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ANALYSIS OF THE ROLE OF CONVECTION IN A SYSTEM DESCRIBING THE TUMOR-INDUCED ANGIOGENESIS*

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Abstract. In this paper, we shall study the initial-boundary value problem of a mathematical model describing the branching of capillary sprouts during angiogenesis in one dimensional space. Under homogeneous Neumann boundary conditions, we show the existence of a unique global classical solution with uniform-in-time bound for all suitably regular initial data. Moreover, we show that the unique solution will exponentially converge to a non-trivial constant steady state as time tends to infinity under some appropriate conditions on the parameters.

Keywords. boundedness; chemotaxis; haptotaxis; convergence rate.

AMS subject classifications. 35A01; 35B40; 35K55; 35Q92; 92C17.

1. Introduction

Angiogenesis describes the formation of blood vessels from pre-existing vasculature. Tumor-induced angiogenesis plays an important role in the tumor invasion of the surrounding host tissue and blood system. Many mathematical models have been proposed to describe tumor angiogenesis, where chemotaxis, as the principal mechanism for cell motion, has been studied in angiogenesis [4, 28]. However, there exists some experimental evidence showing that in the early stage of angiogenesis, haptotaxis and convection play important roles [27, 30, 33]. In fact, during angiogenesis, the endothelial cells (ECs) secrete a matrix consisting of fibronectin, laminin, and collagen IV such that the movement of the ECs can be affected by the distribution of adhesive sites on this matrix [27]. Moreover, as the matrix spreads out, the convection can also play a major role in the transport of ECs [23].

To study the effect of haptotaxis and convection on the endothelial cells within the growing and developing capillary sprouts and secondary branching as they migrate towards the tumor cells, Orme & Chaplain [25] proposed the following system

$$\begin{cases} u_t = d_1\Delta u - \chi\nabla\cdot(u\nabla v) + \kappa_1\nabla\cdot(u\nabla w), \\ v_t = d_2\Delta v + \kappa_2\nabla\cdot(v\nabla w) + \alpha u - \beta v, \\ w_t = d_3\Delta w + \gamma u - \delta w, \end{cases} \tag{1.1}$$

where $u(x,t)$ describes the density of endothelial cells (ECs), $v(x,t)$ is the density of adhesive sites, and $w(x,t)$ stands for the density of the matrix. Here d_1, d_2, d_3 are diffusion coefficients; χ is the rate at which cells move up an adhesive gradient; β, δ are the decay of adhesive sites and matrix, respectively; α, γ describe the rate of secretion of adhesive sites and matrix per cell, respectively. In the model (1.1), the movement of the ECs is governed by a combination of random motility, haptotaxis and convection. More precisely, the cells move up an adhesive gradient (such as fibronectin secreted by the ECs during the angiogenic process), where the convection of cells depending on the spreading of the matrix with the convective flux is proportional to $-\kappa_1\nabla w$. The

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movement of adhesive sites depends on the random diffusion and the convection flux is proportional to $-\kappa_2\nabla w$.

The key feature of (1.1) is the coupling of haptotaxis and convection, which increases the mathematical complexity of (1.1) compared with the classical Keller-Segel (KS) model

$$\begin{cases} u_t = d_1\Delta u - \chi\nabla\cdot(u\nabla v), \\ v_t = d_2\Delta v + \alpha u - \beta v, \end{cases} \tag{1.2}$$

which can be obtained by letting $\kappa_1 = \kappa_2 = 0$ in (1.1). The KS model (1.2) has been extensively studied in various perspectives in the past four decades, one can find the survey articles [2, 7] and references therein for details. One of the most studied topics for the KS model (1.2) is the boundedness and blowup of solutions in two or higher dimensions [8, 24, 31, 32] based on the following Lyapunov function:

$$\mathcal{E}_1(u,v) = d_1\int_\Omega u\ln u - \chi\int_\Omega uv + \frac{\beta\chi}{2\alpha}\int_\Omega v^2 + \frac{\chi d_2}{2\alpha}\int_\Omega |\nabla v|^2.$$

Moreover, if we neglect the convection effect of adhesive sites (i.e., $\kappa_2 = 0$), and then the system (1.1) becomes

$$\begin{cases} u_t = d_1\Delta u - \chi\nabla\cdot(u\nabla v) + \kappa_1\nabla\cdot(u\nabla w), \\ v_t = d_2\Delta v + \alpha u - \beta v, \\ w_t = d_3\Delta w + \gamma u - \delta w, \end{cases} \tag{1.3}$$

which has been used to describe the aggregation of *Microglia* in Alzheimer's disease in [22] and also used to describe the in quorum effect in chemotaxis [26]. Using the L^2-energy estimate method, it has been proved that the system (1.3) has a unique global classical solution with uniform-in-time bound [10, 21] in one dimension. Moreover, based on the Hopf bifurcation theorem as well as the local and global bifurcation theorem, the existence of time-periodic patterns and steady state patterns were studied in [19]. In higher-dimensional spaces ($n \geq 2$), using transformation $s = \kappa_1 w - \chi v$ in [29] to simplify the system or constructing the Lyapunov functional as in [11, 12], it was shown that the solution behavior of (1.3) is essentially determined by the sign of $d_2\kappa_1\gamma - d_3\chi\alpha$. Especially, in two-dimensional spaces, the global bounded solution exists (see [9, 20, 29]) if $d_2\kappa_1\gamma \geq d_3\chi\alpha$, while if $d_2\kappa_1\gamma < d_3\chi\alpha$, a critical mass phenomenon is found in [6, 11, 14]. Furthermore, the global stabilization of homogeneous equilibria was studied in [12, 16–18].

However, for the tumor-induced angiogenesis system (1.1) with $\kappa_2 \neq 0$, the mathematical analysis is very hard due to lack of a Lyapunov functional as constructed for the system (1.3). To the best of our knowledge, there are few results by now on the system (1.1) except some numerical simulation results in [25] and the existence of solutions in one dimension [15]. The purpose of the present paper is to study the existence of global classical solutions and the asymptotic homogenization in the sense of stabilization toward spatially homogeneous equilibria in the large-time limit. More precisely, we first use L^2-energy estimates to assert the global existence of bounded classical solutions in one dimension with arbitrary initial data. Furthermore, under some assumption on the parameter κ_2, we construct an appropriate energy functional, and then, on proper exploitation of the correspondingly obtained information on the decay of the associated dissipation rate functional, show the large-time stabilization toward constant steady states. Precisely, we shall study the following system

$$\begin{cases} u_t = d_1 u_{xx} - \chi (u v_x)_x + \kappa_1 (u w_x)_x, & x \in \Omega, \quad t > 0, \\ v_t = d_2 v_{xx} + \kappa_2 (v w_x)_x + \alpha u - \beta v, & x \in \Omega, \quad t > 0, \\ w_t = d_3 w_{xx} + \gamma u - \delta w, & x \in \Omega, \quad t > 0, \\ u_x = v_x = w_x = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{1.4}$$

Then, we shall state our main results. The first result is concerned with the existence of global classical solutions with uniform-in-time bound for arbitrary initial data.

THEOREM 1.1. *Let Ω be a bounded open interval in $\mathbb{R}$. Suppose that the initial data $u_0 \in H^1(\Omega)$ and $(v_0, w_0) \in [H^2(\Omega)]^2$ with $u_0, v_0, w_0 \gneqq 0$. Then the system (1.4) possesses a unique global classical solution $(u,v,w) \in [C^0(\bar{\Omega} \times [0,\infty); \mathbb{R}^3)]^3 \cap [C^{2,1}(\bar{\Omega} \times (0,\infty); \mathbb{R}^3)]^3$ satisfying $u,v,w > 0$ for all $t > 0$. Moreover, for all $\kappa_2 \in (0,1]$, there exists a constant $M > 0$ independent of t and κ_2 such that*

$$\|u(\cdot,t)\|_{H^1(\Omega)} + \|(v,w)(\cdot,t)\|_{H^2(\Omega)} \leq M.$$

REMARK 1.1. In fact, the global boundedness of solution for (1.4) has been established in [15] in the sense that

$$\|u(\cdot,t)\|_{W^{1,2}(\Omega)} + \|v(\cdot,t)\|_{W^{1,2}(\Omega)} + \|w(\cdot,t)\|_{W^{1,2}(\Omega)} \leq M_0,$$

where M_0 is a positive constant independent of t. However, to study the global stabilization of homogeneous equilibria, we need to show that the constant M_0 remains bounded as $\kappa_2 \to 0$. Under some higher regularity assumptions on the initial data than that in [15], Theorem 1.1 improves the existence of the solution in [15] by obtaining an upper bound of the solution independent of κ_2. In what follows, we assume $0 < \kappa_2 \leq 1$ for simplicity.

