COMMUN. MATH. SCI.
Vol. 19, No. 5, pp. 1269–1300

EXISTENCE OF GLOBAL WEAK SOLUTIONS TO THE COMPRESSIBLE ERICKSEN-LESLIE SYSTEM IN DIMENSION ONE*

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Abstract. We consider the compressible Ericksen-Leslie system of liquid crystal flows in one dimension. A global weak solution is constructed with initial density $\rho_0 \geq 0$ and $\rho_0 \in L^\gamma$ for $\gamma > 1$.

Keywords. Compressible Ericksen-Leslie system; weak solution; nematic liquid crystals.

AMS subject classifications. 35K55; 35D3.

1. Introduction

Nematic liquid crystals are composed of rod-like molecules characterized by average alignment of the long axes of neighboring molecules, which have the simplest structures among various types of liquid crystals. The dynamic theory of nematic liquid crystals had been first proposed by Ericksen [5] and Leslie [15] in the 1960's, which is a macroscopic continuum description of the time evolution of both flow velocity field and orientation order parameter of rod-like liquid crystals.

In this paper, we will study the compressible Ericksen-Leslie system of liquid crystal flows (see [1,20] for modeling). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. The compressible Ericksen-Leslie system is given as follows

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \rho \dot{\mathbf{u}} + \nabla P = \nabla \cdot \sigma - \nabla \cdot \left(\frac{\partial W}{\partial \nabla \mathbf{n}} \otimes \nabla \mathbf{n} \right), \\ \mathbf{g} + \frac{\partial W}{\partial \mathbf{n}} - \nabla \cdot \left(\frac{\partial W}{\partial \nabla \mathbf{n}} \right) = \lambda \mathbf{n}. \end{cases} \tag{1.1}$$

Here, $\rho(\mathbf{x},t): \Omega \times (0,\infty) \to \mathbb{R}$ is the density, $\mathbf{u}(\mathbf{x},t): \Omega \times (0,\infty) \to \mathbb{R}^3$ is the fluid velocity field, $\mathbf{n}(\mathbf{x},t): \Omega \times (0,\infty) \to \mathbb{S}^2$ is the orientation order parameter of nematic material. λ is the Lagrangian multiplier of the constraint $|\mathbf{n}| = 1$, $\dot{f} = f_t + \mathbf{u} \cdot \nabla f$ is the material derivative of function f, and $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T$ for column vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$.

The macrostructure of the crystals has been determined by the Oseen-Frank energy density (cf. [9,21]). One may take the Oseen-Frank energy density in the compressible case as

$$\begin{aligned} 2W(\rho, \mathbf{n}, \nabla \mathbf{n}) = & \frac{2}{\gamma - 1} \rho^\gamma + K_1 (\operatorname{div} \mathbf{n})^2 + K_2 (\mathbf{n} \cdot \operatorname{curl} \mathbf{n})^2 + K_3 |\mathbf{n} \times \operatorname{curl} \mathbf{n}|^2 \\ & + (K_2 + K_4)[\operatorname{tr}(\nabla \mathbf{n})^2 - (\operatorname{div} \mathbf{n})^2], \end{aligned} \tag{1.2}$$

where $\gamma > 1$, and K_j, $j = 1,2,3$, are the positive constants representing splay, twist, and bend effects respectively, with $K_2 \geq |K_4|$, $2K_1 \geq K_2 + K_4$. Then the pressure can be

*Received: June 16, 2020; Accepted (in revised form): January 03, 2021. Communicated by Chun Liu.

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given by the Maxwell relation

$$P(\rho)=\rho W_\rho(\rho,\mathbf{n},\nabla\mathbf{n})-W(\rho,\mathbf{n},\nabla\mathbf{n}).$$

For simplicity, we only consider the case $K_1=K_2=K_3=1,\ K_4=0$ in this paper. The Oseen-Frank energy in the compressible case becomes

$$2W(\rho,\mathbf{n},\nabla\mathbf{n})=\frac{2}{\gamma-1}\rho^\gamma+|\nabla\mathbf{n}|^2.$$

Therefore

$$\nabla\cdot\left(\frac{\partial W}{\partial\nabla\mathbf{n}}\otimes\nabla\mathbf{n}\right)=\nabla\cdot(\nabla\mathbf{n}\odot\nabla\mathbf{n}),\quad \frac{\partial W}{\partial\mathbf{n}}=0,\quad \nabla\cdot\left(\frac{\partial W}{\partial\nabla\mathbf{n}}\right)=\Delta\mathbf{n},\quad P=\rho^\gamma-\frac{1}{2}|\nabla n|^2.$$

Let

$$D=\frac{1}{2}(\nabla\mathbf{u}+\nabla^T\mathbf{u}),\quad \omega=\frac{1}{2}(\nabla\mathbf{u}-\nabla^T\mathbf{u})=\frac{1}{2}\left(\frac{\partial u^i}{\partial x_j}-\frac{\partial u^j}{\partial x_i}\right),\quad N=\dot{\mathbf{n}}-\omega\mathbf{n},$$

represent the rate of strain tensor, skew-symmetric part of the strain rate, and the rigid rotation part of direction-changing rate by fluid vorticity, respectively. The kinematic transport $\mathbf{g}$ is given by

$$\mathbf{g}=\gamma_1 N+\gamma_2 D\mathbf{n}-\gamma_2(\mathbf{n}^T D\mathbf{n})\mathbf{n} \tag{1.3}$$

which represents the effect of the macroscopic flow field on the microscopic structure. The material coefficients γ_1 and γ_2 reflect the molecular shape and the slippery part between fluid and particles. The first term of $\mathbf{g}$ represents the rigid rotation of molecules, while the second term stands for the stretching of molecules by the flow. The viscous (Leslie) stress tensor σ has the following form (cf. [1,16])

$$\begin{aligned}\sigma=&\alpha_0(\mathbf{n}^T D\mathbf{n})\mathbb{I}+\alpha_1(\mathbf{n}^T D\mathbf{n})\mathbf{n}\otimes\mathbf{n}+\alpha_2 N\otimes\mathbf{n}+\alpha_3\mathbf{n}\otimes N\\ &+\alpha_4 D+\alpha_5(D\mathbf{n})\otimes\mathbf{n}+\alpha_6\mathbf{n}\otimes(D\mathbf{n})+\alpha_7(\operatorname{tr}D)\mathbb{I}+\alpha_8(\operatorname{tr}D)\mathbf{n}\otimes\mathbf{n}.\end{aligned} \tag{1.4}$$

