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LONG TIME EXISTENCE OF THE BOUSSINESQ EQUATION WITH LARGE INITIAL DATA IN $\mathbb{R}^N$ *

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Abstract. In this paper, we study the long-time existence of smooth solutions to the Cauchy problem for the Boussinesq equation with large initial data in $\mathbb{R}^n$. Due to the strong dispersive effect in the Boussinesq equation, the method of combining the blowup criterion and Strichartz estimate are used to show that the lifespan of the solutions can be taken arbitrarily large provided that the dispersive coefficient is large enough.

Keywords. Boussinesq equation; dispersion; blowup criterion; Strichartz estimate.

AMS subject classifications. 35Q35; 35B40; 76D99.

1. Introduction

In this paper, the following Cauchy problem of the Boussinesq equation in $\mathbb{R}^n$ is studied

$$\begin{cases} u_{tt}-\alpha^2\Delta u+\alpha^2\Delta^2 u=\Delta f(u), & (x,t)\in\mathbb{R}^n\times\mathbb{R}^+, \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), \end{cases} \tag{1.1}$$

where the dispersive coefficient α is a constant and $f(u)=\pm u^p$ or $f(u)=\pm u|u|^{p-1}$ is the general nonlinear term with an integer $p\geqslant 2$. The first equation of (1.1) was first derived to describe shallow water waves by Boussinesq [1] in 1872. It also covers other various physical phenomena. For example, it was used to model the nonlinear vibrations along a string in [3, 38] and describe two-dimensional irrotational flows of an inviscid liquid in a rectangular channel in [22].

The Cauchy problem (1.1) has been studied by many mathematicians. Various local existence theories in different spaces were established in [2, 3, 10, 11, 19, 22, 33, 35]. For small initial data, the papers [3, 7, 11, 22, 24, 26] established the global smooth existence of solutions. The asymptotic behavior and scattering of solutions were established in [7, 24–26]. For large initial data, the papers [8, 16, 20, 21, 23, 27–31, 37] studied the blow up and singularity. Later, Liu and Xu [28] and Yang and Guo [37] obtained global weak solutions. Farah and Linares [13] got the global mild solutions and Farah [9], Farah and Ferreira [12] established the scattering of mild solutions. Since the Boussinesq equation actually is a nonlinear hyperbolic equation, as it is well known, it is difficult to obtain the global existence of smooth solutions because of the possible singularities. As far as we know, the global existence of smooth solutions to (1.1) for large initial data is still open.

This paper will make a forward step in this direction to get the long-time existence of smooth solutions for large initial data. More precisely, we shall show that for given initial data $(u_0,u_1)\in H^s\times(H^{s-2}\cap\dot{H}^{-1})$ with $s>\frac{n}{2}+2$ and time T, there exists a positive number $\alpha(u_0,u_1,T)$ such that the Boussinesq equation admits a unique smooth solution

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on the time interval $[0,T]$ provided $\alpha \geq \alpha(u_0,u_1,T)$. This also gives a lower bound for the maximal existence time of solutions to (1.1) in terms of the dispersive coefficient α and initial data.

Since the operator $\frac{\Delta}{|\nabla|\sqrt{1-\Delta}}$ is a zero order Fourier multiplier, there is no difficulty of loss of derivatives of the nonlinear term while using the contractive mapping argument to get the following local existence of solutions uniformly with respect to α.

THEOREM 1.1. *Let $s\in\mathbb{R}$ satisfy $s>\frac{n}{2}(n\geq 1)$ and $X=H^s\times(H^{s-2}\cap\dot{H}^{-1})$. Then for any given $\alpha>1$ and $\overrightarrow{u}_0=(u_0,u_1)\in X$, there exists a $T_0\in(0,\infty)$ uniformly with respect to α such that (1.1) admits a unique smooth solution satisfying*

$$u\in\mathcal{C}([0,T_0];H^s).$$

In order to extend the local solutions to be global ones, we derive the high energy estimate

$$\|u(t)\|_{H^s}^2\leq C(\|u_1\|_{H^{s-2}}^2+\|u_0\|_{H^s}^2)\exp\Big(\int_0^T\|u(\tau)\|_{L^\infty}^{p-1}d\tau\Big).$$

Inspired by the works [17, 32] on the Euler equations in the rotational framework (see also [4–6, 15, 34, 36] for other fluid dynamics models in different settings), we will take full advantage of the dispersive effect in the Boussinesq equation to give the control of the time integration $\int_0^T\|u(\tau)\|_{L^\infty}^{p-1}d\tau$ by means of a bootstrap argument with the help of Strichartz estimate (space-time norm). Eventually, we obtain our main result in the present paper by the continuous argument.

