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HYPOCOERCIVITY OF STOCHASTIC GALERKIN FORMULATIONS FOR STABILIZATION OF KINETIC EQUATIONS*

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Abstract. We consider the stabilization of linear kinetic equations with a random relaxation term. The well-known framework of hypocoercivity by J. Dolbeault, C. Mouhot and C. Schmeiser (2015) ensures the stability in the deterministic case. This framework, however, cannot be applied directly for arbitrarily small random relaxation parameters. Therefore, we introduce a Galerkin formulation, which reformulates the stochastic system as a sequence of deterministic ones. We prove for the γ-distribution that the hypocoercivity framework ensures the stability of this series and hence the stochastic stability of the underlying random kinetic equation. The presented approach also yields a convergent numerical approximation.

Keywords. Systems of kinetic and hyperbolic balance laws; exponential stability; asymptotic stability; stochastic Galerkin.

AMS subject classifications. 35B35; 93D20; 37L45; 35B30; 35R60.

1. Introduction

Stabilization of hyperbolic balance laws has been studied intensively in the past years with applications to Euler equations for gas dynamics, the p-system, and shallow water equations, see e.g. [2, 10] and the references therein for an overview on recent results. A well-known approach to prove exponential stability of equilibria is the analysis of dissipative boundary conditions and the construction of suitable Lyapunov functionals [11, 13, 26]. However, general results are so far only available if the source term is sufficiently small [12, 18], diagonally stable [1] or strictly positive definite [3]. For certain balance laws with stiff source term also the limiting behavior has been studied [4, 14, 50].

Kinetic partial differential equations belong to the class of *linear* hyperbolic balance laws and formally the previous results can be applied to study their stabilization. An interesting class of linear kinetic equations are those with an additional *stiff* source (or relaxation) term resulting from linearization of nonlinear problems for stability analysis and optimization.

Commonly, solutions are close to a kinetic equilibrium. Those equilibria typically fulfill hyperbolic conservation laws. However, estimating the rate of the relaxation of the solutions towards an equilibrium is a challenging problem, since the collision term may only act with respect to the velocity space [16].

For linear hyperbolic systems with stiff source term in one dimension that satisfy structural stability conditions, presented in [50], boundary stability has been studied using a weighted Lyapunov functional [32, 51]. We also refer to [30] for boundary control of Vlasov-Fokker-Planck equations and to [23, 39] for boundary control of general kinetic systems. It is remarkable that widely used Lyapunov functionals in boundary control may be improper to characterize the long-time behaviour [25]. This is in particular the case for relaxation systems, when the desired equilibria are not constant in space.

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Then, it has been proven that the solution to the kinetic system may diverge, when the stiffness parameter tends to zero. Therefore, we investigate the use of hypocoercivity for systems with stiff relaxation. In previous works, hypocoercivity has been systematically developed to analyze the large-time behaviour of the solutions and convergence to equilibria. To this end, a modified entropy functional has been proposed to bound the L^2-norm of solutions, see e.g. [16, 17, 28, 29, 47].

Uncertainties commonly arise in hyperbolic and kinetic equations due to modelling errors, measurements and uncertain boundary conditions. In particular, the relaxation parameter is not physically motivated and cannot be obtained by measurements. Hence, it should be modelled by a random parameter taking arbitrarily small values.

Many attempts have been made to stabilize the stochastic systems to obtain a deterministic desired state. Boundary control for hyperbolic systems have been presented in [27, 40] to reduce the variance in the system.

The question if the random solutions to linear kinetic equations converge to the deterministic kinetic equilibirum exponentially fast with respect to a suitable norm has been extensively analyzed [37, 38]. Exponential decay has been established in the parabolic scaling [37, Sec. 5.1] and in the high field scaling [37, Sec. 5.2] for arbitrary random perturbations in the relaxation term and initial data. These results have been extended to nonlinear equations and have been unified with acoustic scalings [35, 38]. Results for the acoustic scaling, however, are only partial, since an arbitrarily small relaxation parameter results in a vanishing decay rate [38, Rem. 2.6, Th. 4.4] and the random input is so far assumed bounded [38, Sec. 6].

This paper is devoted to the acoustic scaling of linear kinetic equations with random, unbounded relaxation that is modelled by the γ-distribution. We prove that the impact of randomness diminishes exponentially fast in time as the solution converges to the kinetic equilibrium.

Typically, one would use Monte-Carlo methods and apply the former hypocoercivity framework for each realisation. In other words, first, the solution is discretized in the stochastic space and then stabilization results are obtained. However, this "first-discretize-then-stabilize" ansatz cannot be directly applied if the relaxation parameter tends to zero.

Instead, the underlying tool to study this problem is the representation of the solution by a series of orthogonal functions, known as generalized polynomial chaos (gPC) expansions [6, 21, 48, 49]. Here, a series expansion of the solutions is substituted into the governing equations and as second step the series is projected to obtain evolution equations for its coefficients. This approach is often applied in uncertainty quantification, where the parameter is interpreted as a random variable. In this direction many results for kinetic equations are available [7, 8, 33, 35, 46, 52, 53]. Recently, also results for hyperbolic equations have been established [9, 15, 19, 20, 22, 34, 36, 43, 45]. For convergence results of the truncated expansions to the true solution smoothness assumptions are required [22, 33, 53]. Similarly to [33, 37, 53], we prove that the solution preserves the regularity of the initial data. Regularity with respect to the relaxation parameter is then found in terms of the decay of the coefficients. To this end, we introduce a weighted sequence space as solution space. It turns out that the hypocoercivity framework [17] can be applied in this sequence space without discretizing the solution in the stochastic space. This allows to obtain the desired convergence and stabilization results.

This paper is organized as follows. Section 2 recalls the hypocoercivity framework from [17]. We illustrate both theoretically and numerically the applicability for a fixed value $\varepsilon > 0$ and its little informative value in the limit $\varepsilon \to 0^+$. Section 3 analyzes a stochastic Galerkin formulation corresponding to the random kinetic equation. An infinite-dimensional weighted sequence space is introduced as solution space. When the hypocoercivity framework from [17] is applied in this space, its informative value for the stabilization of the mean of deviations remains high even if arbitrarily small relaxation parameters occur. We obtain a convergent numerical method by approximating the stochastic Galerkin formulation on a finite-dimensional subspace.

We consider the kinetic equations

$$\partial_t f(t,x,v)+\frac{1}{\varepsilon^\alpha}Tf(t,x,v)=\frac{1}{\varepsilon^{1+\alpha}}Lf(t,x,v) \quad \text{with} \quad (t,x,v)\in\mathbb{R}^+\times\mathbb{R}^2$$

for a distribution function $f(t,x,v)$ subject to the initial data $f(0,x,v)=f_0(x,v)$ and subject to periodic or reflecting boundary conditions. Here, $T:=v\partial_x-\partial_x V(x)\partial_v$ is the transport operator. The external potential $V(x)$ is a possibly space-varying function. The collision operator L is independent of time, for example $Lf=\big(M[f]-f\big)$, where $M[f]$ is the local Maxwellian. The variable v is the velocity and vanishing boundary conditions in the limit $|v|\to\infty$ are imposed.

The parameter $\alpha\geq 0$ describes different regimes. We have $\alpha=1$ for the parabolic scaling and $\alpha=0$ for the acoustic scaling, wherein we are interested in. We denote the linear **kinetic equations with acoustic scaling** as

$$\partial_t f(t,x,v)+Tf(t,x,v)=L_\varepsilon f(t,x,v) \quad \text{with} \quad L_\varepsilon:=\frac{L}{\varepsilon}. \tag{1.1}$$

The relaxation parameter $\varepsilon>0$ is typically unknown and very small. Thus, we replace it by a random variable, defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$, with arbitrarily small positive realizations $\varepsilon(\omega)\in\mathbb{R}^+$ for $\omega\in\Omega$. More precisely, the inverse $\frac{1}{\varepsilon(\omega)}$ is modelled by the γ-distribution that is a family of continuous probability distributions, which includes the exponential, Erlang and χ^2-distribution as special cases. The steady state $F(x,v)$ is independent of each realisation $\varepsilon(\omega)$, but the solution $f(t,x,v;\omega)$ is random as well and depends on the event $\omega\in\Omega$.

This paper addresses the question if there exist positive constants $C,\kappa>0$ such that the random solution $f(t,x,v;\omega)$ converges to the deterministic steady state exponentially fast in the mean squared sense

$$\mathbb{E}\left[\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(f(t,x,v;\omega)-F(x,v)\Big)^2\mathrm{d}x\,\mathrm{d}v\right]\leq Ce^{-\kappa t}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(f_0(x,v)-F(x,v)\Big)^2\mathrm{d}x\,\mathrm{d}v.$$

In the deterministic case, when the value $\varepsilon>0$ is fixed and remains positive, an answer to this question can be given by analysing hypocoercivity of the operators.

2. Hypocoercivity framework

We recall the strategy, proposed in [17], to study the hypocoercivity of the deterministic kinetic Equations (1.1) for a fixed relaxation parameter $\varepsilon>0$. We consider a Hilbert space $\mathcal{H}$ such that the linear operators L_ε, T are closed and generate the strongly continuous semigroup $e^{(L_\varepsilon-T)t}$ on the space $\mathcal{H}$. The orthogonal projection Π from the solution space $\mathcal{D}$ onto the set of local equilibria $\mathcal{N}(L_\varepsilon)$ is defined by

$$\Pi f:=\frac{\rho_f}{\rho_F}F \quad \text{with} \quad \rho_f(t,x):=\int_{\mathbb{R}}f(t,x,v)\mathrm{d}v.$$

We will examine the modified entropy functional $H[f]$

$$\begin{aligned} &H[f] := \frac{1}{2}\|f\|^2 + \gamma\langle Af, f\rangle \quad \text{with} \quad A := \left[1 + (T\Pi)^*(T\Pi)\right]^{-1}(T\Pi)^* \\ &\text{and} \quad \Big\langle f(t,\cdot,\cdot), g(t,\cdot,\cdot)\Big\rangle := \int_{\mathbb{R}^2} f(t,x,v)g(t,x,v)\,\mathrm{d}\mu \quad \text{for} \quad \mu := \frac{\mathrm{d}x\,\mathrm{d}v}{F(x,v)}. \end{aligned} \tag{2.1}$$

Here, $\gamma \in (0,1)$ is a problem-dependent positive parameter. Following [17], we introduce four critical properties.