Next, by constructing some proper energy functions, we show that all the solutions (u,v,w) will exponentially converge to the unique constant steady state $(\bar{u}_0, \frac{\alpha}{\beta}\bar{u}_0, \frac{\gamma}{\delta}\bar{u}_0)$ under some conditions on the parameters. More precisely, we have the following result.

THEOREM 1.2. *Let the conditions in Theorem 1.1 hold and (u,v,w) be the global solution of (1.4). Suppose the parameters satisfy*

$$\frac{\kappa_1 \gamma}{\chi \alpha} > \max\left\{\frac{\beta}{\delta}, \frac{\delta}{\beta}, \frac{d_2}{d_3}, \frac{d_3}{d_2}\right\}. \tag{1.5}$$

Then there exists a constant $\kappa_* \in (0,1]$ such that if $0 < \kappa_2 \leq \kappa_*$, the solution (u,v,w) will converge to the constant steady state $(\bar{u}_0, \frac{\alpha}{\beta}\bar{u}_0, \frac{\gamma}{\delta}\bar{u}_0)$ exponentially in L^∞-norm as $t \to +\infty$, where $\bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0 dx$.

2. Boundedness of solutions

In the following, we shall abbreviate $\int_\Omega f dx$ as $\int_\Omega f$ for simplicity. Moreover, we shall use c_i or $C_i (i = 1,2,3,\cdots)$ to denote a generic constant independent of κ_2, which may vary according to the context.

2.1. Local existence. Using similar arguments as in [10, 21], we shall apply Amann's theory [1] to establish the local existence of solutions.

LEMMA 2.1. *Let Ω be a bounded open interval in $\mathbb{R}$. Suppose that the initial data $u_0 \in H^1(\Omega)$ and $(v_0, w_0) \in [H^2(\Omega)]^2$ with $u_0, v_0, w_0 \geq (\not\equiv) 0$. Then there exists $T_{max} > 0$ such* *that the system* (1.4) *has a unique classical solution* $(u,v,w)\in[C^0(\bar\Omega\times[0,T_{max});\mathbb{R}^3)]^3\cap[C^{2,1}(\bar\Omega\times(0,T_{max});\mathbb{R}^3)]^3$ *satisfying* $u,v,w>0$. *Moreover, we have*

$$\text{either}\quad T_{max}=\infty\quad\text{or}\quad \|u(\cdot,t)\|_{W^{1,2}}+\|v(\cdot,t)\|_{W^{1,2}}+\|w(\cdot,t)\|_{W^{1,2}}\to\infty\quad\text{as}\quad t\nearrow T_{max}.$$

Proof. Define $\Phi=(u,v,w)\in\mathbb{R}^3$. Then the system (1.4) can be rewritten as

$$\begin{cases}\Phi_t-\nabla\cdot(A(\Phi)\nabla\Phi)=B(\Phi), & x\in\Omega,t>0,\\ \frac{\partial\Phi}{\partial\nu}=0, & x\in\partial\Omega,t>0,\\ \Phi(\cdot,0)=(u_0,v_0,w_0), & x\in\partial\Omega,\end{cases}\tag{2.1}$$

where

$$A(\Phi)=\begin{pmatrix} d_1 & -\chi u & \kappa_1 u\\ 0 & d_2 & \kappa_2 v\\ 0 & 0 & d_3\end{pmatrix},\quad B(\Phi)=\begin{pmatrix}0\\ \alpha u-\beta v\\ \gamma u-\delta w\end{pmatrix}.$$

Since all the eigenvalues of $A(\Phi)$ are positive and hence the system (2.1) is normally parabolic. Then using the results in [1, Theorem 14.6], we obtain the local existence of solutions and the extension criterion is a direct consequence of [1, Theorem 15.3]. At last, the positivity of the solutions follows from [1, Theorem 15.1]. ∎

2.2. L^2-estimate of v. To extend the local solution to global one, we need to establish the *a priori* estimates of solution. First, we have the following results.

LEMMA 2.2. *Suppose the conditions in Lemma 2.1 hold and* $\kappa_2\in(0,1]$. *Then the solution* (u,v,w) *of* (1.4) *satisfies*

$$\|u(\cdot,t)\|_{L^1}+\|w(\cdot,t)\|_{H^1}\le C(\|u_0\|_{L^1}+\|w_0\|_{H^1})\tag{2.2}$$

and

$$\|v(\cdot,t)\|_{L^2}\le C(\|u_0\|_{L^1}+\|v_0\|_{L^2}+\|w_0\|_{H^1}),\tag{2.3}$$

where $C>0$ *is a constant independent of* κ_2 *and* t.

Proof. Integrating the first equation of (1.4), we derive that $\frac{d}{dt}\int_\Omega u=0$, which implies that

$$\|u\|_{L^1}=\|u_0\|_{L^1}.\tag{2.4}$$

Using (2.4) and applying the parabolic regularity (see [13, Lemma 1]) to the third equation of (1.4), one has

$$\|w\|_{H^1}\le c_1(\|u_0\|_{L^1}+\|w_0\|_{H^1}).\tag{2.5}$$

To proceed, we integrate the equation of v in (1.4) over Ω to obtain

$$\frac{d}{dt}\int_\Omega v+\beta\int_\Omega v=\alpha\int_\Omega u=\alpha\int_\Omega u_0,$$

which together with Grönwall's inequality yields that

$$\|v\|_{L^1}\le\|v_0\|_{L^1}+\frac{\alpha}{\beta}\|u_0\|_{L^1}.\tag{2.6}$$

Multiplying the second equation of (1.4) by v and using Cauchy-Schwarz inequality, one obtains that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega v^2+\beta\int_\Omega v^2+d_2\int_\Omega v_x^2&=\alpha\int_\Omega uv-\kappa_2\int_\Omega vv_xw_x\\&\le\alpha\|v\|_{L^\infty}\|u\|_{L^1}+\kappa_2\|v\|_{L^\infty}\|v_x\|_{L^2}\|w_x\|_{L^2}\\&\le c_2(\|u_0\|_{L^1}+\|w_0\|_{H^1})(\alpha\|v\|_{L^\infty}+\kappa_2\|v\|_{L^\infty}\|v_x\|_{L^2}).\end{aligned}\tag{2.7}$$

Noting (2.6) and using Gagliardo-Nirenberg inequality, we have

$$\|v\|_{L^\infty}\le c_3(\|v_x\|_{L^2}^{\frac{2}{3}}\|v\|_{L^1}^{\frac{1}{3}}+\|v\|_{L^1})\le c_4\left(\|v_x\|_{L^2}^{\frac{2}{3}}+1\right).\tag{2.8}$$

Then substituting (2.8) into (2.7), and using (2.4)–(2.5), one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega v^2+\beta\int_\Omega v^2+d_2\int_\Omega v_x^2&\le c_5(\kappa_2+1)\left(\|v_x\|_{L^2}^{\frac{5}{3}}+\|v_x\|_{L^2}+\|v_x\|_{L^2}^{\frac{2}{3}}+1\right)\\&\le\frac{d_2}{2}\|v_x\|_{L^2}^2+c_6(\kappa_2^6+1),\end{aligned}$$

which gives

$$\frac{d}{dt}\int_\Omega v^2+2\beta\int_\Omega v^2\le 2c_6(\kappa_2^6+1)\le 4c_6\tag{2.9}$$

due to $0<\kappa_2\le 1$. Applying Grönwall's inequality to (2.9) gives

$$\|v\|_{L^2}^2\le\|v_0\|_{L^2}^2+\frac{2c_6}{\beta},$$

which yields (2.3). The proof of Lemma 2.2 is finished. □

2.3. L^2-estimate of u. In this subsection, we shall use the estimates of u and v to derive the L^2-norm of u.

LEMMA 2.3. *Suppose the conditions in Lemma 2.2 hold. Then we have*

$$\begin{aligned}&\|u(\cdot,t)\|_{L^2}+\|v(\cdot,t)\|_{H^1}+\|w(\cdot,t)\|_{H^2}\\&\le C(\|u_0\|_{L^1}+\|u_0\|_{L^2}+\|v_0\|_{H^1}+\|w_0\|_{H^2})\end{aligned}\tag{2.10}$$

for all $t>0$, where C is a positive constant independent of t and κ_2.

