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ON ENTROPIC SOLUTIONS TO CONSERVATION LAWS COUPLED WITH MOVING BOTTLENECKS*

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Abstract. Moving bottlenecks in road traffic represent an interesting mathematical problem, which can be modeled via coupled PDE-ODE systems. We consider the case of a scalar conservation law modeling the evolution of vehicular traffic and an ODE with discontinuous right-hand side for the bottleneck introduced in [M.L. Delle Monache and P. Goatin, J. Diff. Eqs., 257(11):4015–4029, 2014]. The bottleneck usually corresponds to a slow-moving vehicle influencing the bulk traffic flow via a moving flux pointwise constraint. The definition of solutions requires a special entropy condition selecting non-classical shocks and we prove existence of such solutions for initial data with bounded variation. Approximate solutions are constructed via the wave-front tracking method and their limit are solutions of the Cauchy problem PDE-ODE.

Keywords. Scalar conservation laws with constraints; PDE-ODE coupled system; Wave-front tracking; Traffic flow modeling; Non-classical shocks.

AMS subject classifications. 35L65; 90B20.

1. Introduction

1.1. Modeling of moving bottlenecks. The modeling of the impact of moving bottlenecks on vehicular traffic has been studied by the transportation engineering community [13, 15, 16] as well as the applied mathematics one [8, 9, 14]. In particular, the problem can be modeled via fully coupled PDE-ODE systems. Usually, the PDE models the evolution of vehicular traffic and the ODE represents the trajectory of a slow-moving vehicle. Mathematically speaking, different approaches are used to model the impact of the moving bottleneck. In [14], the authors multiply the usual flux function by a mollifier to represent the capacity drop generated by a slow-moving vehicle. They prove the existence of solutions in the sense of Fillipov [11] using a fractional-step approach and assuming that the slow-moving vehicle travels at maximal speed. In [8, 9], the moving bottleneck influences the bulk traffic via a moving pointwise flux constraint. In [8], the authors defined the constrained Riemann problem for the following coupled PDE-ODE system

$$\partial_t \rho(t,x) + \partial_x(f(\rho(t,x))) = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \quad (1.1a)$$
$$\rho(0,x) = \rho_0(x), \qquad x \in \mathbb{R}, \quad (1.1b)$$
$$f(\rho(t,y(t))) - \dot{y}(t)\rho(t,y(t)) \le F_\alpha(\dot{y}) := \alpha \max_{\rho \in [0,\rho_{\max}]} (f(\rho) - \dot{y}\rho), \qquad t \in \mathbb{R}_+, \quad (1.1c)$$
$$\dot{y}(t) = \min(V_b, v(\rho(t,y(t)+))), \qquad t \in \mathbb{R}_+, \quad (1.1d)$$
$$y(0) = y_0. \quad (1.1e)$$

with $f(\rho) = \rho(1-\rho)$, and proved existence of solutions for the system (1.1), under the assumption of convergence of traces of approximate solutions along the moving bottleneck trajectory (see (1.2).) However, this convergence may be violated due to the

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presence of non-classical shocks (see Remarks 1.1 and 1.2). This paper addresses the existence of solutions for the whole PDE-ODE system (1.1) without additional assumptions, thus including the case when the condition on convergence of traces fails. In [17], a proof of the stability of solutions for (1.1) is given using a wave-front tracking method and the notion of generalized tangent vectors. Some numerical methods have also been developed in [5–7, 10]. In [7], the algorithm used is based on a Godunov scheme combined with a reconstruction technique to avoid diffusion effects and capture non-classical shocks. In [10], the authors use a wave-front tracking algorithm regarding a wave-front as a numerical object. An extension to second order model has been studied in [21]. In [21] the Lighthill-Whitham-Richards (briefly LWR) model (1.1a) has been replaced by the Aw-Rascle-Zhang second order model [1, 22]. The authors define two different Riemann Solvers and they propose numerical methods based on Godunov schemes with reconstruction strategy.

1.2. Basic definitions. We consider a stretch of road $\mathbb{R}$. The quantity $\rho_{\max}$ and $V_{\max}$ stand for the maximum density and the maximum speed of cars allowed on the road respectively. Here we focus on the hybrid PDE-ODE model (1.1) proposed in [8] describing the impact of a slow-moving vehicle on the evolution of vehicular traffic. The first order model (1.1a) with (1.1b) was proposed by Lighthill-Whitham-Richards [18, 19] and this model consists of a single conservation law for the traffic density. The function $\rho=\rho(t,x)\in[0,\rho_{\max}]$ denotes the macroscopic traffic density at time $t\geq 0$ and at the position $x\in\mathbb{R}$. The flux f is given by $f:\rho\in[0,\rho_{\max}]\to\rho v(\rho)$, where $v\in C^2([0,\rho_{\max}];[0,V_{\max}])$ is the average speed of cars. We assume that the flux satisfies the condition

(F) $f\in C^2([0,\rho_{\max}];[0,+\infty))$, $\quad f(0)=f(\rho_{\max})=0$,
f is strictly concave: $-B\leq f''(\rho)\leq-\beta<0$ for all $\rho\in[0,\rho_{\max}]$, for some $\beta,B>0$.

In particular, the speed v is a strictly decreasing function. The ODE (1.1d) with (1.1e) describes the trajectory of the slow-moving vehicle starting at $(t,x)=(0,y_0)$. The slow-moving vehicle moves at its maximum speed $V_b\in(0,V_{\max})$ as long as the downstream traffic moves faster, otherwise it has to adapt its velocity accordingly to the traffic density in front (see Figure 1.1 where $v(\rho)=1-\rho$).

In the sequel, the slow-moving vehicle is regarded as a **M**oving **B**ottleneck (briefly **MB**), see Figure 1.2. It acts on the evolution of vehicular traffic through the moving constraint (1.1c). The left side of (1.1c) represents the flux of cars at the position of the MB in the MB reference frame. The quantity $F_\alpha(\dot y):=\alpha\max_{\rho\in[0,\rho_{\max}]}(f(\rho)-\dot y\rho)$ in the right side of (1.1c) is the reduced maximum flow due to the presence of the MB (see Figure 1.3 and Figure 1.4). For instance, if $v(\rho)=V_{\max}(1-\frac{\rho}{\rho_{\max}})$ then we have $F_\alpha(\dot y):=\frac{\alpha\rho_{\max}}{4V_{\max}}(V_{\max}-\dot y(t))^2$.

For future use, $\check\rho_\alpha$ and $\hat\rho_\alpha$ with $\check\rho_\alpha<\hat\rho_\alpha$ denote the two solutions of the equation $F_\alpha(V_b)+V_b\rho=f(\rho)$ and ρ^* is the solution to $V_b\rho=f(\rho)$ (see Figure 1.3 and Figure 1.4). Since f is strictly concave, $\check\rho_\alpha$, $\hat\rho_\alpha$ and ρ^* are well-defined. In the case where $v(\rho)=V_{\max}(1-\frac{\rho}{\rho_{\max}})$, we have $\check\rho_\alpha=\rho_{\max}(V_{\max}-V_b)(\frac{1-\sqrt{1-\alpha}}{2V_{\max}})$, $\hat\rho_\alpha=\rho_{\max}(V_{\max}-V_b)(\frac{1+\sqrt{1-\alpha}}{2V_{\max}})$ and $\rho^*=\rho_{\max}(1-\frac{V_b}{V_{\max}})$.

Notation: Given $\rho_1,\rho_2\in[0,\rho_{\max}]$, we denote by $\sigma(\rho_1,\rho_2):=\frac{f(\rho_1)-f(\rho_2)}{\rho_1-\rho_2}$ the Rankine-Hugoniot speed of the wave-front (ρ_1,ρ_2).

1.3. Main result. Let's introduce the definition of solutions to the constrained Cauchy problem (1.1).

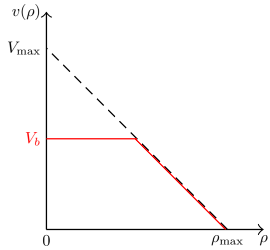

FIG. 1.1. *cars speed (- -) and slow-moving vehicle speed (—) with $v(\rho)=1-\rho$.*

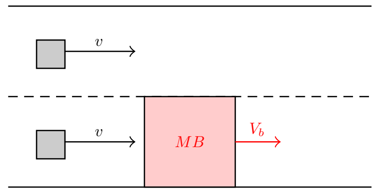

FIG. 1.2. *A slow-moving vehicle regarded as a Moving Bottleneck (MB) blocking one lane.*

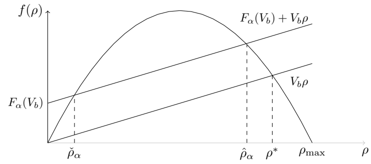

FIG. 1.3. *Flux function for $\dot{y}=V_b$ in a fixed reference frame.*

DEFINITION 1.1. *The couple*

$$(\rho,y)\in C^0\left([0,+\infty[;L^1\cap BV(\mathbb{R};[0,\rho_{\max}])\right)\times W^{1,1}_{loc}([0,+\infty[;\mathbb{R})$$

is a solution to (1.1) if

(i) *The function ρ is a weak solution of (1.1a)-(1.1b), i.e for all $\varphi\in C^1_c(\mathbb{R}^2,\mathbb{R})$,*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}(\rho\partial_t\varphi+f(\rho)\partial_x\varphi)dxdt+\int_{\mathbb{R}}\rho_0(x)\varphi(0,x)dx=0.$$

(ii) *The function ρ is an entropy admissible solution of (1.1a)-(1.1b), i.e for every $k\in[0,\rho_{\max}]$, for all $\varphi\in C^1_c(\mathbb{R}^2,\mathbb{R}_+)$, it holds*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}(|\rho-k|\partial_t\varphi+sgn(\rho-k)(f(\rho)-f(k))\partial_x\varphi)dxdt+\int_{\mathbb{R}}|\rho_0-k|\varphi(0,x)dx$$
$$+2\int_{\mathbb{R}_+}(1-\alpha)(f(k)-k\dot{y}(t))\varphi(t,y(t)))\,dt\geq 0.$$

(iii) *For a.e $t\in\mathbb{R}_+$, $\dot{y}(t)=\min(V_b,v(\rho(t,y(t)+)))$ or for every $t\in\mathbb{R}^+$,*

$$y(t)=y_0+\int_0^t\min\left(V_b,v(\rho(s,y(s)+))\right)ds.$$

(iv) *The constraint (1.1c) is satisfied, in the sense that for a.e. $t\in\mathbb{R}$,*

$$\lim_{x\to y(t)\pm}(f(\rho(t,x))-\dot{y}(t)\rho(t,x))\leq F_\alpha(\dot{y}).$$

The goal of this paper is to prove the existence of solutions for the hybrid PDE-ODE system defined in (1.1).

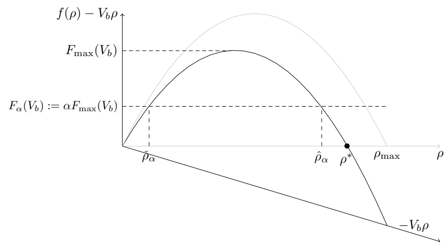

FIG. 1.4. *Flux function for* $\dot{y} = V_b$ *in the* ***MB*** *reference frame.*

THEOREM 1.1. *Let* $\rho_0 \in BV(\mathbb{R}, [0, \rho_{\max}])$*, then the Cauchy problem* (1.1) *admits a solution in the sense of Definition 1.1.*

The proof of Theorem 1.1 is structured as follows. We contruct piecewise-constant approximate solutions (ρ^n, y^n) of (1.1) via the wave-front tracking method described in Section 2.2. By introducing a suitable TV-type functional $\Gamma(t)$ defined in (3.1), we show that there exists $C > 0$ such that, for every $t \in \mathbb{R}_+$, $TV(\rho^n(t,\cdot)) \leq C$ (see Section 3.1). Lemma 3.2 in Section 3.1 is devoted to prove the convergence of the approximate solution (ρ^n, y^n) to (ρ, y) as $n \to \infty$. In Section 3.2, we show that the limit (ρ, y) satisfies Definition 1.1.i and 1.1.ii using Lemma 3.2, ρ^n is a weak solution of (1.1a) and ρ^n is an entropy admissible solution of (1.1a). Moreover, we prove that the limit (ρ, y) verifies Definition 1.1.iv using that both ρ^n are ρ are weak solutions of (1.1a) on $\{(t,x) \in [0,T] \times \mathbb{R} / x < y(t)\}$ and on $\{(t,x) \in [0,T] \times \mathbb{R} / y(t) < x\}$. In Section 3.3, we study the behavior of ρ^n around the point $(t, y^n(t))$ in order to prove that the limit (ρ, y) verifies Definition 1.1.iii.

REMARK 1.1. In [8], the existence theorem for the coupled PDE-ODE system (1.1), with $f(\rho) = \rho(1-\rho)$, is based on the following equality (see [8, (18)]):

$$\lim_{n\to\infty} \rho^n(t, y^n(t)+) = \rho(t, y(t)+). \tag{1.2}$$

More precisely, the authors in [8] prove that the limit (ρ, y) satisfies Definition 1.1.iii and 1.1.iv using (1.2). However, the equality (1.2) may fail due to the presence of non-classical shocks as shown in Section 2.4. This paper provides a general proof, without assuming (1.2), via careful wave estimates for the approximate solution (ρ^n, y^n).

REMARK 1.2. Comparing Definition 1.1.ii with the entropy condition in [8, (12b)], the term $2\int_{\mathbb{R}_+}(1-\alpha)(f(k) - k\dot{y}(t))\varphi(t, y(t))\,dt \geq 0$ in (1.1) is added to select solutions maximizing the flow through the MB trajectory, see [3]. Note that the solution (ρ, y), constructed as limit of wave-front tracking approximation, is the same as the one constructed in [8]. Therefore the solution (ρ, y) verifies Definition 1.1.ii and the entropy condition in [8, (12b)].

2. The Riemann problem of (1.1) and Wave-front tracking method

2.1. The Riemann problem with a moving constraint.

We consider (1.1) with Riemann-type initial data

$$\rho_0(x) = \begin{cases} \rho_L & \text{if} \quad x < 0 \\ \rho_R & \text{if} \quad x > 0 \end{cases} \quad \text{and} \quad y_0 = 0. \tag{2.1}$$

The definition of the Riemann solver for (1.1) and (2.1) is described in [8, Section 3]. We denote by $\mathcal{R}$ the standard Riemann solver for (1.1a)-(1.1b) where ρ_0 is defined in (2.1).