These coefficients α_j $(0\le j\le 8)$, depending on material and temperature, are called Leslie coefficients. The following relations are often assumed in the literature:

$$\gamma_1=\alpha_3-\alpha_2,\quad \gamma_2=\alpha_6-\alpha_5,\quad \alpha_2+\alpha_3=\alpha_6-\alpha_5. \tag{1.5}$$

The first two relations are compatibility conditions, while the third relation is called Parodi's relation, derived from Onsager reciprocal relations expressing the equality of certain relations between flows and forces in thermodynamic systems out of equilibrium (cf. [22]). They also satisfy the following empirical relations (cf. [1,16])

$$\begin{aligned}&\alpha_4>0,\quad 2\alpha_1+3\alpha_4+2\alpha_5+2\alpha_6>0,\quad \gamma_1=\alpha_3-\alpha_2>0,\\ &2\alpha_4+\alpha_5+\alpha_6>0,\quad 4\gamma_1(2\alpha_4+\alpha_5+\alpha_6)>(\alpha_2+\alpha_3+\gamma_2)^2,\\ &\alpha_4+\alpha_7>\alpha_1+\frac{\gamma_2^2}{\gamma_1}\ge 0,\\ &2\alpha_4+\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1}>\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8\ge 0.\end{aligned} \tag{1.6}$$

It is easy to see that an example of coefficients $\alpha_1,\cdots,\alpha_8$ satisfying (1.5) and (1.6) can be taken as follows

$$\alpha_0=\alpha_1=\alpha_5=\alpha_6=\alpha_7=\alpha_8=0,\quad \alpha_2=-1,\quad \alpha_3=\alpha_4=1,$$

so that

$$\gamma_1=\alpha_3-\alpha_2=2>0,\quad \gamma_2=\alpha_6-\alpha_5=\alpha_2+\alpha_3=0.$$

A simplified compressible Ericksen-Leslie system has been recently studied. The idea of simplification was first proposed for the incompressible system by Lin in [17]. In dimension one, the global strong and weak solutions have been constructed in [3] and [4]. In dimension two, under the assumption that the initial data of $\mathbf{n}$ is contained in $\mathbb{S}^2_+$, global weak solutions have been constructed in [12]. In dimension three, the local existence of strong solutions has been studied by [10] and [11], and when the initial data of $\mathbf{n}$ is contained in $\mathbb{S}^2_+$, global weak solutions have been constructed in [18]. The incompressible limit of compressible nematic liquid crystal flows has been studied by [2].

We also mention a related work [13], in which the Ericksen-Leslie parabolic-hyperbolic liquid crystal model has been studied. For small initial data, they have shown the existence of global solutions in dimension three.

1.1. One dimensional model and statement of main results. One of the main motivations of this paper is to investigate the impact of general Leslie stress tensors to the solutions of the compressible Ericksen-Leslie system with coefficients satisfying algebraic conditions (1.5) and (1.6) ensuring the energy dissipation property. Because of the technical complexity of the Ericksen-Leslie system in higher dimensions, we will only consider the following simpler case in one dimension, in which the direction field $\mathbf{n}$ is assumed to map into the equator $\mathbb{S}^1$,

$$\mathbf{u}=\big(u(x,t),\ v(x,t),0\big)^T,\quad \mathbf{n}=\big(\cos n(x,t),\ \sin n(x,t),0\big)^T$$

for any $x\in[0,1]$ and $t\in(0,\infty)$. From the derivation given by Section 2 below, the system (1.1) becomes

$$\begin{cases}\rho_t+(\rho u)_x=0,\\ (\rho u)_t+(\rho u^2)_x+\big(\rho^\gamma\big)_x=J^1-n_{xx}n_x,\\ (\rho v)_t+(\rho uv)_x=J^2,\\ \gamma_1\left(\dot n-\frac12 v_x\right)-\gamma_2\left(u_x\cos n\sin n+\frac12 v_x(1-2\cos^2 n)\right)=n_{xx}.\end{cases}\tag{1.7}$$

Here

$$\begin{aligned}J^1=&(\alpha_0+\alpha_5+\alpha_6+\alpha_8)\big(u_x\cos^2 n\big)_x+\alpha_1\big(u_x\cos^4 n\big)_x-(\alpha_2+\alpha_3)\big(\dot n\cos n\sin n\big)_x\\ &+(\alpha_4+\alpha_7)u_{xx}+\alpha_0\big(v_x\cos n\sin n\big)_x+\alpha_1\big(v_x\cos^3 n\sin n\big)_x\\ &+\frac12(\alpha_2+\alpha_3+\alpha_5+\alpha_6)\big(v_x\cos n\sin n\big)_x,\end{aligned}$$

and

$$\begin{aligned}J^2=&\alpha_1\big(u_x\cos^3 n\sin n\big)_x+\alpha_2\big(\dot n\cos^2 n\big)_x-\alpha_3\big(\dot n\sin^2 n\big)_x+(\alpha_6+\alpha_8)\big(u_x\cos n\sin n\big)_x\\ &+\alpha_1\big(v_x\cos^2 n\sin^2 n\big)_x+\frac12(-\alpha_2+\alpha_5)\big(v_x\cos^2 n\big)_x+\frac12\alpha_4 v_{xx}\\ &+\frac12(\alpha_3+\alpha_6)\big(v_x\sin^2 n\big)_x.\end{aligned}$$

For this system, we consider the following initial and boundary values

$$(\rho, \rho u, \rho v, n)(x,0) = (\rho_0, m_0, l_0, n_0)(x), \tag{1.8}$$
$$u(0,t) = v(0,t) = u(1,t) = v(1,t) = 0, \quad n_x(0,t) = n_x(1,t) = 0. \tag{1.9}$$

The boundary values of n are deduced from the Neumann boundary condition of $\mathbf{n}$.