THEOREM 1.2. *Let $s\in\mathbb{R}$ satisfy $s>\frac{n}{2}$ and $X=H^s\times(H^{s-2}\cap\dot{H}^{-1})$. Then for any $T\in(0,\infty)$ and $\overrightarrow{u}_0=(u_0,u_1)\in X$, there exists a positive $\alpha_0=\alpha_0(s,T,\|\overrightarrow{u}_0\|_X)$ such that if $\alpha>\alpha_0$, the Cauchy problem (1.1) possesses a unique smooth solution satisfying*

$$u\in\mathcal{C}([0,T];H^s).$$

In particular, for $q\geq\max\{4,p-1\}$ there exist a positive constant $C_0(q)$ and a positive constant $C_1=C_1(s,q)$ such that the parameter α_0 can be taken so that

$$\alpha_0\geq C_0\Big[1+\|\overrightarrow{u}_0\|_X^{p-1}T\exp\{C_1T^{1-\frac{p-1}{q}}\|\overrightarrow{u}_0\|_X^{p-1}\}\Big]^q. \tag{1.2}$$

REMARK 1.1. Theorem 1.2 shows that the smooth solution to (1.1) uniquely exists on any arbitrary finite time interval $[0,T]$ for large initial data $\overrightarrow{u}_0\in X$ with large dispersive coefficient α.

REMARK 1.2. Suppose that the dispersive coefficient α is fixed, from the characterization of (1.2), the maximal existence time T_α of the solutions to (1.1) has the lower bound

$$T_\alpha\geq\frac{C_1'}{\|\overrightarrow{u}_0\|_X^{p-1}}\ln\Big(\frac{\alpha}{C_0'}\Big),$$

with some positive constants $C_0'(q)$ and $C_1'(s,q)$.

The outline of this paper is as follows. Section 2 is to prove the local existence. Section 3 aims to derive a Strichartz estimate for the dispersive operator and Section 4 is to establish the blowup criterion for the Boussinesq equation. In Section 5, we establish the long-time existence of smooth solutions.

Throughout this paper, the letter C denotes arbitrary constants which may differ from line to line. In particular, $C=C(\cdot,\cdot,\cdot)$ will denote the constant which depends only on the quantities appearing in parenthesis. The symbols $L^p(1\le p\le\infty)$, $H^s(s\in\mathbb{R})$ and $\dot{H}^s(s\in\mathbb{R})$ denote the Lebesgue spaces, inhomogeneous and homogeneous Sobolev spaces, respectively. The Fourier transform and its inverse are denoted by $\mathcal{F},\mathcal{F}^{-1}$ or $\hat{\cdot},\check{\cdot}$, respectively.

2. Local existence

Now we derive the explicit representation of the solutions by the method of the Green function. Consider the fundamental solution to the Cauchy problem

$$\begin{cases} \partial_{tt}G-\alpha^2\Delta G+\alpha^2\Delta^2 G=0, \\ G(x,0)=0, \partial_t G(x,0)=\delta. \end{cases} \tag{2.1}$$

Taking the Fourier transform in (2.1), we get

$$\begin{cases} \partial_{tt}\hat{G}+\alpha^2|\xi|^2\hat{G}+\alpha^2|\xi|^4\hat{G}=0. \\ \hat{G}(\xi,0)=0, \partial_t\hat{G}(\xi,0)=1. \end{cases} \tag{2.2}$$

The symbol of the Equation (2.2) is

$$\tau^2+\alpha^2|\xi|^2+\alpha^2|\xi|^4=0.$$

By a direct calculation, we obtain

$$\tau=\pm i\alpha p(|\xi|), \quad p(|\xi|)=|\xi|\sqrt{1+|\xi|^2}, \tag{2.3}$$

which reflects the dispersive relation associated to the first equation of (1.1). Then solving the Cauchy problem (2.2), we get

$$\hat{G}(\xi,t)=\frac{\sin(\alpha p(|\xi|)t)}{\alpha p(|\xi|)}=\frac{1}{2\alpha i p(|\xi|)}(e^{i\alpha p(|\xi|)t}-e^{-i\alpha p(|\xi|)t}),$$
$$\partial_t\hat{G}(\xi,t)=\cos(\alpha p(|\xi|)t)=\frac{1}{2}(e^{i\alpha p(|\xi|)t}+e^{-i\alpha p(|\xi|)t}).$$

By the Duhamel principle, we obtain the solutions of Cauchy problem (1.1) in the integral form as follows

$$u(x,t)=\partial_t G(x,t)*u_0+G(x,t)*u_1+\int_0^t G(x,t-\tau)*\Delta f(u)d\tau. \tag{2.4}$$

In what follows, we will use the symbols $|\nabla|^s, g(|\nabla|)$ defined by $\mathcal{F}(|\nabla|^s f)(\xi)=|\xi|^s\mathcal{F}(f)(\xi)$ and $\mathcal{F}(g(|\nabla|)f)(\xi)=g(|\xi|)\mathcal{F}(f)(\xi)$, respectively. We first recall some useful results in order to control the nonlinear term, see [7, 26].

