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AN IMPROVED SMALL DATA THEOREM FOR THE VLASOV-POISSON SYSTEM*

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Abstract. A collisionless plasma is modeled by the Vlasov-Poisson system. Smooth solutions are considered in three spatial dimensions with compactly supported initial data. The main theorem of this work is a small data result that improves an earlier theorem of Bardos and Degond in that it does not require the derivatives of the initial data to be small. Another theorem is presented here that gives a sufficient condition that ensures that the charge density decays as t^{-3}, which is the rate which occurs when asymptotically all particles disperse freely.

Keywords. partial differential equations; kinetic theory; Vlasov equation.

AMS subject classifications. 35L60; 35Q83; 82C22; 82D10.

Dedication: In memory of Robert Glassey.

1. Introduction

Consider the Vlasov-Poisson system:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + eE\cdot\nabla_v f = 0 \\ \rho(t,x) = \int f(t,x,v)dv \\ E = \int \rho(t,y)\frac{x-y}{|x-y|^3}dy \end{cases} \tag{1.1}$$

where $t\geq 0$ is time, $x\in\mathbb{R}^3$ is position, and $v\in\mathbb{R}^3$ is momentum. f is the number density of particles in phase space. Collisional effects are neglected. Since it makes no difference to the methods used here, we consider only one species of particle, with mass one. When $e=+1$ the system describes a collisionless plasma acting under its self induced electrostatic field, E. When $e=-1$ the force is gravitational attraction, and the system is frequently used in modeling a galaxy. The initial condition

$$f(0,x,v) = f_0(x,v)$$

is imposed where f_0 is given.

Let

$$\|f_0\|_\infty = \sup\{|f_0(x,v)| : x,v\in\mathbb{R}^3\},$$

$$\|\rho(t)\|_\infty = \sup\{|\rho(t,x)| : x\in\mathbb{R}^3\},$$

and

$$\mathcal{C}_0 = \{f_0\in C_0^1(\mathbb{R}^6) : \|f_0\|_\infty + \|Df_0\|_\infty \leq C_0,\ f_0\geq 0, \\ \text{and } |x|\geq C_0 \text{ or } |v|\geq C_0 \text{ implies } f_0(x,v)=0\}$$

for some $C_0>0$.

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THEOREM 1.1. *Assume that $f_0 \in \mathcal{C}_0$ and there exist $C>0$ and $p \in (2,3]$ such that*

$$\|\rho(t)\|_\infty \leq C(1+t)^{-p}. \tag{1.2}$$

Then there exists $C_p>0$ such that

$$\|\rho(t)\|_\infty \leq C_p(1+t)^{-3} \tag{1.3}$$

for all $t \geq 0$.

The intuition to this result is that the decay from (1.2) is sufficient to show that after some time, T, the characteristics (particle paths) scatter and actually achieve the full decay obtained in [1]. Thus we call this the scattering result. The condition $p>2$ is used in the proof of Lemma 3.3, which is used in the proof of both theorems.

Our other main theorem is the following small data theorem:

THEOREM 1.2. *There exist $\epsilon_0>0$ and $C>0$ such that if $f_0 \in \mathcal{C}_0$ and*

$$\|f_0\|_\infty \leq \epsilon_0$$

then

$$\|\rho(t)\|_\infty \leq C\|f_0\|_\infty(1+t)^{-3} \tag{1.4}$$

for all $t \geq 0$.

The small data result of [1] requires that $\|Df_0\|_\infty$ be taken sufficiently small, while this theorem does not. Much of what makes this possible is Lemma 2.1 below, which roughly says that a bound on $\|\rho(t)\|_\infty$ ensures that E is "almost Lipschitz" in x.

That solutions remain smooth for all time was established in [22] and independently in [19]. Much less is known about the large-time behavior of solutions. In the plasma physics case some decay estimates follow from an identity developed in [17] (also in [21]). Other time-asymptotic results in three dimensions are obtained in [15, 20, 27]. See [4, 6, 10–12, 26] for results in lower dimension.

A major open question is "under what conditions do all the particles asymptotically disperse freely (leading to (1.3))". It is interest in this question that motivated this paper.

Besides [1] we mention some other small data type results for collisionless kinetic problems: For the relativistic Vlasov-Poisson system see [2]. For the relativistic Vlasov-Maxwell system see [14] and also [5, 13, 24, 28, 30]. For the Vlasov-Einstein system see [8, 25], and [18]. For the Vlasov-Nördstrom system see [7]. For (1.1) we mention also [16] and [29].

Finally we cite [9] and [23] as general references on rigorous kinetic theory.

