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GLOBAL WELL-POSEDNESS OF THE STOCHASTIC CAMASSA-HOLM EQUATION*

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Abstract. We establish the existence of global martingale solutions of the stochastic Camassa-Holm equation in $H^1(\mathbb{R})$. The construction of the solution is based on the regularization method and the stochastic compactness method. Furthermore, we use Borel-Cantelli Lemma to prove the global existence of mild solution of the stochastic Camassa-Holm equation with small noise in $L^2(\mathbb{R})$.

Keywords. stochastic Camassa-Holm equation; martingale solutions; regularization; tightness.

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1. Introduction

The Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \tag{1.1}$$

was derived by Camassa and Holm in [5] as a model of shallow water waves. Here u denotes the fluid velocity in the x direction or, equivalently, the height of the water's free surface above a flat bottom [5, 37, 38, 40]. Equation (1.1) was originally derived by Fuchssteiner and Fokas [28, 29] as a bi-Hamiltonian generalization of KdV. A rigorous justification of the derivation of Equation (1.1) as an approach to the governing equations for water waves was recently provided by Constantin and Lannes [21].

Equation (1.1) is completely integrable [5, 20] as it can be written as a compatibility condition of two linear systems (Lax pair) with a real isospectral parameter λ, and has a bi-Hamiltonian structure [13, 28], which can be written as

$$m_t = -J_1 \frac{\delta H_2}{\delta m} = -J_2 \frac{\delta H_1}{\delta m}, \tag{1.2}$$

where

$$m = u - u_{xx},$$

the Hamiltonian operators

$$J_1 = \partial - \partial^3, \ J_2 = \partial m + m\partial,$$

and the corresponding Hamiltonians

$$H_1 = \frac{1}{2}\int_{\mathbb{R}} (u^2 + u_x^2)dx, \ H_2 = \frac{1}{2}\int_{\mathbb{R}} (u^3 + uu_x^2)dx.$$

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The Cauchy problem for the CH equation has been studied extensively. For initial data $u_0 \in H^s(\mathbb{R}), s > 3/2$, Equation (1.1) is locally well posed [14, 23, 42]. Moreover, Equation (1.1) has global strong solutions [14, 17] and also finite-time blow-up solutions [14, 15, 17, 18, 23, 42]. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$ [3, 4, 12, 16, 19, 32–34, 45]. The ill-posedness of the CH equation in $H^{3/2}$ and in the critical space $B^{3/2}_{2,r}, 1 < r < \infty$ is proved in [31].

Since there are some uncertainties in geophysical and climate dynamics [1, 35], it is widely recognized to take random effect into account in mathematical models. Using stochastic variational method, the stochastic CH equation was derived in [35, 36]. The wellposedness of stochastic CH equation with additive noise in $H^s, s > 3/2$ is proved in [7]. The multiplicative noise case is obtained in [44] in H^s, where $s > 3/2$ for the local wellposedness and $s > 3$ for the global existence. For the general Lévy process, the well-posedness in $H^s, s > 3/2$ is given in [9] as a special example. The wellposedness of stochastic modified CH equation with cubic nonlinearity in $H^s, s > 5/2$ is proved in [8]. In this paper, we will establish the existence of martingale solutions in H^1 and prove the regularization by the multiplicative noise of stochastic CH equation.

1.1. Martingale solutions. Introduce the following Hamiltonian function

$$\tilde{H}_2(m) = \frac{1}{2}\int_{\mathbb{R}} (u^3 + uu_x^2)dx - \frac{1}{2}\partial_x^{-1}\int_{\mathbb{R}} umdx\dot{W}, \tag{1.3}$$

where $\dot{W} = \frac{dW}{dt}$ is a white noise and W is a standard Brownian motion. Putting (1.3) into (1.2) with $H_2(m)$ replaced by $\tilde{H}_2(m)$, we get the following stochastic CH equation

$$dm + (um_x + 2mu_x)dt = mdW(t), t > 0, x \in \mathbb{R}. \tag{1.4}$$

Applying $(1 - \partial_x^2)^{-1}$ to both sides of (1.4), we have

$$du + uu_x dt + v_x dt = udW(t), \tag{1.5}$$

where the source term v is defined as a convolution:

$$v = G(x) * (u^2 + \frac{1}{2}u_x^2),\ G(x) = \frac{1}{2}e^{-|x|}. \tag{1.6}$$

For the initial data, we take

$$u(0,x) = u_0(x). \tag{1.7}$$

We will establish the martingale solution of (1.5)-(1.7), which is defined as follows.

DEFINITION 1.1. *A martingale solution of* (1.5)-(1.7) *is a system* $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}}), \tilde{W}, \tilde{u})$, *which satisfies*

(1) $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ *is a filtered probability space with filtration* $\tilde{\mathcal{F}}_t$,

(2) $\tilde{W}$ *is a* $\tilde{\mathcal{F}}_t$*-standard Brownian motion,*

(3) for almost every t, $\tilde{u}(t)$ *is progressively measurable,*

(4) $\tilde{u} \in L^2(\tilde{\Omega}; C([0,T]; H^1(\mathbb{R})))$. *For* $t \in [0,T], \varphi \in C^\infty$, *the following holds* $\tilde{\mathbb{P}}$*-a.s.*

$$\int_{\mathbb{R}} \tilde{u}(t)\varphi dx = \int_{\mathbb{R}} u_0\varphi dx - \int_0^t \int_{\mathbb{R}} (\tilde{u}\tilde{u}_x)(s)\varphi dxds + \int_0^t \int_{\mathbb{R}} \tilde{v}(s)\varphi_x dxds + \int_0^t \int_{\mathbb{R}} \tilde{u}(s)\varphi dxd\tilde{W},$$

where $\tilde{v} = \frac{1}{2}e^{-|x|} * (\tilde{u}^2 + \frac{1}{2}\tilde{u}_x^2)$.

Denote $\mathcal{M}^+(\mathbb{R})$ as the space of positive regular Borel measures on $\mathbb{R}$ with bounded total variation. The first main result is as follows.

THEOREM 1.1. *Let the initial data $u_0 \in H^1(\mathbb{R})$ and $m_0 = u_0 - u_{0xx} \in \mathcal{M}^+(\mathbb{R})$. Then there exists a global martingale solution of the stochastic Camassa-Holm Equation (1.5)-(1.7).*

Theorem 1.1 will be proved through the following steps.

Step 1: We consider (1.5) with the regularized initial value

$$u^\epsilon(0,x) = u_{0\epsilon}(x), \tag{1.8}$$

where $u_{0\epsilon} = \rho_\epsilon * u_0, 0 < \epsilon \ll 1$ and ρ_ϵ is the Friedrichs' mollifier

$$\rho_\epsilon = \Big(\int_{\mathbb{R}} \rho(\xi)d\xi\Big)^{-1}\epsilon^{-1}\rho(\epsilon^{-1}x),\ x \in \mathbb{R},$$

where $\rho \in C_c^\infty(\mathbb{R})$ is defined by

$$\rho(x) = \begin{cases} e^{1/(x^2-1)}, & \text{for} \quad |x| < 1, \\ 0, & \text{for} \quad |x| \geq 1. \end{cases}$$

The global existence of solution (u^ϵ, v^ϵ) of (1.5), (1.8) in the time interval $[0,T], \forall T > 0$ can be established by Lemma 2.1 and those local results in [7, 9, 44].

Step 2: We establish some uniform estimates of $(u^\epsilon, v^\epsilon, M^\epsilon)$ with $M^\epsilon = \int_0^t u^\epsilon dW$, which are important to get the tightness of the distributions of $(u^\epsilon, v^\epsilon, M^\epsilon)$. These can be obtained mainly by Itô formula and Burkholder-Davis-Gundy (B-D-G) inequality [11, 24, 25]. We also adapt some skills to estimate $\|u_x\|_{L^\infty}$ in Lemma 2.2.

Step 3: We get the tightness results of the random variable $(u^\epsilon, v^\epsilon, M^\epsilon)$ by some lemmas in [43]. Then, from the Jakubowski-Skorohod theorem [39], there exist a probability space $(\Omega^\sharp, \mathcal{F}^\sharp, \mathbb{P}^\sharp)$ and random variables $(\tilde{u}^\epsilon, \tilde{v}^\epsilon, \tilde{M}^\epsilon) \to (\tilde{u}, \tilde{v}, \tilde{M}), \mathbb{P}^\sharp$-a.s., such that the probability distribution of $(\tilde{u}^\epsilon, \tilde{v}^\epsilon, \tilde{M}^\epsilon)$ is the same as that of $(u^\epsilon, v^\epsilon, M^\epsilon)$. Using a cut-off function as in [6], we can show that $(\tilde{u}^\epsilon, \tilde{v}^\epsilon, \tilde{M}^\epsilon)$ satisfies the regularized equation in $(\Omega^\sharp, \mathcal{F}^\sharp, \mathbb{P}^\sharp)$. We also prove the limit $\tilde{M}$ is a martingale and can be expressed as $\tilde{M} = \int_0^t \tilde{u} d\tilde{W}$ in a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ which is an extension of $(\Omega^\sharp, \mathcal{F}^\sharp, \mathbb{P}^\sharp)$.