H1: Microscopic coercivity: The operator L_ε is symmetric and there exists a positive constant $\lambda_m > 0$ such that

$$-\big\langle L_\varepsilon f, f\big\rangle \geq \lambda_m \big\|(1-\Pi)f\big\|^2 \quad \text{for all} \quad f \in \mathcal{D}.$$

H2: Macroscopic coercivity: The operator T is skew-symmetric and there exists a positive constant $\lambda_M > 0$ such that

$$\big\|T\Pi f\big\|^2 \geq \lambda_M \big\|\Pi f\big\|^2 \quad \text{for all} \quad f \in \mathcal{D}.$$

H3: The operator T and L_ε satisfy

$$\Pi T \Pi = 0.$$

H4: The operators $AT(1-\Pi)$ and AL_ε are bounded. There exists a positive constant $C_M > 0$ such that

$$\big\|AT(1-\Pi)f\big\| + \big\|AL_\varepsilon f\big\| \leq C_M \big\|(1-\Pi)f\big\| \quad \text{for all} \quad f \in \mathcal{D}.$$

Microscopic coercivity states that the restriction of the operator L_ε onto the complement $\mathcal{N}(L_\varepsilon)^\perp$ is coercive.

Macroscopic coercivity on the other hand guarantees that the transport operator T is coercive on the nullspace $\mathcal{N}(L_\varepsilon)$. Assumptions (**H3**) and (**H4**) have technical importance. In particular, assumption (**H4**) is slightly stronger than actually required in the proof of [17, Th. 2]. Theorem 3.2 in Section 3 makes use of the following weaker, but sufficient property

$$\big\langle AL_\varepsilon f, f\big\rangle \leq C_M \big\|(1-\Pi)f\big\| \|f\|. \tag{2.2}$$

Theorem 2.1 describes the asymptotic behavior of the deterministic problem (1.1).

THEOREM 2.1 (According to [17, Th. 2]). *Suppose the assumptions* **H1–H4** *hold. For any initial values $f_0 \in \mathcal{D}$ and for any positive relaxation parameter $\varepsilon > 0$ there exist positive constants $C(\varepsilon)$ and $\kappa(\varepsilon)$ that may depend on $\varepsilon > 0$ such that*

$$\big\|f(t,\cdot,\cdot) - F(\cdot,\cdot)\big\|^2 \leq C(\varepsilon)e^{-\kappa(\varepsilon)t}\big\|f_0 - F\big\|^2 \quad \textit{for all} \quad t \geq 0.$$

In particular, we have for some $\delta > 0$ the rate

$$\kappa = \frac{2}{1+\gamma}\min\left\{\lambda_m - \frac{\gamma(C_M+1)}{2\delta}, \frac{\gamma\lambda_M}{1+\lambda_M}\right\} - \frac{\gamma\delta(C_M+1)}{1+\gamma} \quad \textit{with} \quad C = \frac{1+\gamma}{1-\gamma}.$$

Note that periodic and reflecting boundary conditions ensure conservation of mass and the skew-symmetry of the operator T at the boundary. Next, we discuss limitations of the applicability of this strategy in a particular case.

2.1. Applicability of the hypocoercivity framework. As a toy problem, we consider the two-velocity model

$$\begin{aligned} &\partial_t \vec{f}(t,x) + T\vec{f}(t,x) = L_\varepsilon \vec{f}(t,x) \qquad \text{with} \\ &\vec{f}(t,x) = \begin{pmatrix} f^+(t,x) \\ f^-(t,x) \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x, \quad L_\varepsilon = \frac{1}{2\varepsilon} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \end{aligned} \tag{2.3}$$

We obtain the following corollary by applying Theorem 2.1.

Corollary 2.1. *Assume a fixed relaxation parameter $\varepsilon > 0$. Then, there exist positive constants $C(\varepsilon)$ and $\kappa^*(\varepsilon)$ such that the solution to the model (2.3) with acoustic scaling and positive relaxation parameter $\varepsilon > 0$ converges to the steady state $\vec{F}$ exponentially fast, i.e.*

$$\left\| \vec{f}(t,\cdot) - \vec{F}(\cdot) \right\|^2 \leq C(\varepsilon) e^{-\kappa^*(\varepsilon)t} \left\| \vec{f}_0 - \vec{F} \right\|^2 \quad \textit{for all} \quad t \geq 0. \tag{2.4}$$

If the relaxation parameter $\varepsilon > 0$ is sufficiently small, we have the decay rate

$$\kappa^*(\varepsilon) := \max_{0 < \delta < \frac{4\pi^2\varepsilon}{(1+\pi^2)(1+2\varepsilon)}} \left\{ \kappa^*(\delta,\varepsilon) \right\} \quad \textit{for} \quad \kappa^*(\delta,\varepsilon) := \frac{2\delta}{\varepsilon} \frac{4\pi^2\varepsilon - \delta(1+2\varepsilon)(1+\pi^2)}{4\pi^2\varepsilon\delta + (1+2\varepsilon+4\delta)(1+\pi^2)}.$$

Proof. The global steady state and the projection onto the nullspace $\mathcal{N}(L_\varepsilon)$ read

$$\vec{F}(x) = \begin{pmatrix} F^+(x) \\ F^-(x) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \Pi \vec{f} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{f} = \frac{f^+ + f^-}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

It remains to show the properties (**H1**) – (**H4**).

H1: We have $L_\varepsilon = \frac{1}{\varepsilon}(\Pi - 1)$ and $\langle L_\varepsilon \vec{f}, \vec{f} \rangle \leq -\frac{1}{\varepsilon} \left\| (1-\Pi)\vec{f} \right\|^2$. This yields $\lambda_m = \frac{1}{\varepsilon}$.

H2: We have

$$T\Pi\vec{f} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \partial_x \vec{f} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \partial_x \vec{f} = \frac{\partial_x(f^+ + f^-)}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

By applying the Poincaré inequality to the scalar function $\frac{f^+ + f^-}{\sqrt{2}}$, whose average value over the domain $[0,1]$ is zero, we obtain

$$\left\| T\Pi\vec{f} \right\|^2 = \frac{1}{2} \int_0^1 \left[\partial_x (f^+ + f^-) \right]^2 \mathrm{d}x \geq \frac{1}{C_P^2} \left\| \Pi\vec{f} \right\|^2$$

with Poincaré constant $C_P = \frac{1}{\pi}$. Hence, we have $\lambda_M = \pi^2$.

H3: We calculate

$$\Pi T \Pi = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \partial_x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} =: \mathbb{O}.$$

H4: The equality

$$(T\Pi)^* T(1-\Pi) = -\Pi T^2 (1-\Pi) = -\Pi(1-\Pi)\partial_x^2 = \mathbb{O}$$

yields $\left\| AT(1-\Pi)f \right\| = 0$. We conclude $C_M = \frac{1}{2\varepsilon}$.

Theorem 2.1 yields the decay rate

$$\kappa(\varepsilon) = \frac{2}{1+\gamma}\left[\min\left\{\frac{4\delta-\gamma(1+2\varepsilon)}{4\delta\varepsilon}, \frac{\gamma\pi^2}{1+\pi^2}\right\} - \frac{\gamma\delta(1+2\varepsilon)}{4\varepsilon}\right].$$

We make the parameter γ dependent on $\delta,\varepsilon>0$ and we define

$$\begin{aligned}\gamma(\delta,\varepsilon) &:= \operatorname*{argmax}_{\gamma\in(0,1)}\left\{\min\left\{\frac{4\delta-\gamma(1+2\varepsilon)}{4\delta\varepsilon}, \frac{\gamma\pi^2}{1+\pi^2}\right\} - \frac{\gamma\delta(1+2\varepsilon)}{4\varepsilon}\right\} \\ &= \frac{4(1+\pi^2)\delta}{4\pi^2\varepsilon\delta+1+2\varepsilon+\pi^2+2\pi^2\varepsilon}. \end{aligned} \tag{2.5}$$

The existence of a positive value $\gamma(\delta,\varepsilon)>0$ and hence a positive decay rate is guaranteed by the bounds

$$\varepsilon>0 \quad \text{and} \quad 0<\delta<\frac{4\pi^2\varepsilon}{(1+\pi^2)(1+2\varepsilon)}.$$

If the parameter $\varepsilon>0$ is sufficiently small, we have $\gamma(\delta,\varepsilon)<1$. Then, the maximal decay rate $\kappa^*(\delta,\varepsilon)$ in terms of γ is achieved at $\gamma(\delta,\varepsilon)$, i.e

$$\begin{aligned}\kappa^*(\delta,\varepsilon) &= \frac{2}{1+\gamma(\delta,\varepsilon)}\left[\min\left\{\frac{4\delta-\gamma(\delta,\varepsilon)(1+2\varepsilon)}{4\delta\varepsilon}, \frac{\gamma(\delta,\varepsilon)\pi^2}{1+\pi^2}\right\} - \frac{\gamma(\delta,\varepsilon)\delta(1+2\varepsilon)}{4\varepsilon}\right] \\ &= \frac{2\gamma(\delta,\varepsilon)}{1+\gamma(\delta,\varepsilon)}\left[\frac{\pi^2}{1+\pi^2} - \frac{\delta(1+2\varepsilon)}{4\varepsilon}\right] \\ &= \frac{8(1+\pi^2)\delta}{4\pi^2\varepsilon\delta+1+2\varepsilon+\pi^2+2\pi^2\varepsilon+4(1+\pi^2)\delta}\left[\frac{\pi^2}{1+\pi^2} - \frac{\delta(1+2\varepsilon)}{4\varepsilon}\right] \\ &= \frac{2\delta}{\varepsilon}\frac{4\pi^2\varepsilon-\delta(1+2\varepsilon)(1+\pi^2)}{4\pi^2\varepsilon\delta+(1+2\varepsilon+4\delta)(1+\pi^2)}. \end{aligned}$$

□

For the sake of simplicity, we have assumed the parameter $\varepsilon>0$ is sufficiently small. Then, we have $\gamma(\delta,\varepsilon)<1$. However, this assumption is not needed for our novel results in Section 3, since these are not based on the min-max problem (2.5). Furthermore, we refer the interested reader to [37], where this assumption is not required for solving the min-max problem.