The letter, C, is used to denote a generic positive constant which changes from line to line. When a specific constant needs to be named so that it may be referred to later, a numerical subscript is used, such as C_0 in the definition of $\mathcal{C}_0$. C_p denotes a positive constant which depends on p. Throughout the entire paper we assume that $f_0 \in \mathcal{C}_0$. Hence we may bound the L_1 and L_∞ norms of f_0 by a constant, C, which depends on C_0. Since these norms are conserved, these bounds hold at later times also. We define $X(s,t,x,v)$ and $V(s,t,x,v)$ by $X(t,t,x,v)=x$, $V(t,t,x,v)=v$,

$$\frac{dX}{ds}=V$$

and

$$\frac{dV}{ds} = E(s, X).$$

We also define

$$Q(t) = \sup\{|v| : \text{there exists } x \in \mathbb{R}^3 \text{ and } s \in [0,t] \text{ such that } f(s,x,v) \neq 0\}.$$

That $Q(t)$ is finite for all $t \geq 0$ was shown in [22]. This paper is organized as follows: the second Section 2 contains preliminary lemmas. The main lemma of the paper (which both theorems rely on) is in Section 3. Section 4 is the proof of Theorem 1.1. Section 5 is the proof of Theorem 1.2. Finally a few technical arguments are in the Appendix.

The results of this work would hold equally well if multiple species of particles were considered. A single species has been taken to streamline the presentation. Similarly the results are valid for $e = +1$ and for $e = -1$. We will just consider $e = +1$ throughout. Hence from here on the letter e denotes the real number where the natural logarithm is unity. Similarly we expect that the above results may be generalized to solutions that decay as x tends to infinity but do not have compact support in x, but we assume compact support here to streamline the presentation.

2. Preliminaries

In this section we prove some preliminary lemmas. For this entire section we assume that

$$\|\rho(t)\|_\infty \leq \eta(1+t)^{-p} \tag{2.1}$$

holds for $0 \leq t < T_+$ where $T_+ \in (0,\infty) \cup \{+\infty\}$, $p \in (2,3]$, and $0 < \eta \leq C$. Define

$$\log^*(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq e \\ \\ \log(x) \text{ if } e < x. \end{cases}$$

The following lemma may be viewed as a replacement for the Calderon-Zygmund inequality which fails for the L^∞ norm.

LEMMA 2.1. *There exists $C_1 > 0$ such that*

$$|E(t,x+h) - E(t,x)| \leq C_1 \eta^{\frac{1}{2}} (1+t)^{-p_1} |h| \log^*(\frac{1}{|h|}) \tag{2.2}$$

for $0 \leq t < T_+$ and $x, h \in \mathbb{R}^3$ where $p_1 = 2 + \frac{2}{3}(p-2)$.

Proof. We have

$$\begin{aligned} |E(t,x+h) - E(t,x)| &= |\int \rho(t,y)(\frac{x+h-y}{|x+h-y|^3} - \frac{x-y}{|x-y|^3})dy| \\ &\leq \int |\rho(t,x-z)| \, |\frac{z+h}{|z+h|^3} - \frac{z}{|z|^3}|dz. \end{aligned} \tag{2.3}$$

Note that for $|z| \geq 3|h|$, we have $|z+h| \geq \frac{2}{3}|z|$ so

$$|\frac{z+h}{|z+h|^3} - \frac{z}{|z|^3}| = \frac{\Big| |z|^3(z+h) - |z+h|^3 z \Big|}{|z+h|^3 |z|^3}$$

$$\leq \frac{\Big| |z|^3 - |z+h|^3 \Big| \, |z| + |z|^3|h|}{(\frac{2}{3}|z|)^3|z|^3}$$
$$= \frac{27}{8}(|z|^{-5}|\,|z+h|^3 - |z|^3| + |z|^{-3}|h|). \tag{2.4}$$

Using the mean value theorem it may be shown that (for $|z| \geq 3|h|$)

$$\Big| \, |z+h|^3 - |z|^3 \Big| \leq C|z|^2|h|$$

and hence (2.4) yields

$$|\frac{z+h}{|z+h|^3} - \frac{z}{|z|^3}| \leq C|z|^{-3}|h|.$$

Returning to (2.3) we have for $R \geq 3|h|$

$$\begin{aligned}|E(t,x+h) - E(t,x)| \leq &\int_{|z|<3|h|} \|\rho(t)\|_\infty (|z+h|^{-2} + |z|^{-2}) dz \\ &+ \int_{3|h|<|z|<R} \|\rho(t)\|_\infty C|z|^{-3}|h| dz + CR^{-2}\int_{R<|z|} |\rho(t,x-z)| dz \\ &\leq C\|\rho(t)\|_\infty \left(|h| + |h|\log(\frac{R}{3|h|})\right) + C\|\rho(t)\|_1 R^{-2} \\ &\leq C\left(\eta(1+t)^{-p}|h|\log^*(\frac{R}{3|h|}) + R^{-2}\right).\end{aligned} \tag{2.5}$$

Let

$$\overline{R} = (\eta(1+t)^{-p}|h|)^{-1/2}.$$

If $\overline{R} \geq 3|h|$ let $R = \overline{R}$. Then (2.5) yields

$$|E(t,x+h) - E(t,x)| \leq C\eta(1+t)^{-p}|h|\log^*(\frac{(1+t)^{p/2}}{3\eta^{1/2}|h|^{3/2}}). \tag{2.6}$$