Proof. We multiply the first equation of (1.4) by u and integrate the result to obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2+d_1\int_\Omega u_x^2&=\chi\int_\Omega uu_xv_x-\kappa_1\int_\Omega uu_xw_x\\&=-\frac{\chi}{2}\int_\Omega u^2v_{xx}+\frac{\kappa_1}{2}\int_\Omega u^2w_{xx}.\end{aligned}\tag{2.11}$$

Noting the fact $\|u\|_{L^1}+\|w\|_{H^1}\le c_1$ (see (2.2) in Lemma 2.2), we apply Gagliardo-Nirenberg inequality to derive that

$$\|u\|_{L^4}^4\le c_2\left(\|u_x\|_{L^2}^2\|u\|_{L^1}^2+\|u\|_{L^1}^4\right)\le c_3(\|u_x\|_{L^2}^2+1)$$

and

$$\|w_{xx}\|_{L^2}^2 \leq c_4\left(\|w_{xxx}\|_{L^2}\|w_x\|_{L^2}+\|w_x\|_{L^2}^2\right) \leq \frac{d_1^2 d_3^2}{3c_3\gamma^2\kappa_1^2}\|w_{xxx}\|_{L^2}^2+c_5.$$

Then we can use Young's inequality to get that

$$\begin{aligned}
&\int_\Omega u^2-\frac{\chi}{2}\int_\Omega u^2 v_{xx}+\frac{\kappa_1}{2}\int_\Omega u^2 w_{xx}\\
&\leq \frac{d_1}{2c_3}\int_\Omega u^4+\frac{3c_3\chi^2}{8d_1}\int_\Omega v_{xx}^2+\frac{3c_3\kappa_1^2}{8d_1}\int_\Omega w_{xx}^2+\frac{3c_3}{2d_1}|\Omega|\\
&\leq \frac{d_1}{2}\int_\Omega u_x^2+\frac{3c_3\chi^2}{8d_1}\int_\Omega v_{xx}^2+\frac{d_1 d_3^2}{8\gamma^2}\int_\Omega w_{xxx}^2+c_6.
\end{aligned} \tag{2.12}$$

Adding the term $\int_\Omega u^2$ to both sides of (2.11) and combining it with (2.12), one gets

$$\frac{d}{dt}\int_\Omega u^2+2\int_\Omega u^2+d_1\int_\Omega u_x^2 \leq \frac{3c_3\chi^2}{4d_1}\int_\Omega v_{xx}^2+\frac{d_1 d_3^2}{4\gamma^2}\int_\Omega w_{xxx}^2+2c_6. \tag{2.13}$$

Multiplying the equation of v in (1.4) by $-v_{xx}$ and integrating it by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega v_x^2+\beta\int_\Omega v_x^2+d_2\int_\Omega v_{xx}^2\\
&=-\kappa_2\int_\Omega v_x w_x v_{xx}-\kappa_2\int_\Omega v v_{xx} w_{xx}-\alpha\int_\Omega u v_{xx}\\
&\leq \frac{d_2}{4}\int_\Omega v_{xx}^2+\frac{3\kappa_2^2}{d_2}\int_\Omega v_x^2 w_x^2+\frac{3\kappa_2^2}{d_2}\int_\Omega v^2 w_{xx}^2+\frac{3\alpha^2}{d_2}\int_\Omega u^2.
\end{aligned} \tag{2.14}$$

Applying Gagliardo-Nirenberg inequality and Young's inequality, one can derive that

$$\begin{aligned}
\frac{3\kappa_2^2}{d_2}\int_\Omega v_x^2 w_x^2 &\leq \frac{3\kappa_2^2}{d_2}\|v_x\|_{L^2}^2\|w_x\|_{L^\infty}^2\\
&\leq c_7\kappa_2^2\left(\|v_{xx}\|_{L^2}\|v\|_{L^2}+\|v\|_{L^2}^2\right)\left(\|w_{xxx}\|_{L^2}^{\frac{1}{2}}\|w_x\|_{L^2}^{\frac{3}{2}}+\|w_x\|_{L^2}^2\right)\\
&\leq c_8\kappa_2^2(\kappa_2^6+1)(\|v_{xx}\|_{L^2}+1)(\|w_{xxx}\|_{L^2}^{\frac{1}{2}}+1)\\
&\leq \frac{d_2}{8}\|v_{xx}\|_{L^2}^2+\frac{d_1^2 d_2 d_3^2}{6c_3\gamma^2\chi^2}\|w_{xxx}\|_{L^2}^2+c_9(\kappa_2^{20}+1)
\end{aligned} \tag{2.15}$$

where we have used (2.2) and (2.3). Similarly, using Gagliardo-Nirenberg inequality and Cauchy-Schwarz inequality, the third term on the right-hand side of (2.14) can be estimated as follows

$$\begin{aligned}
\frac{3\kappa_2^2}{d_2}\int_\Omega v^2 w_{xx}^2 &\leq \frac{3\kappa_2^2}{d_2}\|v\|_{L^4}^2\|w_{xx}\|_{L^4}^2\\
&\leq c_{10}\kappa_2^2\left(\|v_{xx}\|_{L^2}^{\frac{1}{4}}\|v\|_{L^2}^{\frac{7}{4}}+\|v\|_{L^2}^2\right)\left(\|w_{xxx}\|_{L^2}^{\frac{5}{4}}\|w_x\|_{L^2}^{\frac{3}{4}}+\|w_x\|_{L^2}^2\right)\\
&\leq c_{11}\kappa_2^2\left((\kappa_2^6+1)^{\frac{7}{8}}\|v_{xx}\|_{L^2}^{\frac{1}{4}}+\kappa_2^6+1\right)(\|w_{xxx}\|_{L^2}^{\frac{5}{4}}+1)\\
&\leq \frac{d_2}{8}\|v_{xx}\|_{L^2}^2+\frac{d_1^2 d_2 d_3^2}{6c_3\gamma^2\chi^2}\|w_{xxx}\|_{L^2}^2+c_{12}(\kappa_2^{29}+1).
\end{aligned} \tag{2.16}$$

Moreover, since $\|u\|_{L^1}=\|u_0\|_{L^1}$, the last term on the right-hand side of (2.14) can be estimated as

$$\frac{3\alpha^2}{d_2}\int_\Omega u^2=\frac{3\alpha^2}{d_2}\|u\|_{L^2}^2\le c_{13}\left(\|u_x\|_{L^2}^{\frac{2}{3}}\|u\|_{L^1}^{\frac{4}{3}}+\|u\|_{L^1}^2\right)\le\frac{d_1^2d_3}{6c_3\chi^2}\|u_x\|_{L^2}^2+c_{14}. \tag{2.17}$$