DEFINITION 2.1. *The constrained Riemann solver* $\mathcal{R}^\alpha : [0,\rho_{\max}]^2 \mapsto \boldsymbol{L}^1_{loc}(\mathbb{R};[0,\rho_{\max}])$ *for* (1.1) *and* (2.1) *is defined as follows.*

(1) If $f(\mathcal{R}(\rho_L,\rho_R)(V_b)) > F_\alpha(V_b) + V_b\mathcal{R}(\rho_L,\rho_R)(V_b)$*, then*

$$\mathcal{R}^\alpha(\rho_L,\rho_R)(x/t) = \begin{cases} \mathcal{R}(\rho_L,\hat{\rho}_\alpha)(x/t) & \text{if} \quad x < V_b t, \\ \mathcal{R}(\check{\rho}_\alpha,\rho_R)(x/t) & \text{if} \quad x \geq V_b t, \end{cases} \quad \text{and} \quad y(t) = V_b t.$$

(2) If $V_b\mathcal{R}(\rho_L,\rho_R)(V_b) \leq f(\mathcal{R}(\rho_L,\rho_R)(V_b)) \leq F_\alpha(V_b) + V_b\mathcal{R}(\rho_L,\rho_R)(V_b)$*, then*

$$\mathcal{R}^\alpha(\rho_L,\rho_R) = \mathcal{R}(\rho_L,\rho_R) \quad \text{and} \quad y(t) = V_b t.$$

(3) If $f(\mathcal{R}(\rho_L,\rho_R)(V_b)) < V_b\mathcal{R}(\rho_L,\rho_R)(V_b)$*, then*

$$\mathcal{R}^\alpha(\rho_L,\rho_R) = \mathcal{R}(\rho_L,\rho_R) \quad \text{and} \quad y(t) = v(\rho_R)t.$$

From now on, a wave-front with left density ρ_L and right density ρ_R is called a **shock** if $\rho_L < \rho_R$, a **rarefaction shock** if $\rho_L > \rho_R$ and $\rho_{\max}2^{-n-1} \leq \rho_L - \rho_R \leq 3\rho_{\max}2^{-n-1}$ or a **non-classical shock** if $\rho_L = \hat{\rho}_\alpha$ and $\rho_R = \check{\rho}_\alpha$.

We now describe the collision of two discontinuity waves (see Figure 2.1) and the collision of a discontinuity wave with the MB trajectory (Figure 2.4, Figure 2.7, Figure 2.10 and Figure 2.11). In that case, a new Riemann problem arises and its solution is obtained in the former case using the standard Riemann solver $\mathcal{R}$ and in the latter case using the constrained Riemann solver $\mathcal{R}^\alpha$, see Definition 2.1. There are no other possible interactions (for more details, we refer to [8]).

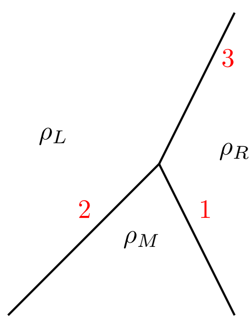

FIG. 2.1. *Two waves interact together producing a third wave.*

From Figure 2.1, Figure 2.4, Figure 2.7, Figure 2.10 and Figure 2.11, we immediately deduce the following Lemma.

LEMMA 2.1. *No rarefaction shock can arise at* $t > 0$.

2.2. Wave-front tracking method. We introduce on $[0,\rho_{\max}]$ the mesh $\widetilde{\mathcal{M}_n} = \{\tilde{\rho}_i^n\}_{i=0}^{2^n}$ defined by

$$\widetilde{\mathcal{M}_n} = \rho_{\max}(2^{-n}\mathbb{N} \cap [0,1]).$$

We add the points $\check{\rho}_\alpha, \hat{\rho}_\alpha$ and ρ^* to the mesh $\widetilde{\mathcal{M}_n}$ as described in [8, Section 4.1].

- If $\min_i |\check{\rho}_\alpha - \tilde{\rho}_i^n| = \rho_{\max}2^{-n-1}$ then we add the point $\check{\rho}_\alpha$ to the mesh

$$\mathcal{M}_n := \widetilde{\mathcal{M}_n} \cup \{\check{\rho}_\alpha\}.$$

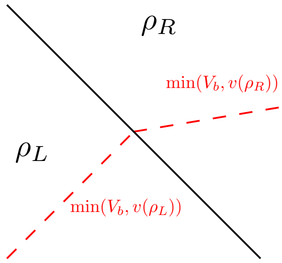

FIG. 2.2. *

Case (a)
$\rho_{\max}2^{-n-1} \leq \rho_L - \rho_R \leq 3\rho_{\max}2^{-n-1}$
and $\rho^* \leq \rho_R < \rho_L$.

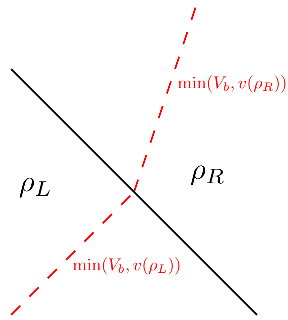

FIG. 2.3. *

Case (b) $\rho^* < \rho_R$ and $\rho_L \in [0, \check{\rho}_\alpha] \cup [\hat{\rho}_\alpha, \rho_R)$.

FIG. 2.4. *Interaction coming from the right with the MB trajectory*

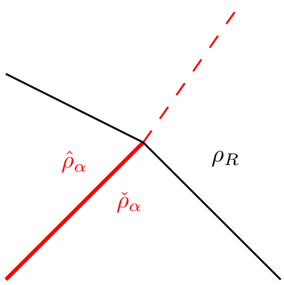

FIG. 2.5. *

Case (a) $\rho_R \in (\hat{\rho}_\alpha, \rho_{\max}]$

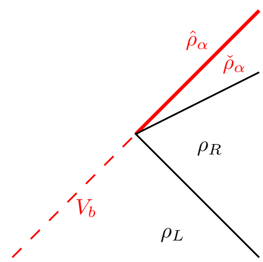

FIG. 2.6. *

Case (b) $\rho_L = \hat{\rho}_\alpha$ and
$\rho_{\max}2^{-n-1} \leq \rho_L - \rho_R \leq 3\rho_{\max}2^{-n-1}$

FIG. 2.7. *Interaction coming from the right with the MB trajectory cancelling (Case (a)) or creating (Case (b)) a non-classical shock.*

- If $|\check{\rho}_\alpha - \tilde{\rho}_l^n| = \min_i |\check{\rho}_\alpha - \tilde{\rho}_i^n| < \rho_{\max}2^{-n-1}$ then we replace $\tilde{\rho}_l^n$ by $\check{\rho}_\alpha$
$$\mathcal{M}_n = \widetilde{\mathcal{M}_n} \cup \{\check{\rho}_\alpha\} \setminus \{\rho_l^n\}.$$
- We perform the same operation for $\hat{\rho}_\alpha$ and for ρ^*.

We denote by $N := \text{card}(\mathcal{M}_n)$. We have $2^n \leq N \leq 2^n + 3$ and the constructed density mesh $\mathcal{M}_n := \{\rho_i^n\}_{i=0}^{N-1}$, sorted in ascending order, includes $\check{\rho}_\alpha$, $\hat{\rho}_\alpha$ and ρ^*. Moreover, for every $i, j \in \{0, \cdots, N-1\}$, we have
$$\rho_{\max}2^{-n-1} \leq |\rho_i^n - \rho_j^n| \leq 3\rho_{\max}2^{-n-1}. \tag{2.2}$$
Let $\rho_0 \in BV(\mathbb{R}, [0,1])$. Since our problem is scalar, we use the very first wave-front tracking algorithm proposed by Dafermos [4]. The initial density ρ_0 is approximated by piecewise-constant functions ρ_0^n verifying $\rho_0^n(x) \in \mathcal{M}_n$ for a.e $x \in \mathbb{R}$. For the sake of clarity, we assume that the $M \in \mathbb{N}$ discontinuous points of ρ_0^n, denoted by $(x_i^n)_{i=1,\cdots,M}$, are different from the initial position of the MB y_0.

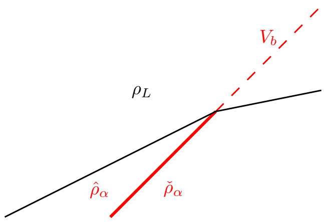

FIG. 2.8. *

Case (a) $\rho_L \in [0, \check{\rho}_\alpha)$

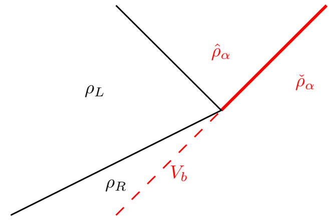

FIG. 2.9. *

Case (b) $\rho_R = \check{\rho}_\alpha$ *and* $\rho_{\max} 2^{-n-1} \le \rho_L - \rho_R \le 3\rho_{\max} 2^{-n-1}$

FIG. 2.10. *Interaction coming from the left with the MB trajectory cancelling (Case (a)) or creating (Case (b)) a non-classical shock.*

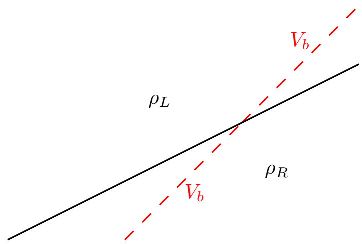

FIG. 2.11. $\rho_L \in [0, \check{\rho}_\alpha]$, $\rho_R \in [0, \check{\rho}_\alpha] \cup [\hat{\rho}_\alpha, \rho^*]$ *and* $\rho_L + \rho_R < \rho^*$. *Interaction coming from the left with the MB trajectory.*

- If $\rho_0^n(x_i^n-) < \rho_0^n(x_i^n+)$, a shock wave $(\rho_0^n(x_i^n-), \rho_0^n(x_i^n+))$ is generated with speed given by the Rankine-Hugoniot condition.
- If $\rho_0^n(x_i^n-) > \rho_0^n(x_i^n+)$, we split the rarefaction wave $(\rho_0^n(x_i^n-), \rho_0^n(x_i^n+))$ into a fan of rarefaction shocks. Since, for almost every $x \in \mathbb{R}$, $\rho_0^n(x) \in \mathcal{M}_n = \{\rho_j^n\}_{j=0}^{N-1}$, there exists $j_0 < j_1$ such that $\rho_0^n(x_i-) = \rho_{j_1}^n$ and $\rho_0^n(x_i+) = \rho_{j_0}^n$. We create $j_1 - j_0$ rarefaction shocks $(\rho_j^n, \rho_{j+1}^n)_{j=j_0,\cdots,j_1-1}$ with speed prescribed by the Rankine-Hugoniot condition. The strength of each rarefaction shock is less than $3\rho_{\max} 2^{-n-1}$ and greater than $\rho_{\max} 2^{-n-1}$.

Thus, solving approximately the Riemann problem at each point of discontinuity of ρ_0^n as described above and piecing solutions together, we construct a solution ρ^n until two waves meet at time t_1. The approximate solution $\rho^n(t_1, \cdot)$ is a piecewise-constant function verifying $\rho^n(t_1, x) \in \mathcal{M}_n$ for a.e $x \in \mathbb{R}$, the corresponding Riemann problems can again be approximately solved as described in Section 2.1 within the class of piecewise-constant functions and so on. We define y^n to be the solution of

$$\begin{cases} \dot{y}(t) = \min(V_b, v(\rho^n(t, y(t)+))), & t \in \mathbb{R}_+, \\ y(0) = y_0, & x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where $\rho^n(t, \cdot)$ is the wave-front tracking approximate solution at time t as described above with initial data ρ_0^n, see also [12, Section 2.6].

2.3. Structure of the approximate solution (ρ^n, y^n)**.** Consider $\rho_1, \rho_2 \in \mathcal{M}_n$, such that $\rho_2 < \rho_1$, and $\bar{t} > 0$. We introduce the following set $\mathcal{A}(\rho_1, \rho_2, \bar{t}) \subset \mathcal{M}_n \times \mathbb{R} \times \mathbb{R}$.

DEFINITION 2.2. *A triplet* (ρ_0^n, x_1, x_2) *belongs to* $\mathcal{A}(\rho_1,\rho_2,\bar t)$ *if the following holds:*

$$\left| \begin{array}{l} x_1 < x_2 \text{ with } \rho^n(\bar t, x_i) = \rho_i,\, i\in\{1,2\},\\ \forall x\in[x_1,x_2],\ \text{either } \rho_{\max}2^{-n-1} \le \rho^n(t,x-)-\rho^n(t,x+) \le 3\rho_{\max}2^{-n-1},\\ \text{or } \rho^n(t,x-)-\rho^n(t,x+)\le 0, \end{array}\right.$$

where $\rho^n(\bar t,\cdot)$ *is the wave-front tracking approximate solution at time* $\bar t$ *with initial data* ρ_0^n.

If $(\rho_0^n,x_1,x_2)\in\mathcal{A}(\rho_1,\rho_2,\bar t)$ then $x\in[x_1,x_2]\to\rho^n(\bar t,x)$ may decrease by a jump of strength at most $3\rho_{\max}2^{-n-1}$. Thus, shocks or rarefaction shocks are the only wave-fronts which are allowed over $\{\bar t\}\times[x_1,x_2]$. In particular, a non-classical shock is not allowed at any $x\in[x_1,x_2]$ since $\rho^n(\bar t,x-)-\rho^n(\bar t,x+)=\hat\rho_\alpha-\check\rho_\alpha>3\rho_{\max}2^{-n-1}$.