If

$$\frac{(1+t)^{p/2}}{3\eta^{1/2}|h|^{3/2}} \leq e$$

then (2.2) follows immediately from (2.6), so consider

$$\frac{(1+t)^{p/2}}{3\eta^{1/2}|h|^{3/2}} > e.$$

Then (2.6) becomes

$$\begin{aligned}|E(t,x+h) - E(t,x)| &\leq C\eta(1+t)^{-p}|h|\tfrac{1}{2}\log(\tfrac{(1+t)^p}{9\eta|h|^3}) \\ &\leq C\eta(1+t)^{-p}|h|\left(\log(1+t) + \log(\tfrac{1}{\eta}) + \log(\tfrac{1}{|h|})\right).\end{aligned}$$

Using $p_1 = 2 + \frac{2}{3}(p-2)$ and $\eta \le C$, this yields

$$
\begin{aligned}
|E(t,x+h) - E(t,x)| &\le C\eta|h|(1+t)^{-p_1} + C(1+t)^{-p}|h|\eta^{1/2} + C\eta(1+t)^{-p}|h|\log(\frac{1}{|h|}) \\
&\le C\eta^{1/2}(1+t)^{-p_1}(|h| + |h|\log(\frac{1}{|h|})) \\
&\le C\eta^{1/2}(1+t)^{-p_1}|h|\log^*(\frac{1}{|h|}),
\end{aligned}
$$

which is (2.2).

Finally if $\overline{R} < 3|h|$, take $R = 3|h|$. Then using $\overline{R} < 3|h|$, (2.5) yields

$$
\begin{aligned}
|E(t,x+h) - E(t,x)| &\le C(\eta(1+t)^{-p}|h| + |h|^{-2}) \\
&\le C\eta(1+t)^{-p}|h|.
\end{aligned}
$$

□

LEMMA 2.2. *Let $t > 0$ and $S \in C^2[0,t]$. Assume that $\frac{dS}{ds}(s)$ vanishes at $s = t$ and that $\alpha \in C[0,t]$ is nonnegative with*

$$
|\frac{d^2S}{ds^2}(s)| \le \alpha(s)|S(s)|\log^*(\frac{1}{|S(s)|})
$$

for all $s \in [0,t]$. Let

$$
A = \exp(-\int_0^t (\tau+1)\alpha(\tau)d\tau),
$$

then for $|S(t)| < \exp(-1/A)$ we have

$$
|S(s)| + |\frac{dS}{ds}(s)| \le |S(t)|^A
$$

for all $s \in [0,t]$.

Proof. Note that

$$
\frac{dS}{ds}(s) = -\int_s^t \frac{d^2S}{d\tau^2}(\tau)d\tau
$$

and

$$
S(s) = S(t) + \int_s^t (\tau - s)\frac{d^2S}{d\tau^2}(\tau)d\tau
$$

so

$$
|S(s)| + |\frac{dS}{ds}(s)| \le |S(t)| + \int_s^t (\tau - s + 1)\alpha(\tau)|S(\tau)|\log^*(\frac{1}{|S(\tau)|})d\tau.
$$

Define $\overline{S}$ on $[0,t]$ by

$$
\overline{S}(s) = |S(t)| + \int_s^t (\tau+1)\alpha(\tau)\overline{S}(\tau)\log^*(\frac{1}{\overline{S}(\tau)})d\tau
$$

and let

$$\overline{T}=\inf\{s\in[0,t]:\overline{S}\leq e^{-1} \text{ on } [s,t]\}.$$

That the set defining $\overline{T}$ is nonempty follows from the assumption that $|S(t)|<\exp(-1/A)$. Then by explicit calculation

$$\overline{S}(s)=|S(t)|^{\exp(-\int_s^t(\tau+1)\alpha(\tau)d\tau)} \tag{2.7}$$

on $[\overline{T},t]$. But then

$$\begin{aligned}\overline{S}(\overline{T}) &= |S(t)|^{\exp(-\int_{\overline{T}}^t(\tau+1)\alpha(\tau)d\tau)}\\ &\leq |S(t)|^A<e^{-1},\end{aligned}$$

so it follows that $\overline{T}=0$. Now since the mapping $x\mapsto x\log^*(\frac{1}{x})$ is increasing it follows that

$$|S(s)|+|\frac{dS}{ds}(s)|\leq\overline{S}(s)$$

on $s\in[0,t]$. Now the lemma follows by (2.7). ◻

Lemma 2.3. *Let C_1 and p_1 be as in Lemma 2.1 and define*

$$\overline{A}=\exp(-C_1\eta^{1/2}(p_1-2)^{-1}).$$

There exists $C>0$ such that for $0\leq t<T_+$ and $|x-y|<e^{-1/\overline{A}}$ we have