The left panel of Figure 2.1 shows the decay rate depending on $\varepsilon>0$ and $\delta>0$. The guaranteed decay rate $\kappa^*(\delta^*,\varepsilon)$ that maximizes the decay rate for each fixed relaxation parameter $\varepsilon>0$ is shown as blue line. The corresponding optimal choice $\gamma(\delta^*,\varepsilon)$ is shown as green (dashed) line in the right panel of Figure 2.1. Since both quantities tend to zero for $\varepsilon\to 0^+$, exponential decay of the system is *not guaranteed* in the small relaxation limit. The reason is the violation of assumption (**H4**).

Note that these observations confirm the findings in [38, Rem. 2.6], where a vanishing decay rate for the acoustic scaling is obtained for $\varepsilon\to 0^+$. In contrast, the decay rate *does not vanish* for the parabolic [37, Th. 3.3] and high field scaling [37, Th. 3.6].

2.2. Numerical simulations. This behaviour is also seen in numerical experiments. Similarly to [31,42,44] we use a first-order implicit-explicit (IMEX) scheme that treats the convective term explicitly and the collision term implicitly due to the stiffness

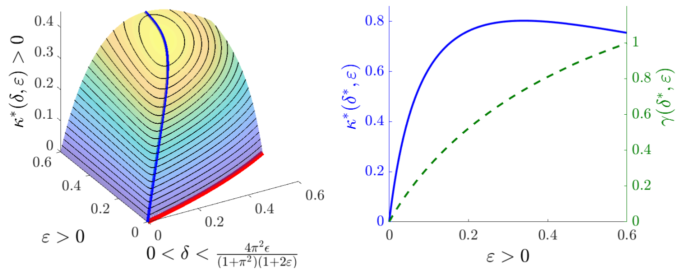

FIGURE 2.1. *Decay rate in Corollary 2.1 depending on $\varepsilon > 0$, $\delta > 0$.*

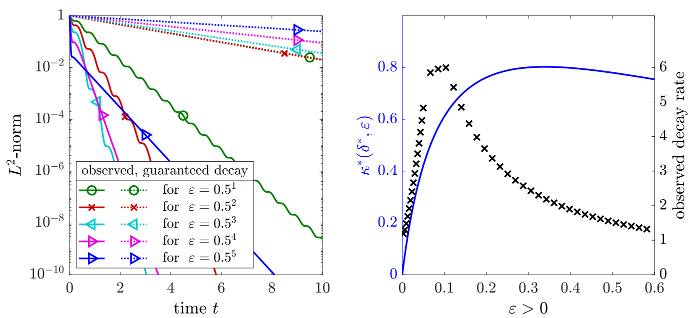

FIGURE 2.2. *Exponential decay depending on the parameter $\varepsilon > 0$.*

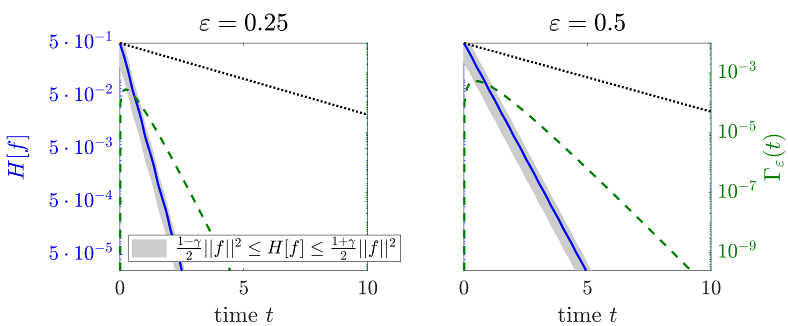

FIGURE 2.3. *Illustration of the entropy functional $H[\vec{f}] = 1/2\|\vec{f}\|^2 + \Gamma_\varepsilon(t)$, defined by Equation (2.1), with $\Gamma_\varepsilon(t) = \gamma(\delta^*,\varepsilon)\left\langle A\vec{f}(t,\cdot), \vec{f}(t,\cdot)\right\rangle$.*

for small values of $\varepsilon > 0$. Then, the IMEX scheme [31] for the two-velocity model (2.3) reads as

$$\vec{f}_i^{n+1} = \frac{\varepsilon}{\varepsilon + \Delta t}\tilde{\vec{f}}_i^n - \Delta t T^{\Delta x}\tilde{\vec{f}}_i^n + \frac{\Delta t}{\varepsilon + \Delta t}\Pi^{\Delta x}\tilde{\vec{f}}_i^n \quad \text{with} \quad \tilde{\vec{f}}_i^n = \frac{\varepsilon \vec{f}_i^n + \Delta t \Pi^{\Delta x}\vec{f}_i^n}{\varepsilon + \Delta t}. \tag{2.6}$$

It is an asymptotic preserving scheme [31, Prop. 1] for any Lipschitz continuous numerical flux function needed for the spatial differentiation of the discrete operator $T^{\Delta x}$. The left panel of Figure 2.2 shows the L^2-norm $\|\vec{f}\|_2^2$ in the logarithm scale for various relaxation parameters $\varepsilon > 0$. The exponential decay guaranteed by Theorem 2.1 and Corollary 2.1 is illustrated using dotted lines in Figure 2.2. The numerically computed rate is shown in solid lines. Different colours are related to different values of the relaxation parameter. The right panel shows the numerically observed decay rate with respect to the right y-axis as black crosses. The numerical experiments suggest that the guaranteed decay rate (blue line) is not sharp as the observed decay rates (black crosses) are much larger. Nevertheless, we observe also numerically that decay rates become smaller, when the relaxation parameter goes to zero, which illustrates the little informative value in the zero limit $\varepsilon \to 0^+$.

Figure 2.3 shows the exponential decay of the entropy functional $H[f]$ as well as the magnitude of the term $\Gamma_\varepsilon(t) := \gamma(\delta^*,\varepsilon)\langle A\vec{f}(t,\cdot), \vec{f}(t,\cdot)\rangle$ in definition (2.1) of the functional $H[f] = {}^1\!/\!_2\|\vec{f}\| + \Gamma_\varepsilon(t)$, where the optimal choice of γ is illustrated in Figure 2.1. The left y-axis shows the exponential decay of the functional $H[f]$ in blue as solid line. The bound by the L^2-norm is gray shaded and the guaranteed decay is shown as dotted line. The term $\Gamma_\varepsilon(t)$, which determines the width of this bound, is shown with respect to the right y-axes in a green dashed line. The value of $\gamma(\delta^*,\varepsilon)$ and hence the contribution of Γ_ε are relatively small.

Summarizing, the parameters λ_m and C_M in the conditions (**H1**) and (**H4**), and hence the decay rate established in Theorem 2.1 depend on the parameter $\varepsilon > 0$. In particular, assumption (**H4**) is violated in the limit $\varepsilon \to 0^+$. Corollary 2.1 explicitly shows that there is *no* guarantee on exponential decay in the limit $\varepsilon \to 0^+$.

3. Stochastic exponential stability

If we consider system (1.1), *without stiff relaxation*, given by

$$\partial_t f(t,x,v) + Tf(t,x,v) = Lf(t,x,v) \quad \text{for} \quad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^2, \tag{3.1}$$

where the operator $L = \varepsilon L_\varepsilon$ is independent of the relaxation parameter $\varepsilon > 0$, then the two systems (1.1) and (3.1) have the same global steady state $F(x,v)$. We consider now the full model (1.1) and we write the term ${}^1\!/\!_\varepsilon > 0$ as

$${}^1\!/\!_\varepsilon = \xi + \eta > 0.$$

We introduce the stochastic system

$$\partial_t f(t,x,v;\xi) + Tf(t,x,v;\xi) = (\xi+\eta)Lf(t,x,v;\xi) \tag{3.2}$$

by considering ξ as a random variable with realizations $\xi(\omega) \in \mathbb{R}_0^+$. The random variable $\xi \sim \mathbb{P}$ is described by the γ-distribution with probability density

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\xi} = \rho(\xi) := \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} \xi^\alpha e^{-\beta\xi} \quad \text{for} \quad \alpha \in \mathbb{R}_0^+ \quad \text{and} \quad \beta \in \mathbb{R}^+.$$