Denote the energy of the system (1.7) by

$$\mathcal{E}(t) := \frac{1}{2}\int_0^1 \rho(u^2+v^2) + \frac{1}{\gamma-1}\int_0^1 \rho^\gamma + \frac{1}{2}\int_0^1 n_x^2.$$

For any smooth solution (ρ,u,v,n), the energy functional satisfies the following energy inequality, whose proof will be provided in Section 3,

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(t) = -\mathcal{D} \\
&:= -\int_0^1 \left[\sqrt{\gamma_1}\dot{n} - \frac{1}{2}\left(\frac{\gamma_2}{\sqrt{\gamma_1}}u_x\sin(2n) + \frac{1}{\sqrt{\gamma_1}}(\gamma_1 - \gamma_2\cos(2n))v_x\right)\right]^2 \\
&\quad - \int_0^1 \left[\frac{1}{4}\left(-\alpha_1 - \frac{\gamma_2^2}{\gamma_1}\right)u_x^2 + (\alpha_4+\alpha_7)u_x^2\right] - \frac{1}{4}\int_0^1 \left(2\alpha_4+\alpha_5+\alpha_6 - \frac{\gamma_2^2}{\gamma_1}\right)v_x^2 \\
&\quad - \frac{1}{4}\left(\alpha_1 + \frac{\gamma_2^2}{\gamma_1}\right)\int_0^1 (u_x\cos(2n) + v_x\sin(2n))^2 \\
&\quad - (\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)\int_0^1 \left[\left(u_x\cos n + \frac{1}{2}v_x\sin n\right)^2 - \frac{1}{4}v_x^2\sin^2 n\right].
\end{aligned} \tag{1.10}$$

By direct computation, the system (1.7) is dissipative when the coefficients satisfy the assumptions (1.6).

Definition 1.1. *For any time $0<T<\infty$, a collection of functions $(\rho,u,v,n)(x,t)$ is a global weak solution to the initial and boundary value problem (1.7)-(1.9) if*

(1) $\rho \geq 0$, *a.e.*, $\quad \rho \in L^\infty(0,T;L^\gamma), \quad \rho u^2, \rho v^2 \in L^\infty(0,T;L^1), \quad u,v \in L^2(0,T;H_0^1)$

$$n \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2), \quad n_t \in L^2(0,T;L^2).$$

(2) The equations of ρ, u, v are satisfied in the weak sense, while the equation of n is valid a.e.. The initial condition (1.8) is satisfied in the weak sense.

(3) The energy inequality is valid for a.e. $t \in (0,T)$

$$\mathcal{E}(t) + \int_0^t \mathcal{D} \leq \mathcal{E}_0 = \frac{1}{2}\int_0^1 \frac{m_0^2+l_0^2}{\rho_0} + \frac{1}{\gamma-1}\int_0^1 \rho_0^\gamma + \frac{1}{2}\int_0^1 (n_0)_x^2.$$

The following is the main result in this paper.

Theorem 1.1. *Assume that the coefficients of Leslie stress tensor satisfy the algebraic conditions (1.5) and (1.6). Then, for any $0<T<\infty$ and any initial data*

$$0 \leq \rho_0 \in L^\gamma, \quad \frac{m_0}{\sqrt{\rho_0}}, \quad \frac{l_0}{\sqrt{\rho_0}} \in L^2, \quad n_0 \in H^1, \tag{1.11}$$

there is a global weak solution $(\rho,u,v,n)(x,t)$ on $(0,1)\times(0,T)$ to the initial and boundary value problem (1.7)-(1.9). Furthermore, $\rho \in L^{2\gamma}((0,1)\times(0,T))$.

The main ideas of the proof utilize and extend those from [8, 14], and [7] in the study of the compressible Navier-Stokes equations, where the quantity called *effective viscous flux* has played crucial roles in controlling the oscillation of the density function ρ. However, the general Leslie stress tensors in the compressible Ericksen-Leslie system (1.7) induce two complicated second-order terms J^1 and J^2 that prohibit direct applications of the method of effective viscous flux. In this paper, we observe that with the algebraic conditions (1.5) and (1.6), the system of $\mathbf{u}=(u,v)^T$ can still be shown to be uniformly parabolic (see (5.26) and (5.27) below), i.e. the coefficient matrix of the second-order terms is uniformly elliptic. Using the inverse of coefficient matrix of the second-order terms, we can then define a modified form of effective viscous flux as in Lemma 5.3, which yields the desired estimates that are necessary in the limiting process of approximated solutions.

The paper is organized as follows. In Section 2, we will sketch a derivation of the system (1.7). In Section 3, we will derive some a priori estimates for smooth solutions of (1.7). In Section 4, an approximated system will be introduced, and the existence of global regular solutions of this approximated system will be proven. In Section 5, we will prove the existence of global weak solutions through some delicate analysis of the convergence process.

2. Derivation of the model in one dimension

This section is devoted to the derivation of the system (1.7) in dimension one. If a solution takes the form

$$\mathbf{u}=\big(u(x,t),\ v(x,t)\big)^T,\quad \mathbf{n}=\big(\cos n(x,t),\ \sin n(x,t)\big)^T,\ (x,t)\in(0,1)\times(0,T),$$

then

$$\nabla\mathbf{u}=\begin{bmatrix} u_x & 0\\ v_x & 0\end{bmatrix},\quad \nabla^T\mathbf{u}=\begin{bmatrix} u_x & v_x\\ 0 & 0\end{bmatrix},$$

so that

$$D=\begin{bmatrix} u_x & \frac12 v_x\\ \frac12 v_x & 0\end{bmatrix},\quad \omega=\begin{bmatrix} 0 & -\frac12 v_x\\ \frac12 v_x & 0\end{bmatrix},$$

$$\operatorname{tr} D=u_x,\quad N=\dot{\mathbf{n}}-\omega\mathbf{n}=\left(\dot n-\frac12 v_x\right)\big(-\sin n,\ \cos n\big)^T.$$

Direct calculations imply that

$$D\mathbf{n}=\left(u_x\cos n+\frac12 v_x\sin n,\ \frac12 v_x\cos n\right)^T,\quad \mathbf{n}^TD\mathbf{n}=u_x\cos^2 n+v_x\cos n\sin n,$$

$$\mathbf{n}\otimes\mathbf{n}=\begin{bmatrix}\cos^2 n & \cos n\sin n\\ \cos n\sin n & \sin^2 n\end{bmatrix},$$

$$(\mathbf{n}^TD\mathbf{n})\mathbf{n}\otimes\mathbf{n}=(u_x\cos^2 n+v_x\cos n\sin n)\begin{bmatrix}\cos^2 n & \cos n\sin n\\ \cos n\sin n & \sin^2 n\end{bmatrix},$$

$$N\otimes\mathbf{n}=\left(\dot n-\frac12 v_x\right)\begin{bmatrix}-\cos n\sin n & -\sin^2 n\\ \cos^2 n & \cos n\sin n\end{bmatrix},$$

$$\mathbf{n}\otimes N=\left(\dot n-\frac12 v_x\right)\begin{bmatrix}-\cos n\sin n & \cos^2 n\\ -\sin^2 n & \cos n\sin n\end{bmatrix},$$

$$(D\mathbf{n})\otimes\mathbf{n}=\begin{bmatrix} u_x\cos^2 n+\frac{1}{2}v_x\cos n\sin n & u_x\cos n\sin n+\frac{1}{2}v_x\sin^2 n \\ \frac{1}{2}v_x\cos^2 n & \frac{1}{2}v_x\cos n\sin n \end{bmatrix},$$

$$\mathbf{n}\otimes(D\mathbf{n})=\begin{bmatrix} u_x\cos^2 n+\frac{1}{2}v_x\cos n\sin n & \frac{1}{2}v_x\cos^2 n \\ u_x\cos n\sin n+\frac{1}{2}v_x\sin^2 n & \frac{1}{2}v_x\cos n\sin n \end{bmatrix}.$$