$$|\rho(t,x)-\rho(t,y)|\leq C|x-y|^{\overline{A}}.$$

Proof. For any $v\in\mathbb{R}^3$ let

$$S(s)=X(s,t,x,v)-X(s,t,y,v).$$

Then by Lemma 2.1

$$\begin{aligned}|\frac{d^2S}{ds^2}(s)|=&|E(s,X(s,t,x,v)-E(s,X(s,t,y,v))|\\ \leq&\alpha(s)|S(s)|\log^*(\frac{1}{|S(s)|})\end{aligned}$$

where

$$\alpha(s)=C_1\eta^{1/2}(1+s)^{-p_1}.$$

Note that

$$\begin{aligned}\int_0^t(\tau+1)\alpha(\tau)d\tau\leq& C_1\eta^{1/2}\int_0^\infty(1+\tau)^{1-p_1}d\tau\\ =&C_1\eta^{1/2}(p_1-2)^{-1}\end{aligned}$$

so

$$\exp(-\int_0^t(1+\tau)\alpha(\tau)d\tau)\geq\overline{A}.$$

By Lemma 2.2 (and since $\frac{dS}{ds}(t)=v-v=0$)

$$|S(0)|+|\frac{dS}{ds}(0)|\leq |S(t)|^{\overline{A}}=|x-y|^{\overline{A}}.$$

Hence

$$\begin{aligned}|f(t,x,v)-f(t,y,v)|=&|f(0,X(0,t,x,v),V(0,t,x,v))-f(0,X(0,t,y,v),V(0,t,y,v))|\\ \leq&\|\nabla_{x,v}f(0)\|_\infty(|S(0)|+|\frac{dS}{ds}(0)|)\leq C|x-y|^{\overline{A}}.\end{aligned} \tag{2.8}$$

Note that for any $s\geq 0, z\in\mathbb{R}^3$, and $R>0$

$$\begin{aligned}|E(s,z)| &\leq \int_{|\tilde{z}-z|<R}\frac{|\rho(s,\tilde{z})|}{|z-\tilde{z}|^2}d\tilde{z}+\int_{|\tilde{z}-z|>R}\frac{|\rho(s,\tilde{z})|}{|z-\tilde{z}|^2}d\tilde{z}\\ &\leq C(\|\rho(s)\|_\infty R+\|\rho(s)\|_1R^{-2})\\ &\leq C(\|\rho(s)\|_\infty R+R^{-2}).\end{aligned}$$

Taking

$$R=\|\rho(s)\|_\infty^{-1/3}$$

and using (2.1) yields

$$\|E(s)\|_\infty\leq C\eta^{2/3}(1+s)^{-2p/3}$$

and hence

$$\begin{aligned}Q(s)\leq& Q(0)+\int_0^s\|E(\tau)\|_\infty d\tau\\ \leq& C+C\eta^{2/3}(\frac{2p}{3}-1)^{-1}\leq C.\end{aligned} \tag{2.9}$$

Now by (2.8)

$$\begin{aligned}|\rho(t,x)-\rho(t,y)| &= |\int_{|v|<Q(t)}(f(t,x,v)-f(t,y,v))dv|\\ &\leq C|x-y|^{\overline{A}}\tfrac{4\pi}{3}Q^3(t)\leq C|x-y|^{\overline{A}},\end{aligned}$$

completing the proof. ∎

Define

$$\|\rho(t)\|_H=\sup\{\frac{|\rho(t,x)-\rho(t,y)|}{|x-y|^{\overline{A}}}:x,y\in\mathbb{R}^3 \text{ with } 0<|x-y|<e^{-1/\overline{A}}\}$$

where $\overline{A}$ is defined in Lemma 2.3.

LEMMA 2.4. *There is $C_p>0$ such that*

$$\|D_xE(t)\|_\infty\leq C_p\log^*(\frac{1+\|\rho(t)\|_H}{\|\rho(t)\|_\infty})\|\rho(t)\|_\infty. \tag{2.10}$$

Proof. From page 345 of [3] we have

$$|\partial_{x_k}E_k(t,x)| = |-\frac{4\pi}{3}\rho(t,x)+\int_{|y-x|<d}(\rho(t,y)-\rho(t,x))\frac{3(y_k-x_k)^2-|y-x|^2}{|y-x|^5}dy \\ +\int_{|y-x|>d}\rho(t,y)\frac{3(y_k-x_k)^2-|y-x|^2}{|y-x|^5}dy|$$

for any $d>0$. So for $0<d\le e^{-1/\overline{A}}$ and $R\ge d$ we have

$$|\partial_{x_k}E_k(t,x)| \le \frac{4\pi}{3}\|\rho(t)\|_\infty+\|\rho(t)\|_H\int_{|y-x|<d}|y-x|^{\overline{A}}4|y-x|^{-3}dy \\ +\|\rho(t)\|_\infty\int_{d<|y-x|<R}4|y-x|^{-3}dy+\int_{R<|y-x|}|\rho(t,y)|4R^{-3}dy \\ \le C_p((1+\log(\frac{R}{d}))\|\rho(t)\|_\infty+\|\rho(t)\|_H d^{\overline{A}}+R^{-3}). \tag{2.11}$$

