G}\left(H^2-K-\frac{|D\log E|^2}{4}\right)\frac{\mathrm{d}S}{E}=0 \tag{3.30}$$

for all $G\in(\varphi,+\infty)$. This implies that the strictly convex equipotential surfaces $\Sigma_G,G\in(\varphi,+\infty)$ are concentric spheres, and that $G(\boldsymbol{r})=\frac{1}{4\pi|\boldsymbol{r}|}$ for $0<|\boldsymbol{r}|<\frac{1}{4\pi\varphi}$. Again, by unique continuation, we conclude that $\partial\Omega$ must be a sphere centered at the origin.

After completing the verification of Theorem 1.1, we can wrap up our main course with the 3-inDc case of Corollary 1.1. Suppose that we have $|D\log E(\boldsymbol r)|=|\boldsymbol n\times\nabla G(\boldsymbol r)|=0,\boldsymbol r\in\Sigma_\varphi$ so that

$$\oint_{\Sigma_\varphi}\frac{(H^2-K)\,\mathrm{d}S}{E}=\oint_{\Sigma_\varphi}\left(H^2-K-\frac{|D\log E|^2}{4}\right)\frac{\mathrm{d}S}{E}\leq 0. \tag{3.31}$$

Since $H^2-K\geq 0$, we are led to $H^2-K\equiv 0$ on Σ_φ. Therefore, equality holds in (3.29), and we are reduced to the scenario in the last paragraph, with the same conclusion about the configuration of $\partial\Omega$.

It is worth noting that Agostiniani and Mazzieri [1, Appendix A] have furnished a general framework for asymptotic analysis of exterior and interior Dirichlet problems involving harmonic potentials, applicable to Euclidean spaces of arbitrary dimensions. I thank an anonymous referee for bringing my attention to their work.

3.3. Related problems in $\mathbb R^d$ $(d\geq 2)$. In the next theorem, we strengthen the first statement of Corollary 1.1 in $\mathbb R^d$ $(d>2)$.

THEOREM 3.1 (A free boundary problem in d-exD). *Let $d\in\mathbb Z_{>2}$. Suppose that $U(\boldsymbol r),\boldsymbol r\in\Omega\subset\mathbb R^d$ solves the Laplace equation in an unbounded domain Ω, whose boundary $\partial\Omega$ is a compact (hyper)surface. This function has asymptotic behavior $U(\boldsymbol r)\sim\frac{\Phi}{d(d-2)\pi^{d/2}|\boldsymbol r|^{d-2}}\int_0^\infty t^{d/2}e^{-t}\mathrm{d}t$ as $|\boldsymbol r|\to+\infty$. If one equipotential surface in Ω is a (hyper)sphere centered at $\boldsymbol r_0\in\mathbb R^d$, then $U(\boldsymbol r)=\frac{\Phi}{d(d-2)\pi^{d/2}|\boldsymbol r-\boldsymbol r_0|^{d-2}}\int_0^\infty t^{d/2}e^{-t}\mathrm{d}t$ holds for all $\boldsymbol r\in\Omega$.*

Proof. Suppose that we have an equipotential surface $|\boldsymbol r-\boldsymbol r_0|=R$. Define

$$V(\boldsymbol r')\equiv V\left(\frac{R^2(\boldsymbol r-\boldsymbol r_0)}{|\boldsymbol r-\boldsymbol r_0|^2}\right):=|\boldsymbol r-\boldsymbol r_0|^{d-2}U(\boldsymbol r),\quad |\boldsymbol r-\boldsymbol r_0|\geq R. \tag{3.32}$$

One can check that this expression extends to a bounded harmonic function $V(\boldsymbol r'),|\boldsymbol r'|\leq R$, whose boundary value is a constant. Such a harmonic function must be a constant function. This proves our claim that all the equipotential surfaces of $U(\boldsymbol r),\boldsymbol r\in\Omega\subset\mathbb R^d$ are (hyper)spherical. ∎

We note that the second half of Corollary 1.1 extends to d-inD (without any convexity requirements) of arbitrary dimensions d, as shown by Payne–Schaeffer [16, Theorem III.2]. Similar results for p-harmonic functions have also been obtained by Alessandrini–Rosset [2, Theorem 1.1], Enciso–Peralta-Salas [9, Theorem 1] and Poggesi [17, Theorem 1.3].