Step 4: We prove the strong convergence of $(\tilde{u}^\epsilon, \tilde{v}^\epsilon)$ in $L^2(\tilde{\Omega}; C([0,T]; L^2_{loc}(\mathbb{R})) \times L^2([0,T]; L^2_{loc}(\mathbb{R})))$ by the uniform integrability criterion and Vitali's convergence theorem. Since there exists $\tilde{u}_x^2$ in $\tilde{v}$, we also need to get the strong convergence of u_x in $L^2(\tilde{\Omega}; C([0,T]; L^2_{loc}(\mathbb{R})))$. It can be solved by the renormalized formulations in the stochastic cases and the stopping time skill. Then in view of the almost sure convergence on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, we can get that $(\tilde{u}, \tilde{W})$ is a martingale solution of (1.5)-(1.7) in the sense of Definition 1.1.

1.2. Mild solutions. If we take

$$\tilde{H}_2(m) = \frac{1}{2}\int_{\mathbb{R}} (u^3 + uu_x^2)dx - \frac{1}{2}\int_{\mathbb{R}} u^2 dx \circ \dot{W},$$

then we can get the following Stratonovich stochastic CH equation

$$dm + (um_x + 2mu_x)dt = m_x \circ dW(t), t > 0, x \in \mathbb{R}. \tag{1.9}$$

Compared with Equation (1.4), Equation (1.9) has the following conserved quantity

$$\|u\|_{H^1}^2 = \int_{\mathbb{R}} (u^2 + u_x^2) dx = \|u_0\|_{H^1}^2.$$

It is also proved in [22] that Equation (1.9) has similar properties as those for the determined case, such as peakon solutions, isospectrality and wave-breaking result.

For this form of noise, Flandoli et al. discovered in [27] that the noise could improve the theory of the linear transport equations. There are also some regularization by noise-type results (e.g. [2, 26, 30]). But, the relevant results for the stochastic nonlinear fluid equation are few. Let us write the Itô form of (1.9) as follows

$$dm + (um_x + 2mu_x)dt = \frac{1}{2} m_{xx} dt + m_x dW(t). \tag{1.10}$$

There is no regularizing effect from $\frac{1}{2}m_{xx}$, which is fully compensated by the Itô term. In fact, let $\eta(x,t) = m(x,t-W(t)) = u(x,t-W(t)) - u_{xx}(x,t-W(t))$, we have

$$\eta_t + u\eta_x + 2\eta u_x = 0,$$

which has the same regularization as the deterministic case. If the noise intensity in (1.10) is small, i.e.

$$dm + (um_x + 2mu_x)dt = \frac{1}{2} m_{xx} dt + \delta m_x dW(t), \tag{1.11}$$

with $\delta \in (0,1)$, we have a regularization from the operator $\frac{1}{2}\partial_x^2$. The effect of a small noise on the stochastic modified Camassa-Holm equation was studied in [10].

Applying $(1-\partial_x^2)^{-1}$ to both sides of (1.11), we have

$$du - \frac{1}{2} u_{xx} dt = -v_x dt + \delta u_x dW(t), \tag{1.12}$$

where v is given by (1.6). The mild solution of (1.12) is given by

$$u(t) = S(t)u_0 - \int_0^t S(t-s) v_x ds + \delta \int_0^t S(t-s) u_x dW(t), \tag{1.13}$$

where $S(t) = \mathcal{F}^{-1}(e^{-\frac{\xi^2}{2}t})$. By the semigroup theory of the stochastic parabolic PDEs [11], we prove the local existence and uniqueness of mild solution of Equation (1.12). Then, we obtain the global existence by the Borel-Cantelli Lemma. The result is as follows.

THEOREM 1.2. *Let the initial data $u_0(x) \in L^2(\mathbb{R})$. Then for any $T > 0$, the stochastic Camassa-Holm Equation (1.12) has a unique solution u such that $u \in L^2(\Omega; C([0,T]; L^2(\mathbb{R})) \cap L^2([0,T]; H^1(\mathbb{R})))$.*

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. In Section 3, we prove Theorem 1.2.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by three subsections. In Subsection 2.1, some estimates of the random variables $(u^\epsilon, v^\epsilon, M^\epsilon)$ are established. In Subsection 2.2, the tightness of the distribution of $(u^\epsilon, v^\epsilon, M^\epsilon)$ is obtained. Finally, the convergence of the random variables is proved in Subsection 2.3.

First, we give some notations. Given $p>1$, $\alpha\in(0,1)$, let $W^{\alpha,p}([0,T];K)$ be the Sobolev space of all $u\in L^p(0,T;K)$ such that

$$\int_0^T\int_0^T\frac{\|u(t)-u(s)\|_K^p}{|t-s|^{1+\alpha p}}dtds<\infty,$$

endowed with the norm

$$\|u\|^2_{W^{\alpha,p}([0,T];K)}=\int_0^T\|u\|_K^2dt+\int_0^T\int_0^T\frac{\|u(t)-u(s)\|_K^p}{|t-s|^{1+\alpha p}}dtds.$$

Denote $W^{1,1}([0,T]\times\mathbb{R})$ as the Sobolev space of $u\in L^1([0,T]\times\mathbb{R})$ and $u_t\in L^1([0,T]\times\mathbb{R})$.

2.1. Uniform estimates. In this subsection, we will construct some estimates of the random variables $(u^\epsilon,v^\epsilon,M^\epsilon)$ of the solution of (1.5), (1.8). To simplify the notation, we will drop ϵ in $(u^\epsilon,v^\epsilon,M^\epsilon)$ throughout this subsection.

LEMMA 2.1. *For $k\geq1$, we have*

$$\mathbb{E}\sup_{0\leq t\leq T}\|u\|^{2k}_{H^1}\leq2\|u_0\|^{2k}_{H^1}e^{CT}. \tag{2.1}$$

Proof. Differentiating (1.5) w.r.t. x one obtains

$$du_x=-(u_x^2+uu_{xx})dt-v_{xx}dt+u_xdW=-\frac{1}{2}(u_x^2+2uu_{xx}-2u^2)dt-vdt+u_xdW. \tag{2.2}$$

By applying the Itô formula to $\|u\|^2_{L^2}$ of Equation (1.4) and $\|u_x\|^2_{L^2}$ of Equation (2.2), we get

$$\|u\|^2_{L^2}=\|u_0\|^2_{L^2}-2\int_0^t(u,v_x)ds+\int_0^t\|u\|^2_{L^2}ds+2\int_0^t(u,u)dW,$$

and

$$\|u_x\|^2_{L^2}=\|u_{0x}\|^2_{L^2}-2\int_0^t(u_x,v)ds+\int_0^t\|u_x\|^2_{L^2}ds+2\int_0^t(u_x,u_x)dW,$$

where $(u,uu_x)=0$ and $(u_x,u_x^2+2uu_{xx}-2u^2)=0$ are used. Then, we have

$$\|u\|^2_{H^1}=\|u\|^2_{L^2}+\|u_x\|^2_{L^2}=\|u_0\|^2_{H^1}+\int_0^t\|u\|^2_{H^1}ds+2\int_0^t\|u\|^2_{H^1}dW, \tag{2.3}$$

from which and applying Itô formula to $\|u\|^{2k}_{H^1}$ with $k\geq1$, we have

$$\begin{aligned}d\|u\|^{2k}_{H^1}=&d(\|u\|^2_{H^1})^k\\=&k\|u\|^{2k-2}_{H^1}d\|u\|^2_{H^1}+\frac{k(k-1)}{2}\|u\|^{2k-4}_{H^1}d\|u\|^2_{H^1}d\|u\|^2_{H^1}\\=&k\|u\|^{2k}_{H^1}dt+2k(k-1)\|u\|^{2k}_{H^1}dt+2k\|u\|^{2k}_{H^1}dW.\end{aligned}$$

By B-D-G inequality and Young inequality, we get

$$\mathbb{E}\sup_{0\leq t\leq T}2k\int_0^t\|u\|^{2k}_{H^1}dW\leq C\mathbb{E}(\int_0^T\|u\|^{4k}_{H^1}ds)^{1/2}$$

$$\leq C\mathbb{E}(\sup_{0\leq t\leq T}\|u(s)\|_{H^1}^{2k}\int_0^T\|u(s)\|_{H^1}^{2k}ds)^{1/2}$$
$$\leq \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T}\|u(s)\|_{H^1}^{2k}+C\mathbb{E}\int_0^T\|u(s)\|_{H^1}^{2k}ds.$$

Then, it follows from the above estimates

$$\mathbb{E}\sup_{0\leq t\leq T}\|u\|_{H^1}^{2k}\leq 2\|u_0\|_{H^1}^{2k}+C\mathbb{E}\int_0^T\|u(s)\|_{H^1}^{2k}ds,$$

from which and Grönwall inequality (2.1) is obtained. □

REMARK 2.1. By the Sobolev embedding theorem, we have for $k\geq 1$

$$\mathbb{E}\sup_{0\leq t\leq T}\|u(t)\|_{L^\infty}^{2k}\leq \mathbb{E}\sup_{0\leq t\leq T}\|u(t)\|_{H^1}^{2k}\leq C. \tag{2.4}$$

LEMMA 2.2. *Suppose $m_0=u_0-u_{0xx}\in\mathcal{M}^+(\mathbb{R})$. Then for $k\geq 1$*

$$\mathbb{E}\sup_{0\leq t\leq T}\|u_x\|_{L^\infty}^{2k}\leq C.$$

Proof. Let $p(t,x)$ be the solution of the following equation for a.e. $\omega\in\Omega$

$$\begin{cases}\partial_t p=u(t,p),\ 0<t<T,\\ p(0,x)=x,\ x\in\mathbb{R}.\end{cases} \tag{2.5}$$

By the well-known results in the theory of ordinary differential equations as that in [18], Equation (2.5) has a unique solution $p\in C^1([0,T]\times\mathbb{R},\mathbb{R})$ for a.e. $\omega\in\Omega$.