LEMMA 2.2. *Let* $\rho_1,\rho_2\in\mathcal{M}_n$ *verifying that* $\rho_2<\rho_1$ *and* $\bar t>0$. *We have*

$$\delta^n(\rho_1,\rho_2,\bar t) := \min_{(\rho_0^n,x_1,x_2)\in\mathcal{A}(\rho_1,\rho_2,\bar t)} x_2-x_1 \ge \bar t\beta(\rho_1-\rho_2-\rho_{\max}2^{-n+1}).$$

Proof. Since $(\rho_0^n,x_1,x_2)\in\mathcal{A}(\rho_1,\rho_2,\bar t)$ and Lemma 2.1, the minimal length in space at time $\bar t$ to go from ρ_1 to ρ_2 in ρ^n is obtained by a fan of rarefaction shocks (ρ_1,ρ_2) coming from $(x,t)=(x_0,0)$ (see Figure 2.12). Since $\rho_1,\rho_2\in\mathcal{M}_n$, there exists $j_2<j_1$ such that $\rho_1=\rho_{j_1}^n$ and $\rho_2=\rho_{j_2}^n$. Thus,

$$\begin{aligned}\delta^n(\rho_1,\rho_2,\bar t) &= \left(\bar t\sigma(\rho_{j_2+1}^n,\rho_{j_2}^n)+x_0\right)-\left(\bar t\sigma(\rho_{j_1}^n,\rho_{j_1-1}^n)+x_0\right),\\ &= \bar t\left(\sigma(\rho_{j_2+1}^n,\rho_{j_2}^n)-\sigma(\rho_{j_1}^n,\rho_{j_1-1}^n)\right).\end{aligned}$$

By definition of σ and using that f is strictly concave, we have

$$f'(\rho_{j_2+1}^n)<\sigma(\rho_{j_2+1}^n,\rho_{j_2}^n)<f'(\rho_{j_2}^n) \quad\text{and}\quad f'(\rho_{j_1}^n)<\sigma(\rho_{j_1}^n,\rho_{j_1-1}^n)<f'(\rho_{j_1-1}^n).$$

Using that $\rho_{\max}2^{-n-1}\le\rho_{j_2+1}^n-\rho_{j_2}^n\le 3\rho_{\max}2^{-n-1}$ and $\rho_{\max}2^{-n-1}\le\rho_{j_1}^n-\rho_{j_1-1}^n\le 3\rho_{\max}2^{-n-1}$, we conclude that

$$\begin{aligned}\delta^n(\rho_1,\rho_2,\bar t) &> \bar t\left(f'(\rho_{j_2+1}^n)-f'(\rho_{j_1-1}^n)\right),\\ &= \bar t f''(c)(\rho_{j_2+1}^n-\rho_{j_1-1}^n),\quad c\in(\rho_{j_2+1}^n,\rho_{j_1-1}^n),\\ &\ge \bar t\beta(\rho_1-\rho_2-\rho_{\max}2^{-n+1}).\end{aligned}$$

∎

REMARK 2.1. $\delta^n(\rho_1,\rho_2,\bar t)$ is the minimal length in space at time $\bar t$ to go from ρ_1 to ρ_2 in ρ^n only using shocks and rarefaction shocks. Lemma 2.2 will play a key role to prove that the limit (ρ,y) satisfies Definition 1.1.iii (see Section 3.3).

2.4. An instructive example. Assuming $f(\rho)=\rho v(\rho)$ with $v(\rho)=1-\rho$. Let $\rho_0(\cdot)=\hat\rho_\alpha\mathbb{1}_{(x_1,x_2)}+\mathbb{1}_{(x_2,+\infty)}$ and $y_0=\frac{x_1+x_2}{2}$ (see Figure 2.15a). We have $V_b=1-\check\rho_\alpha-\hat\rho_\alpha=v(\hat\rho_\alpha)$ and the solution (ρ,y) of (1.1) is

$$\rho(t,x)=\begin{cases}0, & \text{if } (t,x)\in\left[\begin{array}{l}\{(t,x)\in[0,x_2-x_1]\times\mathbb{R}/x<(1-\hat\rho_\alpha)t+x_1\},\\ \{(t,x)\in[x_2-x_1,\infty)\times\mathbb{R}/x<(1-\hat\rho_\alpha)(x_2-x_1)+x_1\},\end{array}\right.\\ \hat\rho_\alpha, & \text{if } (t,x)\in\{(t,x)\in[0,x_2-x_1]\times\mathbb{R}/(1-\hat\rho_\alpha)t+x_1<x<-\hat\rho_\alpha t+x_2\},\\ 1, & \text{if } (t,x)\in\left[\begin{array}{l}\{(t,x)\in[0,x_2-x_1]\times\mathbb{R}/-\hat\rho_\alpha t+x_2<x\},\\ \{(t,x)\in[x_2-x_1,\infty)\times\mathbb{R}/(1-\hat\rho_\alpha)(x_2-x_1)+x_1<x\}.\end{array}\right.\end{cases}$$

and

$$y(t)=\begin{cases}V_bt+y_0, & \text{if } t<\frac{x_2-y_0}{1-\check\rho_\alpha},\\ V_b(\frac{x_2-y_0}{1-\check\rho_\alpha})+y_0, & \text{if } \frac{x_2-y_0}{1-\check\rho_\alpha}<t.\end{cases}$$

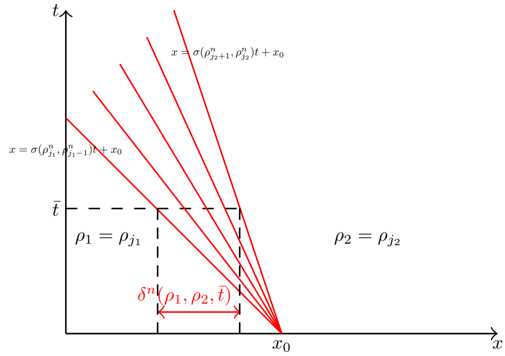

FIG. 2.12. *Illustration of the proof of Lemma 2.2*

Since $\check{\rho}_\alpha \in \mathcal{M}_n$ and $\hat{\rho}_\alpha \in \mathcal{M}_n$ for any $n > 0$, there exist $j_0, j_1 \in \{1, \cdots, N\}$ such that $\check{\rho}_\alpha = \rho^n_{j_0}$, $\hat{\rho}_\alpha = \rho^n_{j_1}$ and

$$\mathcal{M}_n = \{0, 2^{-n}, \cdots, \check{\rho}_\alpha := \rho^n_{j_0}, \rho^n_{j_0+1}, \cdots, \rho^n_{j_1-1}, \hat{\rho}_\alpha := \rho^n_{j_1}, \cdots, 1-2^{-n}, 1\}.$$

Let $\rho^n_0 = 2^{-n}\mathbb{1}_{(-\infty,x_1)} + \hat{\rho}_\alpha \mathbb{1}_{(x_1,y_0)} + \rho^n_{j_1-1}\mathbb{1}_{(y_0,x_2)} + (1-2^{-n})\mathbb{1}_{(x_2,+\infty)}$ (see Figure 2.15b) and $\hat{\rho}_\alpha - 32^{-n-1} \leq \rho^n_{j_1-1} \leq \hat{\rho}_\alpha - 2^{-n-1}$. It is obvious that $\lim_{n\to\infty} \|\rho^n_0 - \rho_0\|_{L^1(\mathbb{R})} = 0$ and $TV(\rho^n_0) = TV(\rho_0)$. Since $\rho^n_{j_1-1} \in (\check{\rho}_\alpha, \hat{\rho}_\alpha)$, a non-classical shock $(\hat{\rho}_\alpha, \check{\rho}_\alpha)$ and a shock wave $(\check{\rho}_\alpha, \rho^n_{j_1-1})$ are created at $(0, y_0)$. The shock wave $(\rho^n_{j_1-1}, 1-2^{-n})$ created at $(0, x_2)$ interacts with the shock wave $(\check{\rho}_\alpha, \rho^n_{j_1-1})$ at time $\bar{t}^n_1 = \frac{x_2 - y_0}{1-\check{\rho}_\alpha - 2^{-n}}$. The resulting shock $(\check{\rho}_\alpha, 1-2^{-n})$ cancels the non-classical shock at time $\bar{t}^n_2 := \left(\frac{\hat{\rho}_\alpha - \rho^n_{j_1-1}}{1-\hat{\rho}_\alpha - 2^{-n}} + 1\right)\bar{t}^n_1$. Moreover, we have $\bar{t} < \bar{t}^n_1 < \bar{t}^n_2$ and $\lim_{n\to\infty} \bar{t}^n_1 = \lim_{n\to\infty} \bar{t}^n_2 = \bar{t}$. We conclude that,

$$\rho(t, y(t)+) = \hat{\rho}_\alpha \text{ and } \rho^n(t, y^n(t)+) = \check{\rho}_\alpha, \quad t \in (0, \bar{t}),$$

$$\rho(t, y(t)+) = 1 \text{ and } \rho^n(t, y^n(t)+) = \check{\rho}_\alpha, \quad t \in [\bar{t}, t^n_2).$$

Thus, for every $t \in (0, \bar{t})$, we have $\lim_{n\to\infty} \rho^n(t, y^n(t)+) = \check{\rho}_\alpha \neq \hat{\rho}_\alpha = \rho(t, y(t)+)$. However, for every $t \in (0, \bar{t})$,

$$\lim_{n\to\infty} \min(V_b, v(\rho^n(t, y^n(t)+))) = V_b = \min(V_b, v(\rho(t, y(t)+))). \tag{2.4}$$

Morever, for every $t \in [\bar{t}, t^n_2)$,

$$\min(V_b, v(\rho^n(t, y^n(t)+))) = V_b \text{ and } \min(V_b, v(\rho(t, y(t)+))) = 0. \tag{2.5}$$

For every $t > t^n_2$,

$$\min(V_b, v(\rho^n(t, y^n(t)+))) = v(1-2^{-n}) \quad \text{and} \quad \min(V_b, v(\rho(t, y(t)+))) = 0. \tag{2.6}$$

Using that $t^n_2 \to \bar{t}$, (2.4), (2.5) and (2.6), we deduce that

$$\lim_{n\to\infty} \min(V_b, v(\rho^n(t, y^n(t)+))) = \min(V_b, v(\rho(t, y(t)+))), \quad \text{for a.e } t \in \mathbb{R}_+.$$

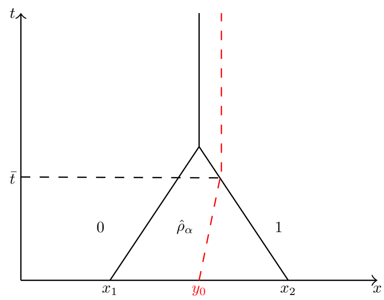

FIG. 2.13. *

(a) Solution (ρ, y) of (1.1) with $(\rho_0(\cdot), y_0) = (\hat{\rho}_\alpha \mathbb{1}_{(x_1,x_2)}(\cdot) + \mathbb{1}_{(x_2,+\infty)}(\cdot), \frac{x_1+x_2}{2})$ and $\bar{t} = \frac{x_2-y_0}{1-\check{\rho}_\alpha}$.

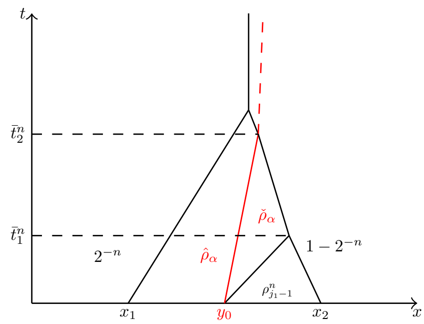

FIG. 2.14. *

(b) Approximate solution (ρ^n, y^n) of (1.1) with $n \in \mathbb{N}^*$, $(\rho_0^n(\cdot), y_0) = (2^{-n}\mathbb{1}_{(-\infty,x_1)} + \hat{\rho}_\alpha \mathbb{1}_{(x_1,y_0)} + \rho^n_{j_1-1}\mathbb{1}_{(y_0,x_2)} + (1-2^{-n})\mathbb{1}_{(x_2,+\infty)}, \frac{x_1+x_2}{2})$ with with $\hat{\rho}_\alpha - 32^{-n-1} \leq \rho^n_{j_1-1} \leq \hat{\rho}_\alpha - 2^{-n-1}$ and $\bar{t}_1^n = \frac{x_2-y_0}{1-\check{\rho}_\alpha - 2^{-n}}$.

FIG. 2.15. *Let* $\bar{t} = \frac{x_2-y_0}{1-\check{\rho}_\alpha}$ *and* $n \in \mathbb{N}^*$. *A case where* $\rho(\bar{t}, y(\bar{t})+) \neq \rho^n(\bar{t}, y^n(\bar{t})+)$ *over* $(0, \bar{t})$.

Example 2.4 shows that the equality $\lim_{n\to\infty} \rho^n(t, y^n(t)+) = \rho(t, y(t)+)$ for almost every $t \in \mathbb{R}_+$ doesn't hold since for every $t \in (0, \bar{t})$, $\rho(t, y(t)+) = \hat{\rho}_\alpha$ and $\rho^n(t, y^n(t)+) = \check{\rho}_\alpha$. To prove Definition 1.1.iii, we construct a measure-zero set $\mathcal{N}$ such that, for every $t \in \mathbb{R} \setminus \mathcal{N}$,

$$\lim_{n\to\infty} \min(V_b, v(\rho^n(t, y^n(t)+))) = \min(V_b, v(\rho(t, y(t)+))).$$

A precise description of $\mathcal{N}$ is given in Section 3.3.

3. Proof of Theorem 1.1

3.1. Convergence of the wave-front tracking approximate solutions (ρ^n, y^n). The proof of convergence follows the same arguments as in [8]. For the sake of completeness, we write the proof in our case where f verifies **(F)**. For a.e $t \in \mathbb{R}$, we define the Total Variation functional

$$\Gamma(t) = \Gamma(\rho^n(t, \cdot)) = TV(\rho^n(t, \cdot)) + \gamma(t), \tag{3.1}$$

where γ is given by

$$\gamma(t) = \begin{cases} -2|\hat{\rho}_\alpha - \check{\rho}_\alpha| \text{ if } & \rho^n(t, y^n(t)-) = \hat{\rho}_\alpha \text{ and } \rho^n(t, y^n(t)+) = \check{\rho}_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Above, $(\rho^n(t, \cdot), y^n(t))$ is an approximate solution of (1.1) at time t constructed by the wave-front tracking method described in Section 2.2.

LEMMA 3.1 ([8, Lemma 2]). *For every* $n \in \mathbb{N}$, *at any interaction, the functional* $\Gamma(t)$ *either decreases by at least* $\rho_{\max} 2^{-n-1}$ *or remains constant and the number of waves does not increase.*

Proof. If no interaction takes place at time $\bar{t}$, we immediately have $\Gamma(\bar{t}+)=\Gamma(\bar{t}-)$ and the number of wave-fronts remains constant. At any interaction time $t=\bar{t}$ either two wave-fronts interact or a wave-front hits the MB trajectory. All the possible interactions are described in Section 2.2.

- Case Figure 2.1; the wave-front (ρ_L,ρ_M) interacts with the wave-front (ρ_M,ρ_R) at time $\bar{t}$. We have

$$\Gamma(\bar{t}+)-\Gamma(\bar{t}-)=|\rho_R-\rho_L|-|\rho_R-\rho_M|-|\rho_M-\rho_L|\leq 0,$$

and the number of wave-fronts decreases by one.

- Case Figure 2.4 and Figure 2.11; a wave interacts at time $\bar{t}$ with a MB without creating or cancelling a non-classical shock. We have

$$\Gamma(\bar{t}+)-\Gamma(\bar{t}-)=|\rho_R-\rho_L|-|\rho_R-\rho_L|=0,$$

and the number of wave-fronts remains constant.