Before closing this article, we state and prove the planar analog of Corollary 1.1 (assuming that the boundary curves $\partial\Omega$ are always smooth Jordan curves), using results from [22].

THEOREM 3.2 (Circular solutions to three free boundary value problems). *If there is a circular equipotential curve in 2-exD, then the boundary $\partial\Omega$ must be a circle. If there is an equipotential curve in 2-exD on which $E(\boldsymbol r)=|\nabla U(\boldsymbol r)|$ remains constant, then $\partial\Omega$ must be a circle. If there is an equipotential curve in 2-inD on which $E(\boldsymbol r)=|\nabla G(\boldsymbol r)|$ remains constant, then $\partial\Omega$ must be a circle centered at the origin.*

Proof. In [22, §2.3], we have demonstrated the following inequality (strict unless $\partial\Omega$ is circular)

$$\oint_\Sigma\left[\boldsymbol n\times\nabla\frac{\kappa}{E}\right]\cdot\left[\boldsymbol n\times\nabla\frac{1}{E}\right]\mathrm{d}s\leq 0 \tag{3.33}$$

for all equipotential curves Σ in 2-exD. Here, the sign convention for curvature has been chosen so that the unit circle has $\kappa=+1$. If an equipotential curve Σ is circular, then we will have

$$\kappa\oint_{\Sigma}\left[\boldsymbol{n}\times\nabla\frac{1}{E}\right]\cdot\left[\boldsymbol{n}\times\nabla\frac{1}{E}\right]\mathrm{d}s\le 0 \tag{3.34}$$

for a positive constant κ. This means that $\boldsymbol{n}\times\nabla\frac{1}{E(\boldsymbol{r})}=\mathbf{0},\boldsymbol{r}\in\Sigma$ and equality holds in (3.33). Therefore, the boundary curve $\partial\Omega$ is indeed a circle.

For two-dimensional electrostatic problems, we have $\kappa=-2H$, $K\equiv 0$ and $\beta^{ij}\equiv 0$. Thus, Proposition 3.1 and Corollary 3.1 reduce to $\Delta\log E=0$ and $\Delta\frac{\boldsymbol{n}}{E}=0$, respectively. The vector Green identity in our proof of Corollary 3.2 then brings us an integral

$$\oint_{\Sigma_\varphi}\frac{\kappa^2-|D\log E|^2}{E}\mathrm{d}s \tag{3.35}$$

that is independent of φ. In [22, §2.2 and §3], we have shown that such a geometric conservation law can be paraphrased as

$$\oint_{\Sigma_\varphi}\left(\frac{\kappa}{E}-\oint_{\Sigma_\varphi}\frac{\kappa}{E}\mathrm{d}\mu\right)^2\mathrm{d}\mu=\oint_{\Sigma_\varphi}\left|D\frac{1}{E}\right|^2\mathrm{d}\mu \tag{3.36}$$

for a probability measure $\mathrm{d}\mu=E\,\mathrm{d}s/\Phi$. In both 2-exD and 2-inD, plugging a constant field intensity $D\frac{1}{E(\boldsymbol{r})}=\mathbf{0},\boldsymbol{r}\in\Sigma_\varphi$ into the right-hand side of the equation above, we may read off from the left-hand side that $D\frac{\kappa}{E}=0$ on the respective equipotential curve. This implies that there is an equality

$$\oint_{\Sigma_\varphi}\left[\boldsymbol{n}\times\nabla\frac{\kappa}{E}\right]\cdot\left[\boldsymbol{n}\times\nabla\frac{1}{E}\right]\mathrm{d}s=0. \tag{3.37}$$

According to our analysis in [22, §2.3 and §3], this can only happen if $\partial\Omega$ is a circle in 2-exD, or $\partial\Omega$ is a circle centered at the origin in 2-inD. □

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