LEMMA 2.1.

(i) *For any* $s\in\mathbb{R}$, $r_1\in(1,\infty], r_2\in(1,\infty)$, $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$, *then we have*

$$\|\nabla^s f(u)\|_{L^r}\le C\|u\|^{p-1}_{L^{(p-1)r_1}}\|\nabla^s u\|_{L^{r_2}}.$$

(ii) *If* $u,v\in H^s\cap L^\infty$ *and* $\|u\|_{L^\infty}\leqslant M, \|v\|_{L^\infty}\leqslant M$, *then*

$$\|f(u)-f(v)\|_{H^s}\le C\Big[\|u-v\|_{L^\infty}(\|u\|_{H^s}+\|v\|_{H^s})(\|u\|^{p-2}_{L^\infty}+\|v\|^{p-2}_{L^\infty}) \\ +\|u-v\|_{H^s}(\|u\|^{p-1}_{L^\infty}+\|v\|^{p-1}_{L^\infty})\Big],$$

where C *depends only on* M.

Then with the help of Lemma 2.1, we can establish the local existence by means of the contractive mapping argument.

Proof. **(The proof of Theorem 1.1.)** To begin with, we construct a suitable metric space

$$Y := \{u(t) \in \mathcal{C}(0,T_0;H^s), \|u\|_{L^\infty(0,T_0;H^s)} \leq 2\|\overrightarrow{u}_0\|\},$$

with the natural metric $d(u,v) = \|u-v\|_{L^\infty(0,T_0;H^s)}$. Then the metric space (Y,d) is complete, which can be proved by using the standard way, one can refer to [7].

According to the representation (2.4) of solutions to (1.1), we introduce the mapping N as follows

$$N(u)(x,t) = \partial_t G(x,t) * u_0 + G(x,t) * u_1 + \int_0^t G(x,t-\tau) * \Delta f(u) d\tau.$$

Next we are going to prove that the mapping $N(u): Y \mapsto Y$ is a contractive one.

Step one: We show $N: Y \mapsto Y$. For any $u \in Y$, by the Plancherel identity, $\alpha > 1$, Lemma 2.1 and the embedding $H^s \hookrightarrow L^\infty$ for $s > \frac{n}{2}$, we have

$$\begin{aligned}
\|N(u)\|_{H^s} &\leq \|\partial_t G(x,t) * u_0\|_{H^s} + \|G(x,t) * u_1\|_{H^s} + \int_0^t \|G(x,t-\tau) * \Delta f(u)\|_{H^s} ds \\
&= \|(1+|\xi|^2)^{s/2} \cos(\alpha p(|\xi|)t)\hat{u}_0\|_{L^2} + \alpha^{-1}\|(1+|\xi|^2)^{s/2} \frac{\sin(\alpha p(|\xi|)t)}{p(|\xi|)} \hat{u}_1\|_{L^2} \\
&\quad + \alpha^{-1} \int_0^t \|(1+|\xi|^2)^{s/2} \frac{\sin(\alpha p(|\xi|)t)}{p(|\xi|)} |\xi|^2 \mathcal{F}(f(u))\|_{L^2} ds \\
&\leq \|u_0\|_{H^s} + \|\frac{1}{|\nabla|} u_1\|_{H^{s-1}} + \int_0^t \|f(u)\|_{H^s} ds \\
&\leq \|\overrightarrow{u}_0\|_X + C\int_0^T \|u\|_{L^\infty}^{p-1} \|u\|_{H^s} ds \\
&\leq \|\overrightarrow{u}_0\|_X + C\int_0^T \|u\|_{H^s}^p ds.
\end{aligned} \tag{2.5}$$

It follows from (2.5) and $u \in Y$ that

$$\begin{aligned}
\|N(u)\|_{L^\infty(0,T_0;H^s)} &\leq \|\overrightarrow{u}_0\|_X + C\int_0^{T_0} \|u\|_{L^\infty(0,T_0;H^s)}^p ds \leq \|\overrightarrow{u}_0\|_X + CT_0 \|u\|_{L^\infty(0,T_0;H^s)}^p \\
&\leq \|\overrightarrow{u}_0\|_X + 2^p CT_0 \|\overrightarrow{u}_0\|_X.
\end{aligned} \tag{2.6}$$

By choosing T_0 such that $2^p CT_0 < 1$, we get from (2.6) that

$$\|N(u)\|_{L^\infty(0,T_0;H^s)} \leq 2\|\overrightarrow{u}_0\|_X,$$

which implies $N: Y \mapsto Y$.