Hence

$$\nabla\cdot\sigma=\left(J^1,J^2\right)^T$$

where

$$\begin{aligned} J^1=&(\alpha_0+\alpha_5+\alpha_6+\alpha_8)\left(u_x\cos^2 n\right)_x+\alpha_1\left(u_x\cos^4 n\right)_x-(\alpha_2+\alpha_3)\left(\dot{n}\cos n\sin n\right)_x \\ &+(\alpha_4+\alpha_7)u_{xx}+\alpha_0\left(v_x\cos n\sin n\right)_x+\alpha_1\left(v_x\cos^3 n\sin n\right)_x \\ &+\frac{1}{2}(\alpha_2+\alpha_3+\alpha_5+\alpha_6)\left(v_x\cos n\sin n\right)_x, \end{aligned}$$

and

$$\begin{aligned} J^2=&\alpha_1\left(u_x\cos^3 n\sin n\right)_x+\alpha_2\left(\dot{n}\cos^2 n\right)_x-\alpha_3\left(\dot{n}\sin^2 n\right)_x+(\alpha_6+\alpha_8)\left(u_x\cos n\sin n\right)_x \\ &+\alpha_1\left(v_x\cos^2 n\sin^2 n\right)_x+\frac{1}{2}(-\alpha_2+\alpha_5)\left(v_x\cos^2 n\right)_x+\frac{1}{2}\alpha_4 v_{xx} \\ &+\frac{1}{2}(\alpha_3+\alpha_6)\left(v_x\sin^2 n\right)_x. \end{aligned}$$

The terms related to $\mathbf{n}$ can be computed as follows

$$\begin{gathered} \mathbf{n}_t=n_t\left(-\sin n,\ \cos n\right)^T, \\ \mathbf{n}_x=n_x\left(-\sin n,\ \cos n\right)^T,\quad |\mathbf{n}_x|^2=(n_x)^2, \\ \mathbf{u}\cdot\mathbf{n}=u\mathbf{n}_x=un_x\left(-\sin n,\ \cos n\right)^T, \\ \mathbf{n}_{xx}=n_{xx}\left(-\sin n,\ \cos n\right)^T+(n_x)^2\left(-\cos n,\ -\sin n\right)^T, \\ \nabla\cdot(\nabla\mathbf{n}\odot\nabla\mathbf{n})-\frac{1}{2}\nabla|\nabla\mathbf{n}|^2=\Delta\mathbf{n}\nabla\mathbf{n}=\left(n_{xx}n_x,\ 0\right)^T. \end{gathered}$$

Therefore, $u(x,t)$ satisfies

$$\rho u_t+\rho u u_x+\left(\rho^\gamma\right)_x=J^1-n_{xx}n_x, \tag{2.1}$$

and $v(x,t)$ satisfies

$$\rho v_t+\rho u v_x=J^2. \tag{2.2}$$

Now we can calculate the equation of n as follows

$$\begin{aligned} \mathbf{g}=&\gamma_1 N+\gamma_2 D\mathbf{n}-\gamma_2(\mathbf{n}^T D\mathbf{n})\mathbf{n} \\ =&\gamma_1\left(\dot{n}-\frac{1}{2}v_x\right)\left(-\sin n,\ \cos n\right)^T+\gamma_2\left(u_x\cos n+\frac{1}{2}v_x\sin n,\ \frac{1}{2}v_x\cos n\right)^T \\ &-\gamma_2\left(u_x\cos^2 n+v_x\cos n\sin n\right)\left(\cos n,\ \sin n\right)^T \\ =&\gamma_1\left(\dot{n}-\frac{1}{2}v_x\right)\left(-\sin n,\ \cos n\right)^T \end{aligned}$$

$$
\begin{aligned}
&+\gamma_2\left(u_x\cos n\sin^2 n+\frac{1}{2}v_x\sin n(1-2\cos^2 n),\ -u_x\cos^2 n\sin n+\frac{1}{2}v_x\cos n(1-2\sin^2 n)\right)^T\\
&=\gamma_1\left(\dot{n}-\frac{1}{2}v_x\right)\left(-\sin n,\ \cos n\right)^T\\
&\quad-\gamma_2\left(u_x\cos n\sin n+\frac{1}{2}v_x(1-2\cos^2 n)\right)\left(-\sin n,\ \cos n\right)^T,
\end{aligned}
$$

$$
\lambda\mathbf{n}=\left(|\nabla\mathbf{n}|^2+\gamma_1 N\cdot\mathbf{n}\right)\mathbf{n}=(n_x)^2\left(\cos n,\ \sin n\right)^T.
$$

Therefore $n(x,t)$ satisfies

$$
\gamma_1\left(\dot{n}-\frac{1}{2}v_x\right)-\gamma_2\left(u_x\cos n\sin n+\frac{1}{2}v_x(1-2\cos^2 n)\right)=n_{xx}. \tag{2.3}
$$

Thus the system (1.1) reduces to (1.7).

3. A priori estimates

In this section, we will prove several useful a priori estimates for smooth solutions of system (1.7).