The constant $\eta > 0$ is arbitrarily small, but deterministic and ensures that the parameterized system cannot simplify to a conservation law in the case $\xi(\omega) = 0$. In the following, we will show that if the system (3.1), *without stiff relaxation*, satisfies the properties (**H1**) – (**H4**), the system (3.2) is exponentially stable in the sense of a weighted, averaged L^2-norm with respect to ξ and any small parameter $\eta > 0$. To be precise, our main result in Theorem 3.2 is to prove the existence of positive constants $C, \kappa > 0$ such that the **mean squared L^2-norm**

$$\mathbb{E}\Big[\big\|f(t,\cdot,\cdot;\xi) - F\big\|^2\Big] = \int_0^\infty \big\|f(t,\cdot,\cdot;\xi) - F\big\|^2 \rho(\xi)\mathrm{d}\xi \le Ce^{-\kappa t}\big\|f_0 - F\big\|^2 \tag{3.3}$$

decays exponentially fast. As basic tool we use generalized Laguerre polynomials defined by the recursion

$$\begin{gathered} L_0^\alpha(\xi) = 1, \quad L_1^\alpha(\xi) = 1 + \alpha - x, \\ (k+1)L_{k+1}^\alpha(\xi) = (2k+1+\alpha-\xi)L_k^\alpha(\xi) - (k+\alpha)L_{k-1}^\alpha(\xi) \end{gathered} \tag{3.4}$$

for $k\geq 1$ and $\alpha\in\mathbb{R}_0^+$. According to [24, Sec. 7.414] the scaling

$$\phi_k(\xi) := \frac{L_k^\alpha(\beta\xi)}{\big\|L_k^\alpha(\beta\xi)\big\|_\rho} \quad \text{with} \quad \big\|L_k^\alpha(\beta\xi)\big\|_\rho = \sqrt{\frac{\Gamma(\alpha+1+k)}{\Gamma(\alpha+1)k!}} \quad \text{and} \quad \beta\in\mathbb{R}^+ \tag{3.5}$$

is orthonormal to the inner product

$$\big\langle \phi_k,\phi_j\big\rangle_\rho := \int_0^\infty \phi_k(\xi)\phi_j(\xi)\ \rho(\xi)\mathrm{d}\xi = \delta_{k,j}.$$

Then, the functional dependence of the solution on the random variable $\xi\sim\mathbb{P}$ is described by the series expansion

$$f(t,x,v;\xi) = \sum_{k=0}^\infty \mathbf{f}_k(t,x,v)\phi_k(\xi) \quad \text{for} \quad \mathbf{f}_k(t,x,v) := \big\langle f(t,x,v;\cdot),\phi_k\big\rangle_\rho.$$

We define the sequence of *infinite* matrices

$$\mathbf{T} := T\mathbb{1}, \quad \mathbf{L}_\eta := L\mathbf{P}_\eta \quad \text{with} \quad \mathbf{P}_\eta := \mathbf{P}+\eta\mathbb{1}, \quad \mathbf{P} := \Big(\big\langle \xi\phi_k(\xi),\phi_j(\xi)\big\rangle_\rho\Big)_{k,j\in\mathbb{N}_0},$$

where $\mathbb{1} := \operatorname{diag}\{1,\dots\}$ denotes a sequence of identity matrices. By projecting the system (3.2) onto the space spanned by the polynomials $\{\phi_0,\phi_1,\dots\}$, we obtain the **stochastic Galerkin formulation**

$$\begin{cases} \partial_t\mathbf{f}(t,x,v) + \mathbf{T}\mathbf{f}(t,x,v) = \mathbf{L}_\eta\mathbf{f}(t,x,v), \\ \mathbf{f}(0,x,v) = \Big(\big\langle f_0(x,v),\phi_k\big\rangle_\rho\Big)_{k\in\mathbb{N}_0} = f_0(x,v)(\delta_{0,k})_{k\in\mathbb{N}_0}, \end{cases} \tag{3.6}$$

where $\mathbf{f} = (\mathbf{f}_1,\mathbf{f}_2,\dots)^{\mathrm{T}}$ is an infinite vector. Due to the orthonormality of the generalized Laguerre polynomials, we have by construction

$$\int_0^\infty \big\|f(t,\cdot,\cdot;\xi)\big\|^2\ \rho(\xi)\mathrm{d}\xi = \sum_{k=0}^\infty \big\|\mathbf{f}_k\big\|^2 = \sum_{k=0}^\infty \int_\mathbb{R}\int_\mathbb{R} \mathbf{f}_k^2(t,x,v)\frac{\mathrm{d}x\,\mathrm{d}v}{F(x,v)}.$$

Therefore, the mean squared L^2-stability (3.3) follows from the L^2-stability of the stochastic Galerkin formulation (3.6). We also consider the truncation

$$\begin{cases} \partial_t\mathbf{p}^{(K)}(t,x,v) + \mathbf{T}^{(K)}\mathbf{p}^{(K)}(t,x,v) = \mathbf{L}_\eta^{(K)}\mathbf{p}^{(K)}(t,x,v), \\ \mathbf{p}^{(K)}(0,x,v) = f_0(x,v)(\delta_{k,0})_{k\in\mathbb{N}_0}. \end{cases} \tag{3.7}$$

Here, the entries of the finite matrices $\mathbf{T}^{(K)},\mathbf{L}^{(K)}\in\mathbb{R}^{(K+1)\times(K+1)}$ satisfy $\mathbf{T}_{i,j}^{(K)} = \mathbf{T}_{i,j}$ and $\mathbf{L}_{i,j}^{(K)} = \mathbf{L}_{i,j}$. The corresponding solution is denoted by

$$p^{(K)}(t,x,v;\xi) = \sum_{k=0}^K \mathbf{p}_k(t,x,v)\phi_k(\xi) \quad \text{with} \quad \mathbf{p}^{(K)} := (\mathbf{p}_0,\dots,\mathbf{p}_K)^{\mathrm{T}}\in\mathbb{R}^{K+1}.$$

In other words, $p^{(K)}(t,x,v;\xi)$ and $\mathbf{p}^{(K)}(t,x,v)$ are approximations to $f(t,x,v;\xi)$ and $\mathbf{f}(t,x,v)$. First, we will show that the solution $\mathbf{f}$ to the infinite system (3.6) belongs for each fixed $t\geq 0$ to the **weighted sequence space**

$$\ell_\sigma^2 := \left\{ \ \mathbf{f} := (\mathbf{f}_k)_{k\in\mathbb{N}_0} \ \middle|\ \big\langle \mathbf{f},\mathbf{g}\big\rangle_{\ell_\sigma^2} := \sum_{k\in\mathbb{N}_0} \sigma_k\Big\langle \mathbf{f}_k(t,\cdot,\cdot)\mathbf{g}_k(t,\cdot,\cdot)\Big\rangle, \ \ \big\|\mathbf{f}\big\|_{\ell_\sigma^2} < \infty \ \right\}$$

with the weights $\sigma_k = k + \dfrac{\sqrt{\alpha+1}}{2\beta\eta}$

and the inner product $\langle\cdot,\cdot\rangle$ defined in Equation (2.1). Note that the inner product depends on time. We write for short $\langle \mathbf{f},\mathbf{g}\rangle_{\ell^2_\sigma}(t) = \langle \mathbf{f},\mathbf{g}\rangle_{\ell^2_\sigma}$ and $\ell^2_\sigma = \ell^2$ in the case $\sigma_k = 1$ with $k \in \mathbb{N}_0$. Then, we show that the hypocoercivity framework can be applied to the space ℓ^2_σ. Finally, we consider stable approximations $\mathbf{p}^{(K)}$ with respect to the stochastic space, which results in a **first-stabilize-then-discretize** framework.

The rest of this section is structured as follows: Section 3.1 states that the solution to the stochastic Galerkin formulation (3.6) is in the weighted sequence space ℓ^2_σ. The main Theorem 3.2 in Section 3.2 shows the exponential stability of the stochastic system (3.2), i.e. the estimate (3.3). Section 3.3 ensures the convergence of solutions using the truncated stochastic Galerkin system (3.7). Finally, Section 3.4 is devoted to the numerical illustration of the theoretical results.

3.1. Characterization of the solution space. First, we calculate the entries of the matrix $\mathbf{P}$ exactly.

LEMMA 3.1. *The scaled, generalized Laguerre polynomials ϕ_k, $k \in \mathbb{N}_0$ satisfy*

$$\beta \mathbf{P}_{0,j} = \begin{cases} 0 & \text{for } j \geq 2, \\ -\sqrt{1+\alpha} & \text{for } j = 1, \\ 1+\alpha & \text{for } j = 0 \end{cases} \qquad \text{and} \quad k = 0,$$

$$\beta \mathbf{P}_{k,j} = \begin{cases} 0 & \text{for } |j-k| \geq 2, \\ -\sqrt{k(k+\alpha)} & \text{for } j = k-1, \\ -\sqrt{(k+1)(k+1+\alpha)} & \text{for } j = k+1, \\ 2k+1+\alpha & \text{for } j = k \end{cases} \qquad \text{and} \quad k \geq 1.$$

Proof. The recurrence relation (3.4) and the normalization (3.5) yield

$$\begin{aligned} &(2k+1+\alpha)\phi_k(\xi) - \sqrt{(k+1)(k+1+\alpha)}\phi_{k+1}(\xi) - \sqrt{k(k+\alpha)}\phi_{k-1}(\xi) \\ &= (2k+1+\alpha)\phi_k(\xi) - (k+1)\frac{\|L^\alpha_{k+1}(\beta\xi)\|_\rho}{\|L^\alpha_k(\beta\xi)\|_\rho}\phi_{k+1}(\xi) - (k+\alpha)\frac{\|L^\alpha_{k-1}(\beta\xi)\|_\rho}{\|L^\alpha_k(\beta\xi)\|_\rho}\phi_{k-1}(\xi) \\ &= \beta\xi\phi_k(\xi) \quad \text{for} \quad k \geq 1. \end{aligned}$$

The claim follows from the orthonormal projection

$$\begin{aligned} \beta\mathbf{P}_{k,j} &= \beta\langle \xi\phi_k(\xi), \phi_j(\xi)\rangle_\rho \\ &= (2k+1+\alpha)\delta_{k,j} - \sqrt{(k+1)(k+1+\alpha)}\delta_{k+1,j} - \sqrt{k(k+\alpha)}\delta_{k-1,j}. \end{aligned}$$

∎

This lemma states that the entries of the matrix $\mathbf{P}$ grow linearly and it allows us to prove the following technical lemma.