Step two: We verify that the mapping N is contractive. For any $u,v \in Y$, it follows from the Plancherel identity and Lemma 2.1 that

$$\begin{aligned}
&\|N(u) - N(v)\|_{H^s} \\
&\leq \int_0^t \|G(\cdot,t-\tau) * \Delta(f(u) - f(v))\|_{H^s} d\tau \leq \int_0^t \|f(u) - f(v)\|_{H^s} d\tau
\end{aligned}$$

$$\leq C\int_0^t [\|u-v\|_{L^\infty}(\|u\|_{H^s}+\|v\|_{H^s})(\|u\|_{L^\infty}^{p-2}+\|v\|_{L^\infty}^{p-2})+\|u-v\|_{H^s}(\|u\|_{L^\infty}^{p-1}+\|v\|_{L^\infty}^{p-1})]ds. \tag{2.7}$$

From (2.7), the embedding $H^s \hookrightarrow L^\infty$ for $s>\frac{n}{2}$ and the fact $u,v\in Y$, we obtain

$$\begin{aligned} d(N(u),N(v)) = \|N(u)-N(v)\|_{L^\infty(0,T_0;H^s)} &\leq C2^{p+1}\|\overrightarrow{u}_0\|_X^{p-1}\int_0^{T_0}\|u-v\|_{L^\infty(0,T_0;H^s)}ds \\ &\leq CT_0 2^{p+1}\|\overrightarrow{u}_0\|_X^{p-1} d(u,v), \end{aligned}$$

which implies that, by choosing T_0 such that $CT_0 2^{p+1}\|\overrightarrow{u}_0\|_X^{p-1}<1$, the mapping N is contractive in the complete metric space (Y,d).

To summarize, it follows from Step one and Step two that, by choosing T_0 independently of α and small enough, the existence and uniqueness of solutions $u\in\mathcal{C}(0,T_0;H^s)$ to the Cauchy problem (1.1) are obtained directly by the contractive mapping principle. □

3. Strichartz estimate

We first recall the definition of the Littlewood-Paley decomposition. The Littlewood-Paley multipliers $\Delta_{j\in\mathbb{Z}}$ are defined by

$$\Delta_j f = \mathcal{F}^{-1}\big(\psi(\frac{\xi}{2^j})\hat{f}\big)$$

where $\psi\in\mathcal{C}_0^\infty(\mathbb{R}^n), supp\psi\subset\{\xi\in\mathbb{R}^n|1/2\leq|\xi|\leq 2\}$ such that

$$\forall \xi\neq 0,\quad \sum_{j\in\mathbb{Z}}\psi(\frac{\xi}{2^j})=1.$$

The low frequency multiplier $\chi(\nabla)$ is defined by

$$\chi(\nabla)f=\mathcal{F}^{-1}[(1-\sum_{j\geq 1}\psi(\frac{\xi}{2^j}))\hat{f}].$$

Then, we recall the definition of the inhomogeneous Besov space.

Definition 3.1. *For $s\in\mathbb{R}$ and $1\leq p,q\leq\infty$, the inhomogeneous Besov space $B_{p,q}^s(\mathbb{R}^n)$ is defined to be the set of all tempered distributions such that the following norm is finite*

$$\|f\|_{B_{p,q}^s}=\|\chi(\nabla)f\|_{L^p}+\|\{2^{sj}\|\Delta_j f\|_{L^p}\}_{j=1}^\infty\|_{l^q}.$$

According to the representations of $G(x,t)$ and $\partial_t G(x,t)$ in the equalities above (2.4), it is necessary to study the dispersive property of the linear propagator $e^{\pm it\alpha p(|\nabla|)}$, which was obtained by Liu and Wang in [26] in the case of $\alpha=1$.

Lemma 3.1. *There exists a positive constant C independent of $(t,x)\in\mathbb{R}^{1+n}$ such that*

$$\Big|\int_{\mathbb{R}^n} e^{ix\xi\pm itp(|\xi|)}\Phi(\xi)d\xi\Big|\leq C(1+|t|)^{-\frac{n}{2}},$$

where $\Phi(\xi)$ is a Schwartz function satisfying supp $\Phi\subset\{\xi\in\mathbb{R}^n|2^{-2}\leq|\xi|\leq 2^2\}$ and $\Phi=1$ on $\{\xi\in\mathbb{R}^n|2^{-1}\leq|\xi|\leq 2\}$.

As an immediate consequence of Lemma 3.1, we have the following lemma.