Define

$$\overline{d}=(\frac{\|\rho(t)\|_\infty}{\|\rho(t)\|_H})^{1/\overline{A}}$$

and

$$\overline{R}=\|\rho(t)\|_\infty^{-1/3}.$$

If $\overline{d}\le e^{-1/\overline{A}}$ and $\overline{R}\ge\overline{d}$ we take $d=\overline{d}$ and $R=\overline{R}$ and obtain

$$\begin{aligned}|\partial_{x_k}E(t,x)| &\le C(1+\log(\tfrac{\overline{R}}{\overline{d}}))\|\rho(t)\|_\infty \\ &\le C\log^*\left(\frac{\|\rho(t)\|_H^{1/\overline{A}}}{\|\rho(t)\|_\infty^{\frac{1}{3}+1/\overline{A}}}\right)\|\rho(t)\|_\infty \\ &\le C\log^*\left(\frac{(1+\|\rho(t)\|_H)^{\frac{1}{3}+1/\overline{A}}}{\|\rho(t)\|_\infty^{\frac{1}{3}+1/\overline{A}}}\right)\|\rho(t)\|_\infty \\ &\le C_p\log^*\left(\frac{1+\|\rho(t)\|_H}{\|\rho(t)\|_\infty}\right)\|\rho(t)\|_\infty,\end{aligned}$$

which is the bound stated in (2.10).

If $\overline{d}>e^{-1/\overline{A}}$ we take

$$d=e^{-1/\overline{A}}$$

and obtain (using $\overline{d}>e^{-1/\overline{A}}$ in (2.11))

$$|\partial_{x_k}E_k(t,x)|\le C((1+\log(Re^{1/\overline{A}}))\|\rho(t)\|_\infty+R^{-3}).$$

If $\overline{R}\ge e^{-1/\overline{A}}$ we take $R=\overline{R}$ and obtain

$$|\partial_{x_k}E_k(t,x)|\le C_p\log^*(\frac{1}{\|\rho(t)\|_\infty})\|\rho(t)\|_\infty$$

and otherwise take $R = e^{-1/\overline{A}}$ which yields

$$|\partial_{x_k} E_k(t,x)| \le C_p \|\rho(t)\|_\infty.$$

In the remaining case ($\overline{d} \le \exp(-1/\overline{A})$ and $\overline{R} < \overline{d}$), we take $d = R = \overline{d}$. Then (2.11) and $\overline{R} < \overline{d}$ yield

$$|\partial_{x_k} E_k(t,x)| \le C_p \|\rho(t)\|_\infty.$$

Thus in all cases

$$|\partial_{x_k} E_k(t,x)| \le C_p \log^*\left(\frac{1+\|\rho(t)\|_H}{\|\rho(t)\|_\infty}\right) \|\rho(t)\|_\infty.$$

For $i \ne k$ from page 346 of [3] we have

$$\begin{aligned} |\partial_{x_i} E_k(t,x)| &= |\textstyle\int_{|y-x|<d} (\rho(t,y)-\rho(t,x)) \frac{3(y_i-x_i)(y_k-x_k)}{|y-x|^5} dy \\ &\quad + \textstyle\int_{d<|y-x|} \rho(t,y) \frac{3(y_i-x_i)(y_k-x_k)}{|y-x|^5} dy| \end{aligned}$$

for any $d > 0$. Proceeding as before leads to the same estimate and (2.10) follows. ∎

Lemma 2.5. *There is $C_p > 0$ such that*

$$\|D_x E(t)\|_\infty \le C_p \log^*(\frac{1+t}{\eta}) \eta (1+t)^{-p} \tag{2.12}$$

for $t \in [0, T_+)$.

Proof. Since $x \mapsto \log^*(\frac{C}{x}) x$ is increasing it follows from Lemma 2.4 and (2.1) that

$$\|D_x E(t)\|_\infty \le C_p \log^*(\frac{1+\|\rho(t)\|_H}{\eta(1+t)^{-p}}) \eta (1+t)^{-p}.$$

But Lemma 2.3 yields

$$\|\rho(t)\|_H \le C.$$

Using this and $\eta \le C$ we have

$$\begin{aligned} \|D_x E(t)\|_\infty &\le C_p \eta (1+t)^{-p} \log^*\left(\tfrac{C(1+t)^p}{\eta}\right) \\ &\le C_p \eta (1+t)^{-p} \log^*\left(\tfrac{C(1+t)^p}{\eta} (\tfrac{C}{\eta})^{p-1}\right) \\ &= C_p \eta (1+t)^{-p} \log^*\left(C(\tfrac{1+t}{\eta})^p\right) \end{aligned}$$

and (2.12) follows. ∎

3. The main lemma

Both Theorems 1.1 and 1.2 rely on Lemma 3.3 below, which appears in [1]. We present a different proof than that of [1] and emphasize that the assumptions on higher order derivatives made in [1] are not made here.