Substituting (2.15)–(2.17) into (2.14), one has

$$\begin{aligned}&\frac{d}{dt}\int_\Omega v_x^2+2\beta\int_\Omega v_x^2+d_2\int_\Omega v_{xx}^2\\&\le\frac{d_1^2d_2}{3c_3\chi^2}\int_\Omega u_x^2+\frac{2d_1^2d_2d_3^2}{3c_3\gamma^2\chi^2}\int_\Omega w_{xxx}^2+c_{15}\left(\kappa_2^{20}+\kappa_2^{29}+1\right).\end{aligned} \tag{2.18}$$

Next, to deal with the term $\int_\Omega w_{xxx}^2$ in (2.13) and (2.18), we differentiate the equation of w in (1.4) and derive that

$$w_{xt}=d_3w_{xxx}+\gamma u_x-\delta w_x. \tag{2.19}$$

Then multiplying (2.19) by $-w_{xxx}$ and integrating it over Ω, one uses Young's inequality to obtain that

$$\frac{d}{dt}\int_\Omega w_{xx}^2+2\delta\int_\Omega w_{xx}^2+d_3\int_\Omega w_{xxx}^2\le\frac{\gamma^2}{d_3}\int_\Omega u_x^2. \tag{2.20}$$

Letting $\mu=\min\{1,\beta,\delta\}$, we multiply (2.13) by $\frac{\gamma^2}{d_1d_3}$, (2.18) by $\frac{3c_3\gamma^2\chi^2}{4d_1^2d_2d_3}$, and (2.20) by $\frac{3}{4}$, respectively, and add the results to obtain that

$$y'+2\mu y\le\frac{3c_3c_{15}\gamma^2\chi^2+8c_6\gamma^2d_1d_2}{4d_1^2d_2d_3}\le c_{16}(\kappa_2^{29}+1)\le 2c_{16}, \tag{2.21}$$

where $y=\frac{\gamma^2}{d_1d_3}\int_\Omega u^2+\frac{3c_3\gamma^2\chi^2}{4d_1^2d_2d_3}\int_\Omega v_x^2+\frac{3}{4}\int_\Omega w_{xx}^2$. Then (2.21) together with Grönwall's inequality implies that

$$y\le y_0+\frac{c_{16}}{\mu},$$

where c_{16} depends on $\|u_0\|_{L^1}$, $\|v_0\|_{L^2}$ and $\|w_0\|_{H^1}$ but is independent of κ_2. As a consequence, we finish the proof. ∎

2.4. H^1-estimate of u. Next, we show the higher-order estimate of u based on energy estimates.

LEMMA 2.4. *Suppose the conditions in Lemma 2.2 hold. Then we can find a constant $C>0$ independent of t and κ_2 such that the solution (u,v,w) fulfills*

$$\|u(\cdot,t)\|_{H^1}+\|v(\cdot,t)\|_{H^2}+\|w(\cdot,t)\|_{H^2}\le C(\|u_0\|_{H^1}+\|v_0\|_{H^2}+\|w_0\|_{H^2}). \tag{2.22}$$

Proof. Differentiating the second equation of (1.4) gives that

$$v_{xt}=d_2v_{xxx}+\kappa_2(v_xw_x+vw_{xx})_x+\alpha u_x-\beta v_x. \tag{2.23}$$

We multiply (2.23) by $-v_{xxx}$ and integrate the result to derive that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_{xx}^2+\beta\int_\Omega v_{xx}^2+d_2\int_\Omega v_{xxx}^2$$

$$
\begin{aligned}
&= -\kappa_2 \int_\Omega v_{xx} v_{xxx} w_x - 2\kappa_2 \int_\Omega v_x v_{xxx} w_{xx} - \kappa_2 \int_\Omega v v_{xxx} w_{xxx} - \alpha \int_\Omega u_x v_{xxx} \\
&\le \frac{d_2}{4} \int_\Omega v_{xxx}^2 + \frac{4\kappa_2^2}{d_2} \int_\Omega v_{xx}^2 w_x^2 + \frac{16\kappa_2^2}{d_2} \int_\Omega v_x^2 w_{xx}^2 + \frac{4\kappa_2^2}{d_2} \int_\Omega v^2 w_{xxx}^2 + \frac{4\alpha^2}{d_2} \int_\Omega u_x^2. \qquad (2.24)
\end{aligned}
$$

Since $\|u\|_{L^2} + \|v\|_{H^1} + \|w\|_{H^2} \le c_1$ (see (2.10) in Lemma 2.3), then we can employ the Sobolev embedding to find constants c_2, $c_3 > 0$ such that

$$
\|v\|_{L^\infty} \le c_2 \qquad \text{and} \qquad \|w\|_{L^\infty} + \|w_x\|_{L^\infty} \le c_3. \qquad (2.25)
$$

Using the Gagliardo-Nirenberg inequality and Young's inequality, we derive that

$$
\begin{aligned}
\frac{4\kappa_2^2}{d_2} \int_\Omega v_{xx}^2 w_x^2 &\le \frac{4\kappa_2^2}{d_2} \|w_x\|_{L^\infty}^2 \|v_{xx}\|_{L^2}^2 \\
&\le \frac{4\kappa_2^2 c_3^2 c_4}{d_2} \left(\|v_{xxx}\|_{L^2} \|v_x\|_{L^2} + \|v_x\|_{L^2}^2 \right) \\
&\le \frac{d_2}{8} \|v_{xxx}\|_{L^2}^2 + c_5 \qquad (2.26)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{16\kappa_2^2}{d_2} \int_\Omega v_x^2 w_{xx}^2 &\le \frac{16\kappa_2^2}{d_2} \|v_x\|_{L^4}^2 \|w_{xx}\|_{L^4}^2 \\
&\le c_6 \kappa_2^2 \left(\|v_{xxx}\|_{L^2}^{\frac14} \|v_x\|_{L^2}^{\frac74} + \|v_x\|_{L^2}^2 \right) \left(\|w_{xxx}\|_{L^2}^{\frac54} \|w_x\|_{L^2}^{\frac34} + \|w_x\|_{L^2}^2 \right) \\
&\le c_7 \kappa_2^2 \left(\|v_{xxx}\|_{L^2}^{\frac14} \|v_x\|_{L^2}^{\frac74} + \|v_x\|_{L^2}^2 \right) \left(\|w_{xxx}\|_{L^2}^{\frac54} + 1 \right) \\
&\le \frac{d_2}{8} \|v_{xxx}\|_{L^2}^2 + \|w_{xxx}\|_{L^2}^2 + c_8 \qquad (2.27)
\end{aligned}
$$

due to $\kappa_2 \in (0,1]$. By (2.25), we can estimate the term $\frac{4\kappa_2^2}{d_2} \int_\Omega v^2 w_{xxx}^2$ as follows

$$
\frac{4\kappa_2^2}{d_2} \int_\Omega v^2 w_{xxx}^2 \le \frac{4\kappa_2^2}{d_2} \|v\|_{L^\infty}^2 \|w_{xxx}\|_{L^2}^2 \le \frac{4c_2^2}{d_2} \|w_{xxx}\|_{L^2}^2. \qquad (2.28)
$$

Moreover, one can find a constant $c_9 > 0$ such that

$$
\frac{4\alpha^2}{d_2} \int_\Omega u_x^2 \le \frac{4c_9\alpha^2}{d_2} \left(\|u_{xx}\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2 \right) \le \frac{d_1}{8} \|u_{xx}\|_{L^2}^2 + c_{10}. \qquad (2.29)
$$

We substitute (2.26)–(2.29) into (2.24) to obtain that

$$
\begin{aligned}
&\frac{d}{dt} \int_\Omega v_{xx}^2 + 2\beta \int_\Omega v_{xx}^2 + d_2 \int_\Omega v_{xxx}^2 - \frac{d_1}{4} \int_\Omega u_{xx}^2 \\
&\le c_{11} \|w_{xxx}\|_{L^2}^2 + c_{12} (\|u_0\|_{L^2}^2 + \|v_0\|_{H^2}^2 + \|w_0\|_{H^2}^2). \qquad (2.30)
\end{aligned}
$$