Lemma 3.2. *For all $f\in L^1(\mathbb{R}^n)$, it holds that*

$$\|e^{\pm itp(|\nabla|)}\Delta_0 f\|_{L^\infty}\le C(1+|t|)^{-\frac{n}{2}}\|\Delta_0 f\|_{L^1}.$$

Proof. Since $\Phi(\xi)=1$ on the support of ψ, we have

$$e^{\pm itp(|\nabla|)}\Delta_0 f(x)=\int_{\mathbb{R}^n} e^{ix\xi\pm itp(|\xi|)}\Phi(\xi)\psi(\xi)\hat{f}d\xi=\mathcal{F}^{-1}(e^{\pm itp(|\xi|)}\Phi(\xi)\psi(\xi)\hat{f}).$$

By the Young inequality, the Hausdorff-Young inequality and Lemma 3.1, we have

$$\begin{aligned}\|e^{\pm itp(|\nabla|)}\Delta_0 f(x)\|_{L^\infty}\le& C\|\mathcal{F}^{-1}(e^{\pm itp(|\xi|)}\Phi(\xi))\|_{L^\infty}\|\mathcal{F}^{-1}(\psi(\xi)\hat{f})\|_{L^1}\\ \le& C(1+|t|)^{-\frac{n}{2}}\|\Delta_0 f\|_{L^1}.\end{aligned}$$

Thus, we complete the proof of Lemma 3.2. □

With the help of the dispersive estimate of the linear propagator $e^{\pm itp(|\nabla|)}$, namely Lemma 3.2, we can derive a suitable Strichartz-type estimate for the linear propagator $e^{\pm it\alpha p(|\nabla|)}$. Firstly, we recall the following classical result obtained by Keel and Tao [18].

Lemma 3.3. *Let $S(t),t\in\mathbb{R}$, be a family of operators. Suppose that, for all $t,s\in\mathbb{R}$,*

$$\begin{aligned}\|S(s)S^*(t)f(x)\|_{L^\infty}&\le(1+|t-s|)^{-\sigma}\|f\|_{L^1},\\ \|S(s)S^*(t)f(x)\|_{L^2}&\le\|f\|_{L^2}.\end{aligned}$$

Then the estimate

$$\|S(t)f(x)\|_{L^q_tL^r_x}\le\|f\|_{L^2},$$

holds for all $2\le q,r\le\infty$ with $(q,r,\sigma)\ne(2,\infty,1)$ satisfying

$$\frac{1}{q}+\frac{\sigma}{r}\le\frac{\sigma}{2}.$$

Now we can obtain the following lemma.

Lemma 3.4. *For $4\le q\le\infty$ and $2\le r\le\infty$ satisfying*

$$\frac{1}{q}+\frac{n}{2r}\le\frac{n}{4},$$

we have the Strichartz estimate

$$\|\Delta_j e^{\pm it\alpha p(|\nabla|)}f\|_{L^q_tL^r_x}\le C\alpha^{-1/q}(2^j)^{n(1/2-1/r)}\|\Delta_j f\|_{L^2}.$$

Proof. By Lemma 3.2, the Plancherel's theorem and Lemma 3.3, we obtain

$$\|e^{\pm itp(|\nabla|)}\Delta_0 f\|_{L^q_tL^r_x}\le C\|\Delta_0 f\|_{L^2}. \tag{3.1}$$

By the change of variable $\xi\mapsto 2^j\xi$, we have for $j\in\mathbb{Z}$

$$e^{\pm itp(|\nabla|)}\Delta_j f=\int_{\mathbb{R}^n} e^{i2^jx\xi\pm itp(|\xi|)}\psi(\xi)\mathcal{F}[f(\frac{\cdot}{2^j})](\xi)d\xi=e^{\pm itp(|\nabla|)}\Delta_0[f(\frac{\cdot}{2^j})](2^jx). \tag{3.2}$$

Hence by (3.1) and (3.2), we have

$$\|\Delta_j e^{\pm itp(|\nabla|)}f(x)\|_{L^q_tL^r_x}\le C(2^j)^{-n/r}\|\Delta_0[f(\frac{\cdot}{2^j})]\|_{L^2}=C(2^j)^{n(1/2-1/r)}\|\Delta_j f\|_{L^2}. \tag{3.3}$$

The scaling in time $t\mapsto\alpha t$ and (3.3) give the desired estimate. □

4. Blowup criterion

In order to extend the local solutions to be global ones, we derive the blowup criterion by the classical energy method.

LEMMA 4.1. *Let $\alpha>1$ and $u(t)$ be a solution of (1.1) defined on a time interval containing $[0,T]$. Then for any $s>2$ we have the estimate*

$$\|u(t)\|_{H^s}^2 \leq C(\|u_1\|_{H^{s-2}}^2+\|u_0\|_{H^s}^2)\exp\Big(\int_0^T \|u(\tau)\|_{L^\infty}^{p-1}d\tau\Big).$$

Proof. We apply the operator ∇^{s-2} on the first equation of (1.1), then multiply the resulting equation by $\nabla^{s-2}u_t$ and integrate to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\big(|\partial_t\nabla^{s-2}u|^2+\alpha^2|\nabla^{s-2}\nabla u|^2+\alpha^2|\nabla^{s-2}\Delta u|^2\big)\,dx=\int_{\mathbb{R}^n}\nabla^{s-2}(\Delta f(u))\nabla^{s-2}u_t dx. \tag{4.1}$$