LEMMA 3.2. *For all $K \in \mathbb{N}_0 \cup \{\infty\}$, the matrices*

$$\begin{aligned} &\mathbf{P}^{(K)}_{\eta,\sigma} := \frac{\sigma^{(K)}\mathbf{P}^{(K)} + \mathbf{P}^{(K)}\sigma^{(K)}}{2} + \eta\sigma^{(K)}, \quad \mathbf{P}_{\eta,\sigma} := \mathbf{P}^{(\infty)}_{\eta,\sigma} \\ &\text{with} \quad \sigma^{(K)} := \operatorname{diag}\{\sigma_0,\dots,\sigma_K\}, \quad \sigma_k = k + \frac{\sqrt{\alpha+1}}{2\beta\eta} \end{aligned} \tag{3.8}$$

are symmetric and positive semidefinite.

Proof. Define the sequences $d_k := \sqrt{k(k+\alpha)}$ and $g_k := 2k - d_k - d_{k+1}$. Then, we have the bound $d_{k+1} - d_k \geq \sqrt{\alpha+1}$ and the sequence g_k is monotonically decreasing with limit $g_k \searrow -(\alpha+1)$ for $k \to \infty$. The nonzero components of the matrix $\mathbf{P}_{\eta,\sigma}^{(K)}$ in the k-th row read for $k \geq 1$ as

$$\left(\mathbf{P}_{\eta,\sigma}^{(K)}\right)_{(k-1,k,k+1)} = \frac{1}{\beta}\left(-d_k \frac{\sigma_{k-1}+\sigma_k}{2}, (2k+1+\alpha)\sigma_k + \beta\eta\sigma_k, -d_{k+1}\frac{\sigma_k+\sigma_{k+1}}{2}\right).$$

The claim follows from Gershgorin circle theorem and the estimate

$$\begin{aligned}
&(2k+1+\alpha)\sigma_k + \beta\eta\sigma_k - d_k\frac{\sigma_{k-1}+\sigma_k}{2} - d_{k+1}\frac{\sigma_k+\sigma_{k+1}}{2} \\
&= \beta\eta b - \frac{d_{k+1}-d_k}{2} + k\eta + (g_k+1+\alpha)(k+b) \\
&\geq \beta\eta b - \frac{\sqrt{\alpha+1}}{2} = 0 \qquad \text{for} \qquad b := \frac{\sqrt{\alpha+1}}{2\beta\eta}.
\end{aligned}$$

∎

Next, we derive the solution space to the stochastic Galerkin formulation (3.6).

THEOREM 3.1. *We define for $K \in \mathbb{N}_0 \cup \{\infty\}$ the possibly infinite matrices*

$$\sigma^{(K)} := \operatorname{diag}\{\sigma_0, \ldots, \sigma_K\} \quad \textit{and} \quad \sigma := \sigma^{(\infty)} := \operatorname{diag}\{\sigma_0, \sigma_1, \ldots\} \quad \textit{with} \quad \sigma_k = k + \frac{\sqrt{\alpha+1}}{2\beta\eta}.$$

Assume initial values f_0 independent of $\varepsilon > 0$. Then, the exact solution $\mathbf{f}(t,x,v)$ to the infinite system (3.6) belongs to the weighted sequences space ℓ^2_σ. In particular, the exact solution and the truncated system (3.7) satisfy the bounds

$$\|\mathbf{f}\|^2_{\ell^2_\sigma} \leq \frac{\sqrt{\alpha+1}}{2\beta\eta}\|f_0\|^2 \quad \textit{and} \quad \|\mathbf{p}^{(K)}\|^2_{\ell^2_\sigma} \leq \frac{\sqrt{\alpha+1}}{2\beta\eta}\|f_0\|^2 \quad \textit{for all} \quad t \geq 0. \tag{3.9}$$

Proof. According to Lemma 3.2 the symmetric matrix $\mathbf{P}_{\eta,\sigma}^{(K)}$ is positive semidefinite and hence, the square root $(\mathbf{P}_{\eta,\sigma}^{(K)})^{1/2}$ exists. Thus, we obtain

$$\begin{aligned}
\left\langle \mathbf{p}^{(K)}, \mathbf{L}_\eta^{(K)}\mathbf{p}^{(K)} \right\rangle_{\ell^2_\sigma} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{p}^{(K)}(t,x,v)^{\mathrm{T}} \frac{\sigma^{(K)}\mathbf{L}_\eta^{(K)} + \mathbf{L}_\eta^{(K)}\sigma^{(K)}}{2} \mathbf{p}^{(K)}(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left((\mathbf{P}_{\eta,\sigma}^{(K)})^{1/2}\mathbf{p}^{(K)}(t,x,v)\right)^{\mathrm{T}} L\left((\mathbf{P}_{\eta,\sigma}^{(K)})^{1/2}\mathbf{p}^{(K)}(t,x,v)\right)\mathrm{d}x\,\mathrm{d}v \\
&\leq 0 \quad \text{for all} \quad t \geq 0.
\end{aligned}$$

The system (3.7) implies

$$\begin{aligned}
&0 = \left\langle \mathbf{p}^{(K)}, \partial_t\mathbf{p}^{(K)} + \mathbf{T}^{(K)}\mathbf{p}^{(K)} - \mathbf{L}_\eta^{(K)}\mathbf{p}^{(K)} \right\rangle_{\ell^2_\sigma} \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathbf{p}^{(K)}\right\|^2_{\ell^2_\sigma} \\
&\Rightarrow \quad \|\mathbf{p}^{(K)}(t,\cdot,\cdot)\|^2_{\ell^2_\sigma} \leq \|\mathbf{p}^{(K)}(0,\cdot,\cdot)\|^2_{\ell^2_\sigma} = \frac{\sqrt{\alpha+1}}{2\beta\eta}\|f_0\|^2 < \infty \quad \text{for all} \quad t \geq 0.
\end{aligned}$$

As Lemma 3.2 holds for all $K \in \mathbb{N}_0 \cup \{\infty\}$, the bound is valid in the limit $K \to \infty$. ∎

3.2. Hypocoercivity framework. Using Theorem 3.1, we show that the solution $\mathbf{f}\in\ell^2_\sigma$ satisfies the inequality $\|\mathbf{f}\|_{\ell^2_\sigma}\le c_\sigma\|\mathbf{f}\|_{\ell^2}$ for some constant $c_\sigma>0$. Note that this inequality *does not hold* for general elements $\mathbf{f}\in\ell^2$. The restriction $\mathbf{f}\in\ell^2_\sigma\subsetneq\ell^2$ is necessary.

COROLLARY 3.1. *Consider a solution $\mathbf{f}\in\ell^2_\sigma$ to the stochastic Galerkin formulation (3.6). Then, there exists a constant $c_\sigma>0$ that satisfies the inequality.*

$$\|\mathbf{f}\|_{\ell^2_\sigma}\le c_\sigma\|\mathbf{f}\|_{\ell^2}.$$

Proof. Theorem 3.1 states that the solution satisfies $\mathbf{f}(t,\cdot,\cdot)\in\ell^2_\sigma$, $\sigma_k=k+\frac{\sqrt{\alpha+1}}{2\beta\eta}$ for all $t\ge 0$. Thus, we have the bound

$$\big\|\mathbf{f}(t,\cdot,\cdot)\big\|^2_{\ell^2}\le\frac{\sqrt{\alpha+1}}{2\beta\eta}\big\|f_0\big\|^2<\infty\quad\text{for all}\quad\mathbf{f}(t,\cdot,\cdot)\in\ell^2_\sigma\quad\text{and}\quad t\ge 0.\tag{3.10}$$

The bound (3.10) implies that the space ℓ^2_σ is continuously embedded into ℓ^2. Furthermore, the spaces $\big(\ell^2_\sigma,\|\cdot\|_{\ell^2_\sigma}\big)$, $\big(\ell^2,\|\cdot\|_{\ell^2}\big)$ are Hilbert spaces [41]. Thus, the identity

$$\mathrm{id}\;:\;\big(\ell^2_\sigma,\|\cdot\|_{\ell^2_\sigma}\big)\to\big(\ell^2_\sigma,\|\cdot\|_{\ell^2}\big)$$

is linear, bounded and bijective. The open mapping theorem [5] states that also the inverse

$$\mathrm{id}^{-1}\;:\;\big(\ell^2_\sigma,\|\cdot\|_{\ell^2}\big)\to\big(\ell^2_\sigma,\|\cdot\|_{\ell^2_\sigma}\big)$$

is linear and bounded, which means

$$\big\|\mathbf{f}(t,\cdot,\cdot)\big\|_{\ell^2_\sigma}=\big\|\mathrm{id}^{-1}\big[\mathbf{f}(t,\cdot,\cdot)\big]\big\|_{\ell^2_\sigma}\le c_\sigma\big\|\mathbf{f}(t,\cdot,\cdot)\big\|_{\ell^2}\quad\text{for all}\quad\mathbf{f}(t,\cdot,\cdot)\in\ell^2_\sigma\quad\text{and}\quad t\ge 0.$$

$\square$

Finally, we state our main theorem.

THEOREM 3.2. *Assume there exist positive constants $\lambda_m,\lambda_M,C_M>0$ such that the deterministic system (3.1) satisfies the properties **(H1)** – **(H4)**. Then, for any given parameter $\eta>0$ there exist positive constants $C>0$ and $\kappa>0$, independent of $\varepsilon>0$, such that the random solution to the system (3.2) with $1/\varepsilon=\xi+\eta>0$ decays exponentially fast in the mean squared sense*

$$\mathbb{E}\Big[\big\|f(t,\cdot,\cdot;\xi)-F\big\|^2\Big]=\int_0^\infty\big\|f(t,\cdot,\cdot;\xi)-F\big\|^2\;\rho(\xi)\mathrm{d}\xi\le Ce^{-\kappa t}\big\|f_0-F\big\|^2.\tag{3.11}$$

Proof. Let $\mathbb{1}$ be the identity matrix of infinite dimension and define the matrices

$$\mathbf{L}:=L\mathbb{1},\qquad\qquad \mathbf{\Pi}:=\Pi\mathbb{1},$$
$$\mathbf{A}:=\Big[\mathbb{1}+(\mathbf{T\Pi})^*(\mathbf{T\Pi})\Big]^{-1}(\mathbf{T\Pi})^*,\qquad \sigma:=\mathrm{diag}\bigg\{\frac{\sqrt{\alpha+1}}{2\beta\eta},1+\frac{\sqrt{\alpha+1}}{2\beta\eta},\dots\bigg\},$$

which fulfill $\mathbf{L}_\eta=\mathbf{L}\mathbf{P}_\eta=\mathbf{P}_\eta\mathbf{L}$. The augmented system (3.6) satisfies the properties (**H2**) and (**H3**), since these are independent of $\varepsilon>0$. It remains to prove the properties (**H1**) and (**H4**).