Moreover, we multiply the equation $w_{xt} = d_3 w_{xxx} + \gamma u_x - \delta w_x$ by $-w_{xxx}$ and integrate the result to get that

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega w_{xx}^2 + 2\delta \int_\Omega w_{xx}^2 + d_3 \int_\Omega w_{xxx}^2 &\le \frac{\gamma^2}{d_3} \|u_x\|_{L^2}^2 \\
&\le c_{13} \left(\|u_{xx}\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2 \right)
\end{aligned}
$$

$$\leq \frac{d_1 d_3}{8c_{11}}\|u_{xx}\|_{L^2}^2 + c_{14}(\|u_0\|_{L^2}^2 + \|v_0\|_{H^2}^2 + \|w_0\|_{H^2}^2). \tag{2.31}$$

To control the term $\|u_{xx}\|_{L^2}$ on the right hand in (2.31), we apply $-u_{xx}$ as a test function to the first equation of (1.4) and integrate the result to obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u_x^2 + \int_\Omega u_x^2 + d_1\int_\Omega u_{xx}^2\\
=&\chi\int_\Omega u_x u_{xx} v_x + \chi\int_\Omega u u_{xx} v_{xx} - \kappa_1\int_\Omega u_x u_{xx} w_x - \kappa_1\int_\Omega u u_{xx} w_{xx} + \int_\Omega u_x^2\\
\leq&\frac{d_1}{8}\|u_{xx}\|_{L^2}^2 + \frac{8\chi^2}{d_1}\|u_x\|_{L^2}^2\|v_x\|_{L^\infty}^2 + \frac{8\chi^2}{d_1}\|u\|_{L^4}^2\|v_{xx}\|_{L^4}^2 + \frac{8\kappa_1^2}{d_1}\|u_x\|_{L^2}^2\|w_x\|_{L^\infty}^2\\
&+\frac{8\kappa_1^2}{d_1}\|u\|_{L^\infty}^2\|w_{xx}\|_{L^2}^2 + \|u_x\|_{L^2}^2.
\end{aligned} \tag{2.32}$$

By using Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
&\frac{8\chi^2}{d_1}\|u_x\|_{L^2}^2\|v_x\|_{L^\infty}^2\\
\leq&\frac{8c_{15}\chi^2}{d_1}\left(\|u_{xx}\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2\right)\left(\|v_{xxx}\|_{L^2}^{\frac{1}{2}}\|v_x\|_{L^2}^{\frac{3}{2}} + \|v_x\|_{L^2}^2\right)\\
\leq&\frac{d_1}{16}\|u_{xx}\|_{L^2}^2 + \frac{d_2}{8}\|v_{xxx}\|_{L^2}^2 + c_{16}
\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}
&\frac{8\chi^2}{d_1}\|u\|_{L^4}^2\|v_{xx}\|_{L^4}^2\\
\leq&\frac{8c_{17}\chi^2}{d_1}\left(\|u_{xx}\|_{L^2}^{\frac{1}{4}}\|u\|_{L^2}^{\frac{7}{4}} + \|u\|_{L^2}^2\right)\left(\|v_{xxx}\|_{L^2}^{\frac{5}{4}}\|v_x\|_{L^2}^{\frac{3}{4}} + \|v_x\|_{L^2}^2\right)\\
\leq&\frac{d_1}{16}\|u_{xx}\|_{L^2}^2 + \frac{d_2}{8}\|v_{xxx}\|_{L^2}^2 + c_{18}.
\end{aligned} \tag{2.34}$$

Besides, we can use Gagliardo-Nirenberg inequality and Young's inequality to obtain that

$$\frac{8\kappa_1^2}{d_1}\|u_x\|_{L^2}^2\|w_x\|_{L^\infty}^2 \leq \frac{8c_3^2 c_{19}\kappa_1^2}{d_1}\left(\|u_{xx}\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2\right) \leq \frac{d_1}{16}\|u_{xx}\|_{L^2}^2 + c_{20} \tag{2.35}$$

and

$$\frac{8\kappa_1^2}{d_1}\|u\|_{L^\infty}^2\|w_{xx}\|_{L^2}^2 \leq \frac{8c_1^2 c_{21}\kappa_1^2}{d_1}\left(\|u_{xx}\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{3}{2}} + \|u\|_{L^2}^2\right) \leq \frac{d_1}{16}\|u_{xx}\|_{L^2}^2 + c_{22} \tag{2.36}$$

as well as

$$\|u_x\|_{L^2}^2 \leq c_{23}\left(\|u_{xx}\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2\right) \leq \frac{d_1}{8}\|u_{xx}\|_{L^2}^2 + c_{24}. \tag{2.37}$$

Substituting (2.33)–(2.37) into (2.32), we arrive at

$$\frac{d}{dt}\int_\Omega u_x^2 + 2\int_\Omega u_x^2 + d_1\int_\Omega u_{xx}^2 \leq \frac{d_2}{2}\int_\Omega v_{xxx}^2 + c_{25}(\|u_0\|_{L^2}^2 + \|v_0\|_{H^2}^2 + \|w_0\|_{H^2}^2). \tag{2.38}$$

The combination of (2.30), (2.31) and (2.38) gives that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(u_x^2+v_{xx}^2+\frac{2c_{11}}{d_3}w_{xx}^2\right)+2\int_\Omega\left(u_x^2+\beta v_{xx}^2+\frac{2\delta c_{11}}{d_3}w_{xx}^2\right)\\&\quad+\frac{1}{2}\int_\Omega(d_1u_{xx}^2+d_2v_{xxx}^2+2c_{11}w_{xxx}^2)\\&\le c_{26}(\|u_0\|_{L^2}^2+\|v_0\|_{H^2}^2+\|w_0\|_{H^2}^2).\end{aligned}$$

Then, by Grönwall's inequality, we arrive at

$$\|u_x\|_{L^2}^2+\|v_{xx}\|_{L^2}^2+\frac{2c_{11}}{d_3}\|w_{xx}\|_{L^2}^2\le c_{27}(\|u_0\|_{H^1}^2+\|v_0\|_{H^2}^2+\|w_0\|_{H^2}^2),$$

which together with (2.10) yields (2.22). Then the proof of this Lemma 2.4 is completed. ◻

Proof. **(Proof of Theorem 1.1.)** Based on the results in Lemma 2.4, we derive that

$$\|u(\cdot,t)\|_{W^{1,2}}+\|v(\cdot,t)\|_{W^{1,2}}+\|w(\cdot,t)\|_{W^{1,2}}\le c_1(\|u_0\|_{H^1}+\|v_0\|_{H^2}^2+\|w_0\|_{H^2}^2).$$

Then using Lemma 2.1, we obtain Theorem 1.1 directly. ◻

3. Convergence of constant steady state

In this section, motivated by some ideas in [12], we shall study the asymptotic behavior of solutions for (1.4) by constructing some energy functional. First, we define an energy functional

$$\mathcal{E}(t):=\frac{\theta_1}{2\kappa_1\chi}\int_\Omega u\ln\frac{u}{\bar{u}}+\frac{\theta_2}{4\kappa_1\alpha}\int_\Omega v_x^2+\frac{\theta_2}{4\gamma\chi}\int_\Omega w_x^2-\int_\Omega v_xw_x, \tag{3.1}$$

where $\bar{u}=\frac{1}{|\Omega|}\int_\Omega u_0$ and

$$\theta_1:=\kappa_1\gamma-\chi\alpha,\qquad \theta_2:=\kappa_1\gamma+\chi\alpha. \tag{3.2}$$

Let M be the constant in Theorem 1.1, then one has $\|v\|_{L^\infty}+\|v_x\|_{L^\infty}\le M$. We have the following results.