By the Hölder inequality, Lemma 2.1 and the Cauchy inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\nabla^{s-2}(\Delta f(u))\nabla^{s-2}u_t dx &\leq C\|\nabla^{s-2}(\Delta f(u))\|_{L^2}\|\nabla^{s-2}u_t\|_{L^2}\\ &\leq C\|\nabla^s f(u)\|_{L^2}\|\nabla^{s-2}u_t\|_{L^2}\\ &\leq C\|u\|_{L^\infty}^{p-1}\|\nabla^s u\|_{L^2}\|\nabla^{s-2}u_t\|_{L^2}\\ &\leq C\|u\|_{L^\infty}^{p-1}\big(\|\nabla^s u\|_{L^2}^2+\|\nabla^{s-2}u_t\|_{L^2}^2\big).\end{aligned} \tag{4.2}$$

It follows from (4.1)-(4.2) and $\alpha>1$ that

$$\begin{aligned}&\frac{d}{dt}(\|\partial_t u\|_{H^{s-2}}^2+\alpha^2\|u\|_{H^{s-1}}^2+\alpha^2\|u\|_{H^s}^2)\\ \leq& C\|u\|_{L^\infty}^{p-1}\big(\|\partial_t u\|_{H^{s-2}}^2+\alpha^2\|u\|_{H^{s-1}}^2+\alpha^2\|u\|_{H^s}^2\big).\end{aligned} \tag{4.3}$$

Applying the Gronwall inequality to (4.3), we obtain

$$\|\partial_t u\|_{H^{s-2}}^2+\alpha^2\|u\|_{H^{s-1}}^2+\alpha^2\|u\|_{H^s}^2\leq C\big(\|u_1\|_{H^{s-2}}^2+\alpha^2\|u_0\|_{H^s}^2\big)\exp\left(\int_0^t\|u(\tau)\|_{L^\infty}^{p-1}d\tau\right),$$

which implies

$$\|u\|_{H^s}^2\leq C(\|u_1\|_{H^{s-2}}^2+\|u_0\|_{H^s}^2)\exp(\int_0^t\|u(\tau)\|_{L^\infty}^{p-1}d\tau).$$

□

5. Long-time existence

To end this paper, we give the proof of Theorem 1.2 in this section. We first recall an important estimate on the Littlewood-Paley multipliers, namely Bernstein inequality (see [14])

$$\|\nabla^s\Delta_j f\|_{L^l}\leq C2^{js}\|\Delta_j f\|_{L^l}\leq C2^{js}\|f\|_{L^l},$$

for all $s\geq 0$, and all $1\leq l\leq\infty$.

Proof. **(The proof of Theorem 1.2.)** By Lemma 3.4, we find that for $q\geq 4$,

$$\|\Delta_j e^{\pm i\alpha t p(|\nabla|)} f\|_{L_t^q L_x^\infty} \leq C\alpha^{-\frac{1}{q}}(2^j)^{\frac{n}{2}}\|\Delta_j f\|_{L^2}. \tag{5.1}$$

In the following, for $\beta>0$, we are going to derive an estimate for

$$\|u\|_{L_t^q B^\beta_{\infty,\infty}} = \left\| \|\chi(\nabla)u\|_{L_x^\infty} + \sup_{j\geq 1}\left(2^{j\beta}\|\Delta_j u\|_{L_x^\infty}\right)\right\|_{L_t^q}.$$

By (5.1) and the Bernstein inequality, we have for $j\geq 1$

$$\left\|2^{j\beta}\|\Delta_j e^{\pm i\alpha p(|\nabla|)t}u_0\|_{L_x^\infty}\right\|_{L_t^q} \leq C\alpha^{-\frac{1}{q}}2^{j(\frac{n}{2}+\beta)}\|\Delta_j u_0\|_{L^2} \leq C\alpha^{-\frac{1}{q}}\|u_0\|_{H^{\frac{n}{2}+\beta}}. \tag{5.2}$$

Similarly,

$$\begin{aligned}\left\|2^{j\beta}\|\Delta_j e^{\pm i\alpha p(|\nabla|)t}\frac{1}{\alpha|\nabla|\sqrt{1+|\nabla|^2}}u_1\|_{L_x^\infty}\right\|_{L_t^q} &\leq C\alpha^{-1-\frac{1}{q}}2^{j(\frac{n}{2}+\beta)}\|\Delta_j u_1\|_{L^2}\\ &\leq C\alpha^{-1-\frac{1}{q}}\|u_1\|_{H^{\frac{n}{2}+\beta}}.\end{aligned} \tag{5.3}$$