- Case Figure 2.7(a); a non-classical shock $(\hat{\rho}_\alpha,\check{\rho}_\alpha)$ is cancelled at time $\bar{t}$ by a shock $(\check{\rho}_\alpha,\rho_R)$ coming from the right of the MB trajectory. Since $\rho_R>\hat{\rho}_\alpha$, we have

$$\Gamma(\bar{t}+)-\Gamma(\bar{t}-)=|\rho_R-\hat{\rho}_\alpha|-(|\rho_R-\check{\rho}_\alpha|+|\check{\rho}_\alpha-\hat{\rho}_\alpha|-2|\check{\rho}_\alpha-\hat{\rho}_\alpha|)=0,$$

and since a non-classical shock is cancelled, the number of wave-fronts decreases by one.

- Case Figure 2.7(b); a non-classical shock $(\hat{\rho}_\alpha,\check{\rho}_\alpha)$ is created at time $\bar{t}$ by a rarefaction shock (ρ_L,ρ_R) coming from the right of the MB trajectory. Since $\rho_L=\hat{\rho}_\alpha$ and $\rho_{\max}2^{-n-1}\leq\rho_L-\rho_R\leq 3\rho_{\max}2^{-n-1}$, we have

$$\begin{aligned}\Gamma(\bar{t}+)-\Gamma(\bar{t}-)&=(|\rho_R-\check{\rho}_\alpha|+|\check{\rho}_\alpha-\hat{\rho}_\alpha|-2|\check{\rho}_\alpha-\hat{\rho}_\alpha|)-|\rho_R-\rho_L|,\\&\leq -2|\rho_R-\rho_L|,\\&\leq -\rho_{\max}2^{-n-1},\end{aligned}$$

and since a non-classical shock is created, the number of wave-fronts increases by one.

- Case Figure 2.10(a); a non-classical shock $(\hat{\rho}_\alpha,\check{\rho}_\alpha)$ is cancelled at time $\bar{t}$ by a shock $(\rho_L,\hat{\rho}_\alpha)$ coming from the left of the MB trajectory. Since $\rho_L\in[0,\check{\rho}_\alpha)$, we have

$$\Gamma(\bar{t}+)-\Gamma(\bar{t}-)=|\rho_L-\check{\rho}_\alpha|-(|\rho_L-\hat{\rho}_\alpha|+|\check{\rho}_\alpha-\hat{\rho}_\alpha|-2|\check{\rho}_\alpha-\hat{\rho}_\alpha|)=0,$$

and since a non-classical shock is cancelled, the number of wave-fronts decreases by one.

- Case Figure 2.10(b); a non-classical shock $(\hat{\rho}_\alpha,\check{\rho}_\alpha)$ is created at time $\bar{t}$ by a rarefaction shock (ρ_L,ρ_R) coming from the left of the MB trajectory. Since $\rho_R=\check{\rho}_\alpha$ and $\rho_{\max}2^{-n-1}\leq\rho_L-\rho_R\leq 3\rho_{\max}2^{-n-1}$, we have

$$\begin{aligned}\Gamma(\bar{t}+)-\Gamma(\bar{t}-)&=(|\rho_L-\hat{\rho}_\alpha|+|\check{\rho}_\alpha-\hat{\rho}_\alpha|-2|\check{\rho}_\alpha-\hat{\rho}_\alpha|)-|\rho_R-\rho_L|,\\&\leq -2|\rho_R-\rho_L|,\\&\leq -\rho_{\max}2^{-n-1},\end{aligned}$$

and since a non-classical shock is created, the number of wave-fronts increases by one.

∎

From Lemma 3.1, we conclude that the wave front tracking procedure can be prolonged to any time $T>0$ and for every $n\in\mathbb{N}$, for every $t\in\mathbb{R}_+$,

$$TV(\rho^n(t,\cdot))\leq TV(\rho_0)+\gamma(0)-\gamma(t)\leq TV(\rho_0)+2|\check{\rho}_\alpha-\hat{\rho}_\alpha|. \tag{3.2}$$

The inequality (3.2) is the key point to prove the convergence of the wave-front tracking approximate solution (ρ^n,y^n).

LEMMA 3.2. *Let (ρ^n,y^n) be an approximate solution of (1.1) constructed by the wave-front tracking method described in Section 2.2. Then, up to a subsequence, we have the following convergences*

$$\rho^n\to\rho,\quad in\, L^1_{loc}(\mathbb{R}_+\times\mathbb{R};[0,\rho_{\max}]),$$

$$y^n(\cdot)\to y(\cdot),\quad in\, L^\infty([0,T];\mathbb{R})\, for\ all\ T>0,$$

$$\dot{y}^n(\cdot)\to \dot{y}(\cdot),\quad in\, L^1([0,T];\mathbb{R})\, for\ all\ T>0,$$

for some $\rho\in C^0(\mathbb{R}_+;L^1\cap BV(\mathbb{R};[0,\rho_{\max}]))$ and $y\in W^{1,1}([0,T];\mathbb{R})\cap C^0([0,T];\mathbb{R})$ with Lipschitz constant V_b.

Proof. From (3.2) and using Helly's Theorem (see [2, Theorem 2.4]), there exists a function $\rho\in C^0\left([0,T];\left(L^1\cap BV\right)(\mathbb{R};[0,\rho_{\max}])\right)$ and a subsequence of $(\rho^n)_n$, still denoted by $(\rho^n)_n$, such that $\rho^n\to\rho$ in $L^1_{\text{loc}}(\mathbb{R}_+\times\mathbb{R};[0,\rho_{\max}])$. By construction of y^n (see Section 2.2), we deduce that

$$0\leq\dot{y}^n(t)\leq V_b, \tag{3.3}$$

for a.e. $t>0$ and $n\in\mathbb{N}\setminus\{0\}$. Hence Ascoli Theorem [20, Theorem 7.25] implies that there exists a function $y\in C^0([0,T];\mathbb{R})$ and a subsequence of $(y^n)_n$, still denoted by $(y^n)_n$, such that y^n converges to y uniformly in $C^0([0,T];\mathbb{R})$. Moreover, y is a Lipschitz function with Lipschitz constant V_b. Thus, we have $y^n(\cdot)\to y(\cdot)$ in $L^\infty([0,T];\mathbb{R})$ for all $T>0$.

To prove that $\dot{y}^n(\cdot)\to\dot{y}(\cdot)$ in $L^1([0,T];\mathbb{R})$ for all $T>0$, we show that $TV(\dot{y}^n)$ is uniformly bounded. Since $\|\dot{y}^n\|_{L^\infty}\leq V_b$, it is sufficient to estimate the positive variation of $\dot{y}^n$ over $[0,T]$, denoted by $PV(\dot{y}^n;[0,T])$. More precisely, we have

$$TV(\dot{y}^n;[0,T])\leq 2PV(\dot{y}^n;[0,T])+\|\dot{y}^n\|_{L^\infty}. \tag{3.4}$$

From Figure 2.1, Figure 2.4, Figure 2.7, Figure 2.10 and Figure 2.11 in Section 2.3, the speed of the MB is increased only by interactions with rarefaction waves coming from the right of the MB trajectory. Since all rarefaction shocks start at $t=0$, we have $PV(\dot{y}^n;[0,T])\leq TV(\rho_0)$. From (3.4), we deduce that

$$TV(\dot{y}^n;[0,T])\leq 2TV(\rho_0)+V_b,$$

which concludes the proof of Lemma 3.2. ∎

3.2. The limit (ρ,y) verifies the points i-ii-iv of Definition 1.1. From (3.2), for every $(t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}$, there exist $\lim_{x\to x_0,x>x_0}\rho^n(t_0,x):=\rho^n(t_0,x_0+)$ and $\lim_{x\to x_0,x<x_0}\rho^n(t_0,x):=\rho^n(t_0,x_0-)$ and from Lemma 3.2 there exist $\lim_{x\to x_0,x>x_0}\rho(t_0,x):=\rho(t_0,x_0+)$ and $\lim_{x\to x_0,x<x_0}\rho(t_0,x):=\rho(t_0,x_0-)$.

We start by proving that the limit (ρ,y) defined in Lemma 3.2 verifies Definition 1.1.i-ii. Since ρ^n is a weak solution of (1.1a) with initial density ρ_0^n, then, for every $\varphi\in C_c^1(\mathbb{R}^2;\mathbb{R})$,

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}}(\rho^n\partial_t\varphi+f(\rho^n)\partial_x\varphi)\,dx\,dt+\int_{\mathbb{R}}\rho_0^n(x)\varphi(0,x)\,dx=0. \tag{3.5}$$

From Lemma 3.2, by passing to the limit in (3.5) as $n\to+\infty$, we conclude that ρ is a weak solution of (1.1a)-(1.1b). Since ρ^n is also an entropy admissible solution of (1.1a)-(1.1b) by construction, we prove similarly that the limit ρ verifies the point (ii) of Definition 1.1.

We introduce the following sets

$$\begin{aligned} D_n^-:=\{(t,x)\in(0,T)\times\mathbb{R}:x<y^n(t)\}\quad D_n^+:=\{(t,x)\in(0,T)\times\mathbb{R}:x>y^n(t)\}\\ D^-:=\{(t,x)\in(0,T)\times\mathbb{R}:x<y(t)\}\quad D^+:=\{(t,x)\in(0,T)\times\mathbb{R}:x>y(t)\}.\end{aligned}$$

Let $T>0$ and fix $\varphi\in C_c^1((0,T)\times\mathbb{R};\mathbb{R}_+)$. Let $\epsilon>0$ and $a\in\mathbb{R}$ small enough to have, for every $t\in(0,T)$, for every $x\in(\infty,a+2\epsilon)$, $\varphi(t,x)=0$. We introduce the two following Lipschitz continuous functions $\varphi_\epsilon^1:\mathbb{R}_+\mapsto\mathbb{R}$ and $\varphi_\epsilon^2:\mathbb{R}_+\times\mathbb{R}\mapsto\mathbb{R}$ defined by

$$\varphi_\epsilon^1(t)=\begin{cases}0, & \text{if } 0\le t\le\epsilon,\\ \frac{t}{\epsilon}-1, & \text{if } \epsilon\le t\le 2\epsilon,\\ 1, & \text{if } 2\epsilon\le t\le T-2\epsilon,\\ \frac{T-t}{\epsilon}-1, & \text{if } T-2\epsilon\le t\le T-\epsilon,\\ 0, & \text{if } T-\epsilon\le t\le T,\end{cases}$$

and

$$\varphi_\epsilon^2(t,x)=\begin{cases}0, & \text{if } x\le a,\\ \frac{x-(a+\epsilon)}{\epsilon}+1, & \text{if } a\le x\le a+\epsilon,\\ 1, & \text{if } a+\epsilon\le x\le y_\nu(t)-2\epsilon,\\ \frac{y_\nu(t)-x}{\epsilon}-1, & \text{if } y_\nu(t)-2\epsilon\le x\le y_\nu(t)-\epsilon,\\ 0, & \text{if } y_\nu(t)+\epsilon\le x.\end{cases}$$

In particular $\varphi_\epsilon:(t,x)\mapsto\varphi_\epsilon^1(t)\varphi_\epsilon^2(t,x)\varphi(t,x)\in C_c^1(\mathcal{D}_n^-)$. Since ρ^n is a weak solution of (1.1a), for every $\epsilon>0$, we have

$$\int_0^T\int_{\mathbb{R}}(\rho^n\partial_t\varphi_\epsilon+f(\rho^n)\partial_x\varphi_\epsilon)\,dt\,dx=0. \tag{3.6}$$

By straightforward computations and by passing to the limit as $\epsilon \to 0$ in (3.6), we deduce that

$$\int_{\mathcal{D}_n^-} (\rho^n \partial_t \varphi + f(\rho^n)\partial_x \varphi) dt dx = \int_0^T (f(\rho^n(t,y^n(t)-)) - \rho^n(t,y^n(t)-)\dot{y}^n(t))\varphi(t,y^n(t))dt, \tag{3.7}$$

Since ρ is also a weak solution of (1.1a), replacing ρ^n and y^n by ρ and y respectively in the proof of (3.7), we also have

$$\int_{D^-} (\rho\partial_t\varphi + f(\rho)\partial_x\varphi)\, dt\, dx = \int_0^T (f(\rho(t,y(t)-)) - \rho(t,y(t)-)\dot{y}(t))\,\varphi(t,y(t))dt. \tag{3.8}$$

The construction of (ρ^n, y^n) and the fact that $\varphi \geq 0$ imply

$$\int_0^T [f(\rho^n(t,y^n(t)-)) - \rho^n(t,y^n(t)-)\dot{y}^n(t)]\,\varphi(t,y^n(t))dt \leq \int_0^T F_\alpha(\dot{y}^n(t))\varphi(t,y^n(t))dt. \tag{3.9}$$

Lemma 3.2 and the Dominated Convergence Theorem imply

$$\lim_{n\to\infty} \int_{D_n^-} (\rho^n\partial_t\varphi + f(\rho^n)\partial_x\varphi)\, dt\, dx = \int_{D^-} (\rho\partial_t\varphi + f(\rho)\partial_x\varphi)\, dt\, dx, \tag{3.10}$$

and

$$\lim_{n\to\infty} \int_0^T F_\alpha(\dot{y}^n(t))\varphi(t,y^n(t))dt = \int_0^T F_\alpha(\dot{y}(t))\varphi(t,y(t))dt. \tag{3.11}$$

Therefore, using (3.7), (3.8), (3.9), (3.10), (3.11), we get

$$\begin{aligned}\int_0^T (f(\rho(t,y(t)-)) - \rho(t,y(t)-)\dot{y}(t))\,\varphi(t,y(t))dt &= \int_{D^-} (\rho\partial_t\varphi + f(\rho)\partial_x\varphi)\, dt\, dx \\ &= \lim_{n\to+\infty} \int_{D_n^-} (\rho^n\partial_t\varphi + f(\rho^n)\partial_x\varphi)\, dt\, dx \\ &\leq \lim_{n\to+\infty} \int_0^T F_\alpha(\dot{y}^n(t))\varphi(t,y^n(t))dt, \\ &= \int_0^T F_\alpha(\dot{y}(t))\varphi(t,y(t))dt.\end{aligned}$$

Analogously,

$$\int_0^T [f(\rho(t,y(t)+)) - \rho(t,y(t)+)\dot{y}(t)]\,\varphi(t,y(t))dt \leq \int_0^T F_\alpha(\dot{y}(t))\varphi(t,y(t))dt.$$

By the arbitrariness of φ, we deduce that

$$f(\rho(t,y(t)\pm)) - \rho(t,y(t)\pm)\dot{y}(t) \leq F_\alpha(\dot{y}(t)),$$

for a.e $t \in (0,T)$. Thus the couple (ρ, y) satisfies point (iv) of Definition 1.1.