By Itô multiplicative formula, we have

$$\begin{aligned}d[m(t,p)p_x^2]&=[dm+m_x dp]p_x^2+2mp_x dp_x\\ &=[dm+m_x u dt]p_x^2+2mp_x^2u_x dt=mp_x^2dW,\end{aligned}$$

from which we get

$$m(t,p)p_x^2=m_0e^{W(t)-\frac{t}{2}}. \tag{2.6}$$

Since $u=\frac{1}{2}e^{-|x|}*m$, we have

$$u(t,x)=\frac{1}{2}e^{-x}\int_{-\infty}^x e^y m(t,y)dy+\frac{1}{2}e^x\int_x^\infty e^{-y}m(t,y)dy, \tag{2.7}$$

from which we deduce that

$$u_x(t,x)=-\frac{1}{2}e^{-x}\int_{-\infty}^x e^y m(t,y)dy+\frac{1}{2}e^x\int_x^\infty e^{-y}m(t,y)dy.$$

Consequently,

$$u(t,x)+u_x(t,x)=e^x\int_x^\infty e^{-y}m(t,y)dy, \tag{2.8}$$

$$u(t,x)-u_x(t,x)=e^{-x}\int_{-\infty}^{x}e^{y}m(t,y)dy. \tag{2.9}$$

If m_0 does not change sign on $\mathbb{R}$, then by (2.6), so does m. Since $m_0=u_0-u_{0xx}\in\mathcal{M}^+(\mathbb{R})$, by Lemmas 3.3 and 3.4 in [16], we have $m_{0\epsilon}\geq 0$ and $u_{0\epsilon}\rightharpoonup u_0$ weakly in H^1. Then by (2.7), $u\geq 0$ and by (2.8)-(2.9),

$$-u(t,x)\leq u_x(t,x)\leq u(t,x),$$

from which and (2.4) we obtain that

$$\mathbb{E}\sup_{0\leq t\leq T}\|u_x\|_{L^\infty}^{2k}\leq \mathbb{E}\sup_{0\leq t\leq T}\|u\|_{L^\infty}^{2k}\leq C, \tag{2.10}$$

where $k\geq 1$. □

We give some estimates for the nonlinear term v defined by (1.6).

LEMMA 2.3. *For $j=1$ or ∞, and $k\geq 1$, we have*

$$\mathbb{E}\sup_{0\leq t\leq T}\|v\|_{W^{1,j}}^{k}\leq C, \tag{2.11}$$

$$\mathbb{E}\|v\|_{W^{1,1}([0,T]\times\mathbb{R})}^{k}\leq C. \tag{2.12}$$

Proof. Let $G(x)=\frac{1}{2}e^{-|x|}$. Then $\|G\|_{W^{1,j}}\leq C$ for $j=1$ or ∞. By Young inequality,

$$\begin{aligned}\|v\|_{W^{1,j}}^{k}&=\|\int_{\mathbb{R}}G(x-y)(u^2+\frac{1}{2}u_x^2)(y)dy\|_{W^{1,j}}^{k}\\&\leq\|G\|_{W^{1,j}}\|u^2+\frac{1}{2}u_x^2\|_{L^1}^{k}\leq C\|u\|_{H^1}^{2k},\end{aligned}$$

from which and Lemma 2.1, (2.11) is obtained.

Next, we prove (2.12). By Itô multiplicative formula, we have

$$\begin{aligned}&\|\frac{d}{dt}v\|_{L^1([0,T]\times\mathbb{R})}\\&=\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}G(x-y)[2udu+u_xdu_x+\frac{du}{dt}\frac{du}{dt}+\frac{1}{2}\frac{du_x}{dt}\frac{du_x}{dt}]dydxdt\\&=2\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}G(x-y)[u(-uu_x-v_x+u\frac{dW}{dt})]dydxdt\\&\quad+\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}G(x-y)\{u_x[-\frac{1}{2}(u_x^2+2uu_{xx}-2u^2)-v+u_x\frac{dW}{dt}]\}dydxdt\\&\quad+\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}G(x-y)[u^2+\frac{1}{2}u_x^2]dydxdt\\&=:I_1+I_2+I_3.\end{aligned}$$

By Young and Hölder inequalities, we have

$$\begin{aligned}I_1&\leq C\int_0^T\|G\|_{L^\infty}(\|u^2u_x\|_{L^1}+\|uv_x\|_{L^1})dt+C\int_0^T\|G\|_{L^\infty}\|u^2\|_{L^1}dW\\&\leq C\int_0^T(\|u\|_{L^\infty}\|u\|_{L^2}\|u_x\|_{L^2}+\|u\|_{L^2}\|v_x\|_{L^2})dt+C\int_0^T\|u\|_{L^2}^2dW\end{aligned}$$

$$\leq C\int_0^T (\|u\|_{L^2}^2 + \|u\|_{L^\infty}^2\|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2)dt + C\int_0^T \|u\|_{L^2}^2 dW$$
$$\leq C\int_0^T (\|u\|_{L^2}^2 + \|u\|_{H^1}^4 + \|v_x\|_{L^2}^2)dt + C\int_0^T \|u\|_{L^2}^2 dW.$$

We can rewrite I_2 as follows

$$I_2 = \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} G(x-y)[-\frac{1}{2}(uu_x^2)_x + u_x u^2) - u_x v + u_x^2 \frac{dW}{dt}]dydxdt$$
$$= \frac{1}{2}\int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{dG(x-y)}{dx} uu_x^2 dydxdt - \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} G(x-y)u_x v dydxdt$$
$$+ \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} G(x-y)u_x^2 dydxdW.$$

Then Young and Hölder inequalities imply

$$I_2 \leq C\int_0^T \|G\|_{W^{1,\infty}}\|uu_x^2\|_{L^1}dt + \int_0^T \|G\|_{W^{1,\infty}}\|u_x v\|_{L^1}dt + \int_0^T \|G\|_{W^{1,\infty}}\|u_x^2\|_{L^1}dW$$
$$\leq C\int_0^T (\|u\|_{L^\infty}^2 + \|u_x^2\|_{L^1}^2 + \|u_x\|_{L^2}\|v\|_{L^2})dt + \int_0^T \|u_x\|_{L^2}^2 dW$$
$$\leq C\int_0^T (\|u\|_{H^1}^2 + \|u\|_{H^1}^4 + \|v\|_{L^2}^2)dt + \int_0^T \|u_x\|_{L^2}^2 dW.$$

Similarly, we have

$$I_3 \leq C\int_0^T \|u\|_{H^1}^2 dt.$$

Hence, it follows from the above estimates that

$$\mathbb{E}\|\frac{d}{dt}v\|_{L^1([0,T]\times\mathbb{R})}^k \leq C\mathbb{E}[\int_0^T (\|u\|_{H^1}^2 + \|u\|_{H^1}^4 + \|v\|_{W^{1,2}}^2)dt]^k$$
$$\leq C\mathbb{E}\int_0^T (\|u\|_{H^1}^{2k} + \|u\|_{H^1}^{4k} + \|v\|_{W^{1,2}}^{2k})dt \leq C,$$

where the last inequality follows from Lemma 2.1 and (2.11). ∎

2.2. Tightness. In this subsection, we obtain the tightness of M^ϵ, u^ϵ and v^ϵ.

LEMMA 2.4 (Tightness). *Define*

$$S = C([0,T];L_{loc}^2(\mathbb{R})) \times L^p([0,T];L_{loc}^p(\mathbb{R})) \times C([0,T];H_{loc}^1(\mathbb{R})),\ p \geq 1,$$

equipped with its Borel σ-algebra. Let μ^ϵ be the probability measure on S which is the image of $\mathbb{P}$ on Ω by the map: $\omega \to (u^\epsilon(\omega,\cdot), v^\epsilon(\omega,\cdot), M^\epsilon(\omega,\cdot))$, that is, for any $B \subset S$,

$$\mu^\epsilon(B) = \mathbb{P}(\omega \in \Omega : (u^\epsilon(\omega,\cdot), v^\epsilon(\omega,\cdot), M^\epsilon(\omega,\cdot)) \in B).$$

Then the sequence of the probability measure μ^ϵ is tight.

Proof.
Step 1: We will show for each $\eta>0$, there is a compact subset $\mathcal{K}_1^\eta$ of $C([0,T];H^1_{loc}(\mathbb{R}))$ such that $\mathbb{P}(M^\epsilon\notin\mathcal{K}_1^\eta)\le\frac{\eta}{3}$.