We will use the follwing two technical facts whose proofs are in the Appendix:

LEMMA 3.1. *Let $A,B\in\mathbb{R}^3$ with $A_k\geq 0$, $B_k\geq 0$, and*

$$B_k-\frac{1}{12}|B|\leq A_k$$

for $k=1,2,3$. Then

$$|A|\geq\frac{2}{5}|B|. \tag{3.1}$$

LEMMA 3.2. *Let $M\in\mathbb{R}^3\times\mathbb{R}^3$ and $B>0$ such that*

$$|Mx|\geq B|x|$$

for all $x\in\mathbb{R}^3$. Then

$$|\det(M)|\geq B^3. \tag{3.2}$$

LEMMA 3.3. *For any $p\in(2,3]$ there exists $\epsilon_p>0$ such that if*

$$\|D_xE(t)\|_\infty\leq\epsilon_p(1+t-T)^{-p} \tag{3.3}$$

holds on $[T,T_+)$ with $0\leq T$, $T_+\in(T,\infty)\cup\{\infty\}$, then

$$|\frac{\partial X}{\partial v}(s,t,x,v)-(s-t)I|\leq\frac{1}{12}(t-s), \tag{3.4}$$

$$|X(s,t,x,v)-X(s,t,x,w)|\geq\frac{2}{5}(t-s)|v-w|, \tag{3.5}$$

and

$$|\det(\frac{\partial X}{\partial v}(s,t,x,v))|\geq(\frac{2}{5}(t-s))^3 \tag{3.6}$$

for $T\leq s\leq t<T_+$ and $x,v,w\in\mathbb{R}^3$. Also

$$\|\rho(t)\|_\infty\leq C\|f_0\|_\infty(1+T)^3Q^3(T)(t-T)^{-3} \tag{3.7}$$

for $t\in(T,T_+)$.

Proof. Let us temporarily suppress the dependence on (t,x,v) and write

$$R(s)=\frac{\partial X}{\partial v}(s,t,x,v)-(s-t)I.$$

Then

$$R(t)=\frac{dR}{ds}(t)=0$$

and

$$\frac{d^2R}{ds^2}(s)=D_xE(s,X(s))(R(s)+(s-t)I)$$

so

$$|R(s)|=|\int_s^t(\tau-s)D_xE(\tau,X(\tau))(R(\tau)+(\tau-t)I)d\tau|.$$

Assume that

$$\|D_xE(s)\|_\infty \leq \epsilon(1+s-T)^{-p}$$

holds on $[T,T_+)$. Restrictions on ϵ will be imposed as needed. Then

$$|R(s)| \leq \epsilon \int_s^t \frac{\tau - s}{(1+\tau-T)^p}(|R(\tau)|+t-\tau)d\tau.$$

Integration by parts yields

$$\begin{aligned} |\textstyle\int_s^t \frac{(\tau-s)(t-\tau)}{(1+\tau-T)^p}d\tau| &= |(p-1)^{-1}(p-2)^{-1}(\frac{t-s}{(1+s-T)^{p-2}} \\ &\quad -\frac{s-t}{(1+t-T)^{p-2}} - 2\textstyle\int_s^t \frac{d\tau}{(1+\tau-T)^{p-2}})| \\ &\leq C_p(t-s) \end{aligned}$$

so

$$|R(s)| \leq \alpha(s) + \int_s^t \beta(\tau)|R(\tau)|d\tau$$

where

$$\alpha(s) = C_p\epsilon(t-s)$$

and

$$\beta(\tau) = \epsilon(1+\tau-T)^{1-p} \geq \epsilon\frac{\tau-s}{(1+\tau-T)^p}.$$

By Grönwall's inequality

$$\begin{aligned} |R(s)| &\leq \alpha(s) + \textstyle\int_s^t \alpha(\tau)\beta(\tau)e^{\int_s^\tau \beta(u)du}d\tau \\ &\leq C_p\epsilon(t-s) + C_p\epsilon(t-s)\textstyle\int_T^\infty \beta(\tau)e^{\int_T^\infty \beta(u)du}d\tau \\ &= C_p\epsilon(t-s)(1+C_p\epsilon e^{C_p\epsilon}). \end{aligned}$$

Hence for ϵ sufficiently small (3.4) holds.

To prove (3.5) let us write

$$X(v) = X(s,t,x,v).$$

Let

$$A_k = |X_k(v) - X_k(w)|$$

and

$$B_k = (t-s)|v_k - w_k|.$$

By the mean value theorem there is ξ such that

$$\begin{aligned} A_k &= |\nabla X_k(\xi)\cdot(v-w)| \\ &\geq B_k - |\nabla X_k(\xi)\cdot(v-w)-(s-t)(v_k-w_k)|. \end{aligned}$$

By (3.4) it follows that

$$A_k \geq B_k - \frac{1}{12}(t-s)|v-w| = B_k - \frac{1}{12}|B|,$$

so (3.5) follows by Lemma 3.1.