3.3. The limit (ρ, y) verifies the point (iii) of Definition 1.1. Let $\epsilon > 0$, from Lemma 3.2 and using the fact that (ρ^n, y^n) satisfies (2.3), there exists a measure-zero set $\mathcal{N}$ such that, for every $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$,

- $\lim_{n\to\infty} \rho^n(\bar{t}, x) = \rho(\bar{t}, x)$ for almost every $x \in \mathbb{R}$,
- $y(\cdot)$ is a differentiable function at $t = \bar{t}$,
- $\lim_{n\to\infty} \dot{y}^n(\bar{t}) = \dot{y}(\bar{t})$. That is to say, for n large enough, $|y^n(\bar{t}) - y(\bar{t})| \leq \frac{\beta \bar{t} \epsilon}{2}$.
- For every $n \in \mathbb{N}$, $\dot{y}^n(\bar{t}) = \min(V_b, v(\rho^n(\bar{t}, y^n(\bar{t})+)).$

We will prove that for every $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$,

$$\lim_{n\to\infty} \min(V_b, v(\rho^n(\bar{t}, y^n(\bar{t})+))) = \min(V_b, v(\rho(\bar{t}, y(\bar{t})+))).$$

From now on, we denote by $\rho_+ := \lim_{x \to y(\bar{t}), x > y(\bar{t})} \rho(\bar{t}, x)$ and $\rho_- := \lim_{x \to y(\bar{t}), x < y(\bar{t})} \rho(\bar{t}, x)$. The following lemma gives the range of ρ^n and ρ in a neighbourhood of $(\bar{t}, y(\bar{t}))$, see Figure 3.1.

LEMMA 3.3. *Fix $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho_-, \rho_+ \in [0, \rho_{\max}]$. There exists $\delta > 0$ such that*

$$\rho(\bar{t}, x) \in \begin{cases} (\max(\rho_- - \frac{\epsilon}{2}, 0), \min(\rho_- + \frac{\epsilon}{2}, \rho_{\max})), \ \forall x \in (y(\bar{t}) - \delta, y(\bar{t})), \\ (\max(\rho_+ - \frac{\epsilon}{2}, 0), \min(\rho_+ + \frac{\epsilon}{2}, \rho_{\max})), \ \forall x \in (y(\bar{t}), y(\bar{t}) + \delta), \end{cases} \tag{3.12}$$

and there exists $0 < \tilde{\delta} < \delta$ such that, for $n \in \mathbb{N}$ large enough,

$$\rho^n(\bar{t}, x) \in \begin{cases} (\max(\rho_- - \epsilon, 0), \min(\rho_- + \epsilon, \rho_{\max})), \ \forall x \in (\min(y(\bar{t}), y^n(\bar{t})) - \tilde{\delta}, \min(y(\bar{t}), y^n(\bar{t}))), \\ (\max(\rho_+ - \epsilon, 0), \min(\rho_+ + \epsilon, \rho_{\max})), \ \forall x \in (\max(y(\bar{t}), y^n(\bar{t})), \max(y(\bar{t}), y^n(\bar{t})) + \tilde{\delta}). \end{cases} \tag{3.13}$$

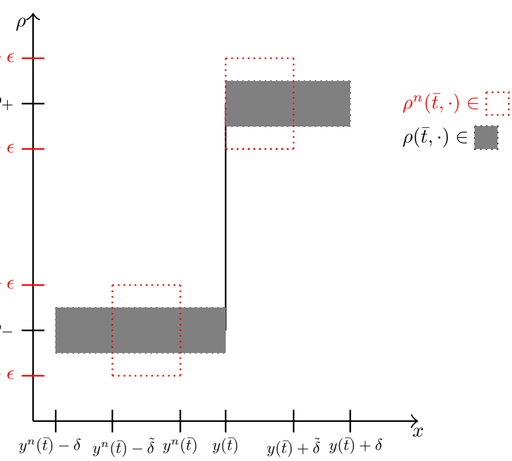

FIG. 3.1. *Illustration of Lemma 3.3; $\rho_-, \rho_+ \in [0, \rho_{\max}]$ with $y^n(\bar{t}) < y(\bar{t})$. The approximate density $\rho^n(\bar{t}, \cdot)$ over $[y^n(\bar{t}) - \tilde{\delta}, y^n(\bar{t})] \cup [y(\bar{t}), y(\bar{t}) + \tilde{\delta}]$ belongs to the area surrounded by the dotted lines (...) and $\rho(\bar{t}, \cdot)$ over $[y(\bar{t}) - \delta, y(\bar{t}) + \delta]$ belongs to the shaded zone.*

Proof. From Lemma 3.2, there exists $C > 0$ such that $TV(\rho(\bar{t}, \cdot)) < C$. Thus, we have for every $\epsilon > 0$, there exists $\delta > 0$ such that $TV(\rho_{|(y(\bar{t}), y(\bar{t}) + \delta)}) < \frac{\epsilon}{2}$ and

$TV(\rho_{|(y(\bar{t})-\delta,y(\bar{t}))}) < \frac{\epsilon}{2}$. This implies (3.12). We argue by contradiction to prove that there exists $\tilde{\delta}$ verifying $0<\tilde{\delta}<\delta$ such that, for n large enough,

$$\rho^n(\bar{t},x) \in (\rho_+ - \epsilon, \rho_+ + \epsilon),$$

for every $x \in (\max(y^n(\bar{t}),y(\bar{t})), \max(y^n(\bar{t}),y(\bar{t}))+\tilde{\delta})$. We assume that for every $\tilde{\delta}>0$ with $0<\tilde{\delta}<\delta$, for every $n_0 \in \mathbb{N}$, there exists $n \geq n_0$ and $x_n \in (\max(y^n(\bar{t}),y(\bar{t})), \max(y^n(\bar{t}),y(\bar{t}))+\tilde{\delta})$ such that $\rho^n(\bar{t},x_n) \in [0,\rho_+-\epsilon] \cup [\rho_+ + \epsilon, \rho_{\max}]$. In particular, choosing $\tilde{\delta} = \frac{\delta}{n}$, we construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that

* $\lim_{n\to\infty} x_n = y(\bar{t})$,
* $x_n > \max(y^n(\bar{t}),y(\bar{t}))$,
* $\rho^n(\bar{t},x_n) \in [0,\rho_+-\epsilon] \cup [\rho_+ + \epsilon, \rho_{\max}]$.

From Lemma 3.2, there exists a sequence $(z_m)_{m\in\mathbb{N}}$ such that $z_m > y(\bar{t})$, $\lim_{m\to\infty} z_m = y(\bar{t})$ and $\lim_{n\to\infty} \rho^n(\bar{t},z_m) = \rho(\bar{t},z_m) \in (\rho_+ - \frac{\epsilon}{2}, \rho_+ + \frac{\epsilon}{2})$. Thus, for n large enough, $\rho^n(\bar{t},z_m) \in (\rho_+ - \frac{3\epsilon}{4}, \rho_+ + \frac{3\epsilon}{4})$.

- If $\rho^n(\bar{t},x_n) \in [0,\rho_+-\epsilon]$, by diagonal method, we construct $(z_n)_{n\in\mathbb{N}}$ such that
  - $\max(y^n(\bar{t}),y(\bar{t})) < z_n < x_n$,
  - $\lim_{n\to\infty} z_n = y(\bar{t})$,
  - $\rho^n(t,z_n) \in (\rho_+ - \frac{3\epsilon}{4}, \rho_+ + \frac{3\epsilon}{4})$.

  Since $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n = y(\bar{t})$, for n large enough, we have $x_n - z_n \leq \frac{\bar{t}\beta\epsilon}{8}$. Since $z_n > y^n(\bar{t})$ and $x_n > y^n(\bar{t})$, to go from $\rho^n(\bar{t},z_n)$ to $\rho^n(\bar{t},x_n)$, we only have shocks or rarefaction shocks. From Lemma 2.2, the minimal length in space at time $\bar{t}$ to go from $\rho^n(\bar{t},z_n)$ to $\rho^n(\bar{t},x_n)$ is

  $$\delta^n(\rho^n(\bar{t},z_n),\rho^n(\bar{t},x_n),\bar{t}) \geq \bar{t}\beta(\rho^n(\bar{t},z_n) - \rho^n(\bar{t},x_n) - \rho_{\max}2^{-n+1}).$$

  Therefore, for n large enough, $\delta^n(\rho^n(\bar{t},z_n),\rho^n(\bar{t},x_n),\bar{t}) > \frac{\bar{t}\beta\epsilon}{8}$. Since $x_n - z_n \leq \frac{\beta\bar{t}\epsilon}{8}$ we have $\delta^n(\rho^n(\bar{t},z_n),\rho^n(\bar{t},x_n),\bar{t}) > x_n - z_n$, whence the contradiction.
- If $\rho^n(\bar{t},x_n) \in [\rho_+ + \epsilon, \rho_{\max}]$, by diagonal method, we construct $(z_n)_{n\in\mathbb{N}}$ such that
  - $\max(y^n(\bar{t}),y(\bar{t})) < x_n < z_n$,
  - $\lim_{n\to\infty} z_n = y(\bar{t})$,
  - $\rho^n(t,z_n) \in (\rho_+ - \frac{3\epsilon}{4}, \rho_+ + \frac{3\epsilon}{4})$.

  Since $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n = y(\bar{t})$, for n large enough, $z_n - x_n \leq \frac{\bar{t}\beta\epsilon}{8}$. Since $z_n > y^n(\bar{t})$ and $x_n > y^n(\bar{t})$, to go from $\rho^n(\bar{t},x_n)$ to $\rho^n(\bar{t},z_n)$, we only have shocks or rarefaction shocks. From Lemma 2.2, the minimal length in space at time $\bar{t}$ to go from $\rho^n(\bar{t},x_n)$ to $\rho^n(\bar{t},z_n)$ is

  $$\delta^n(\rho^n(\bar{t},x_n),\rho^n(\bar{t},z_n),\bar{t}) \geq \bar{t}\beta(\rho^n(\bar{t},x_n) - \rho^n(\bar{t},z_n) - \rho_{\max}2^{-n+1}).$$

  Therefore, for n large enough, $\delta^n(\rho^n(\bar{t},x_n),\rho^n(\bar{t},z_n),\bar{t}) > \frac{\bar{t}\beta\epsilon}{8}$. Since $z_n - x_n \leq \frac{\beta\bar{t}\epsilon}{8}$ we have $\delta(\rho^n(\bar{t},x_n),\rho^n(\bar{t},z_n),\bar{t}) > z_n - x_n$, whence the contradiction.

Using the same strategy as above, we also show that there exists $\tilde{\delta}$ verifying $0<\tilde{\delta}<\delta$ such that, for n large enough and for every $x \in (\min(y^n(\bar{t}),y(\bar{t}))-\tilde{\delta}, \min(y^n(\bar{t}),y(\bar{t})))$, $\rho^n(\bar{t},x) \in (\rho_- - \epsilon, \rho_- + \epsilon)$. ∎

3.3.1. Case $(\rho_-,\rho_+) \in [\rho^*,\rho_{\max}]$.

LEMMA 3.4. *Fix $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\epsilon > 0$. If $(\rho_-,\rho_+) \in [\rho^*,\rho_{\max}]$, the only possible case is $\rho_- \leq \rho_+$.*

Proof. Assume that $\rho^* \le \rho_+ < \rho_-$. We have $\hat{\rho}_\alpha < \rho^*$ and for ϵ small enough, $\rho_+ < \rho_- - 3\epsilon$. From Lemma 3.3, we have

$$\rho^n(\bar{t}, \min(y^n(\bar{t}), y(\bar{t}))-) \in (\rho_- - \epsilon, \min(\rho_- + \epsilon, \rho_{\max})) \subset (\rho^*, \rho_{\max}], \tag{3.14}$$

and

$$\rho^n(\bar{t}, \max(y^n(\bar{t}), y(\bar{t}))+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon) \subset (\rho^* - \epsilon, \rho_{\max}]. \tag{3.15}$$

Since $\hat{\rho}_\alpha < \rho_+ + \epsilon$, to go from $\rho_- - \epsilon$ to $\rho_+ + \epsilon$ in ρ^n we only have shocks and rarefaction shocks. Therefore, from Lemma 2.2 and for n large enough

$$\delta^n(\rho_- - \epsilon, \rho_+ + \epsilon, \bar{t}) > \frac{\beta \bar{t} \epsilon}{2}. \tag{3.16}$$

Using that $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$, we have $|y^n(\bar{t}) - y(\bar{t})| \le \frac{\bar{t}\beta\epsilon}{2}$. Therefore, from (3.14), (3.15) and (3.16), we conclude that, for n large enough,

$$\delta^n(\rho^n(\bar{t}, \min(y^n(\bar{t}), y(\bar{t}))-), \rho^n(\bar{t}, \max(y^n(\bar{t}), y(\bar{t}))+), \bar{t}) > |y^n(\bar{t}) - y(\bar{t})|,$$

whence the contradiction. ∎

LEMMA 3.5. *Fix $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho^* \le \rho_- \le \rho_+$. Then for $n \in \mathbb{N}^*$ large enough,*

$$\rho^n(\bar{t}, x) \in (\rho_- - 2\epsilon, \min(\rho_+ + 2\epsilon, \rho_{\max})),$$

for every $x \in (\min(y(\bar{t}), y^n(\bar{t})), \max(y(\bar{t}), y^n(\bar{t})))$.

An illustration of Lemma 3.5 is given in Figure 3.2.