By B-D-G inequality and Lemma 2.1, we have for $k\ge 1$,

$$\mathbb{E}\sup_{0\le t\le T}\|M^\epsilon(t)\|_{H^1}^{2k}=\mathbb{E}\sup_{0\le t\le T}\|\int_0^t u^\epsilon dW\|_{H^1}^{2k}$$
$$\le C\mathbb{E}(\int_0^T\|u^\epsilon\|_{H^1}^2dt)^k\le C\mathbb{E}\int_0^T\|u^\epsilon\|_{H^1}^{2k}dt\le CT. \tag{2.13}$$

By Itô formula,

$$\|M^\epsilon(t)-M^\epsilon(s)\|_{H^1}^2\le\int_s^t\|u^\epsilon(r)\|_{H^1}\|M^\epsilon(r)-M^\epsilon(s)\|_{H^1}dW(r)+\int_s^t\|u^\epsilon(r)\|_{H^1}^2dr.$$

By Lemma 2.1, B-D-G inequality, Hölder and Young inequalities, we obtain

$$\mathbb{E}\sup_{s\le\tau\le t}(\int_s^\tau\|u^\epsilon(r)\|_{H^1}\|M^\epsilon(r)-M^\epsilon(s)\|_{H^1}dW(r))^2$$
$$\le\mathbb{E}\int_s^t\|u(r)\|_{H^1}^2\|M^\epsilon(r)-M^\epsilon(s)\|_{H^1}^2dr$$
$$\le\mathbb{E}\sup_{s\le r\le t}\|M^\epsilon(r)-M^\epsilon(s)\|_{H^1}^2\|u(r)\|_{H^1}^2(t-s)$$
$$\le\frac{1}{2}\mathbb{E}\sup_{s\le r\le t}\|M^\epsilon(r)-M^\epsilon(s)\|_{H^1}^4+C(t-s)^2.$$

From the above two estimates, we have

$$\mathbb{E}\sup_{s\le r\le t}\|M^\epsilon(r)-M^\epsilon(s)\|_{H^1}^4\le C(t-s)^2+2\mathbb{E}\sup_{s\le r\le t}\|u^\epsilon(r)\|_{H^1}^4(t-s)^2\le C(t-s)^2.$$

Hence,

$$\mathbb{E}\|M^\epsilon\|^4_{W^{\frac{3}{8},4}(0,T;H^1)}=\mathbb{E}\int_0^T\int_0^T\frac{\|M^\epsilon(r)-M^\epsilon(s)\|_{H^1}^4}{|t-s|^{\frac{5}{2}}}dtds\le C, \tag{2.14}$$

where C is independent of ϵ.

Let

$$\mathcal{K}_1^\eta=\{g\in C([0,T];H^1):\|g\|_{W^{\frac{3}{8},4}([0,T];H^1)}\le R\}.$$

Then $\mathcal{K}_1^\eta$ is a compact subset of $C([0,T];H^1_{loc})$ by Corollary 2 in [43]. It follows from (2.14) and Chebyshev inequality that

$$\mathbb{P}(M^\epsilon\notin\mathcal{K}_1^\eta)=\mathbb{P}(\|M^\epsilon\|_{W^{\frac{3}{8},4}([0,T];H^1)}\ge R)\le\frac{\mathbb{E}\|M^\epsilon\|^4_{W^{\frac{3}{8},4}(0,T;H^1)}}{R^4}\le\frac{C}{R^4}.$$

Choosing $R^4=3C\eta^{-1}$, we get

$$\mathbb{P}(M^\epsilon\in\mathcal{K}_1^\eta)\ge 1-\frac{\eta}{3}. \tag{2.15}$$

Step 2: Find a compact subset $\mathcal{K}_2^\eta$ of $C([0,T];L^2_{loc}(\mathbb{R}))$ such that $\mathbb{P}(u^\epsilon \notin \mathcal{K}_2^\eta) \le \frac{\eta}{3}$.

Let

$$\tilde{\mathcal{K}}_2^\eta = \{g \in C([0,T];H^1) : \|g\|_{C([0,T];H^1)} \le R, \|\partial_t g\|_{C([0,T];L^2)} \le R\}.$$

Then by Lemma A.1, $\tilde{\mathcal{K}}_2^\eta$ is a compact subset of $C([0,T];L^2_{loc}(\mathbb{R}))$.

From (1.5), Hölder, Young and interpolation inequalities,

$$\begin{aligned}\|\partial_t(u^\epsilon - M^\epsilon)\|_{L^2} &\le \|u^\epsilon u_x^\epsilon\|_{L^2} + \|v_x^\epsilon\|_{L^2} \\ &\le C\|u^\epsilon\|_{L^\infty}\|u^\epsilon\|_{H^1} + C\|v_x^\epsilon\|_{L^1}^{1/2}\|v_x^\epsilon\|_{L^\infty}^{1/2} \\ &\le C(\|u^\epsilon\|_{L^\infty}^2 + \|u^\epsilon\|_{H^1}^2 + \|v_x^\epsilon\|_{L^1} + \|v_x^\epsilon\|_{L^\infty}).\end{aligned}$$

which combined with (2.4), Lemmas 2.1 and 2.3 imply

$$\mathbb{E}\sup_{0\le t\le T}\|\partial_t(u^\epsilon - M^\epsilon)\|_{L^2} \le C. \tag{2.16}$$

It follows from Lemma 2.1, (2.13), (2.16) and Chebyshev inequality that

$$\begin{aligned}\mathbb{P}(u^\epsilon - M^\epsilon \notin \tilde{\mathcal{K}}_2^\eta) &\le \mathbb{P}(\|u^\epsilon - M^\epsilon\|_{C([0,T];H^1)} \ge R) + \mathbb{P}(\|\partial_t(u^\epsilon - M^\epsilon)\|_{C([0,T];H^1)} \ge R) \\ &\le \frac{\mathbb{E}\|u^\epsilon - M^\epsilon\|_{C([0,T];H^1)}^2 + \mathbb{E}\|\partial_t(u^\epsilon - M^\epsilon)\|_{C([0,T];H^1)}^2}{R^2} \\ &\le \frac{C}{R^2}.\end{aligned}$$

Choosing $R^2 = 3C\eta^{-1}$, we have

$$\mathbb{P}(u^\epsilon - M^\epsilon \in \tilde{\mathcal{K}}_2^\eta) \ge 1 - \frac{\eta}{3}. \tag{2.17}$$

It follows from (2.15) and (2.17) that there exists a compact subset $\mathcal{K}_2^\eta$ of $C([0,T];L^2_{loc}(\mathbb{R}))$ such that

$$\mathbb{P}(u^\epsilon \in \mathcal{K}_2^\eta) \ge 1 - \frac{\eta}{3}.$$

Step 3: Find a compact subset $\mathcal{K}_3^\eta$ of $L^p([0,T];L^p_{loc}(\mathbb{R}))$ such that $\mathbb{P}(v^\epsilon \notin \mathcal{K}_3^\eta) \le \frac{\eta}{3}$.

Let

$$\mathcal{K}_3^\eta = \{v \in C([0,T];H^1) : \|g\|_{C([0,T];W^{1,\infty})} \le R, \|g\|_{W^{1,1}([0,T]\times\mathbb{R})} \le R\}.$$

Since $W^{1,\infty}(\mathbb{R}) \subset\subset L^p_{loc}(\mathbb{R}) \subset L^1(\mathbb{R})$, then by Lemma A.1, $\mathcal{K}_3^\eta$ is a compact subset of $L^p([0,T];L^p_{loc}(\mathbb{R}))$ with $p \ge 1$. It follows from Lemma 2.3 and Chebyshev inequality that

$$\begin{aligned}\mathbb{P}(v^\epsilon \notin \mathcal{K}_3^\eta) &\le \mathbb{P}(\|v^\epsilon\|_{C([0,T];W^{1,\infty})} \ge R) + \mathbb{P}(\|v^\epsilon\|_{W^{1,1}([0,T]\times\mathbb{R})} \ge R) \\ &\le \frac{\mathbb{E}\|v^\epsilon\|_{C([0,T];W^{1,\infty})}^2 + \mathbb{E}\|v^\epsilon\|_{W^{1,1}([0,T]\times\mathbb{R})}^2}{R^2} \\ &\le \frac{C}{R^2}.\end{aligned}$$

Choosing $R^2 = 3C\eta^{-1}$, we have

$$\mathbb{P}(v^\epsilon \in \mathcal{K}_3^\eta) \ge 1 - \frac{\eta}{3}.$$

In conclusion, for any $\eta > 0$, there exists compact subset $\mathcal{K}_1^\eta \times \mathcal{K}_2^\eta \times \mathcal{K}_3^\eta$ of S such that

$$\mathbb{P}(\omega : M^\epsilon \in \mathcal{K}_1^\eta, u^\epsilon \in \mathcal{K}_2^\eta, v^\epsilon \in \mathcal{K}_3^\eta) \geq 1 - \epsilon.$$