H1: The smallest eigenvalue of the matrix $\mathbf{P}_\eta$, which is symmetric and positive definite, is bounded from below by $\eta>0$. We define $\tilde{\mathbf{f}}:=\mathbf{P}_\eta^{1/2}\mathbf{f}$ to obtain

$$\begin{aligned}-\big\langle\mathbf{L}_\eta\mathbf{f},\mathbf{f}\big\rangle_{\ell^2}=-\big\langle\mathbf{L}\tilde{\mathbf{f}},\tilde{\mathbf{f}}\big\rangle_{\ell^2}\ge\lambda_m\big\|(\mathbb{1}-\mathbf{\Pi})\tilde{\mathbf{f}}\big\|^2_{\ell^2}=\lambda_m\big\|\mathbf{P}_\eta^{1/2}(\mathbb{1}-\mathbf{\Pi})\mathbf{f}\big\|^2_{\ell^2}\\ \ge\lambda_m\eta\big\|(\mathbb{1}-\mathbf{\Pi})\mathbf{f}\big\|^2_{\ell^2}\quad\text{for all}\quad\mathbf{f}\in\ell^2.\end{aligned}$$

H4: Since the term $AT(1-\Pi)f$ is independent of $\varepsilon>0$, we also have

$$\left\|\mathbf{AT}(\mathbb{1}-\mathbf{\Pi})\mathbf{f}\right\|_{\ell^2} \leq C_M \left\|(\mathbb{1}-\mathbf{\Pi})\mathbf{f}\right\|_{\ell^2}.$$

According to Theorem 3.1 the solution satisfies $\mathbf{f}(t,\cdot,\cdot)\in\ell^2_\sigma$ for all $t\geq 0$. Thus, Corollary 3.1 implies that there exists a constant $c_\sigma>0$ such that

$$\left\|\mathbf{f}\right\|_{\ell^2_\sigma} \leq c_\sigma \left\|\mathbf{f}\right\|_{\ell^2}, \quad \left\|(\mathbb{1}-\mathbf{\Pi})\mathbf{f}\right\|_{\ell^2_\sigma} \leq c_\sigma \left\|(\mathbb{1}-\mathbf{\Pi})\mathbf{f}\right\|_{\ell^2} \quad \text{for all} \quad \mathbf{f}\in\ell^2_\sigma.$$

The linear growth of the entries in the matrix $\mathbf{P}_\eta$, stated in Lemma 3.1, is bounded by the scaling σ^{-1}. Thus, there exists a constant $c_P>0$ such that $\left\|\mathbf{P}_\eta\sigma^{-1}\right\|_{\ell^2}<c_P$. We use the assumption (**H4**) on the deterministic system (3.1), i.e. $\|ALf\|\leq C_M\left\|(1-\Pi)f\right\|$, to prove the weaker version (2.2) of assumption (**H4**) for the stochastic Galerkin formulation. Then, we have

$$\begin{aligned}
\left\langle \mathbf{AL}_\eta\mathbf{f},\mathbf{f}\right\rangle_{\ell^2} &= \left\langle \mathbf{ALP}_\eta\sigma^{-1}\sigma^{1/2}\mathbf{f},\sigma^{1/2}\mathbf{f}\right\rangle_{\ell^2} \leq c_P\left\|\mathbf{AL}\sigma^{1/2}\mathbf{f}\right\|_{\ell^2}\left\|\sigma^{1/2}\mathbf{f}\right\|_{\ell^2}\\
&\leq c_P C_M\left\|(\mathbb{1}-\mathbf{\Pi})\sigma^{1/2}\mathbf{f}\right\|\left\|\sigma^{1/2}\mathbf{f}\right\|_{\ell^2} = c_P C_M\left\|(\mathbb{1}-\mathbf{\Pi})\mathbf{f}\right\|_{\ell^2_\sigma}\left\|\mathbf{f}\right\|_{\ell^2_\sigma}\\
&\leq c_P C_M c_\sigma^2\left\|(\mathbb{1}-\mathbf{\Pi})\mathbf{f}\right\|_{\ell^2}\left\|\mathbf{f}\right\|_{\ell^2}.
\end{aligned}$$

∎

Theorem 3.2 guarantees that the impact of randomness diminishes exponentially fast in time as the solution converges to the kinetic equilibrium. However, it does not state an explicit decay rate in contrast to the results in [37] for the parabolic and high field scaling or the results in [38] for the acoustic scaling with a fixed relaxation parameter.

3.3. Discretization in the stochastic space. The previous analysis is constructive, since it leads to a numerical approach to compute the mean squared deviations (3.3) by truncating the stochastic Galerkin system (3.6). Whereas Theorem 3.1 states the boundedness of solutions with respect to the stronger weighted ℓ^2_σ-norm, the next theorem is devoted to the decay of the gPC coefficients which is used to ensure the convergence for $K\to\infty$.

THEOREM 3.3. *For the approximate solutions $p^{(K)}$ and $\mathbf{p}^{(K)}$ generated by the truncated system (3.7), there exist constants $c_{(\alpha,\beta,\eta)}>0$ and $d_{(\alpha,\beta,\eta)}>2$ – independent of $\varepsilon>0$ and $K\in\mathbb{N}_0$ – that satisfy the bound*

$$\left\|\mathbf{p}_k\right\|^2 \leq c_{(\alpha,\beta,\eta)}\left(k+\frac{\sqrt{\alpha+1}}{2\beta\eta}\right)^{-d_{(\alpha,\beta,\eta)}} \quad \textit{for all} \quad k=0,\ldots,K \quad \textit{and} \quad t\geq 0. \tag{3.12}$$

Furthermore, the mean squared error is a priori bounded by

$$\begin{aligned}
&\mathbb{E}\left[\left\|\left(p^{(K)}-f\right)(t,\cdot,\cdot;\xi)\right\|^2\right] = \int_0^\infty \left\|\left(p^{(K)}-f\right)(t,\cdot,\cdot;\xi)\right\|^2 \rho(\xi)\mathrm{d}\xi\\
&\leq c_{(\alpha,\beta,\eta)}\|L\|^2\left(K+1+\frac{\sqrt{\alpha+1}}{2\beta\eta}\right)^{2-d_{(\alpha,\beta,\eta)}} t^2 + \frac{c_{(\alpha,\beta,\eta)}}{d_{(\alpha,\beta,\eta)}-1}\left(K+1+\frac{\sqrt{\alpha+1}}{2\beta\eta}\right)^{1-d_{(\alpha,\beta,\eta)}}.
\end{aligned} \tag{3.13}$$

Proof. The existence of constants $c_{(\alpha,\beta,\eta)}>0$, $d_{(\alpha,\beta,\eta)}>2$ that satisfy the bound (3.12) follows from the hyperharmonic series and the bound (3.9), since we have

$$\left\|\mathbf{p}^{(K)}(t,\cdot,\cdot)\right\|^2_{\ell^2_\sigma} = \sum_{k=0}^{K}\left(k+\frac{\sqrt{\alpha+1}}{2\beta\eta}\right)\left\|\mathbf{p}_k(t,\cdot,\cdot)\right\|^2 \leq \frac{\sqrt{\alpha+1}}{2\beta\eta}\left\|f_0\right\|^2.$$

Due to the linearity of the system (3.2), the approximation $\mathbf{p}^{(K)} \in \mathbb{R}^{K+1}$, described by the system (3.7), and the truncated exact solution $\mathbf{f}^{(K)} := \left(\mathbf{f}_0, \dots, \mathbf{f}_K\right)^{\mathrm{T}}$ satisfy

$$\begin{aligned}
&\partial_t \mathbf{p}^{(K)}(t,x,v) + \mathbf{T}^{(K)} \mathbf{p}^{(K)}(t,x,v) = \mathbf{L}_\eta^{(K)} \mathbf{p}^{(K)}(t,x,v), \\
&\partial_t \mathbf{f}^{(K)}(t,x,v) + \mathbf{T}^{(K)} \mathbf{f}^{(K)}(t,x,v) = \mathbf{L}_\eta^{(K)} \mathbf{f}^{(K)}(t,x,v) + \mathcal{R}^{(K)}(t,x,v) \\
&\text{with residual} \quad \mathcal{R}_k^{(K)}(t,x,v) = L \sum_{j=K+1}^{\infty} \mathbf{f}_j(t,x) \Big\langle (\xi+\eta)\phi_j(\xi), \phi_k(\xi) \Big\rangle_\rho \\
&\qquad\qquad\qquad\qquad\qquad\quad = \begin{cases} 0 & \text{if } \ 0 \le k \le K-1, \\ \frac{\sqrt{(K+1)(K+1+\alpha)}}{\beta} L \mathbf{f}_{K+1}(t,x,v) & \text{if } \ k = K. \end{cases}
\end{aligned}$$