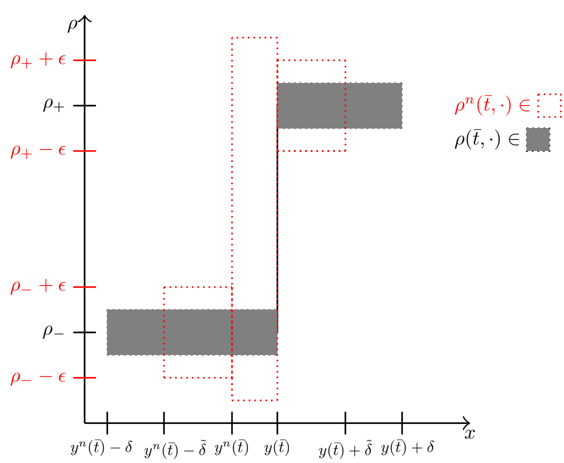

FIG. 3.2. *Illustration of Lemma 3.5; $\rho^* \le \rho_- \le \rho_+$ with $y^n(\bar{t}) < y(\bar{t})$. The approximate density $\rho^n(\bar{t}, \cdot)$ over $(y^n(\bar{t}) - \tilde{\delta}, y(\bar{t}) + \tilde{\delta})$ belongs to the area surrounded by the dotted lines (...) and $\rho(\bar{t}, \cdot)$ over $(y(\bar{t}) - \delta, y(\bar{t}) + \delta)$ belongs to the shaded zone.*

Proof. We argue by contradiction. In the same spirit of Proof of Lemma 3.3, we construct a sequence $(x_n)_{n \in \mathbb{N}}$ such that

* $\lim_{n\to\infty} x_n = y(\bar{t})$,
* $\min(y^n(\bar{t}), y(\bar{t})) < x_n < \max(y^n(\bar{t}), y(\bar{t}))$,
* $\rho^n(\bar{t}, x_n) \in [0, \rho_- - 2\epsilon] \cup [\min(\rho_+ + 2\epsilon, \rho_{\max}), \rho_{\max}]$.

From Lemma 3.3, $\rho^n(t, \min(y^n(\bar{t}), y(\bar{t}))-) \in (\rho_- - \epsilon, \min(\rho_- + \epsilon, \rho_{\max}))$ and $\rho^n(t, \max(y^n(\bar{t}), y(\bar{t}))+) \in (\rho_+ - \epsilon, \min(\rho_+ + \epsilon, \rho_{\max}))$. By construction of $(x_n)_{n\in\mathbb{N}}$ and using that $\bar{t} \in \mathbb{R}\backslash\mathcal{N}$, we have

$$x_n - \min(y^n(\bar{t}), y(\bar{t})) \le |y^n(\bar{t}) - y(\bar{t})| \le \frac{\beta\bar{t}\epsilon}{2}, \tag{3.17}$$

and

$$\max(y^n(\bar{t}), y(\bar{t})) - x_n \le |y^n(\bar{t}) - y(\bar{t})| \le \frac{\beta\bar{t}\epsilon}{2}. \tag{3.18}$$

- Assuming that $\rho^n(\bar{t}, x_n) \in [0, \rho_- - 2\epsilon]$. Since $\hat{\rho}_\alpha < \rho_- - 2\epsilon$, to go from $\rho_- - \epsilon$ to $\rho_- - 2\epsilon$ in ρ^n we only have shocks or rarefaction shocks. Therefore, from Lemma 2.2, for n large enough,

$$\delta^n(\rho_- - \epsilon, \rho_- - 2\epsilon, \bar{t}) > \frac{\bar{t}\beta\epsilon}{2}. \tag{3.19}$$

  From (3.17) and (3.19), for n large enough, we have $\delta^n(\rho_- - \epsilon, \rho_- - 2\epsilon, \bar{t}) > x_n - \min(y^n(\bar{t}), y(\bar{t}))$. Using that $\rho^n(\bar{t}, x_n) \in [0, \rho_- - 2\epsilon]$ and from Lemma 3.3, we deduce that

$$\rho^n(t, \min(y^n(\bar{t}), y(\bar{t}))-) \in (\rho_- - \epsilon, \min(\rho_- + \epsilon, \rho_{\max}),$$

  whence a contradiction.
- Assuming that $\rho_+ < \rho_{\max}$ and $\rho^n(\bar{t}, x_n) \in [\rho_+ + 2\epsilon, \rho_{\max}]$. Since $\hat{\rho}_\alpha < \rho_+ + \epsilon$, to go from $\rho_+ + 2\epsilon$ to $\rho_+ + \epsilon$ in ρ^n we only have shocks or rarefaction shocks. Therefore, from Lemma 2.2, for n large enough,

$$\delta^n(\rho_+ + 2\epsilon, \rho_+ + \epsilon, \bar{t}) > \frac{\bar{t}\beta\epsilon}{2}. \tag{3.20}$$

  From (3.18) and (3.20), for n large enough, we have $\delta^n(\rho_+ + 2\epsilon, \rho_+ + \epsilon, \bar{t}) > \max(y^n(\bar{t}), y(\bar{t})) - x_n$. Using that $\rho^n(\bar{t}, x_n) \in [\rho_+ + 2\epsilon, \rho_{\max}]$ and from Lemma 3.3, we deduce that

$$\rho^n(t, \max(y^n(\bar{t}), y(\bar{t}))+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon),$$

  whence a contradiction.

□

Case $(\rho_-, \rho_+) \in [\rho^*, \rho_{\max}]$**:** From Lemma 3.4, the only possible case is $\rho^* \le \rho_- \le \rho_+$.

- If $\rho_+ = \rho_-$; using Lemma 3.3 and Lemma 3.5, we have

$$v(\min(\rho_+ + 2\epsilon, \rho_{\max})) \le \min(V_b, v(\rho^n(\bar{t}, y^n(\bar{t})+))) := \dot{y}^n(\bar{t}) \le \min(V_b, v(\rho_+ - 2\epsilon)). \tag{3.21}$$

  Since $\bar{t} \in \mathbb{R}_+^* \backslash \mathcal{N}$, by passing to the limit in (3.21) as $n \to \infty$, we deduce that for arbitrary ϵ

$$\dot{y}(\bar{t}) = v(\rho_+) := \min(V_b, v(\rho(\bar{t}, y(\bar{t})+))). \tag{3.22}$$

- If $\rho_+ \neq \rho_-$ and $y(\bar{t}) \leq y^n(\bar{t})$ for an infinite set of indices n; from Lemma 3.3 we have

$$v(\min(\rho_+ + \epsilon, \rho_{\max})) \leq \min(V_b, v(\rho^n(t, y^n(\bar{t})+))) := \dot{y}^n(\bar{t}) \leq v(\rho_+ - \epsilon). \quad (3.23)$$

Since $\bar{t} \in \mathbb{R}_+^* \setminus \mathcal{N}$, the equality (3.22) holds by passing to the limit in (3.23) as $n \to \infty$.

- If $\rho_+ \neq \rho_-$ and $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n; in this case, from Lemma 3.3 and Lemma 3.5, $\rho^n(\bar{t}, y^n(\bar{t})+) \in (\rho_- - 2\epsilon, \rho_+ + 2\epsilon)$. We study the behavior of the approximate solution (ρ^n, y^n) in the triangle $\mathcal{T}_0$ defined by

$$\mathcal{T}_0 := \Big\{(t,x) \in [\bar{t}, t_f[\times] v(\rho_- - 2\epsilon)(t - \bar{t}) + y^n(\bar{t}) - \tilde{\delta}, f'(\rho_+ + 2\epsilon)(t - \bar{t}) + y(\bar{t}) + \tilde{\delta}[\Big\}, \quad (3.24)$$

with $t_f = \frac{y(\bar{t}) - y^n(\bar{t}) + 2\tilde{\delta}}{v(\rho_- - 2\epsilon) - f'(\rho_+ + 2\epsilon)}$. The structure of the proof is illustrated in Figure 3.3.

LEMMA 3.6. *Fix $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho^* \leq \rho_- < \rho_+$ and $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n. There exists a piecewise-linear function $\xi^n(\cdot)$ such that for every $t \in [\bar{t}, t_f^\xi)$,*

$$(t, \xi^n(t)) \in \mathcal{T}_0, \quad (3.25)$$

and extending $\xi^n(\cdot)$ to $\mathbb{R}_+$ by imposing that $\xi^n(t) = \xi^n(t_f^\xi)$ for every $t \in [t_f^\xi, \infty)$, we have

$$\rho^n(t, x+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon), \quad \forall (t,x) \in \{(t,x) \in [\bar{t}, +\infty) \times \mathbb{R}, x > \xi^n(t)\} \cap \mathcal{T}_0. \quad (3.26)$$

We denote by t_f^ξ and $t_f^{y^n}$ the time when $\xi^n(\cdot)$ and $y^n(\cdot)$ exit the triangle $\mathcal{T}_0$ respectively. Then we have $\min(t_f^{y^n}, t_f^\xi) \geq \bar{t} + c$ with $c > 0$ independent of n and there exists $t_n \in [\bar{t}, \min(t_f^{y^n}, t_f^\xi))$ such that $y^n(t_n) = \xi^n(t_n)$ and $\lim_{n \to \infty} t_n = \bar{t}$.

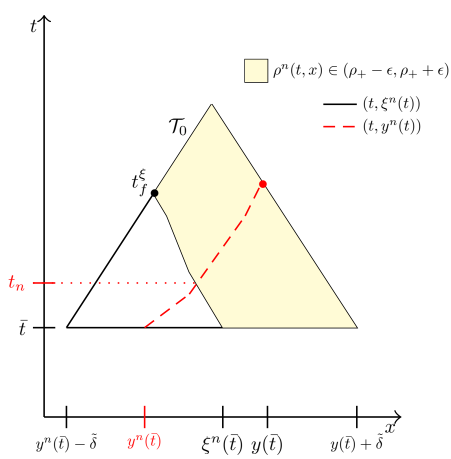

FIG. 3.3. *$\rho^* \leq \rho_- < \rho_+ \leq \rho_{\max}$ with $y^n(\bar{t}) < y(\bar{t})$, $n \in \mathbb{N}$.*

The proof of Lemma 3.6 is postponed to Appendix A. From Lemma 3.2, for a.e $t>\bar{t}$

$$y^n(t)-y^n(\bar{t})=\int_{\bar{t}}^t \dot{y}^n(s)ds, \tag{3.27}$$

and

$$\lim_{n\to\infty} y^n(t)=y(t). \tag{3.28}$$

We fix $t\in(\bar{t},\bar{t}+c]$ with c defined in Lemma 3.6 such that (3.27) and (3.28) hold. For n large enough, $t>t_n$ and $\rho^n(s,y^n(s)+)\in(\rho_+-\epsilon,\rho_++\epsilon)$ for every $s\in[t_n,t]$. Hence, we have $\dot{y}^n(s)\in(v(\rho_++\epsilon),v(\rho_+-\epsilon))$ for every $s\in[t_n,t]$. By passing to the limit in (3.27), we have for a.e $t\in(\bar{t},\bar{t}+c]$

$$\frac{y(t)-y(\bar{t})}{t-\bar{t}}\in[v(\rho_++\epsilon),v(\rho_+-\epsilon)]. \tag{3.29}$$

Using that y is differentiable at time $\bar{t}$ and the arbitrariness of ϵ, we have

$$\dot{y}(\bar{t})=v(\rho_+)=\min(V_b,v(\rho(\bar{t},y(\bar{t})+))).$$

3.3.2. Case $(\rho_-,\rho_+)\in[0,\rho^*]$.

LEMMA 3.7. *Fix $\bar{t}\in\mathbb{R}_+\setminus\mathcal{N}$ and $\epsilon>0$. Assume that $(\rho_-,\rho_+)\in[0,\rho^*]$. For $n\in\mathbb{N}^*$ large enough, for every $x\in(\min(y^n(\bar{t}),y(\bar{t}))-\tilde{\delta}),\max(y^n(\bar{t}),y(\bar{t}))+\tilde{\delta}))$,*

$$\rho^n(\bar{t},x)\in(0,\rho^*+2\epsilon).$$

Proof. From Lemma 3.3 and using that $(\rho_-,\rho_+)\in[0,\rho^*]$, for every $x\in\Big(\min(y^n(\bar{t}),y(\bar{t}))-\tilde{\delta}),\min(y^n(\bar{t}),y(\bar{t}))\Big)\cup\Big(\max(y^n(\bar{t}),y(\bar{t}))),\max(y^n(\bar{t}),y(\bar{t}))+\tilde{\delta})\Big)$.

$$0\le\rho^n(\bar{t},x)<\rho^*+\epsilon. \tag{3.30}$$

To prove Lemma 3.7, we argue by contradiction: assuming that there exists a sequence $(x_n)_{n\in\mathbb{N}^*}$ such that, for every $n\in\mathbb{N}$,

$$x_n\in[\min(y^n(\bar{t}),y(\bar{t})),\max(y^n(\bar{t}),y(\bar{t}))]\quad\text{and}\quad\rho^n(\bar{t},x_n)\in[\rho^*+2\epsilon,\rho_{\max}]. \tag{3.31}$$

Since $\hat{\rho}_\alpha<\rho^*+\epsilon$, to go from $\rho^*+2\epsilon$ to $\rho^*+\epsilon$ in ρ^n we can only have shocks or rarefaction shocks. Therefore, from Lemma 2.2, for n large enough,

$$\delta^n(\rho^*+2\epsilon,\rho^*+\epsilon,\bar{t})>\frac{t\beta\epsilon}{2}. \tag{3.32}$$

From (3.31) and (3.32) and $\bar{t}\in\mathbb{R}_+^*\setminus\mathcal{N}$, we have $\delta^n(\rho_++2\epsilon,\rho_++\epsilon,\bar{t})>\max(y^n(\bar{t}),y(\bar{t}))-x_n$ and $\delta^n(\rho_++2\epsilon,\rho_++\epsilon,\bar{t})>x_n-\min(y^n(\bar{t}),y(\bar{t}))$. Using that $\rho^n(\bar{t},x_n)\in[\rho^*+2\epsilon,\rho_{\max}]$ and (3.30), we have a contradiction. ◻

Proof of point (iii) of Definition 1.1 when $(\rho_-,\rho_+)\in[0,\rho^*]$: Since $\bar{t}\in\mathbb{R}_+^*\setminus\mathcal{N}$,

$$\dot{y}(\bar{t})=\lim_{n\to\infty}\dot{y}^n(\bar{t})=\lim_{n\to\infty}\min(V_b,v(\rho^n(\bar{t},y^n(\bar{t})+)).$$

From Lemma 3.7, $v(\rho^*+\epsilon)\le\min(V_b,v(\rho^n(\bar{t},y^n(\bar{t})))\le V_b$. Since $\rho_+\in[0,\rho^*]$, for arbitrary ϵ, we conclude that

$$\dot{y}(\bar{t})=V_b=\min(V_b,v(\rho(\bar{t},y(\bar{t})+))).$$

3.3.3. Point (iii) of Definition 1.1 when $\rho_- < \rho^* < \rho_+$ or $\rho_+ < \rho^* < \rho_-$.

LEMMA 3.8. *The only possible case is $\rho_- < \rho^* < \rho_+$*

Proof. Assuming that $\rho_+ < \rho^* < \rho_-$. From Lemma 3.3, we have $\rho^n(\bar{t}, \min(y^n(\bar{t}), y(\bar{t}))-) \in (\min(\rho_- - \epsilon, 0), \rho_- + \epsilon) \subset (\rho^* + \frac{\epsilon}{2}, \rho_{\max})$ and $\rho^n(\bar{t}, \max(y^n(\bar{t}), y(\bar{t}))+) \in [\rho_+ - \epsilon, \rho_+ + \epsilon] \subset (0, \rho^* - \frac{\epsilon}{2})$. Since $\hat{\rho}_\alpha < \rho^* - \frac{\epsilon}{2}$, to go from $\rho^* + \frac{\epsilon}{2}$ to $\rho^* - \frac{\epsilon}{2}$ in ρ^n we only have shocks and rarefaction shocks. Therefore, from Lemma 2.2, for n large enough,

$$\delta^n\left(\rho^* + \frac{\epsilon}{2}, \rho^* - \frac{\epsilon}{2}, \bar{t}\right) > \frac{\bar{t}\beta\epsilon}{2}. \tag{3.33}$$

Using that $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$, we have $|y^n(\bar{t}) - y(\bar{t})| \le \frac{\bar{t}\beta\epsilon}{2}$. Therefore, from (3.33), we conclude that

$$\delta^n\left(\rho^n(\bar{t}, \min(y^n(\bar{t}), y(\bar{t})-)), \rho^n(\bar{t}, \max(y^n(\bar{t}), y(\bar{t})+)), \bar{t}\right) > |y^n(\bar{t}) - y(\bar{t})|,$$

whence the contradiction. ◻

Proof of point (iii) of Definition 1.1 when $\rho_- < \rho^* < \rho_+$ or $\rho_+ < \rho^* < \rho_-$: From Lemma 3.8, the only possible case is $\rho_- < \rho^* < \rho_+$.