LEMMA 3.1. *Any smooth solution to the system (1.7) satisfies the following energy inequality*

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}(t)=&-\int_0^1\left[\sqrt{\gamma_1}\dot{n}-\frac{1}{2}\left(\frac{\gamma_2}{\sqrt{\gamma_1}}u_x\sin(2n)+\frac{1}{\sqrt{\gamma_1}}(\gamma_1-\gamma_2\cos(2n))v_x\right)\right]^2\\
&-\int_0^1\left[\frac{1}{4}\left(-\alpha_1-\frac{\gamma_2^2}{\gamma_1}\right)u_x^2+(\alpha_4+\alpha_7)u_x^2\right]\\
&-\frac{1}{4}\int_0^1\left(2\alpha_4+\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1}\right)v_x^2\\
&-\frac{1}{4}\left(\alpha_1+\frac{\gamma_2^2}{\gamma_1}\right)\int_0^1\left(u_x\cos(2n)+v_x\sin(2n)\right)^2\\
&-(\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)\int_0^1\left[\left(u_x\cos n+\frac{1}{2}v_x\sin n\right)^2-\frac{1}{4}v_x^2\sin^2 n\right].
\end{aligned}\tag{3.1}
$$

Proof. Multiplying the second equation by u, the third equation by v and integrating over $[0,1]$, we have

$$
\frac{1}{2}\frac{d}{dt}\int_0^1\rho(u^2+v^2)+\frac{1}{\gamma-1}\frac{d}{dt}\int_0^1\rho^\gamma=\int_0^1\left(J^1u+J^2v-un_{xx}n_x\right).
$$

Multiplying the last equation by $\dot{n}$ and integrating over $[0,1]$, we obtain

$$
\frac{d}{dt}\frac{1}{2}\int_0^1(n_x)^2+\gamma_1\int_0^1\dot{n}^2=\int_0^1\left[\frac{1}{2}\gamma_2u_x\sin(2n)\dot{n}+\frac{1}{2}(\gamma_1-\gamma_2\cos(2n))v_x\dot{n}+un_{xx}n_x\right].
$$

Adding these two equations together, we have

$$
\frac{1}{2}\frac{d}{dt}\int_0^1\rho(u^2+v^2)+\frac{1}{\gamma-1}\frac{d}{dt}\int_0^1\rho^\gamma+\frac{1}{2}\frac{d}{dt}\int_0^1(n_x)^2
$$

$$=\int_0^1 (J^1u+J^2v)-\gamma_1\int_0^1 \dot{n}^2+\int_0^1 \frac{1}{2}[\gamma_2 u_x\sin(2n)\dot{n}+(\gamma_1-\gamma_2\cos(2n))v_x\dot{n}]. \tag{3.2}$$

By integrating by parts, we can estimate the term related to J^1, J^2 as follows

$$\begin{aligned}&\int_0^1 J^1u\\ =&-\int_0^1 [(\alpha_0+\alpha_5+\alpha_6+\alpha_8)u_x^2\cos^2 n+\alpha_1 u_x^2\cos^4 n+(\alpha_4+\alpha_7)u_x^2]\\ &-\int_0^1 \left[\alpha_1 u_x v_x\cos^3 n\sin n+\left(\alpha_0+\frac{1}{2}(\alpha_2+\alpha_3+\alpha_5+\alpha_6)\right)u_x v_x\cos n\sin n\right]\\ &+\int_0^1 (\alpha_2+\alpha_3)u_x\dot{n}\cos n\sin n,\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\int_0^1 J^2v\\ =&-\int_0^1 \left[\alpha_1 v_x^2\cos^2 n\sin^2 n+\frac{1}{2}(-\alpha_2+\alpha_5)v_x^2\cos^2 n+\frac{1}{2}(\alpha_3+\alpha_6)v_x^2\sin^2 n+\frac{1}{2}\alpha_4 v_x^2\right]\\ &-\int_0^1 [\alpha_1 u_x v_x\cos^3 n\sin n+(\alpha_6+\alpha_8)u_x v_x\cos n\sin n]\\ &-\int_0^1 [\alpha_2 v_x\dot{n}\cos^2 n-\alpha_3 v_x\dot{n}\sin^2 n].\end{aligned} \tag{3.4}$$

First notice that all the terms related to α_1 in (3.3) and (3.4) can be written as

$$\begin{aligned}&-\alpha_1\int_0^1 [u_x^2\cos^4 n+2u_x v_x\cos^3 n\sin n+v_x^2\cos^2 n\sin^2 n]\\ =&-\alpha_1\int_0^1 [u_x\cos^2 n+v_x\cos n\sin n]^2.\end{aligned} \tag{3.5}$$

The other term related to $u_x v_x$ in (3.3) and (3.4) (without terms with α_1) can be written as

$$\begin{aligned}&-\int_0^1 u_x v_x\cos n\sin n\left[\alpha_0+\frac{1}{2}(\alpha_2+\alpha_3+\alpha_5+\alpha_6)+(\alpha_6+\alpha_8)\right]\\ =&-\int_0^1 (\alpha_0+2\alpha_6+\alpha_8)u_x v_x\cos n\sin n,\end{aligned} \tag{3.6}$$

where we have used $\alpha_2+\alpha_3=\alpha_6-\alpha_5$. The terms related to u_x^2, v_x^2 in (3.3) and (3.4) (without terms with α_1) can be written as

$$\begin{aligned}&-\int_0^1 [(\alpha_0+\alpha_5+\alpha_6+\alpha_8)u_x^2\cos^2 n+(\alpha_4+\alpha_7)u_x^2]\\ &-\int_0^1 \left[\frac{1}{4}(2\alpha_4-\alpha_2+\alpha_5+\alpha_3+\alpha_6)v_x^2-\frac{1}{2}\gamma_2 v_x^2\cos(2n)\right].\end{aligned} \tag{3.7}$$

What is left in (3.2)-(3.4) are all terms related to $u_x\dot{n}$ and $v_x\dot{n}$

$$\int_0^1 \left[\frac{1}{2}\gamma_2 u_x\sin(2n)\dot{n}+(\alpha_2+\alpha_3)u_x\dot{n}\cos n\sin n\right]$$

$$
\begin{aligned}
&+\int_0^1\left[\frac{1}{2}(\gamma_1-\gamma_2\cos(2n))v_x\dot{n}-\alpha_2 v_x\dot{n}\cos^2 n+\alpha_3 v_x\dot{n}\sin^2 n\right]\\
=&\int_0^1\gamma_2 u_x\dot{n}\sin(2n)+\int_0^1(\gamma_1-\gamma_2\cos(2n))v_x\dot{n}, \qquad (3.8)
\end{aligned}
$$

where we have used $\gamma_1=\alpha_3-\alpha_2$ and $\gamma_2=\alpha_2+\alpha_3=\alpha_6-\alpha_5$. Therefore, putting (3.5)-(3.8) into (3.2), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\rho(u^2+v^2)+\frac{1}{\gamma-1}\frac{d}{dt}\int_0^1\rho^\gamma+\frac{d}{dt}\frac{1}{2}\int_0^1(n_x)^2\\
=&-\alpha_1\int_0^1\left[u_x\cos^2 n+v_x\cos n\sin n\right]^2-\int_0^1 u_x v_x\cos n\sin n(\alpha_0+2\alpha_6+\alpha_8)\\
&-\int_0^1\left[(\alpha_0+\alpha_5+\alpha_6+\alpha_8)u_x^2\cos^2 n+(\alpha_4+\alpha_7)u_x^2\right]\\
&-\int_0^1\left[\frac{1}{4}(2\alpha_4+\alpha_5+\alpha_6+\gamma_1)v_x^2-\frac{1}{2}\gamma_2 v_x^2\cos(2n)\right]\\
&-\gamma_1\int_0^1\dot{n}^2+\int_0^1\gamma_2 u_x\dot{n}\sin(2n)+\int_0^1(\gamma_1-\gamma_2\cos(2n))v_x\dot{n}. \qquad (3.9)
\end{aligned}
$$