Hence, the tightness property of μ^ϵ is proved. ∎

From the tightness property in Lemma 2.4 and Prokhorov's theorem, there exists a subsequence such that $\mu^\epsilon \to \mu$ weakly, where μ is a probability on S. According to Skorokhod's theorem, there exists a probability space $(\Omega^\sharp, \mathcal{F}^\sharp, \mathbb{P}^\sharp)$ and random variables $(\tilde{u}^\epsilon, \tilde{v}^\epsilon, \tilde{M}^\epsilon)$, and $(\tilde{u}, \tilde{v}, \tilde{M})$ with values in S such that

$$\mathcal{L}(u^\epsilon) = \mathcal{L}(\tilde{u}^\epsilon),\ \mathcal{L}(v^\epsilon) = \mathcal{L}(\tilde{v}^\epsilon),\ \mathcal{L}(M^\epsilon) = \mathcal{L}(\tilde{M}^\epsilon), \tag{2.18}$$

where $\mathcal{L}(\cdot)$ denotes the probability law of $(\cdot)$ and

$$(\tilde{u}^\epsilon, \tilde{v}^\epsilon, \tilde{M}^\epsilon) \to (\tilde{u}, \tilde{v}, \tilde{M}) \text{ in } S,\ \mathbb{P}^\sharp - a.s. \tag{2.19}$$

Next, we need to prove that $(\tilde{u}^\epsilon, \tilde{v}^\epsilon, \tilde{M}^\epsilon)$ satisfies the following equation:

$$\tilde{u}^\epsilon(t) = u_{0\epsilon} - \int_0^t \tilde{u}^\epsilon \tilde{u}_x^\epsilon ds - \int_0^t \tilde{v}_x^\epsilon ds + \tilde{M}^\epsilon. \tag{2.20}$$

In order to prove (2.20), we define

$$\gamma^\epsilon(t) \triangleq \int_0^T \|u^\epsilon(t) - u_{0\epsilon} + \int_0^t u^\epsilon u_x^\epsilon ds + \int_0^t v_x^\epsilon ds - M^\epsilon\|_{H^{-1}}^2 dt.$$

Of course

$$\gamma^\epsilon = 0,\ \mathbb{P} - a.s. \tag{2.21}$$

Similarly, we denote

$$\tilde{\gamma}^\epsilon(t) \triangleq \int_0^T \|\tilde{u}^\epsilon(t) - u_{0\epsilon} + \int_0^t \tilde{u}^\epsilon \tilde{u}_x^\epsilon ds + \int_0^t \tilde{v}_x^\epsilon ds - \tilde{M}^\epsilon\|_{H^{-1}}^2 dt. \tag{2.22}$$

We have the following lemma.

Lemma 2.5. *For $\tilde{\gamma}^\epsilon$ defined in (2.22), we have $\tilde{\gamma}^\epsilon = 0$, $\mathbb{P}^\sharp - a.s.$. That is, $(\tilde{u}^\epsilon, \tilde{v}^\epsilon, \tilde{M}^\epsilon)$ satisfies (2.20).*

Proof. By (2.18) and the continuity of $\tilde{\gamma}^\epsilon$, we have that the distribution of $\tilde{\gamma}^\epsilon$ is equal to the distribution of γ^ϵ on $\mathbb{R}^+$, that is

$$\mathbb{E}^\sharp \phi(\tilde{\gamma}^\epsilon) = \mathbb{E}\phi(\gamma^\epsilon), \tag{2.23}$$

for any $\phi \in C_b(\mathbb{R}^+)$, which is the space of continuous bounded functions on $\mathbb{R}^+$. Now, for $\forall \eta > 0$, define $\phi_\eta \in C_b(\mathbb{R}^+)$ by

$$\phi_\eta(y) = \begin{cases} \frac{y}{\eta}, & \text{for} \quad 0 \leq y < \eta, \\ 1_{[\eta,\infty)}(y), & \text{for} \quad y \geq \eta. \end{cases}$$

Then by (2.21) and (2.23),

$$\mathbb{P}^\sharp(\tilde{\gamma}^\epsilon \geq \eta) = \int_{\Omega^\sharp} 1_{[\eta,\infty)} \tilde{\gamma}^\epsilon d\mathbb{P}^\sharp$$

$$\begin{aligned}&\leq \int_{\Omega^\sharp} 1_{[0,\eta]}\frac{\tilde{\gamma}^\epsilon}{\eta} d\mathbb{P}^\sharp + \int_{\Omega^\sharp} 1_{[\eta,\infty)}\tilde{\gamma}^\epsilon d\mathbb{P}^\sharp \\ &= \mathbb{E}^\sharp \phi_\eta(\tilde{\gamma}^\epsilon) = \mathbb{E}\phi_\eta(\gamma^\epsilon) = 0.\end{aligned} \tag{2.24}$$

Since η is arbitrary, we can infer from (2.24) that

$$\tilde{\gamma}^\epsilon = 0,\ \mathbb{P}^\sharp - a.s.,$$

from which we get that $(\tilde{u}^\epsilon, \tilde{v}^\epsilon, \tilde{M}^\epsilon)$ satisfies (2.20). ∎

LEMMA 2.6. *The limit process $\tilde{M}$ in (2.19) is an H^1-valued continuous martingale. Moreover, there exists a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that*

$$\tilde{M}(t) = \int_0^t \tilde{u} d\tilde{W}, \tag{2.25}$$

where $\tilde{W}$ is a standard Brownian motion over the basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

Proof. By Fatou's lemma, (2.18) and (2.13),

$$\begin{aligned}\mathbb{E}^\sharp \|\tilde{M}\|^4_{C([0,T];H^1)} &\leq \underline{lim}_{\epsilon\to 0} \mathbb{E}^\sharp \|\tilde{M}^\epsilon\|^4_{C([0,T];H^1)} \\ &= \underline{lim}_{\epsilon\to 0} \mathbb{E}\|M^\epsilon\|^4_{C([0,T];H^1)} \leq C.\end{aligned} \tag{2.26}$$

For any bounded continuous function φ on $H^1 \times L^2$ and $0 \leq r \leq s \leq t \leq T$, it holds

$$\mathbb{E}((M^\epsilon(t) - M^\epsilon(s))\varphi(u^\epsilon(r), v^\epsilon(r))) = 0,$$

which yields

$$\mathbb{E}^\sharp((\tilde{M}^\epsilon(t) - \tilde{M}^\epsilon(s))\varphi(\tilde{u}^\epsilon(r), \tilde{v}^\epsilon(r))) = 0.$$

Hence

$$\mathbb{E}^\sharp((\tilde{M}(t) - \tilde{M}(s))\varphi(\tilde{u}(r), \tilde{v}(r))) = 0. \tag{2.27}$$

Let $\hat{\mathcal{F}}_t$ be the σ-algebra generated by $(\tilde{u}(r), \tilde{v}(r), \tilde{M}(r)), 0 \leq r \leq t$, and all $\mathbb{P}^\sharp$-negligible sets in $\mathcal{F}^\sharp$. Then, set

$$\mathcal{F}^\sharp_t = \bigcap_{\eta > 0} \hat{\mathcal{F}}_{t+\eta},\ 0 \leq t < T.$$

By (2.26)-(2.27), $\tilde{M}$ is an H^1-valued continuous martingale with respect to $\{\mathcal{F}^\sharp_t\}$.

For $a, b \in H^1, 0 \leq t \leq T$ and almost all $\tilde{\omega} \in \tilde{\Omega}$, we find as $\epsilon \to 0$,

$$\begin{gathered}(\tilde{M}^\epsilon(t), a)_{H^1}(\tilde{M}^\epsilon(t), b)_{H^1} \to (\tilde{M}(t), a)_{H^1}(\tilde{M}(t), b)_{H^1}, \\ \int_0^t (a, \tilde{u}^\epsilon)_{H^1}(b, \tilde{u}^\epsilon)_{H^1} ds \to \int_0^t (a, \tilde{u})_{H^1}(b, \tilde{u})_{H^1} ds.\end{gathered}$$

Hence, the quadratic variation of $\tilde{M}$ is given by

$$\langle \tilde{M} \rangle_t = \int_0^t \sum_{j=1}^\infty (e_j, \tilde{u})^2_{H^1} ds = \int_0^t \|\tilde{u}\|^2_{H^1} ds$$

where $\{e_j\}$ is an orthonormal basis for H^1. By Theorem 8.2 in [24], there exists a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ which is an extension of $(\Omega^\sharp, \mathcal{F}^\sharp, \mathbb{P}^\sharp)$ such that (2.25) holds. ∎

2.3. Convergence. In this subsection, we get some strong convergence in $\tilde{\Omega}\times[0,T]\times\mathbb{R}$. Let $q^\epsilon=\tilde{u}^\epsilon_x$. Then from (2.20), q^ϵ satisfies

$$dq^\epsilon=(-(\tilde{u}^\epsilon q^\epsilon)_x-\tilde{u}^{\epsilon 2}-\frac{1}{2}q^{\epsilon 2}+\tilde{v}^\epsilon)dt+d\tilde{M}^\epsilon_x. \tag{2.28}$$