On the other hand, we have from the Minkowski inequality, (5.1) and the Cauchy inequality that

$$\begin{aligned}\|\chi(\nabla)e^{\pm i\alpha p(|\nabla|)t}u_0\|_{L_t^q L_x^\infty} &\leq C\left\|\sum_{j=-\infty}^{2}\|\Delta_j e^{\pm i\alpha p(|\nabla|)t}u_0\|_{L_x^\infty}\right\|_{L_t^q}\\ &\leq C\sum_{j=-\infty}^{2}\|\Delta_j e^{\pm i\alpha p(|\nabla|)t}u_0\|_{L_t^q L_x^\infty}\\ &\leq C\alpha^{-\frac{1}{q}}\sum_{j=-\infty}^{2}2^{\frac{n}{2}j}\|\Delta_j u_0\|_{L^2}\\ &\leq C\alpha^{-\frac{1}{q}}(\sum_{j=-\infty}^{2}2^{nj})^{\frac{1}{2}}(\sum_{j=-\infty}^{2}\|\Delta_j u_0\|_{L^2}^2)^{\frac{1}{2}} \leq C\alpha^{-\frac{1}{q}}\|u_0\|_{L^2}.\end{aligned} \tag{5.4}$$

Similarly,

$$\begin{aligned}\|\chi(\nabla)e^{\pm i\alpha p(|\nabla|)t}\frac{1}{\alpha|\nabla|\sqrt{1+|\nabla|^2}}u_1\|_{L_t^q L_x^\infty} &\leq C\alpha^{-1-\frac{1}{q}}(\sum_{j=-\infty}^{2}2^{nj})^{\frac{1}{2}}(\sum_{j=-\infty}^{2}\|\Delta_j\frac{1}{|\nabla|}u_1\|_{L^2}^2)^{\frac{1}{2}}\\ &\leq C\alpha^{-1-\frac{1}{q}}\|u_1\|_{\dot{H}^{-1}}.\end{aligned} \tag{5.5}$$

Thus (5.2)-(5.5) and $\alpha>1$ give

$$\|\partial_t G(x,t)*u_0 + G(x,t)*u_1\|_{L_t^q B^\beta_{\infty,\infty}} \leq C\alpha^{-\frac{1}{q}}\|\vec{u}_0\|_X. \tag{5.6}$$

By the Minkowski inequality and (4.1), the property of zero order pseudo-differential operator, Lemma 2.1 and the embedding $H^{\frac{n}{2}+\beta}\hookrightarrow L^\infty$, we have, for $t\in[0,T]$,

$$\left\|2^{j\beta}\|\Delta_j\int_0^t e^{\pm i\alpha p(|\nabla|)(t-\tau)}\frac{\Delta}{\alpha|\nabla|\sqrt{1+|\nabla|^2}}f(u)d\tau\|_{L_x^\infty}\right\|_{L_t^q}$$

$$
\begin{aligned}
&\le \alpha^{-1}\int_0^T \Big\|2^{j\beta}\|\Delta_j e^{\pm i\alpha p(|\nabla|)(t-\tau)}\frac{\Delta}{|\nabla|\sqrt{1+|\nabla|^2}}f(u)\|_{L^\infty}\Big\|_{L^q_t}d\tau\\
&\le C\alpha^{-1}\int_0^T 2^{j(\frac{n}{2}+\beta)}\alpha^{-\frac{1}{q}}\|\Delta_j e^{\mp ip(|\nabla|)\tau}\frac{\Delta}{|\nabla|\sqrt{1+|\nabla|^2}}f(u)(\tau)\|_{L^2}d\tau\\
&\le C\alpha^{-1-\frac{1}{q}}\int_0^T \|f(u)(\tau)\|_{H^{\frac{n}{2}+\beta}}d\tau \le C\alpha^{-1-\frac{1}{q}}\int_0^T \|u(\tau)\|^{p-1}_{L^\infty}\|u(\tau)\|_{H^{\frac{n}{2}+\beta}}d\tau\\
&\le C\alpha^{-1-\frac{1}{q}}\int_0^T \|u(\tau)\|^p_{H^{\frac{n}{2}+\beta}}d\tau.
\end{aligned}
\tag{5.7}
$$

By the Minkowski inequality, Lemma 2.1, the Cauchy inequality and the property of zero order pseudo-differential operator, we have, for $t\in[0,T]$,

$$
\begin{aligned}
&\|\chi(\nabla)\int_0^t e^{\pm i\alpha p(|\nabla|)(t-\tau)}\frac{\Delta}{\alpha|\nabla|\sqrt{1+|\nabla|^2}}f(u)d\tau\|_{L^q_tL^\infty_x}\\
&\le C\sum_{j=-\infty}^{2}\int_0^T \|\Delta_j e^{\pm i\alpha p(|\nabla|)(t-\tau)}\frac{\Delta}{\alpha|\nabla|\sqrt{1+|\nabla|^2}}f(u)\|_{L^q_tL^\infty_x}d\tau\\
&\le C\alpha^{-1-\frac{1}{q}}\int_0^T\sum_{j=-\infty}^{2}(2^j)^{\frac{n}{2}}\|\Delta_j e^{\mp i\alpha p(|\nabla|)\tau}\frac{\Delta}{|\nabla|\sqrt{1+|\nabla|^2}}f(u)(\tau)\|_{L^2}d\tau\\
&\le C\alpha^{-1-\frac{1}{q}}\int_0^T \|f(u)(\tau)\|_{L^2}d\tau\\
&\le C\alpha^{-1-\frac{1}{q}}\int_0^T \|u\|^p_{H^{\frac{n}{2}+\beta}}d\tau.
\end{aligned}
\tag{5.8}
$$