We first complete the square for all terms with $\dot{n}$ in (3.5)

$$
\begin{aligned}
&\gamma_1\int_0^1\dot{n}^2-\int_0^1\gamma_2 u_x\dot{n}\sin(2n)-\int_0^1(\gamma_1-\gamma_2\cos(2n))v_x\dot{n}\\
=&\gamma_1\int_0^1\dot{n}^2-2\cdot\frac{1}{2}\int_0^1\sqrt{\gamma_1}\dot{n}\left(\frac{\gamma_2}{\sqrt{\gamma_1}}u_x\sin(2n)+\frac{1}{\sqrt{\gamma_1}}(\gamma_1-\gamma_2\cos(2n))v_x\right)\\
=&\int_0^1\left[\sqrt{\gamma_1}\dot{n}-\frac{1}{2}\left(\frac{\gamma_2}{\sqrt{\gamma_1}}u_x\sin(2n)+\frac{1}{\sqrt{\gamma_1}}(\gamma_1-\gamma_2\cos(2n))v_x\right)\right]^2\\
&-\frac{1}{4}\int_0^1\left(\frac{\gamma_2}{\sqrt{\gamma_1}}u_x\sin(2n)+\frac{1}{\sqrt{\gamma_1}}(\gamma_1-\gamma_2\cos(2n))v_x\right)^2. \qquad (3.10)
\end{aligned}
$$

The last term in (3.10) can also be rewritten as follows

$$
\begin{aligned}
&\left(\frac{\gamma_2}{\sqrt{\gamma_1}}u_x\sin(2n)+\frac{1}{\sqrt{\gamma_1}}(\gamma_1-\gamma_2\cos(2n))v_x\right)^2\\
=&\frac{\gamma_2^2}{\gamma_1}u_x^2\sin^2(2n)+2\frac{\gamma_2}{\gamma_1}u_x v_x\sin(2n)(\gamma_1-\gamma_2\cos(2n))+\frac{1}{\gamma_1}(\gamma_1-\gamma_2\cos(2n))^2 v_x^2\\
=&\frac{\gamma_2^2}{\gamma_1}u_x^2\sin^2(2n)+2u_x v_x\sin(2n)\left(\gamma_2-\frac{\gamma_2^2}{\gamma_1}\cos(2n)\right)\\
&+\left(\gamma_1-2\gamma_2\cos(2n)+\frac{\gamma_2^2}{\gamma_1}\cos^2(2n)\right)v_x^2. \qquad (3.11)
\end{aligned}
$$

To complete the square for the remaining terms, we first investigate the terms containing $u_x v_x$ in (3.10) and (3.11):

$$
\frac{1}{2}\alpha_1\int_0^1 u_x v_x\sin(2n)(1+\cos(2n))+\frac{1}{2}\int_0^1(\alpha_0+2\alpha_6+\alpha_8)u_x v_x\sin(2n)
$$

$$
\begin{aligned}
&-\frac{1}{2}\int_0^1 u_x v_x \sin(2n)\left(\gamma_2-\frac{\gamma_2^2}{\gamma_1}\cos(2n)\right)\\
=&\frac{1}{2}\int_0^1(\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)u_x v_x\sin(2n)+\frac{1}{2}\int_0^1\left(\alpha_1+\frac{\gamma_2^2}{\gamma_1}\right)u_x v_x\sin(2n)\cos(2n)\\
=&\int_0^1(\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)u_x v_x\sin n\cos n\\
&+\frac{1}{2}\int_0^1\left(\alpha_1+\frac{\gamma_2^2}{\gamma_1}\right)u_x v_x\sin(2n)\cos(2n).
\end{aligned}\tag{3.12}
$$

Thus we can calculate the terms containing u_x^2 in (3.10) and (3.11) as follows

$$
\begin{aligned}
&\frac{1}{4}\int_0^1\left[\alpha_1 u_x^2(1+\cos(2n))^2-\frac{\gamma_2^2}{\gamma_1}u_x^2\sin^2(2n)\right]\\
&+\int_0^1\left[(\alpha_0+\alpha_5+\alpha_6+\alpha_8)u_x^2\cos^2 n+(\alpha_4+\alpha_7)u_x^2\right]\\
=&\frac{1}{4}\int_0^1\left[\alpha_1 u_x^2(1+2\cos(2n)+\cos^2(2n))-\frac{\gamma_2^2}{\gamma_1}u_x^2+\frac{\gamma_2^2}{\gamma_1}u_x^2\cos^2(2n)\right]\\
&+\int_0^1\left[(\alpha_0+\alpha_5+\alpha_6+\alpha_8)u_x^2\cos^2 n+2(\alpha_4+\alpha_7)u_x^2\right]\\
=&\frac{1}{4}\int_0^1\left(\alpha_1+\frac{\gamma_2^2}{\gamma_1}\right)u_x^2\cos^2(2n)+\int_0^1(\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)u_x^2\cos^2 n\\
&+\int_0^1\left[\frac{1}{4}\left(-\alpha_1-\frac{\gamma_2^2}{\gamma_1}\right)u_x^2+(\alpha_4+\alpha_7)u_x^2\right].
\end{aligned}\tag{3.13}
$$

Similarly, the terms involving v_x^2 in (3.10) and (3.11) can be calculated as follows

$$
\begin{aligned}
&\frac{1}{4}\int_0^1\alpha_1 v_x^2\sin^2(2n)+\int_0^1\left[\frac{1}{4}(2\alpha_4+\alpha_5+\alpha_6+\gamma_1)v_x^2-\frac{1}{2}\gamma_2 v_x^2\cos(2n)\right]\\
&-\frac{1}{4}\int_0^1\left(\gamma_1-2\gamma_2\cos(2n)+\frac{\gamma_2^2}{\gamma_1}\cos^2(2n)\right)v_x^2\\
=&\frac{1}{4}\int_0^1\alpha_1 v_x^2\sin^2(2n)+\frac{1}{4}\int_0^1\left(2\alpha_4+\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1}\cos^2(2n)\right)v_x^2\\
=&\frac{1}{8}\int_0^1(2\alpha_1+3\alpha_4+2\alpha_5+2\alpha_6)v_x^2\sin^2(2n)+\frac{1}{8}\int_0^1\alpha_4 v_x^2\sin^2(2n)\\
&+\frac{1}{4}\int_0^1\left(2\alpha_4+\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1}\right)v_x^2\cos^2(2n).
\end{aligned}\tag{3.14}
$$