LEMMA 2.7 (Convergence). *The following convergences hold*

$$\tilde{u}^\epsilon\to\tilde{u} \text{ strongly in } L^2(\tilde{\Omega};C([0,T];L^2_{loc}(\mathbb{R}))), \tag{2.29}$$

$$\tilde{v}^\epsilon\to\tilde{v} \text{ strongly in } L^2(\tilde{\Omega};L^2([0,T];L^2_{loc}(\mathbb{R}))), \tag{2.30}$$

$$q^\epsilon\rightharpoonup q \text{ weakly in } L^k(\tilde{\Omega};C([0,T];L^2(\mathbb{R}))), \tag{2.31}$$

$$q^{\epsilon 2}\rightharpoonup\overline{q^2} \text{ weakly in } L^k(\tilde{\Omega};C([0,T];L^p(\mathbb{R}))), \tag{2.32}$$

where $1\le k<\infty, 1\le p<\infty$. *Moreover,*

$$q^2(\omega,t,x)\le\overline{q^2}(\omega,t,x), \text{ for almost all } (\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}, \tag{2.33}$$

$$q=\tilde{u}_x, \text{ in the sense of distributions on } \tilde{\Omega}\times[0,T]\times\mathbb{R}. \tag{2.34}$$

Proof. Let us consider the positive nondecreasing function $f(x)=x^2$, which satisfies $\lim_{x\to\infty}\frac{f(x)}{x}=\infty$. Since $\tilde{u}^\epsilon$ and u^ϵ has the same distribution, by Lemma 2.1, we have

$$\tilde{\mathbb{E}}\sup_{0\le t\le T}f(\|\tilde{u}^\epsilon\|^2_{L^2})=\tilde{\mathbb{E}}\sup_{0\le t\le T}\|\tilde{u}^\epsilon\|^4_{H^1}=\mathbb{E}\sup_{0\le t\le T}\|u^\epsilon\|^4_{H^1}\le C.$$

Thus, by Lemma A.2 and (2.19), we have (2.29).

By (2.18), Young inequality and Lemma 2.1, we have

$$\begin{aligned}\tilde{\mathbb{E}}\|\tilde{v}^\epsilon\|^4_{L^2([0,T]\times\mathbb{R})}=&\mathbb{E}\|v^\epsilon\|^4_{L^2([0,T]\times\mathbb{R})}\\ &=\mathbb{E}\int_0^T\|\int_{\mathbb{R}}G(x-y)(u^2+\frac{1}{2}u_x^2)(y)dy\|^2_{L^2}dt\\ &\le C\mathbb{E}\int_0^T\|G\|^2_{L^2}\|u^2+\frac{1}{2}u_x^2\|^2_{L^1}dt\\ &\le C\mathbb{E}\int_0^T\|u\|^4_{H^1}dt\le CT,\end{aligned}$$

which, combined with Lemma A.2 and (2.19), imply (2.30).

From Lemmas 2.1, 2.2 and (2.18), we have for $k\ge 1$

$$\tilde{\mathbb{E}}\sup_{t\in[0,T]}\|q^\epsilon\|^{2k}_{L^2}\le C, \tag{2.35}$$

$$\tilde{\mathbb{E}}\sup_{t\in[0,T]}\|q^\epsilon\|^{2k}_{L^\infty}\le C. \tag{2.36}$$

By (2.35) we can infer that the sequence q^ϵ contains a subsequence, still denoted by q^ϵ, that satisfies (2.31).

By interpolation and Young inequalities, for $2\le p<\infty$

$$\|q^\epsilon\|_{L^p}\le \|q^\epsilon\|_{L^2}^{\frac{2}{p}}\|q^\epsilon\|_{L^\infty}^{1-\frac{2}{p}}\le \|q^\epsilon\|_{L^2}+\|q^\epsilon\|_{L^\infty}.$$

from which and (2.35)-(2.36) imply

$$\tilde{\mathbb{E}}\sup_{t\in[0,T]}\|q^{\epsilon 2}\|_{L^{p/2}}^k=\tilde{\mathbb{E}}\sup_{t\in[0,T]}\|q^\epsilon\|_{L^p}^{2k}\le C,$$

which implies the weak convergence of (2.32).

Inequality (2.33) is true thanks to the weak convergence in (2.32). Finally, (2.34) is a consequence of the definition of q^ϵ, (2.29) and (2.31). ∎

Taking $\epsilon\to 0$ in (2.20) and (2.28), it follows from Lemmas 2.6 and 2.7 that

$$\tilde{u}(t)=u_0-\int_0^t \tilde{u}\tilde{u}_x+\tilde{v}_x ds+\int_0^t \tilde{u}d\tilde{W}, \tag{2.37}$$

$$q(t)=u_{0x}-\int_0^t ((\tilde{u}q)_x+\tilde{u}^2+\frac{1}{2}\overline{q^2}-\tilde{v})ds+\int_0^t qd\tilde{W}, \tag{2.38}$$

hold in the sense of distribution in $[0,T]\times\mathbb{R}$ for almost all $\tilde{\omega}\in\tilde{\Omega}$.

Since we have the nonlinear term $\tilde{u}_x^2=q^2$ in $\tilde{v}$, we need to show that the strong convergence of q^ϵ in $L^2(\tilde{\Omega};L^2([0,T];L^2_{loc}(\mathbb{R})))$. First, we give the following lemma.

LEMMA 2.8. *The following limits hold*

$$\lim_{t\to 0+}\tilde{\mathbb{E}}\int_{\mathbb{R}} q^2(t,x)dx=\lim_{t\to 0+}\tilde{\mathbb{E}}\int_{\mathbb{R}}\overline{q^2}(t,x)dx=\int_{\mathbb{R}} u_{0x}^2(x)dx. \tag{2.39}$$

Proof. Since $\tilde{u}\in L^2(\tilde{\Omega};C([0,T];H^1))$ and (2.34), we have $q(t)\rightharpoonup u_{0x}$ in L^2 as $t\to 0+$, so that

$$\liminf_{t\to 0+}\tilde{\mathbb{E}}\int_{\mathbb{R}} q^2(t,x)dx\ge\int_{\mathbb{R}} u_{0x}^2 dx. \tag{2.40}$$

Since u^ϵ and $\tilde{u}^\epsilon$ have the same distribution, taking expectation on (2.3), we can get

$$\mathbb{E}\|\tilde{u}^\epsilon\|_{H^1}^2=\|u_{0\epsilon}\|_{H^1}^2+\mathbb{E}\int_0^t\|\tilde{u}^\epsilon\|_{H^1}^2 ds,$$

from which and Grönwall inequality we have

$$\mathbb{E}(\|\tilde{u}^\epsilon\|_{L^2}^2+\|\tilde{u}_x^\epsilon\|_{L^2}^2)\le(\|u_{0\epsilon}\|_{L^2}^2+\|u_{0\epsilon x}\|_{L^2}^2)e^t,$$

which combined with Lemma 2.7 imply,

$$\lim_{t\to 0+}\tilde{\mathbb{E}}(\|\tilde{u}\|_{L^2}^2+\int_{\mathbb{R}}\overline{q^2}dx)\le\lim_{t\to 0+}\liminf_{\epsilon\to 0}\tilde{\mathbb{E}}(\|\tilde{u}^\epsilon\|_{L^2}^2+\int_{\mathbb{R}} q^{\epsilon 2}dx)\le\|u_0\|_{L^2}^2+\|u_{0x}\|_{L^2}^2. \tag{2.41}$$

Since $\tilde{u}\in L^2(\tilde{\Omega};C([0,T];H^1))$, it follows from (2.33), (2.40) and (2.41) that

$$\int_{\mathbb{R}} u_{0x}^2 dx\le\liminf_{t\to 0+}\tilde{\mathbb{E}}\int_{\mathbb{R}} q^2(t,x)dx\le\lim_{t\to 0+}\tilde{\mathbb{E}}\int_{\mathbb{R}}\overline{q^2}dx\le\int_{\mathbb{R}} u_{0x}^2 dx,$$

from which we get (2.39). ∎

Now, we prove the strong convergence of q^ϵ.

LEMMA 2.9 (Convergence). *Let $q^\epsilon = \tilde{u}_x^\epsilon$ and $q = \tilde{u}_x$. Then, we have*

$$q^2(t) = \overline{q^2}(t), \text{ almost everywhere in } \tilde{\Omega} \times [0,T] \times \mathbb{R}. \tag{2.42}$$

Proof. From (2.28) and (2.38), by Itô formula, we have

$$\begin{aligned} dq^2 =& 2qdq + \langle dq, dq \rangle = -(2q(\tilde{u}q)_x + 2q\tilde{u}^2 + q\overline{q^2} - 2q\tilde{v} - q^2)dt + 2q^2 d\tilde{W} \\ =& -((\tilde{u}q^2)_x + q(\overline{q^2} - q^2) - 2q(\tilde{v} - \tilde{u}^2) - q^2)dt + 2q^2 d\tilde{W}, \end{aligned} \tag{2.43}$$

and

$$\begin{aligned} dq^{\epsilon 2} =& 2q^\epsilon dq^\epsilon + \langle dq^\epsilon, dq^\epsilon \rangle = -(2q^\epsilon(\tilde{u}^\epsilon q^\epsilon)_x + 2q^\epsilon \tilde{u}^{\epsilon 2} + q^{\epsilon 3} - 2q^\epsilon \tilde{v}^\epsilon - q^{\epsilon 2})dt + 2q^{\epsilon 2} d\tilde{W} \\ =& -((\tilde{u}^\epsilon q^{\epsilon 2})_x - 2q(\tilde{v}^\epsilon - \tilde{u}^{\epsilon 2}) - q^{\epsilon 2})dt + 2q^{\epsilon 2} d\tilde{W}. \end{aligned} \tag{2.44}$$