Estimates (5.7) and (5.8) give, for $t\in[0,T]$,

$$
\Big\|\int_0^t e^{\pm i\alpha p(|\nabla|)(t-\tau)}\frac{\Delta}{\alpha|\nabla|\sqrt{1+|\nabla|^2}}f(u)\Big\|_{L^q_tB^\beta_{\infty,\infty}} \le C\alpha^{-1-\frac{1}{q}}\int_0^T \|u(\tau)\|^p_{H^{\frac{n}{2}+\beta}}d\tau. \tag{5.9}
$$

Combining (2.4), (5.6) and (5.9) yields, for $t\in[0,T]$,

$$
\|u\|_{L^q_tB^\beta_{\infty,\infty}} \le C\Big(\alpha^{-\frac{1}{q}}\|\vec{u}_0\|_X+\alpha^{-1-\frac{1}{q}}\int_0^T \|u(\tau)\|^p_{H^{\frac{n}{2}+\beta}}d\tau\Big). \tag{5.10}
$$

Define

$$
M(T)=\int_0^T \|u\|^{p-1}_{L^\infty}dt.
$$

We get from the Young inequality, the embedding $B^\beta_{\infty,\infty}\hookrightarrow L^\infty$, (5.10) and Lemma 4.1 that

$$
\begin{aligned}
M(T)&\le T^{1-\frac{p-1}{q}}\Big(\int_0^T \|u\|^{p-1\cdot\frac{q}{p-1}}_{L^\infty}dt\Big)^{\frac{p-1}{q}} = T^{1-\frac{p-1}{q}}\|u\|^{p-1}_{L^q_tL^\infty_x}\\
&\le CT^{1-\frac{p-1}{q}}\Big(\alpha^{-\frac{1}{q}}\|\vec{u}_0\|_X+\alpha^{-1-\frac{1}{q}}\int_0^T \|u(t)\|^p_{H^{\frac{n}{2}+\beta}}dt\Big)^{p-1}\\
&\le CT^{1-\frac{p-1}{q}}\Big(\alpha^{-\frac{1}{q}}\|\vec{u}_0\|_X+\alpha^{-1-\frac{1}{q}}\int_0^T \|\vec{u}_0\|^p_X e^{\frac{p}{2}M(t)}dt\Big)^{p-1}
\end{aligned}
$$

$$\leq CT^{1-\frac{p-1}{q}}\left(\alpha^{-\frac{1}{q}}\|\overrightarrow{u}_0\|_X+\alpha^{-1-\frac{1}{q}}\|\overrightarrow{u}_0\|_X^p Te^{\frac{p}{2}M(t)}\right)^{p-1}$$

$$\leq C_1T^{1-\frac{p-1}{q}}\alpha^{-\frac{p-1}{q}}\|\overrightarrow{u}_0\|_X^{p-1}\left(1+\|\overrightarrow{u}_0\|_X^{p-1}Te^{\frac{p}{2}M(t)}\right)^{p-1}. \tag{5.11}$$

Suppose that

$$M(T)\leq C_1T^{1-\frac{p-1}{q}}\|\overrightarrow{u}_0\|_X^{p-1}. \tag{5.12}$$

By (5.12), we can choose $\alpha>1$ sufficiently large such that

$$\alpha^{-\frac{p-1}{q}}\left(1+\|\overrightarrow{u}_0\|_X^{p-1}Te^{\frac{p}{2}M(T)}\right)^{p-1}\leq\alpha^{-\frac{p-1}{q}}\left(1+\|\overrightarrow{u}_0\|_X^{p-1}Te^{C_1T^{1-\frac{p-1}{q}}\|\overrightarrow{u}_0\|_X^{p-1}}\right)^{p-1}\leq\frac{1}{2}. \tag{5.13}$$

It follows from (5.11) and (5.13) that

$$M(T)\leq\frac{1}{2}C_1T^{1-\frac{p-1}{q}}\|\overrightarrow{u}_0\|_X^{p-1}. \tag{5.14}$$

By the bootstrap principle, we deduce from (5.14) that (5.12) actually holds. Thus from the local existence Theorem 1.1 and the blowup criterion Lemma 4.1, the Cauchy problem (1.1) admits a unique smooth solution satisfying $u\in\mathcal{C}([0,T];H^s)$. ∎

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