For the terms with coefficient $\alpha_1+\frac{\gamma_2^2}{\gamma_1}$ in (3.12) and (3.13), we have

$$
\begin{aligned}
&\frac{1}{4}\int_0^1\left(\alpha_1+\frac{\gamma_2^2}{\gamma_1}\right)u_x^2\cos^2(2n)+\frac{1}{2}\int_0^1\left(\alpha_1+\frac{\gamma_2^2}{\gamma_1}\right)u_x v_x\sin(2n)\cos(2n)\\
=&\frac{1}{4}\left(\alpha_1+\frac{\gamma_2^2}{\gamma_1}\right)\int_0^1\left[(u_x\cos(2n)+v_x\sin(2n))^2-v_x^2\sin^2(2n)\right].
\end{aligned}\tag{3.15}
$$

The terms with coefficient $\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8$ in (3.12) and (3.13) can be written as

$$
(\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)\int_0^1(u_x^2\cos^2 n+u_x v_x\sin n\cos n)
$$

$$=(\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)\int_0^1\left[\left(u_x\cos n+\frac{1}{2}v_x\sin n\right)^2-\frac{1}{4}v_x^2\sin^2 n\right]. \tag{3.16}$$

Collecting all the terms involving v_x^2 in (3.14)-(3.16), we have

$$\begin{aligned}&\frac{1}{8}\int_0^1(2\alpha_1+3\alpha_4+2\alpha_5+2\alpha_6)v_x^2\sin^2(2n)+\frac{1}{8}\int_0^1\alpha_4v_x^2\sin^2(2n)\\&\quad+\frac{1}{4}\int_0^1\left(2\alpha_4+\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1}\right)v_x^2\cos^2(2n)-\frac{1}{4}\left(\alpha_1+\frac{\gamma_2^2}{\gamma_1}\right)\int_0^1v_x^2\sin^2(2n)\\&\quad-\frac{1}{4}(\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)\int_0^1v_x^2\sin^2 n\\&=\frac{1}{4}\int_0^1\left(2\alpha_4+\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1}\right)v_x^2-\frac{1}{4}(\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)\int_0^1v_x^2\sin^2 n. \end{aligned}\tag{3.17}$$

Therefore, putting the identities (3.10), (3.15)-(3.17) into (3.9) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\rho(u^2+v^2)+\frac{1}{\gamma-1}\frac{d}{dt}\int_0^1\rho^\gamma+\frac{d}{dt}\frac{1}{2}\int_0^1(n_x)^2\\&=-\int_0^1\left[\sqrt{\gamma_1}\dot{n}-\frac{1}{2}\left(\frac{\gamma_2}{\sqrt{\gamma_1}}u_x\sin(2n)+\frac{1}{\sqrt{\gamma_1}}(\gamma_1-\gamma_2\cos(2n))v_x\right)\right]^2\\&\quad-\int_0^1\left[\frac{1}{4}\left(-\alpha_1-\frac{\gamma_2^2}{\gamma_1}\right)u_x^2+(\alpha_4+\alpha_7)u_x^2\right]-\frac{1}{4}\int_0^1\left(2\alpha_4+\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1}\right)v_x^2\\&\quad-\frac{1}{4}\left(\alpha_1+\frac{\gamma_2^2}{\gamma_1}\right)\int_0^1(u_x\cos(2n)+v_x\sin(2n))^2\\&\quad-(\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)\int_0^1\left[\left(u_x\cos n+\frac{1}{2}v_x\sin n\right)^2-\frac{1}{4}v_x^2\sin^2 n\right],\end{aligned}$$

which completes the proof of lemma. □

From the energy inequality above, we can obtain the following estimates for n.

LEMMA 3.2. *For any smooth solution to the system (1.7), it holds that*

$$\|n_{xx}\|_{L^2(0,T;L^2)}+\|n_t\|_{L^2(0,T;L^2)}\le C(\mathcal{E}_0,T). \tag{3.18}$$

Proof. First notice that the equation of n is

$$\gamma_1\left(\dot{n}-\frac{1}{2}v_x\right)-\gamma_2\left(u_x\cos n\sin n+\frac{1}{2}v_x(1-2\cos^2 n)\right)=n_{xx}. \tag{3.19}$$

It is not hard to see that

$$\begin{aligned}&\gamma_1\left(\dot{n}-\frac{1}{2}v_x\right)-\gamma_2\left(u_x\cos n\sin n+\frac{1}{2}v_x(1-2\cos^2 n)\right)\\&=\gamma_1\dot{n}-\frac{1}{2}\gamma_2u_x\sin(2n)-\frac{1}{2}(\gamma_1-\gamma_2\cos(2n))v_x.\end{aligned}$$

By the energy inequality, we obtain the estimates for n_{xx}. Next, by the equation of n and the energy inequality, we obtain the estimate for n_t. □

We also need to show the higher integrability of ρ, which is inspired by the argument in [4].

LEMMA 3.3. *For any smooth solution to the system (1.7), it holds that*

$$\|\rho\|_{L^{2\gamma}([0,1]\times[0,T];)} \leq C(\mathcal{E}_0,T). \tag{3.20}$$

Proof. First set

$$G(x,t) := \int_0^x \rho^\gamma - x\int_0^1 \rho^\gamma.$$

It is easy to see that

$$\frac{\partial G}{\partial x} = \rho^\gamma - \int_0^1 \rho^\gamma, \quad G(0,t) = G(1,t) = 0.$$

Notice that the equation of u can be written as

$$(\rho u)_t + (\rho u^2)_x + \left(\rho^\gamma\right)_x = J^1 - \frac{1}{2}((n_x)^2)_x$$

where

$$\begin{aligned} J^1 =& (\alpha_0+\alpha_5+\alpha_6+\alpha_8)\left(u_x\cos^2 n\right)_x + \alpha_1\left(u_x\cos^4 n\right)_x - (\alpha_2+\alpha_3)\left(\dot{n}\cos n\sin n\right)_x \\ &+ (\alpha_4+\alpha_7)u_{xx} + \alpha_0\left(v_x\cos n\sin n\right)_x + \alpha_1\left(v_x\cos^3 n\sin n\right)_x \\ &+ \frac{1}{2}(\alpha_2+\alpha_3+\alpha_5+\alpha_6)\left(v_x\cos n\sin n\right)_x. \end{aligned}$$