- If $y(\bar{t}) \le y^n(\bar{t})$ for an infinite set of indices n; from Lemma 3.3 we have

$$v(\min(\rho_+ + \epsilon, \rho_{\max})) \le \min(V_b, v(\rho^n(\bar{t}, y^n(\bar{t})+))) := \dot{y}^n(\bar{t}) \le v(\rho_+ - \epsilon). \tag{3.34}$$

  Since $\bar{t} \in \mathbb{R}_+^* \setminus \mathcal{N}$, the equality (3.22) holds by passing to the limit in (3.34) as $n \to \infty$ and using the arbitrariness of ϵ.

- If $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n; we study the behavior of the approximate solution (ρ^n, y^n) in the triangle $\mathcal{T}_1$ defined by

$$\mathcal{T}_1 := \left\{(t,x) \in [\bar{t}, t_f[\times]v(0)(t-\bar{t}) + y^n(\bar{t}) - \tilde{\delta}, f'(\rho_+ + 2\epsilon)(t - \bar{t}) + y(\bar{t}) + \tilde{\delta}[\right\}, \tag{3.35}$$

  with $t_f = \frac{y(\bar{t}) - y^n(\bar{t}) + 2\tilde{\delta}}{v(0) - f'(\rho_+ + 2\epsilon)}$. The structure of the proof is illustrated in Figure 3.6.

  LEMMA 3.9. *Fix $\bar{t} \in \mathbb{R}_+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho_- < \rho^* < \rho_+$ and $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n. There exists a piecewise-linear function $\xi_1^n(\cdot)$ such that for every $t \in [\bar{t}, t_f^{\xi_1})$,*

$$(t, \xi_1^n(t)) \in \mathcal{T}_1, \tag{3.36}$$

  and extending $\xi_1^n(\cdot)$ to $\mathbb{R}_+$ by imposing that $\xi_1^n(t) = \xi_1^n(t_f^{\xi_1})$ for every $t \in [t_f^{\xi_1}, \infty)$, we have

$$\rho^n(\bar{t}, x+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon), \quad \forall (t,x) \in \{(t,x) \in [\bar{t}, +\infty) \times \mathbb{R}, x > \xi_1^n(t)\} \cap \mathcal{T}_1. \tag{3.37}$$

  We denote by $t_f^{\xi_1}$ and $t_f^{y^n}$ the time when $\xi_1^n(\cdot)$ and $y^n(\cdot)$ exit the triangle $\mathcal{T}_1$ respectively. Then we have $\min(t_f^{y^n}, t_f^{\xi_1}) > \bar{t} + c$ with $c > 0$ independent of n and there exists $t_n \in [\bar{t}, \min(t_f^{y^n}, t_f^{\xi_1})]$ such that $y^n(t_n) = \xi_1^n(t_n)$ and $\lim_{n\to\infty} t_n = \bar{t}$.

  The proof of Lemma 3.9 is postponed to Appendix B.

  Following the same argument as in Section 3.3.2, (3.27), (3.28) and (3.29) hold. Using that y is differentiable at time $\bar{t}$ and the arbitrariness of ϵ, we have

$$\dot{y}(\bar{t}) = v(\rho_+) = \min(V_b, v(\rho(\bar{t}, y(\bar{t})+))).$$

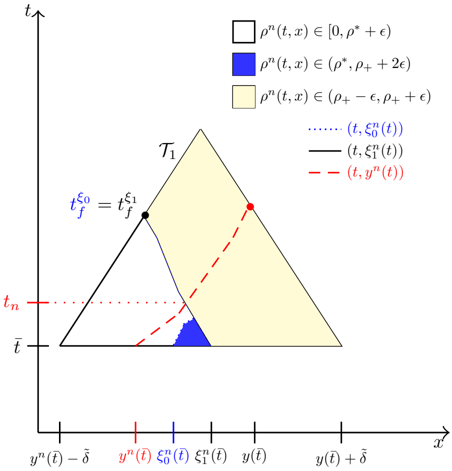

FIG. 3.4. $\xi_0^n(\cdot)$ *interacts with* $\xi_1^n(\cdot)$ *before meeting* $y^n(\cdot)$.

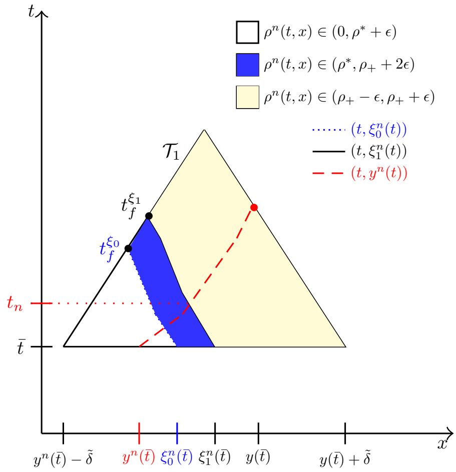

FIG. 3.5. $\xi_0^n(\cdot)$ *doesn't interact with* $\xi_1^n(\cdot)$.

FIG. 3.6. *Illustration of Lemma 3.9;* $\rho_- < \rho^* < \rho_+ < \rho_{\max}$ *and* $y^n(\bar{t}) < y(\bar{t})$.

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Appendix A. Proof of Lemma 3.6. We have $\rho_-,\rho_+ \in [\rho^*,\rho_{\max}]$, $\rho_- < \rho_+$ and $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n. There exists a subsequence of $(y^n)_{n\in\mathbb{N}}$, still denoted by $(y^n)_{n\in\mathbb{N}}$ such that for every $n\in\mathbb{N}$, $y^n(\bar{t}) < y(\bar{t})$. The construction of $\xi^n(\cdot)$ is based on the two following lemmas:

LEMMA A.1. *For every* $(t,x)\in\mathcal{T}_0$, $\rho^n(t,x)\in[\rho_- - 2\epsilon, \min(\rho_+ + 2\epsilon, \rho_{\max})]$.

Proof. From Lemma 3.3 and Lemma 3.5, for every $x\in(y^n(\bar{t})-\tilde{\delta}, y(\bar{t})+\tilde{\delta})$, we have $\rho^n(\bar{t},x)\in[\rho_- - 2\epsilon, \rho_+ + 2\epsilon]$. Since for every $\rho\in[0,\rho_{\max}]$, $\sigma(\rho,\rho_- - 2\epsilon)\le v(\rho_- - 2\epsilon)$ and $f'(\rho_+ + 2\epsilon)\le\sigma(\rho,\rho_- - 2\epsilon)$, an outside wave-front of $\mathcal{T}_0$ cannot enter in the triangle $\mathcal{T}_0$. Thus, all discontinuity waves in $\mathcal{T}_0$ are coming from the segment $\{\bar{t}\}\times[y^n(\bar{t})-\tilde{\delta}, y(\bar{t})+\tilde{\delta}]$. Since, $\hat{\rho}_\alpha < \rho_- - 2\epsilon$, we deduce that we have $\rho^n(t,x)\in[\rho_- - 2\epsilon, \rho_+ + 2\epsilon]$ for every $(t,x)\in\mathcal{T}_0$ and a non-classical shock cannot appear along the trajectory of y^n in the triangle $\mathcal{T}_0$. ∎

By construction of ρ^n via the wave-front tracking method, $\rho^n(\bar{t},\cdot)$ has $N(\bar{t},n)$ points of discontinuity $x_1^n < \cdots < x_j^n < \cdots < x_{N(\bar{t},n)}^n$ such that for every $j\in\{1,\cdots,N(\bar{t},n)\}$,

$\rho^n(\bar t,x_j^n-)\in\mathcal{M}_n$ and $\rho^n(\bar t,x_j^n+)\in\mathcal{M}_n$.

Lemma A.2. *There exists $j_0\in\{1,\cdots,N(\bar t,n)\}$ such that $x_{j_0}^n\in[y^n(\bar t),y(\bar t)]$ and for every $j\ge j_0$*

$$\rho^n(\bar t,x_j^n+)\in(\rho_+-\epsilon,\rho_++\epsilon). \tag{A.1}$$

with $x_j^n<y(\bar t)+\tilde\delta$.

Proof. From Lemma 3.3 we have $\rho^n(\bar t,\cdot)\in(\rho_+-\epsilon,\rho_++\epsilon)$ over $(y(\bar t),y(\bar t)+\tilde\delta)$. In particular, we have $\rho^n(\bar t,y(\bar t)+)\in(\rho_+-\epsilon,\rho_++\epsilon)$. Moreover, there exists $j_0\in\{1,\cdots,N(\bar t,n)\}$ such that $x_{j_0}^n\le y(\bar t)<x_{j_0+1}^n$. Thus, $\rho^n(\bar t,x_{j_0}^n+)=\rho^n(\bar t,y(\bar t)+)\in(\rho_+-\epsilon,\rho_++\epsilon)$ and for every $j\ge j_0$, $x_{j_0}^n\le x_j^n\le y(\bar t)+\tilde\delta$, whence $\rho^n(\bar t,x_j^n+)\in(\rho_+-\epsilon,\rho_++\epsilon)$. From Lemma 3.3 and using $\rho_-<\rho_+$, $\rho^n(\bar t,y^n(\bar t)-)\in(\rho_--\epsilon,\rho_-+\epsilon)$. Thus, $y^n(\bar t)\le x_{j_0}^n$. ∎

The proof of Lemma 3.6 is illustrated in Figure 3.3. We track forward in time the wave-front denoted by $\xi^n(\cdot)$ constructed by a wave front tracking method and starting at $\xi^n(0)=x_{j_0}^n$; for every $t\in[\bar t,t_1]$,

$$\xi^n(t)=x_{j_0}^n+(t-\bar t)\sigma(\rho^n(\bar t,x_{j_0}^n-),\rho^n(\bar t,x_{j_0}^n+)),$$

where t_1 is defined as follows:

- if $\xi^n(\cdot)$ never interacts with a wave-front in the triangle $\mathcal{T}_0$ then t_1 is the time when $\xi^n(\cdot)$ exits the triangle $\mathcal{T}_0$.
- otherwise, t_1 is the first time when $\xi^n(\cdot)$ interacts with a wave-front. By construction of ρ^n, two waves interacting together produces a third one (see Figure 2.1). Thus, for every $t\in[t_1,t_2]$,

  $$\xi^n(t)=\xi^n(t_1)+(t-t_1)\sigma(\rho^n(t_1,\xi^n(t_1)-),\rho^n(t_1,\xi^n(t_1)+)),$$

  where t_2 is defined as follows:
  - if $t\in(t_1,\infty]\mapsto\xi^n(t)$ never interacts with a wave-front in the triangle $\mathcal{T}_0$, t_2 is the time when $\xi^n(\cdot)$ exits the triangle $\mathcal{T}_0$.
  - otherwise, t_2 is the first time where $\xi^n:(t_1,\infty)\to\mathbb{R}$ interacts with a wave-front and so on.

By induction, we construct a piecewise-linear function $\xi^n(\cdot)$ such that for every $t\in[\bar t,t_f^\xi)$, $(t,\xi^n(t))\in\mathcal{T}_0$ with $t_f^\xi=\sup_{t\in[\bar t,\infty],\,(t,\xi^n(t))\in\mathcal{T}} t$. We extend $\xi^n(\cdot)$ to $\mathbb{R}_+$ by imposing that, for every $t\in[t_f^\xi,\infty)$, $\xi^n(t)=\xi^n(t_f^\xi)$. Since an outside wave-front of $\mathcal{T}_0$ cannot enter in $\mathcal{T}_0$ and from Lemma A.2, we conclude that for every $(t,x)\in\{(t,x)\in[\bar t,+\infty)\times\mathbb{R},x>\xi^n(t)\}\cap\mathcal{T}_0$

$$\rho^n(t,x+)\in(\rho_+-\epsilon,\rho_++\epsilon). \tag{A.2}$$

From Lemma A.1 and (A.2), we have for a.e $t\in(\bar t,t_f^\xi)$

$$\sigma(\rho_++\epsilon,\rho_++2\epsilon)\le\dot\xi^n(t)\le\sigma(\rho_--2\epsilon,\rho_+-\epsilon). \tag{A.3}$$

Let $t_f^{y^n}:=\sup_{t\in[\bar t,\infty],(t,y^n(t))\in\mathcal{T}_0} t$ be the time when $y^n(\cdot)$ exits the triangle $\mathcal{T}_0$. From Lemma A.1, for every $t\in[\bar t,t_f^{y^n})$, we have

$$(t,y^n(t))\in\mathcal{T}_0 \quad\text{and}\quad v(\rho_++2\epsilon)\le\dot y^n(t)\le v(\rho_--2\epsilon). \tag{A.4}$$

Using (A.3), we have

$$t_f^\xi > \bar{t} + \min\left(\frac{\tilde{\delta}}{v(\rho_- - 2\epsilon) - \sigma(\rho_+ + \epsilon, \rho_+ + 2\epsilon)}, \frac{\tilde{\delta}}{\sigma(\rho_- - 2\epsilon, \rho_+ - \epsilon) - f'(\rho_+ + 2\epsilon)}\right) \quad \text{(A.5)}$$

and using (A.4)

$$t_f^{y^n} \geq \bar{t} + \min\left(\frac{\tilde{\delta}}{v(\rho_- - 2\epsilon) - f'(\rho_+ + 2\epsilon)}, \frac{\tilde{\delta}}{v(\rho_- - 2\epsilon) - v(\rho_+ + 2\epsilon)}\right) \quad \text{(A.6)}$$

From (A.5) and (A.6), ther exists $c > 0$ independent of n such that

$$\min(t_f^\xi, t_f^{y^n}) \geq \bar{t} + c.$$

From (A.3) and (A.4),

$$\dot{y}^n(t) - \dot{\xi}^n(t) \geq v(\rho_+ + 2\epsilon) - \sigma(\rho_- - 2\epsilon, \rho_+ - \epsilon) > 0. \quad \text{(A.7)}$$

Using (A.7), $y^n(\cdot)$ interacts with $\xi^n(\cdot)$ at time $t_n > \bar{t}$ and

$$t_n \leq \frac{\xi^n(\bar{t}) - y^n(\bar{t})}{v(\rho_+ + 2\epsilon) - \sigma(\rho_- - 2\epsilon, \rho_+ - \epsilon)}. \quad \text{(A.8)}$$

Using that $\lim_{n\to\infty} y^n(\bar{t}) = y(\bar{t})$ and $y^n(\bar{t}) \leq \xi^n(\bar{t}) \leq y(\bar{t})$ and (A.8), we have $\lim_{n\to\infty} t_n = 0$.