LEMMA 3.1. *Suppose the condition (1.5) holds. Let (u,v,w) be the solution of (1.4). Then the energy functional $\mathcal{E}(t)$ defined in (3.1) is nonnegative and satisfies*

$$\frac{d}{dt}\mathcal{E}(t)+\mathcal{F}(t)\le 0, \tag{3.3}$$

where $\mathcal{F}(t)$ is defined as

$$\begin{aligned}\mathcal{F}(t):=&\frac{d_1\theta_1}{2\kappa_1\chi}\int_\Omega\frac{u_x^2}{u}+\frac{d_2\theta_2}{2\kappa_1\alpha}\int_\Omega v_{xx}^2+\left(\frac{d_3\theta_2}{2\gamma\chi}-\kappa_2M\right)\int_\Omega w_{xx}^2+\frac{\beta\theta_2}{2\kappa_1\alpha}\int_\Omega v_x^2\\&+\frac{\delta\theta_2}{2\gamma\chi}\int_\Omega w_x^2-(\beta+\delta)\int_\Omega v_xw_x-\frac{\kappa_2\theta_2M}{2\kappa_1\alpha}\int_\Omega|v_{xx}||w_x|\\&-\left(\frac{\kappa_2\theta_2M}{2\kappa_1\alpha}+d_2+d_3\right)\int_\Omega|v_{xx}||w_{xx}|-\kappa_2M\int_\Omega|w_x||w_{xx}|.\end{aligned} \tag{3.4}$$

Proof. First, we show $\mathcal{E}(t)$ and $\mathcal{F}(t)$ satisfy (3.3). To this end, we multiply the first equation of (1.4) by $\ln\frac{u}{\bar u}$ and integrate the result to derive that

$$\frac{d}{dt}\int_\Omega u\ln\frac{u}{\bar u} = -d_1\int_\Omega \frac{u_x^2}{u} + \chi\int_\Omega u_x v_x - \kappa_1\int_\Omega u_x w_x. \tag{3.5}$$

Employing $-v_{xx}$ as a test function to the second equation of (1.4) and integrating the result, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_x^2 + \beta\int_\Omega v_x^2 + d_2\int_\Omega v_{xx}^2 = \alpha\int_\Omega u_x v_x - \kappa_2\int_\Omega (vw_x)_x v_{xx}. \tag{3.6}$$

Similarly, it follows from the third equation of (1.4) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w_x^2 + \delta\int_\Omega w_x^2 + d_3\int_\Omega w_{xx}^2 = \gamma\int_\Omega u_x w_x. \tag{3.7}$$

We multiply (3.5) by $\frac{\theta_1}{2\kappa_1\chi}$, (3.6) by $\frac{\theta_2}{2\kappa_1\alpha}$ and (3.7) by $\frac{\theta_2}{2\gamma\chi}$, respectively, and add the results to derive that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\theta_1}{2\kappa_1\chi}\int_\Omega u\ln\frac{u}{\bar u} + \frac{\theta_2}{4\kappa_1\alpha}\int_\Omega v_x^2 + \frac{\theta_2}{4\gamma\chi}\int_\Omega w_x^2\right) + \frac{d_1\theta_1}{2\kappa_1\chi}\int_\Omega \frac{u_x^2}{u}\\
&\quad + \frac{d_2\theta_2}{2\kappa_1\alpha}\int_\Omega v_{xx}^2 + \frac{d_3\theta_2}{2\gamma\chi}\int_\Omega w_{xx}^2 + \frac{\beta\theta_2}{2\kappa_1\alpha}\int_\Omega v_x^2 + \frac{\delta\theta_2}{2\gamma\chi}\int_\Omega w_x^2\\
&= \gamma\int_\Omega u_x v_x + \alpha\int_\Omega u_x w_x - \frac{\kappa_2\theta_2}{2\kappa_1\alpha}\int_\Omega (vw_x)_x v_{xx}.
\end{aligned} \tag{3.8}$$

Recalling the third equation of (1.4), we have $\gamma u_x = w_{xt} - d_3 w_{xxx} + \delta w_x$. Then the term $\gamma\int_\Omega u_x v_x$ on the right-hand side of (3.8) can be rewritten as

$$\begin{aligned}
\gamma\int_\Omega u_x v_x &= \int_\Omega (w_{xt} - d_3 w_{xxx} + \delta w_x) v_x\\
&= \frac{d}{dt}\int_\Omega v_x w_x - \int_\Omega v_{xt} w_x + d_3\int_\Omega v_{xx} w_{xx} + \delta\int_\Omega v_x w_x.
\end{aligned} \tag{3.9}$$

In view of the equation of v in (1.4), we substitute $v_{xt} = d_2 v_{xxx} + \kappa_2 (vw_x)_{xx} + \alpha u_x - \beta v_x$ into (3.9) and integrate the result by parts to obtain

$$\begin{aligned}
\gamma\int_\Omega u_x v_x = &\frac{d}{dt}\int_\Omega v_x w_x + (d_2+d_3)\int_\Omega v_{xx} w_{xx} + (\beta+\delta)\int_\Omega v_x w_x + \kappa_2\int_\Omega (vw_x)_x w_{xx}\\
&-\alpha\int_\Omega u_x w_x,
\end{aligned}$$

which together with (3.8) gives that

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t) + \frac{d_1\theta_1}{2\kappa_1\chi}\int_\Omega \frac{u_x^2}{u} + \frac{d_2\theta_2}{2\kappa_1\alpha}\int_\Omega v_{xx}^2 + \frac{d_3\theta_2}{2\gamma\chi}\int_\Omega w_{xx}^2 + \frac{\beta\theta_2}{2\kappa_1\alpha}\int_\Omega v_x^2\\
&\quad + \frac{\delta\theta_2}{2\gamma\chi}\int_\Omega w_x^2 - (d_2+d_3)\int_\Omega v_{xx} w_{xx} - (\beta+\delta)\int_\Omega w_x v_x\\
&= \kappa_2\int_\Omega (vw_x)_x w_{xx} - \frac{\kappa_2\theta_2}{2\kappa_1\alpha}\int_\Omega (vw_x)_x v_{xx}.
\end{aligned} \tag{3.10}$$

Since $\|v\|_{L^\infty}+\|v_x\|_{L^\infty}\le M$, one has

$$\begin{aligned}&\kappa_2\int_\Omega (vw_x)_x w_{xx}-\frac{\kappa_2\theta_2}{2\kappa_1\alpha}\int_\Omega (vw_x)_x v_{xx}\\ =&\kappa_2\int_\Omega v_x w_x w_{xx}+\kappa_2\int_\Omega v w_{xx}^2-\frac{\kappa_2\theta_2}{2\kappa_1\alpha}\int_\Omega v_x w_x v_{xx}-\frac{\kappa_2\theta_2}{2\kappa_1\alpha}\int_\Omega v w_{xx} v_{xx}\\ \le&\frac{\kappa_2\theta_2 M}{2\kappa_1\alpha}\int_\Omega (|w_x|+|w_{xx}|)|v_{xx}|+\kappa_2 M\int_\Omega |w_x||w_{xx}|+\kappa_2 M\int_\Omega w_{xx}^2,\end{aligned}$$

which together with (3.10) yields (3.3).