Next, from (3.5) it follows that

$$|(D_v X(v))u| \geq \frac{2}{5}(t-s)|u|$$

for any $v,u \in \mathbb{R}^3$ and now (3.6) follows from Lemma 3.2.

Writing

$$X(v) = X(T,t,x,v),\ V(v) = V(T,t,x,v)$$

we have

$$\int f(t,x,v)dv = \int_S f(T,X(v),V(v)) \frac{\det(\frac{\partial X}{\partial v})}{\det(\frac{\partial X}{\partial v})} dv$$

where

$$S = \{v : f(t,x,v) \neq 0\}.$$

Now by (3.6)

$$\begin{aligned}\int f(t,x,v)dv &\leq \|f(T)\|_\infty \int_S \frac{|\det(\frac{\partial X}{\partial v})|}{(\frac{2}{5}(t-T))^3} dv \\ &= (\frac{5}{2})^3 \|f_0\|_\infty (t-T)^{-3} \int_{\{X(v):v\in S\}} dy. \end{aligned} \tag{3.8}$$

Consider $v \in S$, then

$$f(t,x,v) = f_0(X(0,t,x,v),V(0,t,x,v)) \neq 0$$

so

$$|X(0,t,x,v)| \leq C.$$

Hence

$$\begin{aligned}|X(v)| &= |X(0,t,x,v) + \textstyle\int_0^T V(s,t,x,v)ds| \\ &\leq C + \textstyle\int_0^T Q(s)ds \leq C + TQ(T) \\ &\leq C(1+T)Q(T).\end{aligned}$$

Now by (3.8)

$$\int f(t,x,v)dv \leq C\|f_0\|_\infty (1+T)^3 Q^3(T)(t-T)^{-3}$$

and (3.7) follows. ∎

4. The scattering result

Assume that

$$\|\rho(t)\|_\infty \leq C(1+t)^{-p} \tag{4.1}$$

for all $t \geq 0$ with $p \in (2,3]$. Then taking $\eta = C$, (2.1) holds and Lemma 2.5 yields

$$\|D_x E(t)\|_\infty \leq C_p \log^*(\frac{1+t}{C})(1+t)^{-p}.$$

Let

$$p_1 = 2 + \frac{2}{3}(p-2)$$

and choose ϵ_{p_1} as in Lemma 3.3. Choose $T_p \geq 0$ such that

$$C_p \log^*(\frac{1+t}{C})(1+t)^{p_1 - p} \leq \epsilon_{p_1}$$

for all $T \geq T_p$. Then for $t > T_p$

$$\begin{aligned}\|D_x E(t)\|_\infty &\leq C_p \log^*(\tfrac{1+t}{C})(1+t)^{-p} \\ &\leq \epsilon_{p_1}(1+t)^{-p_1} \leq \epsilon_{p_1}(1+t-T_p)^{-p_1}.\end{aligned}$$

Now by Lemma 3.3 (with $T_+ = \infty$)

$$\|\rho(t)\|_\infty \leq C\|f_0\|_\infty (1+T_p)^3 Q^3(T_p)(t-T_p)^{-3}$$

for $t > T_p$. Since $T_p \leq C_p$ we have

$$(1+T_p)^3 Q^3(T_p) \leq C_p$$

and for $t > 2T_p + 1$

$$\|\rho(t)\|_\infty \leq C_p(t-T_p)^{-3} \leq C_p(t+1)^{-3}.$$

But from (4.1)

$$\|\rho(t)\|_\infty \leq C \leq C_p(t+1)^{-3}$$

on $0 \leq t \leq T_p$, so (1.3) follows.

5. The small data result

Let $p = \frac{5}{2}$ and choose ϵ_p as in Lemma 3.3. By Lemma 2.4 we have

$$\|D_x E(0)\|_\infty < \epsilon_p$$

for $\|f_0\|_\infty$ sufficiently small. Define

$$T_+ = \sup\{\tau > 0 : \|D_x E(t)\|_\infty \leq \epsilon_p (1+t)^{-p} \text{ for all } t \in [0,\tau]\}.$$

We assume T_+ is finite and derive a contradiction. Since T_+ is finite we have

$$\|D_x E(T_+)\|_\infty = \epsilon_p(1+T_+)^{-p}. \tag{5.1}$$

Taking $T=0$ we have

$$\|D_xE(t)\|_\infty \le \epsilon_p(1+t)^{-p}$$

on $t\in[0,T_+)$ so by Lemma 3.3

$$\|\rho(t)\|_\infty \le C\|f_0\|_\infty t^{-3} \tag{5.2}$$

for $t\in(0,T_+)$. For $0\le t\le 1$

$$\begin{aligned}|\rho(t,x)| &\le \int_{|v|<Q(t)} \|f(t)\|_\infty dv\\ &= \frac{4\pi}{3}Q^3(t)\|f_0\|_\infty \le \frac{4\pi}{3}Q^3(1)\|f_0\|_\infty\\ &\le C\|f_0\|_\infty.\end{aligned} \tag{5.3}$$