Subtracting these equations yields the system

$$\partial_t \mathbf{e}^{(K)}(t,x,v) + \mathbf{T}^{(K)} \mathbf{e}^{(K)}(t,x,v) = \mathbf{L}_\eta^{(K)} \mathbf{e}^{(K)}(t,x,v) + \mathcal{R}^{(K)}(t,x,v)$$

for the error $\mathbf{e}^{(K)} = \mathbf{f}^{(K)} - \mathbf{p}^{(K)} \in \mathbb{R}^{K+1}$. Due to

$$\frac{\sqrt{(K+1)(K+1+\alpha)}}{\beta} \le (K+1)\frac{\sqrt{1+\alpha}}{\beta} \le \max\{1,\eta\} \left[\frac{K+1}{2} + \frac{\sqrt{1+\alpha}}{2\beta\eta} \right] \le \max\{1,\eta\}\sigma_{K+1}$$

and Cauchy-Schwarz inequality we have

$$\begin{aligned}
0 &= \Big\langle \mathbf{e}^{(K)}, \partial_t \mathbf{e}^{(K)} + \mathbf{T}^{(K)} \mathbf{e}^{(K)} - \mathbf{L}_\eta^{(K)} \mathbf{e}^{(K)} - \mathcal{R}^{(K)} \Big\rangle_{\ell^2} \\
&\ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\| \mathbf{e}^{(K)} \right\|_{\ell^2}^2 - \Big\langle \mathbf{e}^{(K)}, \mathcal{R}^{(K)} \Big\rangle_{\ell^2} \\
&\ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \left\| \mathbf{e}^{(K)} \right\|_{\ell^2}^2 - \left\| \mathbf{e}^{(K)} \right\|_{\ell^2} \max\{1,\eta\}\sigma_{K+1} \|L\| \|\mathbf{f}_{K+1}\|.
\end{aligned}$$

Since the initial error $\mathbf{e}^{(K)}(0,x,v)$ is zero, the bounds (3.12) yield

$$\begin{aligned}
\|\mathbf{e}^{(K)}\|_{\ell^2} &\le \|\mathbf{e}^{(K)}(0,\cdot,\cdot)\|_{\ell^2} + \max\{1,\eta\}\sigma_{K+1}\|L\| \int_0^t \|\mathbf{f}_{K+1}(\tau,\cdot,\cdot)\| \mathrm{d}\tau \\
&\le \tilde{c}_{(\alpha,\beta,\eta)}^{1/2} \|L\| \sigma_{K+1}^{1-\frac{d_{(\alpha,\beta,\eta)}}{2}} t
\end{aligned} \tag{3.14}$$

with $\tilde{c}_{(\alpha,\beta,\eta)}^{1/2} := \max\{1,\eta\} c_{(\alpha,\beta,\eta)}^{1/2}$. Furthermore, the bounds (3.12) imply

$$\begin{aligned}
\sum_{k=K+1}^{\infty} \|\mathbf{f}_k\|^2 \le c_{(\alpha,\beta,\eta)} &\sum_{k=K+1}^{\infty} \left(k + \frac{\sqrt{\alpha+1}}{2\beta\eta} \right)^{-d_{(\alpha,\beta,\eta)}} \\
&\le c_{(\alpha,\beta,\eta)} \int_{K+1}^{\infty} \left(\tau + \frac{\sqrt{\alpha+1}}{2\beta\eta} \right)^{-d_{(\alpha,\beta,\eta)}} \mathrm{d}\tau \\
&= \frac{c_{(\alpha,\beta,\eta)}}{d_{(\alpha,\beta,\eta)} - 1} \left(K+1+\frac{\sqrt{\alpha+1}}{2\beta\eta} \right)^{1-d_{(\alpha,\beta,\eta)}}.
\end{aligned} \tag{3.15}$$

Then, estimates (3.14) and (3.15) give the a priori estimate

$$\begin{aligned}
&\int_0^\infty \left\| \left(p^{(K)} - f\right)(t,\cdot,\cdot;\xi) \right\|^2 \rho(\xi)\mathrm{d}\xi = \|\mathbf{e}^{(K)}\|_{\ell^2}^2 + \sum_{k=K+1}^{\infty} \|\mathbf{f}_k\|^2 \\
&\le \tilde{c}_{(\alpha,\beta,\eta)} \|L\|^2 \sigma_{K+1}^{2-d_{(\alpha,\beta,\eta)}} t^2 + \frac{\tilde{c}_{(\alpha,\beta,\eta)}}{d_{(\alpha,\beta,\eta)} - 1} \sigma_{K+1}^{1-d_{(\alpha,\beta,\eta)}}.
\end{aligned}$$

□

3.4. Numerical results. To compute the solution to the truncated system (3.7) for $K\in\mathbb{N}_0$, the IMEX scheme (2.6) is applied. The discretization reads as

$$\begin{aligned}\mathbf{f}_i^{n+1} &= \Big(\mathbb{1}^{(K)}+\Delta t\mathbf{P}_\eta^{(K)}\Big)^{-1}\mathbf{f}_i^n - \Delta t\mathbf{T}^{\Delta x}\tilde{f}_i^n + \Delta t\Big(\mathbb{1}^{(K)}+\Delta t\mathbf{P}_\eta^{(K)}\Big)^{-1}\mathbf{P}_\eta^{(K)}\mathbf{\Pi}^{(K)}\mathbf{f}_i^n,\\ \tilde{f}_i^n &= \Big(\mathbb{1}^{(K)}+\Delta t\mathbf{P}_\eta^{(K)}\Big)^{-1}\Big(\mathbf{f}_i^n+\Delta t\mathbf{P}_\eta^{(K)}\mathbf{\Pi}^{(K)}\mathbf{f}_i^n\Big)\end{aligned}$$

with $\mathbb{1}^{(K)}:=\operatorname{diag}\{1,\dots,1\}\in\mathbb{R}^{K+1}$ and $\mathbf{\Pi}^{(K)}:=\Pi\mathbb{1}^{(K)}$. Upwinding in the numerical flux is used for the spatial differential operator $\mathbf{T}^{\Delta x}$. Theoretical results are illustrated by means of the two-velocity model with initial values $f^{\pm}(0,x)=\pm\cos(2\pi x)$ for the exponential distribution $\rho(\xi)=e^{-\xi}$ with parameter $\alpha=0$ and $\beta=1$.

3.4.1. Reference solution. One may try to deduce the averaged L^2-stability *directly* from Theorem 2.1, where the constant $C(\varepsilon)>0$ and the decay rate $\kappa(\varepsilon)$ depend on the relaxation parameter $\varepsilon>0$. By applying Theorem 2.1 for each fixed relaxation parameter $\varepsilon=(\xi+\eta)^{-1}$, we obtain

$$\int_0^\infty \big\|f(t,\cdot,\cdot;\xi)\big\|^2\,\rho(\xi)\mathrm{d}\xi \le \bar{E}(t)\big\|f_0\big\|^2,\quad \bar{E}(t):=\int_0^\infty C\left(\frac{1}{\xi+\eta}\right)e^{-\kappa\left(\frac{1}{\xi+\eta}\right)t}\,\rho(\xi)\mathrm{d}\xi.$$

However, a possible violation of assumption (**H4**) in the limit $\varepsilon\to0^+$ prevents to deduce the *exponential decay* of the function $\bar{E}(t)$. We can only obtain the bound $\bar{E}(t)\le\bar{E}(0)$ due to

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{E}(t)=-\int_0^\infty \kappa\Big(\frac{1}{\xi+\eta}\Big)C\Big(\frac{1}{\xi+\eta}\Big)e^{-\kappa\left(\frac{1}{\xi+\eta}\right)t}\,\rho(\xi)\mathrm{d}\xi \le -\inf_{\xi\in[0,\infty)}\left\{\kappa\Big(\frac{1}{\xi+\eta}\Big)\right\}\bar{E}(t)=0.$$

Still, we may use the decay rate $\kappa^*(\varepsilon)$ derived in Corollary 2.1 to compute numerically a reference decay rate. The quadrature rule, however, does not take the limit $\varepsilon\to0^+$ into account. Hence, there is no convergence result in this "first-discretize-then-stabilize" framework. To justify at least numerically that the function $\bar{E}(t)$ with decay rate $\kappa^*(\varepsilon)$ decays exponentially with a rate $\kappa_{\bar{E}}>0$, we neglect the constant C and we consider the expression

$$\bar{E}(t):=\int_0^\infty \exp\left(-\kappa^*\Big(\delta^*,\frac{1}{\xi+\eta}\Big)t\right)\,\rho(\xi)\mathrm{d}\xi\le e^{-\kappa_{\bar{E}}t}\quad\Leftrightarrow\quad -\frac{1}{t}\ln\Big(\bar{E}(t)\Big)\ge\kappa_{\bar{E}}. \tag{3.16}$$

The left panel of Figure 3.1 illustrates the exponential decay of $\bar{E}(t)$, where the scale is indicated on the left y-axis. The dotted function illustrates that the decay rate $\kappa_{\bar{E}}=0.01$ yields an upper bound, i.e. $\bar{E}(t)\le\exp\big(-\kappa_{\bar{E}}t\big)\bar{E}(0)$. The integrals are computed using Gaussian quadrature with 100 nodes. The right panel is devoted to a toy problem with decay rate $\kappa(\xi):=\frac{1}{2}\exp(-\xi)$. Such a decay rate may arise from a pointwise application of Theorem 2.1 to each sample of relaxation parameters. Namely, this theorem states only for each sample a positive decay rate that may vanish in the relaxation limit. This choice yields

$$\begin{aligned}&\bar{E}(t)=\frac{1-e^{-t}}{t},\quad \bar{E}'(t)=\frac{e^{-t}(t-e^t+1)}{t^2}\\ \Rightarrow\qquad &\lim_{t\to\infty}\frac{\bar{E}'(t)}{\bar{E}(t)}=\lim_{t\to\infty}\frac{1}{e^t-1}-\frac{1}{t}=0.\end{aligned}\tag{3.17}$$

Due to limit (3.17) there exists *no strictly positive decay rate* $\kappa_{\bar{E}} > 0$ such that

$$\frac{\bar{E}'(t)}{\bar{E}(t)} \leq -\kappa_{\bar{E}} \quad \Leftrightarrow \quad e(t) \leq e(0)e^{-\kappa_{\bar{E}} t} \quad \text{for all} \quad t \geq 0.$$

We have circumvented this issue by considering augmented systems with solutions in the weighted sequence space ℓ^2_σ.