Next, we show the nonnegativity of $\mathcal{E}(t)$ to complete the proof of this lemma. Let $X=(v_x,w_x)$, then we rewrite $\mathcal{E}(t)$ as

$$\mathcal{E}(t)=\frac{\theta_1}{2\kappa_1\chi}\int_\Omega u\ln\frac{u}{\bar u}+\int_\Omega XAX^\top,$$

where

$$A:=\begin{pmatrix}\frac{\theta_2}{4\kappa_1\alpha} & -\frac12\\ -\frac12 & \frac{\theta_2}{4\gamma\chi}\end{pmatrix}.$$

Using (1.5) and the definition of θ_i $(i=1,2)$ in (3.2), we can check $\theta_1>0$ and $\theta_2>0$ directly. Then, one can derive that

$$|A|=\frac{\theta_2^2}{16\kappa_1\gamma\chi\alpha}-\frac14=\frac{\theta_1^2}{16\kappa_1\gamma\chi\alpha}>0.$$

Hence, the matrix A is positive-definite by Sylvester's criterion. Then we can find a constant $c_1>0$ such that

$$\int_\Omega XAX^\top\ge c_1\int_\Omega\left(v_x^2+w_x^2\right).$$

Moreover, we have $\frac{\theta_1}{2\kappa_1\chi}\int_\Omega u\ln\frac{u}{\bar u}\ge 0$ by applying the inequality (see [3, 5])

$$\int_\Omega f\ln\frac{f}{\bar f}\ge\frac{1}{2\bar f}\|f-\bar f\|_{L^1}^2\ge 0,\quad \forall f>0,$$

where $\bar f=\frac{1}{|\Omega|}\int_\Omega f$. In summary, we obtain

$$\mathcal{E}(t)\ge\frac{\theta_1}{2\kappa_1\chi}\int_\Omega u\ln\frac{u}{\bar u}+c_1\int_\Omega\left(v_x^2+w_x^2\right)\ge 0. \tag{3.11}$$

Thus the proof of Lemma 3.1 is completed. □

Next, we show that the energy functional $\mathcal{E}(t)$ defined in (3.1) is monotone decreasing under certain conditions on the parameters. To this end, we must show the nonnegativity of $\mathcal{F}(t)$.

LEMMA 3.2. *Suppose the conditions in Lemma 3.1 hold. Then there exists a constant $\kappa_*>0$ such that if $0<\kappa_2\le\kappa_*$, we have*

$$\mathcal{E}(t)\le\eta\mathcal{F}(t), \tag{3.12}$$

for some $\eta>0$ independent of κ_2 and t.

Proof. By the definition of $\mathcal{E}(t)$ and $\mathcal{F}(t)$ (see (3.1) and (3.4)), we have

$$\eta\mathcal{F}(t)-\mathcal{E}(t)\geq \frac{\theta_1}{2\kappa_1\chi}\int_\Omega\left(d_1\eta\frac{u_x^2}{u}-u\ln\frac{u}{\bar{u}}\right)+J,$$

where

$$J=\eta J_1+J_2-\eta\kappa_2 J_3 \tag{3.13}$$

with

$$\begin{aligned}
J_1&=\frac{d_2\theta_2}{2\kappa_1\alpha}\int_\Omega v_{xx}^2+\frac{d_3\theta_2}{2\gamma\chi}\int_\Omega w_{xx}^2-(d_2+d_3)\int_\Omega|v_{xx}||w_{xx}|,\\
J_2&=\frac{\theta_2(2\beta\eta-1)}{4\kappa_1\alpha}\int_\Omega v_x^2+\frac{\theta_2(2\delta\eta-1)}{4\gamma\chi}\int_\Omega w_x^2-(1+\eta(\beta+\delta))\int_\Omega|v_x||w_x|,\\
J_3&=M\int_\Omega w_{xx}^2+M\int_\Omega|w_x||w_{xx}|+\frac{M\theta_2}{2\kappa_1\alpha}\int_\Omega|v_{xx}||w_x|+\frac{M\theta_2}{2\kappa_1\alpha}\int_\Omega|v_{xx}||w_{xx}|.
\end{aligned}$$

Since $\int_\Omega\bar{u}=\int_\Omega u=\int_\Omega u_0>0$, one can use the inequality (see [3, 5])

$$0\leq\frac{1}{2\bar{f}}\|f-\bar{f}\|_{L^1}^2\leq\int_\Omega f\ln\frac{f}{\bar{f}}\leq\frac{1}{\bar{f}}\|f-\bar{f}\|_{L^2}^2,\quad\forall f>0 \tag{3.14}$$

and the Poincaré inequality to derive that

$$\int_\Omega u\ln\frac{u}{\bar{u}}\leq\frac{1}{\bar{u}}\|u-\bar{u}\|_{L^2}^2\leq c_1\|u_x\|_{L^2}^2\leq c_1\|u\|_{L^\infty}\int_\Omega\frac{u_x^2}{u}\leq c_2\int_\Omega\frac{u_x^2}{u}, \tag{3.15}$$

where the boundedness of u is used. Letting $\eta\geq\eta_1:=\frac{c_2}{d_1}$, it follows from (3.15) that

$$\frac{\theta_1}{2\kappa_1\chi}\int_\Omega\left(d_1\eta\frac{|u_x|^2}{u}-u\ln\frac{u}{\bar{u}}\right)\geq 0. \tag{3.16}$$

Next, we prove $J\geq 0$. To this end, we first rewrite J_1 as $J_1=\int_\Omega X_1A_1X_1^\top$, where

$$X_1=(|v_{xx}|,|w_{xx}|)\quad\text{and}\quad A_1:=\begin{pmatrix}\frac{d_2\theta_2}{2\kappa_1\alpha} & -\frac{d_2+d_3}{2}\\ -\frac{d_2+d_3}{2} & \frac{d_3\theta_2}{2\gamma\chi}\end{pmatrix}.$$

Using the condition (1.5), we can check that

$$|A_1|=\frac{d_2d_3\theta_2^2}{4\gamma\chi\alpha\kappa_1}-\frac{(d_2+d_3)^2}{4}=\frac{d_2d_3\chi\alpha}{4\kappa_1\gamma}\left(\frac{\kappa_1\gamma}{\chi\alpha}-\frac{d_2}{d_3}\right)\left(\frac{\kappa_1\gamma}{\chi\alpha}-\frac{d_3}{d_2}\right)>0.$$

Hence, we derive the positivity of matrix A_1 based on Sylvester's criterion. As a result, one can find a constant $c_3>0$ independent of κ_2 such that

$$J_1\geq c_3\int_\Omega(v_{xx}^2+w_{xx}^2). \tag{3.17}$$

On the other hand, choosing $\eta>\eta_2:=\max\{\frac{2}{3\beta},\frac{2}{3\delta}\}$ and letting $X_2=(|v_x|,|w_x|)$, then we rewrite J_2 as

$$J_2=\eta\int_\Omega X_2A_2X_2^\top-\frac{\theta_2}{4\kappa_1\alpha}\int_\Omega v_x^2-\frac{\theta_2}{4\gamma\chi}\int_\Omega w_x^2-\int_\Omega|v_x||w_x|$$

with

$$A_2 := \begin{pmatrix} \frac{\beta\theta_2}{2\kappa_1\alpha} & -\frac{\beta+\delta}{2} \\ -\frac{\beta+\delta}{2} & \frac{\delta\theta_2}{2\gamma\chi} \end{pmatrix}.$$

Condition (1.5) guarantees that

$$|A_2| = \frac{\delta\beta\theta_2^2}{4\kappa_1\gamma\chi\alpha} - \frac{(\delta+\beta)^2}{4} = \frac{\delta\beta\chi\alpha}{4\kappa_1\gamma}\left(\frac{\kappa_1\gamma}{\chi\alpha} - \frac{\delta}{\beta}\right)\left(\frac{\kappa_1\gamma}{\chi\alpha} - \frac{\beta}{\delta}\right) > 0.$$