Taking $\epsilon \to 0$ in (2.44) and by Lemma 2.7,

$$d\overline{q^2} = -((\tilde{u}\overline{q^2})_x - 2q(\tilde{v} - \tilde{u}^2) - \overline{q^2})dt + 2\overline{q^2} d\tilde{W}. \tag{2.45}$$

Let $f = q^2 - \overline{q^2}$. Then, it follows from (2.43) and (2.45) that

$$df = -((\tilde{u}f)_x + qf - f)dt + 2f d\tilde{W},$$

Define the stopping time

$$\tau_R = \inf\{t \in [0,T] : \sup_{t \in [0,T]} \|q\|_{L^\infty} < R\}.$$

Denote $t \wedge \tau_R = \min\{t, \tau_R\}$. Taking integrations over $\mathbb{R} \times [\eta, t \wedge \tau_R]$, we have

$$\begin{aligned} \int_{\mathbb{R}} f(t \wedge \tau_R) dx =& \int_{\mathbb{R}} f(\eta) dx - \int_\eta^{t \wedge \tau_R} \int_{\mathbb{R}} ((\tilde{u}f)_x + qf - f) dx ds + 2 \int_\eta^{t \wedge \tau_R} \int_{\mathbb{R}} f d\tilde{W} \\ \leq& \int_{\mathbb{R}} f(\eta) dx + C(R) \int_\eta^{t \wedge \tau_R} \int_{\mathbb{R}} f dx ds + 2 \int_\eta^{t \wedge \tau_R} \int_{\mathbb{R}} f dx d\tilde{W}. \end{aligned}$$

By B-D-G inequality,

$$\begin{aligned} 2\tilde{\mathbb{E}} \sup_{t \in [0,T]} \left| \int_\eta^{t \wedge \tau_R} \int_{\mathbb{R}} f dx \tilde{W} \right| \leq& C\tilde{\mathbb{E}} \left(\int_\eta^{T \wedge \tau_R} \left(\int_{\mathbb{R}} f dx \right)^2 dt \right)^{1/2} \\ \leq& C\tilde{\mathbb{E}} \left(\sup_{t \in [0,T]} \int_{\mathbb{R}} f(t) dx \int_\eta^{T \wedge \tau_R} \int_{\mathbb{R}} f dx dt \right)^{1/2} \\ \leq& \frac{1}{2} \tilde{\mathbb{E}} \sup_{t \in [0,T]} \int_{\mathbb{R}} f(t) dx + C\tilde{\mathbb{E}} \int_\eta^{T \wedge \tau_R} \int_{\mathbb{R}} f dx dt. \end{aligned}$$

Let $\eta \to 0$ and by Lemma 2.9, we have

$$\tilde{\mathbb{E}} \sup_{t \in [0,T]} \int_{\mathbb{R}} f(t \wedge \tau_R) dx \leq C(R) \tilde{\mathbb{E}} \sup_{t \in [0,T]} \int_0^{t \wedge \tau_R} \int_{\mathbb{R}} f dx ds,$$

from which, Grönwall inequality and (2.33) imply that

$$q^2(t\wedge\tau_R)=\overline{q^2}(t\wedge\tau_R),\ \text{almost everywhere in}\ \tilde{\Omega}\times[0,T]\times\mathbb{R}.$$

Let $R\to\infty$ and by Lemma 2.2, we can get (2.42). □

Proof. **(Proof of Theorem 1.1.)** By Lemmas 2.7 and 2.9,

$$\begin{aligned}&\tilde{u}^\epsilon\to\tilde{u}\ \text{strongly in}\ L^2(\tilde{\Omega};C([0,T];L^2_{loc}(\mathbb{R}))),\\&\tilde{u}^\epsilon_x\to\tilde{u}_x\ \text{strongly in}\ L^2(\tilde{\Omega};L^2([0,T];L^2(\mathbb{R}))).\end{aligned}$$

Then, taking $\epsilon\to0$ in (2.20), we have

$$\tilde{u}(t)=u_0-\int_0^t\tilde{u}\tilde{u}_x+\tilde{v}_xds+\int_0^t\tilde{u}d\tilde{W},$$

holds in the sense of distribution in $[0,T]\times\mathbb{R}$ for almost all $\tilde{\omega}\in\tilde{\Omega}$ and $\tilde{v}=\frac{1}{2}e^{-|x|}*(\tilde{u}^2+\frac{1}{2}\tilde{u}_x^2)$. Thus, the proof is complete. □

3. Proof of Theorem 1.2

In this section, we will prove the global existence and uniqueness of the stochastic CH Equation (1.11) by contraction mapping theorem.

Since v is local Lipschitz, we need to consider the truncated equation of (1.13)

$$u(t)=S(t)u_0-\int_0^tS(t-s)v^n_xds+\delta\int_0^tS(t-s)u_xdW(t),\tag{3.1}$$

where

$$v^n=\eta_n(\|u\|_{H^1})v\tag{3.2}$$

and for $n>0$, $\eta_n:[0,\infty)\to[0,1]$ is a mollifer C^∞-function such that $\eta_n(r)=1$ for $0\le r\le n$ and $\eta_n(r)=0$ for $r\ge 2n$.

Proof. **(Proof of Theorem 1.2.)** Introduce a Banach space Y_T equipped with the norm

$$\|u\|_T^2=\mathbb{E}\{\sup_{0\le t\le T}\|u\|^2_{L^2}+\int_0^T\|u\|^2_{H^1}dt\}.\tag{3.3}$$

Denote Φ be a mapping in Y_T defined by

$$\Phi u=S(t)u_0-\int_0^tS(t-s)v^n_xds+\delta\int_0^tS(t-s)u_xdW(t).\tag{3.4}$$

Step 1: $\Phi:Y_T\to Y_T$ is well defined and bounded.

The first term in (3.4) can be estimated as follows

$$\begin{aligned}\|S(t)u_0\|_T=&\mathbb{E}\{\sup_{0\le t\le T}\|S(t)u_0\|^2_{L^2}+\int_0^T\|S(t)u_0\|^2_{H^1}dt\}\\\le&\mathbb{E}\{\|u_0\|^2_{L^2}+\int_0^T\int_{\mathbb{R}}\xi^2e^{-\xi^2t}\hat{u}_0^2(\xi)d\xi dt\}\\\le&2\|u_0\|^2_{L^2}.\end{aligned}\tag{3.5}$$

By Young inequality, we have

$$\begin{aligned}\|\int_0^t S(t-s)v_x^n ds\|_{L^2}^2 \leq & t\int_0^t \|v_x^n\|_{L^2}^2 ds \\ \leq & t\|G_x\|_{L^2}^2\eta_n(\|u\|_{H^1})\int_0^t \|u^2+\frac{1}{2}u_x^2\|_{L^1}^2 ds \\ \leq & C_nT\int_0^T \|u\|_{H^1}^2 dt. \end{aligned} \tag{3.6}$$

Denote $I=\int_0^t S(t-s)v_x^n ds$. Then I is the solution of the following equation

$$\begin{cases}\partial_t I - \frac{1}{2}I_{xx} = v_x^n, \\ I(x,0)=0.\end{cases} \tag{3.7}$$

By the standard energy estimate on (3.7), Hölder inequality and (3.6), we have

$$\begin{aligned}\|I\|_{L^2}^2+\int_0^t \|I(s)\|_{H^1}^2 ds = & 2\int_0^t (v_x^n, I)dr \leq \int_0^t 2\|I\|_{L^2}\|v_x^n\|_{L^2} dr \\ \leq & 2\sup_{0\leq t\leq T}\|I(t)\|_{L^2}\int_0^t \|v_x^n\|_{L^2} dr \\ \leq & C_n\sqrt{T}(\int_0^T \|u\|_{H^1}^2 dt)^{1/2}\eta_n(\|u\|_{H^1})\int_0^t \|G_x\|_{L^2}\|u^2+\frac{1}{2}u_x^2\|_{L^1} dr \\ \leq & C_nT\int_0^T \|u\|_{H^1}^2 dt, \end{aligned}$$

from which implies

$$\int_0^T \|\int_0^t S(t-s)v_x^n ds\|_{H^1}^2 ds \leq C_nT\int_0^T \|u\|_{H^1}^2 dt. \tag{3.8}$$

By B-D-G inequality, we have

$$\delta\mathbb{E}\sup_{0\leq t\leq T}\|\int_0^t S(t-s)u_x dW(t)\|_{L^2}^2 \leq \delta\mathbb{E}\int_0^T \|u_x\|_{L^2}^2 dt \leq \delta\mathbb{E}\int_0^T \|u\|_{H^1}^2 dt, \tag{3.9}$$

and by Itô isometry,

$$\begin{aligned}\delta\mathbb{E}\int_0^T \|\int_0^t S(t-s)u_x dW(t)\|_{H^1}^2 dt = & \delta\mathbb{E}\int_0^T\int_0^t\int_{\mathbb{R}} \xi^2 e^{-\xi^2(t-s)}\hat{u}_\xi(\xi,s)d\xi ds dt \\ = & \delta\mathbb{E}\int_{\mathbb{R}}\int_0^T\int_s^T \xi^2 e^{-\xi^2(t-s)}\hat{u}_\xi(\xi,s)dt ds d\xi \\ \leq & \delta\mathbb{E}\int_0^T \|u\|_{H^1}^2 ds. \end{aligned} \tag{3.10}$$

Taking (3.5)-(3.10) into account, we can find a constant $C_n(T)$ such that

$$\|\Phi u\|_T^2 \leq C_n(T)(\|u_0\|_{L^2}^2+\|u\|_T^2).$$

Therefore the operator $\Phi: Y_T \to Y_T$ is well defined and bounded.