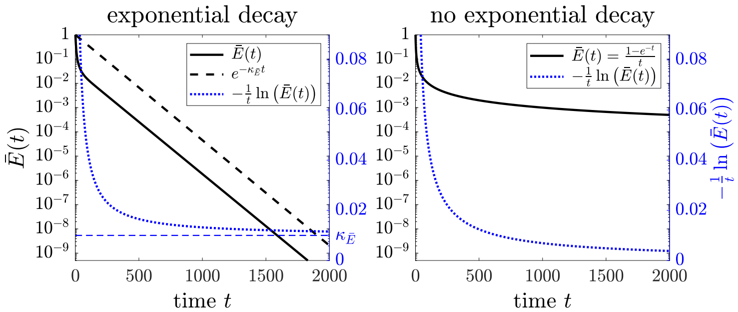

FIGURE 3.1. *Left panel: Exponential decay* (3.16) *for the reference solution with* $\eta = 10^{-8}$*; Right panel: No exponential decay for the pointwise decay rate* $\kappa(\xi) = \frac{1}{2}\exp(-\xi)$.

3.4.2. Illustration of the solution space. Figure 3.2 illustrates the main idea of Theorem 3.1. Namely, the exact solution belongs to the weighted sequence space ℓ^2_σ. Its proof is mainly based on Lemma 3.2, where the positive semidefiniteness of the matrix $\mathbf{P}^{(K)}_{\eta,\sigma}$ is ensured for all $K \in \mathbb{N}_0$ and $\eta > 0$. This property is illustrated by the left panel of Figure 3.2. The smallest eigenvalue decays, however, it remains positive. It is important to note that a strictly positive parameter $\eta > 0$ in Equation (3.2) is necessary. Otherwise, one would obtain the matrix

$$\tilde{\mathbf{P}}^{(K)}_{0,\sigma} := \frac{1}{2}\left(\sigma^{(K)}\mathbf{P}^{(K)} + \mathbf{P}^{(K)}\sigma^{(K)}\right)$$

that is in general indefinite, since the smallest eigenvalue may become negative, which is also illustrated in the left panel. The failure of our results for $\eta = 0$, however, is a desired property, because an exponential decay cannot be expected. In this case, the stochastic system (3.2) is a conservation law for the realizations $\xi(\omega) + \eta = 0$.

The middle and right panels illustrate the statement of Theorem 3.1, in particular the weighted ℓ^2_σ-norm $\|\mathbf{f}\|_{\ell^2_\sigma}$ is bounded by the inequality (3.9). Furthermore, we note that the theorem *does not* state an exponential decay in the *stronger* weighted ℓ^2_σ-norm $\|\mathbf{f}\|_{\ell^2_\sigma}$, which can be misleadingly directly deduced from Figure 3.2. A zoom on the time horizon $t \in [9, 10]$ reveals that the slope can be close to zero for some $t \in \mathbb{R}^+_0$ as highlighted by black, dotted horizontal lines. This issue becomes more clear for the small value $\eta = 10^{-8}$ in the right panel, where additionally an exponentially decaying function (blue, dashed) is shown.

3.4.3. Exponential decay in the mean squared sense. This section is devoted to Theorem 3.2. Here, we demonstrate that the hypocoercivity framework [17] implies the exponential decay of solutions $\mathbf{f} \in \ell^2_\sigma$ to the stochastic Galerkin formulation (3.6) with respect to the ℓ^2-norm provided that they are bounded with respect to the stronger weighted ℓ^2_σ-norm. Figure 3.3 shows the decay of the mean squared

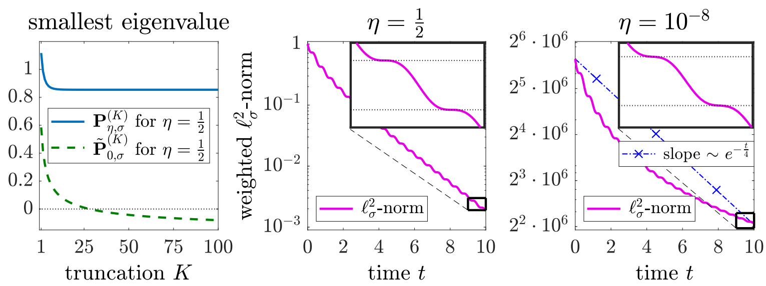

FIGURE 3.2. *Boundedness of the weighted ℓ^2_σ-norm $\left\|\mathbf{p}^{(K)}\right\|^2_{\ell^2_\sigma} \leq \frac{\sqrt{\alpha+1}}{2\beta\eta}\left\|f_0\right\|^2$ for $\eta=10^{-8}$ (right), $\eta=1/2$ (middle) and $K=20$. The left panel shows the smallest eigenvalue of the matrix $\mathbf{P}^{(K)}_{\eta,\sigma}$, defined in Lemma 3.2, and the smallest eigenvalue of $\tilde{\mathbf{P}}^{(K)}_{0,\sigma} = \frac{\sigma^{(K)}\mathbf{P}^{(K)}+\mathbf{P}^{(K)}\sigma^{(K)}}{2}$.*

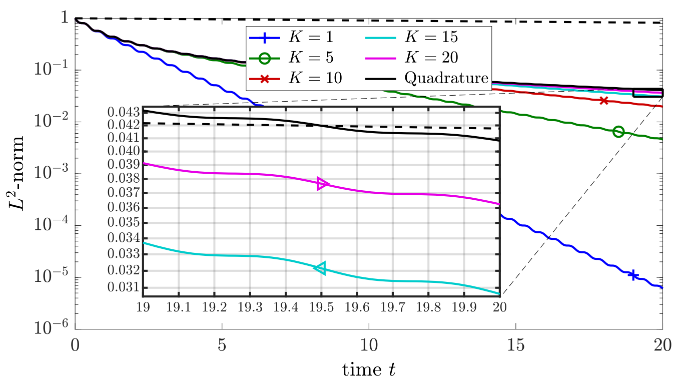

FIGURE 3.3. *Exponential decay of the mean squared L^2-norm according to Theorem 3.2 for $\eta=10^{-8}$.*

L^2-norm according to Theorem 3.2. It is approximated by

$$\int_0^\infty \left\|f(t,\cdot,\cdot;\xi)\right\|^2 \rho(\xi)\mathrm{d}\xi \approx \left\|\mathbf{p}^{(K)}\right\|^2. \tag{3.18}$$

As reference solution (black line), the integral (3.18) is computed using Gaussian quadrature with 100 nodes. As reference decay rate (black, dashed), the decay $\exp(-\kappa_{\bar{E}}t)$ is shown, which is deduced in Section 3.4.1. The mean squared L^2-norm decays for all choices of $K \in \mathbb{N}_0$ and approaches the reference solution from below.

3.4.4. Truncation errors. Figure 3.4 shows the mean squared error defined in Equation (3.13). The integrals are computed using Gaussian quadrature with 100 quadrature points, and spatial discretization $\Delta x = 2^{-8}$. The plot in the upper half of the figure consists of the error in time for different truncations K. We observe an increase of the truncation error that is bounded by Theorem 3.3. The second and third plots show the truncation error at $t=5$ and $t=20$, which decays if the number of basis functions K increases.

4. Summary and conclusion

We have applied the hypocoercivity framework from [17] to stabilize stochastic linear kinetic equations with random, stiff source terms, which are modelled by the

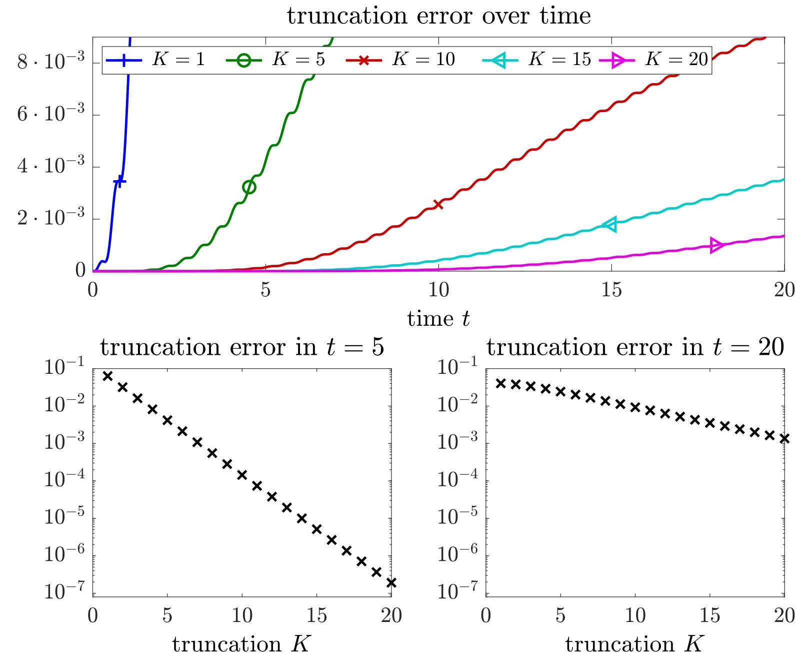

FIGURE 3.4. *Truncation error* (3.13) *in Theorem 3.3 for* $\eta = 10^{-8}$.

γ-distribution. We have shown in Corollary 2.1 that a direct application of this framework is not possible, since the realizations of the source term may be arbitrarily large. Therefore, the solution space of a stochastic Galerkin formulation has been derived in Theorem 3.1 and hypocoercivity in this solution space has been shown in Theorem 3.2. The presented approach yields also a stable numerical approximation, whose convergence is proven in Theorem 3.3.

It is the subject of future research to extend this framework to more general probability distributions and to study the limits of its applicability. In particular, our main theorem states the existence of a positive decay rate, but not an explicit estimate. Moreover, the stabilization of nonlinear systems is in general an unresolved problem.

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