From (5.2)and (5.3) it follows that

$$\|\rho(t)\|_\infty \le C\|f_0\|_\infty(1+t)^{-3} \tag{5.4}$$

for $t\in[0,T_+)$. This is (2.1) with

$$\eta = C\|f_0\|_\infty,\ p=3$$

so Lemma 2.5 yields

$$\|D_xE(t)\|_\infty \le C\log^*\left(\frac{1+t}{C\|f_0\|_\infty}\right)C\|f_0\|_\infty(1+t)^{-3}$$

for $t\in[0,T_+)$. It follows that

$$\|D_xE(T_+)\|_\infty \le C\log^*\left(\frac{1+T_+}{C\|f_0\|_\infty}\right)C\|f_0\|_\infty(1+T_+)^{-3}.$$

We take $\|f_0\|_\infty$ small enough that

$$C\|f_0\|_\infty \le e^{-1},$$

then

$$\begin{aligned}\log^*\left(\tfrac{1+T_+}{C\|f_0\|_\infty}\right)(1+T_+)^{-1/2} &= (1+T_+)^{-1/2}\log\left(\tfrac{1+T_+}{C\|f_0\|_\infty}\right)\\ &\le x^{-1/2}\log\left(\tfrac{x}{C\|f_0\|_\infty}\right)\Big|_{x=e^2C\|f_0\|_\infty}\\ &= C\|f_0\|_\infty^{-1/2}.\end{aligned}$$

Hence

$$\begin{aligned}\|D_xE(T_+)\|_\infty &\le C\|f_0\|_\infty^{-1/2}C\|f_0\|_\infty(1+T_+)^{-5/2}\\ &= C\|f_0\|_\infty^{1/2}(1+T_+)^{-5/2}.\end{aligned}$$

For $\|f_0\|_\infty$ sufficiently small this contradicts (5.1). Hence $T_+=+\infty$ for ϵ sufficiently small and (5.4) holds for all $t\ge 0$ completing the proof.

Appendix. To prove Lemma 3.1 define

$$\mathcal{G}=\{k:B_k\geq \frac{1}{6}|B|\}.$$

Then

$$\sum_{\mathcal{G}} B_k^2=|B|-\sum_{\mathcal{G}^c} B_k^2\geq |B|^2-\sum_{\mathcal{G}^c}(\frac{1}{6}|B|)^2\geq \frac{11}{12}|B|^2.$$

Also, for $k\in\mathcal{G}$

$$A_k^2\geq (B_k-\frac{1}{12}|B|)^2=\left(\frac{1}{2}B_k+\frac{1}{2}(B_k-\frac{1}{6}|B|)\right)^2\geq \frac{1}{4}B_k^2$$

so

$$|A|^2\geq \sum_{\mathcal{G}} A_k^2\geq \frac{1}{4}\sum_{\mathcal{G}} B_k^2\geq \frac{1}{4}(\frac{11}{12}|B|^2)$$

and (3.1) follows.

To prove Lemma 3.2 define $e^{(k)}\in\mathbb{R}^3$ by $e_i^{(k)}=1$ if $i=k$ and 0 otherwise. Let

$$P^{(1)}=Me^{(1)}$$

and choose $c_1,c_2,c_3\in\mathbb{R}$ and $P^{(2)},P^{(3)}\in\mathbb{R}^3$ such that

$$Me^{(2)}=c_1P^{(1)}+P^{(2)},$$

$$Me^{(3)}=c_2P^{(1)}+c_3P^{(2)}+P^{(3)},$$

and

$$P^{(1)}\cdot P^{(2)}=P^{(1)}\cdot P^{(3)}=P^{(2)}\cdot P^{(3)}=0.$$

Then

$$|\det M|=|P^{(1)}||P^{(2)}||P^{(3)}|.$$

But

$$|P^{(1)}|=|Me^{(1)}|\geq B|e^{(1)}|=B,$$

$$\begin{aligned}|P^{(2)}| &= |Me^{(2)}-c_1P^{(1)}|=|M(e^{(2)}-c_1e^{(1)})|\\ &\geq B|e^{(2)}-c_1e^{(1)}|\geq B,\end{aligned}$$

and

$$\begin{aligned}|P^{(3)}| &= |Me^{(3)}-c_2P^{(1)}-c_3P^{(2)}|\\ &=|Me^{(3)}-c_2Me^{(1)}-c_3M(e^{(2)}-c_1e^{(1)})|\\ &=|M(e^{(3)}+(c_1c_3-c_2)e^{(1)}-c_3e^{(2)})|\\ &\geq B|e^{(3)}+(c_1c_3-c_2)e^{(1)}-c_3e^{(2)}|\\ &\geq B,\end{aligned}$$

so (3.2) follows.

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