Step 2: $\Phi:Y_T\to Y_T$ is a contraction.

To this end, for some technical reason to be seen, we need to introduce an equivalent norm in Y_T, depending on a parameter $\mu>0$, defined as follows

$$\|u\|^2_{\mu,T}=\mathbb{E}\{\sup_{0\le t\le T}\|u\|^2_{L^2}+\mu\int_0^T\|u\|^2_{H^1}dt\}. \tag{3.11}$$

Let $u_1,u_2\in Y_T$. Then in view of (3.1), $g=u_1-u_2$ satisfies

$$g(t)=-\int_0^t S(t-s)(v^n_{1x}-v^n_{2x})ds+\delta\int_0^t g_x(s)dW(s). \tag{3.12}$$

Without loss of generality, let $\|u_1\|_{H^1}>\|u_2\|_{H^1}$. Then

$$\begin{aligned}&\|v^n_{1x}-v^n_{2x}\|^2_{L^2}\\ =&\|\eta_n(\|u_1\|_{H^1})G(x)*\partial_x(u_1^2+\frac{1}{2}u^2_{1x})-\eta_n(\|u_2\|_{H^1})G(x)*\partial_x(u_2^2+\frac{1}{2}u^2_{2x})\|^2_{L^2}\\ \le&\|\eta_n(\|u_1\|_{H^1})G(x)*\partial_x[g(u_1+u_2)+\frac{1}{2}g_x(u_{1x}+u_{2x})\|^2_{L^2}\\ &+\|(\eta_n(\|u_1\|_{H^1})-\eta_n(\|u_2\|_{H^1}))G(x)*\partial_x(u_2^2+\frac{1}{2}u^2_{2x})\|^2_{L^2}\\ \le&C\|G_x\|^2_{L^2}\|\eta_n(\|u_1\|_{H^1})[g(u_1+u_2)+\frac{1}{2}g_x(u_{1x}+u_{2x})\|^2_{L^1}\\ &+\eta_n^{'}(\cdot)(\|u_1\|_{H^1}-\|u_2\|_{H^1})^2\|G_x\|^2_{L^2}\|u_2^2+\frac{1}{2}u^2_{2x}\|^2_{L^1}\\ \le&C_n\|g\|^2_{H^1}.\end{aligned} \tag{3.13}$$

Using (3.12) and the simple inequality $(a+b)^2\le C_\varepsilon a^2+(1+\varepsilon)b^2$ with $C_\varepsilon=(1+\varepsilon)/\varepsilon$, for any $\varepsilon>0$, we get

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\|\Phi g\|^2_{L^2}\le&\mathbb{E}\sup_{0\le t\le T}\{C_\varepsilon\|\int_0^t S(t-s)(v^n_{1x}-v^n_{2x})ds\|^2_{L^2}\\ &+(1+\varepsilon)\delta\|\int_0^t g_x(s)dW(s)\|^2_{L^2}\},\end{aligned} \tag{3.14}$$

similarly,

$$\begin{aligned}\mathbb{E}\int_0^T\|\Phi g\|^2_{H^1}dt\le&\mathbb{E}\{C_\varepsilon\int_0^T\|\int_0^t S(t-s)(v^n_{1x}-v^n_{2x})ds\|^2_{H^1}dt\\ &+(1+\varepsilon)\delta\int_0^T\|\int_0^t g_x(s)dW(s)\|^2_{H^1}dt\}.\end{aligned} \tag{3.15}$$

Applying the estimates (3.6), (3.8), (3.9) and (3.10) to (3.14) and (3.15), we obtain

$$\|\Phi g\|^2_{\mu,T}\le C_nC_\varepsilon T^2(1+\mu)\mathbb{E}\sup_{0\le t\le T}\|u\|^2_{H^1}+\mu(1+\varepsilon)(1+\frac{1}{\mu})\delta\mathbb{E}\int_0^T\|g\|^2_{H^1}dt. \tag{3.16}$$

Choose $\mu=\frac{1}{\delta},\varepsilon=\sqrt{(1+\delta)/2\delta}-1$ and sufficiently small T so that

$$\|\Phi g\|^2_{\mu,T}\le\rho\|g\|^2_{\mu,T}, \tag{3.17}$$

for some $\rho\in(0,1)$. Therefore, Φ is a contraction in Y_T and it has a unique solution u^n of Equation (3.1) in Y_T for a small T. Since T does not depend on the initial value u_0, that solution may be extended to any interval $[0,T_0]$ with $\forall T_0>0$. We write $T_0=T$ in the following.

Introducing a stopping time τ_n defined by

$$\tau_n=\inf\{t>0:\|u^n\|_{H^1}>n\}$$

if it exists, and set $\tau_n=T$ otherwise. Then, for $t<\tau_n, u(t)=u^n(t)$ is the solution of Equation (1.12). Since τ_n is increasing in n, let $\tau_\infty=\lim_{n\to\infty}\tau_n$ a.s.. For $t<\tau_\infty$, we have $t<\tau_n$ for some $n>0$, and define $u(t)=u^n(t)$. Then $\lim_{t\to\tau_\infty}\|u\|_{H^1}=\infty$ if $\tau_\infty<T$ and hence u is a local solution. For the uniqueness, suppose that there is another solution $\tilde{u}(t), t<\tau$ for a stopping τ. Then $\tilde{u}(t)=u^n(t)$ for $t<\tau_n$. It follows that $\tilde{u}(t)=u(t)$ for $t<\tau_\infty$ and $\tau=\tau_\infty$.

Step 3: Global solution. Using Itô formula to $\|u(T\wedge\tau_n)\|_{H^1}$, we have

$$\|u(T\wedge\tau_n)\|_{H^1}^2=\|u_0\|_{H^1}^2. \tag{3.18}$$

On the other hand, we have

$$\mathbb{E}\|u(T\wedge\tau_n)\|_{H^1}^2\geq\mathbb{E}\{I(\tau_n\leq T)\|u(T\wedge\tau_n)\|_{H^1}^2\}\geq n^2\mathbb{P}\{\tau_n\leq T\}, \tag{3.19}$$

where $I(\cdot)$ is the indicator function. In view of (3.18)-(3.19), we have

$$\mathbb{P}\{\tau_n\leq T\}\leq\frac{1}{n^2}$$

so that, by the Borel-Cantelli lemma,

$$\mathbb{P}\{\tau_\infty>T\}=1,$$

for any $T>0$. Hence, $u=\lim_{n\to\infty}u^n$ is a global solution. ∎

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Appendix. Some lemmas. The following lemma is proved in Theorems 5 and 7 in [43].

LEMMA A.1 ([43]). *Let X,Y and Z be Banach spaces such that $X\subset\subset Y\subset Z$.*

(1) Assume $1\leq p\leq\infty$, $\mathcal{K}$ is a bounded set in $L^p(0,T;X)$ and for $u\in\mathcal{K}$, $\|u(t+\delta)-u(t)\|_{L^p(0,T-\delta;Z)}\to0$ as $\delta\to0$. Then $\mathcal{K}$ is relatively compact in $L^p(0,T;Y)$ (and in $C(0,T;Y)$ if $p=\infty$).

(2) Assume Y be intermediate space of class θ with respect to X and Z, that is to say there exists θ such that

$$\|u\|_Y\leq C\|u\|_X^{1-\theta}\|u\|_Z^\theta,\ \forall u\in X\cap Z, 0<\theta<1.$$

Assume $1 \leq p_i \leq \infty, i = 1,2$, $\mathcal{K}$ *is a bounded set in* $L^{p_1}(0,T;X)$ *and for* $u \in \mathcal{K}$, $\|u(t+\delta) - u(t)\|_{L^{p_2}(0,T-\delta;Z)} \to 0$ *as* $\delta \to 0$. *Then* $\mathcal{K}$ *is relatively compact in* $L^p(0,T;Y)$ *with* $1/p = (1-\theta)/p_1 + \theta/p_2$.

The following lemmas are proved in [41].

LEMMA A.2 (Uniform integrability [41]). *If there exists a nonnegative measurable function* f *in* $\mathbb{E}^+$, *such that* $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$ *and* $\sup_{\alpha\in\Gamma} \mathbb{E}[f(|X_\alpha|)] < \infty$. *Then* $\{X_\alpha, \alpha \in \Gamma\}$ *are uniformly integrable.*

LEMMA A.3 (Vitali's convergence theorem [41]). *Suppose* $p \in [1,\infty)$, $\{v^\epsilon\} \in L^p$ *and* $\{v^\epsilon\}$ *converges to* v *in probability. Then the following are equivalent:*

(1) $v^\epsilon \to v$ *in* L^p;

(2) the variables $|v^\epsilon|^p$ *are uniformly integrable;*

(3) $\mathbb{E}(|v^\epsilon|^p) \to \mathbb{E}(|v|^p)$.

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