Applying Sylvester's criterion, we conclude that the matrix A_2 is positive-definite, which entails us to find a constant $c_4 > 0$ independent of κ_2 such that

$$\begin{aligned} J_2 &\geq \eta c_4 \int_\Omega (v_x^2 + w_x^2) - \frac{\theta_2}{4\kappa_1\alpha}\int_\Omega v_x^2 - \frac{\theta_2}{4\gamma\chi}\int_\Omega w_x^2 - \int_\Omega |v_x||w_x| \\ &\geq \left(\eta c_4 - \frac{\theta_2}{4\kappa_1\alpha} - \frac{1}{2}\right)\int_\Omega v_x^2 + \left(\eta c_4 - \frac{\theta_2}{4\gamma\chi} - \frac{1}{2}\right)\int_\Omega w_x^2 \\ &\geq \frac{\eta c_4}{2}\int_\Omega (v_x^2 + w_x^2) \end{aligned} \tag{3.18}$$

by selecting $\eta > \eta_3 := \max\{\frac{2\kappa_1\alpha+\theta_2}{2c_4\kappa_1\alpha}, \frac{2\gamma\chi+\theta_2}{2c_4\gamma\chi}\}$. Furthermore, one uses Young's inequality to derive that

$$J_3 \leq \xi \int_\Omega (v_{xx}^2 + w_x^2 + w_{xx}^2), \tag{3.19}$$

where $\xi := M(2 + \frac{\theta_2}{2\kappa_1\alpha})$. Then letting $\eta > \max\{\eta_1, \eta_2, \eta_3\}$, and substituting (3.17), (3.18) and (3.19) into (3.13), one has

$$\begin{aligned} J = \eta J_1 + J_2 - \eta\kappa_2 J_3 \geq & c_3\eta \int_\Omega (v_{xx}^2 + w_{xx}^2) + \frac{\eta c_4}{2}\int_\Omega (v_x^2 + w_x^2) \\ & - \eta\kappa_2\xi \int_\Omega (v_{xx}^2 + w_x^2 + w_{xx}^2) \\ \geq & \eta\Big(c_3 - \kappa_2\xi\Big)\int_\Omega (v_{xx}^2 + w_{xx}^2) + \eta\Big(\frac{c_4}{2} - \kappa_2\xi\Big)\int_\Omega w_x^2. \end{aligned} \tag{3.20}$$

Letting $\kappa_* := \min\{\frac{c_3}{\xi}, \frac{c_4}{2\xi}, 1\}$ and noting that the constants c_3, c_4 and ξ are independent of κ_2, then if $\kappa_2 \leq \kappa_*$, from (3.20) we can derive that $J \geq 0$, which together with (3.16) gives (3.12). ∎

LEMMA 3.3. *Let the conditions in Lemma 3.2 hold. Then it holds that*

$$\|u(\cdot,t) - \bar{u}\|_{L^\infty} \leq Ce^{-\frac{t}{3\eta}}, \tag{3.21}$$

where C and η are positive constants independent of t.

Proof. The combination of (3.3) and (3.12) gives that

$$\frac{d}{dt}\mathcal{E}(t) + \frac{1}{\eta}\mathcal{E}(t) \leq 0,$$

which implies that

$$\mathcal{E}(t) \leq c_1 e^{-\frac{t}{\eta}}. \tag{3.22}$$

Recalling the fact (3.11) and (3.14), (3.22) yields

$$\|u(\cdot,t)-\bar{u}\|_{L^1}^2 \leq \frac{4c_1\bar{u}\kappa_1\chi}{\theta_1}e^{-\frac{t}{\eta}}.$$

By the Gagliardo-Nirenberg inequality and Theorem 1.1, we obtain that

$$\|u-\bar{u}\|_{L^\infty} \leq c_2(\|u_x\|_{L^2}^{\frac{2}{3}}\|u-\bar{u}\|_{L^1}^{\frac{1}{3}}+\|u-\bar{u}\|_{L^1}) \leq c_3\|u-\bar{u}\|_{L^1}^{\frac{1}{3}} \leq c_4 e^{-\frac{t}{3\eta}},$$

which yields (3.21). ∎

Lemma 3.4. *Suppose that the assumptions in Lemma 3.2 hold. Then there exist two positive constants C,λ_1 independent of t such that*

$$\|v(\cdot,t)-\frac{\alpha}{\beta}\bar{u}\|_{L^\infty}+\|w(\cdot,t)-\frac{\gamma}{\delta}\bar{u}\|_{L^\infty} \leq Ce^{-\lambda_1 t}.$$

Proof. From Theorem 1.1, we have $\|u\|_{L^\infty}+\|v\|_{L^\infty}+\|w\|_{L^\infty} \leq c_1$. Then we multiply the second equation of (1.4) by $v-\frac{\alpha}{\beta}\bar{u}$ and integrate the result to derive that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left(v-\frac{\alpha}{\beta}\bar{u}\right)^2\\
&= -d_2\int_\Omega v_x^2-\kappa_2\int_\Omega vv_xw_x+\int_\Omega(\alpha u-\beta v)\left(v-\frac{\alpha}{\beta}\bar{u}\right)\\
&\leq -\frac{d_2}{2}\int_\Omega v_x^2+\frac{c_1^2\kappa_2^2}{2d_2}\int_\Omega w_x^2+\alpha\int_\Omega(u-\bar{u})\left(v-\frac{\alpha}{\beta}\bar{u}\right)-\beta\int_\Omega\left(v-\frac{\alpha}{\beta}\bar{u}\right)^2\\
&\leq -\frac{d_2}{2}\int_\Omega v_x^2+\frac{c_1^2\kappa_2^2}{2d_2}\int_\Omega w_x^2-\frac{\beta}{2}\int_\Omega\left(v-\frac{\alpha}{\beta}\bar{u}\right)^2+\frac{\alpha^2}{2\beta}\int_\Omega(u-\bar{u})^2,
\end{aligned}$$

which implies that

$$\frac{d}{dt}\int_\Omega\left(v-\frac{\alpha}{\beta}\bar{u}\right)^2+\beta\int_\Omega\left(v-\frac{\alpha}{\beta}\bar{u}\right)^2+d_2\int_\Omega v_x^2 \leq \frac{c_1^2\kappa_2^2}{d_2}\int_\Omega w_x^2+\frac{\alpha^2}{\beta}\int_\Omega(u-\bar{u})^2. \tag{3.23}$$

Similarly, it follows from the third equation of (1.4) that

$$\frac{d}{dt}\int_\Omega\left(w-\frac{\gamma}{\delta}\bar{u}\right)^2+\delta\int_\Omega\left(w-\frac{\gamma}{\delta}\bar{u}\right)^2+2d_3\int_\Omega w_x^2 \leq \frac{\gamma^2}{\delta}\int_\Omega(u-\bar{u})^2. \tag{3.24}$$

Multiplying (3.24) by $\frac{c_1^2\kappa_2^2}{d_2d_3}$ and adding it to (3.23), letting $y(t):=\int_\Omega(v-\frac{\alpha}{\beta}\bar{u})^2+\frac{c_1^2\kappa_2^2}{d_2d_3}\int_\Omega\left(w-\frac{\gamma}{\delta}\bar{u}\right)^2$ and $c_2:=\min\{\beta,\delta\}$, one can derive that

$$y'(t)+c_2y(t) \leq c_3\|u-\bar{u}\|_{L^\infty}^2 \leq c_4e^{-\frac{2t}{3\eta}}, \tag{3.25}$$

where the last inequality holds due to (3.21). Then solving (3.25) gives two positive constants c_5 and λ_0 such that

$$y(t) \leq c_5e^{-\lambda_0 t},$$

which, together with the definition of $y(t)$, implies

$$\|v-\frac{\alpha}{\beta}\bar{u}\|_{L^2}^2+\|w-\frac{\gamma}{\delta}\bar{u}\|_{L^2}^2 \leq c_6e^{-\lambda_0 t}.$$

Then using Gagliardo-Nirenberg inequality, we can find a constant $\lambda_1 > 0$ such that

$$\|v-\frac{\alpha}{\beta}\bar{u}\|_{L^\infty}+\|w-\frac{\gamma}{\delta}\bar{u}\|_{L^\infty}\le c_7\left(\|v-\frac{\alpha}{\beta}\bar{u}\|_{L^2}^{\frac{1}{2}}+\|w-\frac{\gamma}{\delta}\bar{u}\|_{L^2}^{\frac{1}{2}}\right)\le c_8e^{-\lambda_1 t}.$$

Thus, we complete the proof of this lemma. ∎

Proof. **(Proof of Theorem 1.2.)** Theorem 1.2 is a consequence of the combination of Lemma 3.3 and Lemma 3.4. ∎

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