Appendix B. Proof of Lemma 3.9. We have $\rho_- < \rho^* < \rho_+$ and $y^n(\bar{t}) < y(\bar{t})$ for an infinite set of indices n. There exists a subsequence of $(y^n)_{n\in\mathbb{N}}$, still denoted by $(y^n)_{n\in\mathbb{N}}$, such that for every $n \in \mathbb{N}$ $y^n(\bar{t}) < y(\bar{t})$. By construction of ρ^n in Section 2.2, $\rho^n(\bar{t}, \cdot)$ has $N(\bar{t}, n)$ points of discontinuity $x_1^n < \cdots < x_j^n < \cdots < x_{N(\bar{t},n)}^n$ such that for every $j \in \{1, \cdots, N(\bar{t}, n)\}$, $\rho^n(\bar{t}, x_j^n-) \in \mathcal{M}_n$ and $\rho^n(\bar{t}, x_j^n+) \in \mathcal{M}_n$.

Lemma B.1. *There exists $j_1 \in \{1, \cdots, N(\bar{t}, n)\}$ such that*

$$x_{j_1}^n \in [y^n(\bar{t}), y(\bar{t})] \quad \textit{and} \quad \rho^n(\bar{t}, x_j^n+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon),$$

for $j \geq j_1$ such that $x_j^n < y(\bar{t}) + \tilde{\delta}$.

Proof. From Lemma 3.3, we have $\rho^n(\bar{t}, \cdot) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$ over $(y(\bar{t}), y(\bar{t}) + \tilde{\delta})$. In particular, we have $\rho^n(\bar{t}, y(\bar{t})+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$. Moreover, there exists $j_1 \in \{1, \cdots, N(\bar{t}, n)\}$ such that $x_{j_1}^n \leq y(\bar{t}) < x_{j_1+1}^n$ and $\rho^n(\bar{t}, x_{j_1}+) = \rho^n(\bar{t}, y(\bar{t})+)$. For every $j \geq j_1$, $x_{j_1}^n \leq x_j^n \leq y(\bar{t}) + \tilde{\delta}$ and $\rho^n(\bar{t}, x_j^n+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$. From Lemma 3.3 and using $\rho_- < \rho^* < \rho_+$, $\rho^n(\bar{t}, y^n(\bar{t})-) \in (\max(0, \rho_- - \epsilon), \rho_- + \epsilon)$. Thus, $y^n(\bar{t}) \leq x_{j_1}^n$. ∎

Lemma B.2. *There exists $j_0 \in \{1, \cdots, N(\bar{t}, n)\}$ such that*

$$x_{j_0}^n \in [y^n(\bar{t}), y(\bar{t})] \quad \textit{and} \quad \rho^n(\bar{t}, x_j^n+) \in (\rho^*, \rho_+ + 2\epsilon),$$

for $j \geq j_0$ such that $x_j^n < y(\bar{t}) + \tilde{\delta}$.

Proof. From Lemma 3.3, there exists $j_0 \in \{1, \cdots, N(\bar{t}, n)\}$ such that $\rho^n(\bar{t}, x_{j_0}^n-) \leq \rho^*$ and $\rho^n(\bar{t}, x_{j_0}^n+) > \rho^*$ with $x_{j_0}^n \in [y^n(\bar{t}), y(\bar{t})]$ and for every $j > j_0$, $\rho^n(\bar{t}, x_j^n+) > \rho^*$. We assume that there exists $k > j_0$ such that $\rho^n(\bar{t}, x_k^n+) \geq \rho_+ + 2\epsilon$. Using $\rho^* < \rho_+$ and Lemma

B.1, we have $\rho^* < \rho^n(\bar{t}, x^n_{j_1}+)$. Thus, we only have shocks and rarefaction shocks to go from $\rho^n(\bar{t}, x^n_k+)$ to $\rho^n(\bar{t}, x^n_{j_1}+)$. From Lemma 2.2, for n large enough,

$$\delta^n(\rho^n(\bar{t}, x^n_k+), \rho^n(\bar{t}, x^n_{j_1}+)) > \frac{\beta\bar{t}\epsilon}{2}.$$

Using that $x^n_k, x^n_{j_1} \in [y^n(\bar{t}), y(\bar{t})]$ and $|y^n(\bar{t}) - y(\bar{t})| \leq \frac{\beta\bar{t}\epsilon}{2}$, we have a contradiction. ∎

The proof of Lemma 3.9 is illustrated in Figure 3.6. We track forward in time two wave-fronts denoted by $\xi^n_0(\cdot)$ and $\xi^n_1(\cdot)$ constructed by a wave front tracking method and starting at $\xi^n_0(0) = x^n_{j_0}$ and $\xi^n_1(0) = x^n_{j_1}$; for $i \in \{0,1\}$, since $x^n_{j_i}$ is a discontinuity point of $\rho^n(\bar{t}, \cdot)$, a wave-front $\xi^n_i(\cdot)$ such that $\xi^n_i(0) = x^n_{j_i}$ is constructed via the wave-front tracking method and we follow it until it interacts with another wave-front or $y^n(\cdot)$ at time t^1_i. By construction of $\mathcal{T}_1$ defined in (3.35), other wave-fronts out of the triangle $\mathcal{T}_1$ cannot interact with a wave-front in the triangle $\mathcal{T}_1$. Thus, from Lemma B.2, for every $t \in [0, t^1_0]$, for every $x \in [\xi^n_0(t), f'(\rho_+ + 2\epsilon)(t - \bar{t}) + y(\bar{t}) + \tilde{\delta}]$,

$$\rho^n(t, x+) \in (\rho^*, \rho_+ + 2\epsilon) \tag{B.1}$$

and from Lemma B.1, for every $t \in [0, t^1_1]$, for every $x \in [\xi^n_1(t), f'(\rho_+ + 2\epsilon)(t - \bar{t}) + y(\bar{t}) + \tilde{\delta}]$,

$$\rho^n(t, x+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon). \tag{B.2}$$

- If $\xi^n_i(\cdot)$ interacts with a shock or a rarefaction shock at time t^1_i; we follow the unique wave-front produced (see Figure 2.1). Moreover, $\rho^n(t^1_0, x+) \in (\rho^*, \rho_+ + 2\epsilon)$ for every $x \in [\xi^n_0(t^1_0), f'(\rho_+ + 2\epsilon)(t - \bar{t}) + y(\bar{t}) + \tilde{\delta}]$ and $\rho^n(t^1_1, x+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon)$ for every $x \in [\xi^n_1(t), f'(\rho_+ + 2\epsilon)(t - \bar{t}) + y(\bar{t}) + \tilde{\delta}]$.
- If $\xi^n_i(\cdot)$ interacts with $y^n(\cdot)$ at time t^1_i; from (B.1), (B.2) and using that all the possible interaction between a wave-front and $y^n(\cdot)$ is described in Figure 2.4, Figure 2.7, Figure 2.10 and Figure 2.11, we deduce that only the cases illustrated in Figure 2.4 and Figure 2.7(a) are possible. Thus, a unique wave-front is produced. Moreover, $\rho^n(t^1_0, \xi^n_0(t^1_0)+) \in (\rho^*, \rho_+ + 2\epsilon)$, $\rho^n(t^1_1, \xi^n_1(t^1_1)+) \in (\rho_+ - 2\epsilon, \rho_+ + \epsilon)$ and $\dot{y}^n(t^1_i) = v(\rho(t^1_i, y^n(t^1_i)+))$.

By an iteration procedure, we construct $\xi^n_0(\cdot)$ and $\xi^n_1(\cdot)$ over $[\bar{t}, t^{\xi_0}_f)$ and $[\bar{t}, t^{\xi_1}_f)$ respectively. For $i = 1, 2$, we extend $\xi^n_i(\cdot)$ to $\mathbb{R}_+$ by imposing that, for every $t \in [t^{\xi_i}_f, \infty)$, $\xi^n_i(t) = \xi^n_i(t^{\xi_i}_f)$. We conclude that, for every $(t,x) \in \{(t,x) \in [\bar{t}, +\infty) \times \mathbb{R}, x < \xi^n_0(t)\} \cap \mathcal{T}_1$,

$$\rho^n(t, x+) \in (0, \rho^* + \epsilon), \tag{B.3}$$

for every $(t,x) \in \{(t,x) \in [\bar{t}, +\infty) \times \mathbb{R}, x > \xi^n_0(t)\} \cap \mathcal{T}_1$,

$$\rho^n(t, x+) \in (\rho^*, \rho_+ + 2\epsilon), \tag{B.4}$$

and for every $(t,x) \in \{(t,x) \in [\bar{t}, +\infty) \times \mathbb{R}, x > \xi^n_1(t)\} \cap \mathcal{T}_1$,

$$\rho^n(t, x+) \in (\rho_+ - \epsilon, \rho_+ + \epsilon). \tag{B.5}$$

For $i = 1, 2$, $t^{\xi_i}_f$ and $t^{y^n}_f$ are the time when $\xi^n_i(\cdot)$ and $y^n(\cdot)$ exits the triangle $\mathcal{T}_1$ respectively. Note that, for every $t \in \mathbb{R}$, $\xi^n_0(t) \leq \xi^n_1(t)$ and as soon as there exists $t_1 \geq \bar{t}$ such that $\xi_0^n(t_1)=\xi_1^n(t_1)$, we have for every $t\in[t_1,+\infty]$ $\xi_0^n(t)=\xi_1^n(t)$. From (B.3), (B.4) and (B.5), we have

$$t_f^{\xi_0}\geq \bar{t}+\min\left(\frac{\tilde{\delta}}{v(0)-\sigma(\rho^*+\epsilon,\rho_++2\epsilon)},\frac{\tilde{\delta}}{v(\rho_+-\epsilon)-f'(\rho_++2\epsilon)}\right), \tag{B.6}$$

and

$$t_f^{\xi_1}\geq \bar{t}+\min\left(\frac{\tilde{\delta}}{v(0)-\sigma(\rho_++2\epsilon,\rho_++\epsilon)},\frac{\tilde{\delta}}{v(\rho_+-\epsilon)-f'(\rho_++2\epsilon)}\right). \tag{B.7}$$

Therefore, using that $\lim_{n\to\infty}y^n(\bar{t})=y(\bar{t})$, $y^n(\bar{t})\in[y^n(\bar{t})-\tilde{\delta},y(\bar{t})+\tilde{\delta}]$ and the finite speed of y^n, there exists $c>0$ independent of n such that

$$\min(t_f^{y^n},t_f^{\xi_0},t_f^{\xi_1})\geq \bar{t}+c.$$

From (B.3), (B.4) and (B.5), for every $t>\bar{t}$ such that $(t,\xi_0^n(t))\in\mathcal{T}_1$, $(t,\xi_1^n(t))\in\mathcal{T}_1$ and $(t,y^n(t))\in\mathcal{T}_1$, if $y^n(\cdot)$ belongs to the area A_1 defined by for every $(t,x)\in A_1$, $\rho^n(t,x)\in(\rho^*,\rho_++2\epsilon)$ (see the shaded zone in Figure 3.6) then $v(\rho_++2\epsilon)\leq\dot{y}^n(t)\leq v(\rho^*)$ and we have

$$\sigma(\rho_++2\epsilon,\rho_++\epsilon)\leq\dot{\xi}_1^n(t)\leq\sigma(\rho^*,\rho_+-\epsilon), \tag{B.8}$$

and if $y^n(\cdot)$ belongs to the area A_2 defined by for every $(t,x)\in A_2$, $\rho^n(t,x)\in(0,\rho^*+\epsilon)$ (see white zone in Figure 3.6) then $v(\rho^*+\epsilon)\leq\dot{y}^n(t)\leq V_b$ then either (B.8) holds or

$$\sigma(\rho^*+\epsilon,\rho_++\epsilon)\leq\dot{\xi}_1^n(t)\leq v(\rho_+-\epsilon). \tag{B.9}$$

From (B.8), (B.9) and using that f is strictly concave

$$\dot{y}^n(t)-\dot{\xi}_1^n(t)\geq v(\rho^*+\epsilon)-\sigma(\rho^*,\rho_+-\epsilon)>0. \tag{B.10}$$

Using (B.10), $y^n(\cdot)$ interacts with $\xi_1^n(\cdot)$ at time $t_n>\bar{t}$ and

$$t_n\leq\frac{\xi_1^n(\bar{t})-y^n(\bar{t})}{v(\rho^*+\epsilon)-\sigma(\rho^*,\rho_+-\epsilon)}. \tag{B.11}$$

Using that $\lim_{n\to\infty}y^n(\bar{t})=y(\bar{t})$ and $y^n(\bar{t})\leq\xi_1^n(\bar{t})\leq y(\bar{t})$ and (B.11), $\lim_{n\to\infty}t_n=0$.

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