Multiplying this equation by $G(x,t)$, integrating over $[0,1]\times(0,T)$, and using integration by parts, we obtain that

$$\begin{aligned} \int_0^T\int_0^1 \rho^{2\gamma} =& \int_0^T\left(\int_0^1 \rho^\gamma\right)^2 + \int_0^T\int_0^1 (\rho u)_t G(x,t) - \int_0^T\int_0^1 \rho u^2 \frac{\partial G(x,t)}{\partial x} \\ &- \int_0^T\int_0^1 J_1 G(x,t) - \frac{1}{2}\int_0^T\int_0^1 |n_x|^2 \frac{\partial G(x,t)}{\partial x} \\ =& \sum_{i=1}^5 I_i. \end{aligned} \tag{3.21}$$

For the first term, it is easy to estimate by energy inequality

$$I_1 \leq C(\mathcal{E}_0,T).$$

For the second term, we need to use integration by parts with respect to t to obtain

$$\begin{aligned} I_2 =& \int_0^1 \rho u G(x,T) - \int_0^1 \rho u G(x,0) - \int_0^T\int_0^1 \rho u G_t(x,t) \\ \leq& C\sup_{0\leq t\leq T}\left(\int_0^1 \rho|u|\int_0^1 \rho^\gamma\right) - \int_0^T\int_0^1 \rho u G_t(x,t) \\ \leq& C\sup_{0\leq t\leq T}\left(\int_0^1 \rho|u|^2\int_0^1 \rho^\gamma + \int_0^1 \rho\int_0^1 \rho^\gamma\right) - \int_0^T\int_0^1 \rho u G_t(x,t) \\ \leq& C(\mathcal{E}_0,T) - \int_0^T\int_0^1 \rho u G_t(x,t). \end{aligned}$$

To estimate the last term here, we multiply the equation of ρ by $\gamma\rho^{\gamma-1}$ to get

$$(\rho^\gamma)_t+(\rho^\gamma u)_x+(\gamma-1)\rho^\gamma u_x=0.$$

Then it holds

$$\begin{aligned}
&-\int_0^T\int_0^1 \rho u G_t(x,t)\\
=&-\int_0^T\int_0^1 \rho u\left(\int_0^x \rho_t^\gamma - x\int_0^1\rho_t^\gamma\right)\\
=&\int_0^T\int_0^1\rho u\int_0^x((\rho^\gamma u)_x+(\gamma-1)\rho^\gamma u_x)-\int_0^T\int_0^1 x\rho u\int_0^1((\rho^\gamma u)_x+(\gamma-1)\rho^\gamma u_x)\\
=&\int_0^T\int_0^1\rho^{\gamma+1}u^2+(\gamma-1)\int_0^T\int_0^1\rho u\left(\int_0^x\rho^\gamma u_x - x\int_0^1\rho^\gamma u_x\right)\\
\le&\int_0^T\int_0^1\rho^{\gamma+1}u^2+C\int_0^T\int_0^1\rho|u|\int_0^1\rho^\gamma|u_x|\\
\le&\int_0^T\int_0^1\rho^{\gamma+1}u^2+C\int_0^T\left(\int_0^1(\rho+\rho|u|^2)\left(\int_0^1\rho^{2\gamma}\right)^{\frac12}\left(\int_0^1|u_x|^2\right)^{\frac12}\right)\\
\le&\int_0^T\int_0^1\rho^{\gamma+1}u^2+C(\mathcal{E}_0,T)\int_0^T\left(\left(\int_0^1\rho^{2\gamma}\right)^{\frac12}\left(\int_0^1|u_x|^2\right)^{\frac12}\right)\\
\le&\int_0^T\int_0^1\rho^{\gamma+1}u^2+\frac14\int_0^T\int_0^1\rho^{2\gamma}+C(\mathcal{E}_0,T)\int_0^T\int_0^1|u_x|^2\\
\le&\int_0^T\int_0^1\rho^{\gamma+1}u^2+\frac14\int_0^T\int_0^1\rho^{2\gamma}+C(\mathcal{E}_0,T),
\end{aligned}$$

where we have used the Cauchy inequality, the Hölder inequality, the Young inequality and the energy inequality. Hence we obtain

$$I_2\le\int_0^T\int_0^1\rho^{\gamma+1}u^2+\frac14\int_0^T\int_0^1\rho^{2\gamma}+C(\mathcal{E}_0,T).$$

For the third term in (3.21), it holds

$$I_3=-\int_0^T\int_0^1\rho u^2\left(\rho^\gamma-\int_0^1\rho^\gamma\right)=-\int_0^T\int_0^1\rho^{\gamma+1}u^2+C(\mathcal{E}_0,T).$$

Then

$$I_2+I_3\le\frac14\int_0^T\int_0^1\rho^{2\gamma}+C(\mathcal{E}_0,T).$$

For the fourth term in (3.21), by integration by parts it holds

$$\begin{aligned}
I_4\le&\int_0^T\int_0^1(|u_x|+|\dot n|+|v_x|)\rho^\gamma+\int_0^T\int_0^1(|u_x|+|n_t|+|v_x|)\int_0^1\rho^\gamma\\
\le&\frac14\int_0^T\int_0^1\rho^{2\gamma}+C\int_0^T\int_0^1\left(|u_x|^2+|\dot n|^2+|v_x|^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+C(\mathcal{E}_0,T)\int_0^T\int_0^1\left(|u_x|^2+|n_t|^2+|v_x|^2\right)+C(\mathcal{E}_0,T)\\
\leq&\frac{1}{4}\int_0^T\int_0^1\rho^{2\gamma}+C(\mathcal{E}_0,T).
\end{aligned}
$$

For the last term in (3.21), it holds

$$
I_5=-\frac{1}{2}\int_0^T\int_0^1|n_x|^2\left(\rho^\gamma-\int_0^1\rho^\gamma\right)\leq C(\mathcal{E}_0,T).
$$

Therefore, by adding all the estimates together in (3.21) we obtain

$$
\int_0^T\int_0^1\rho^{2\gamma}\leq\frac{1}{2}\int_0^T\int_0^1\rho^{2\gamma}+C(\mathcal{E}_0,T),
$$

which implies the estimate (3.20). □

4. Approximated solutions

In this section, we first consider the case that the initial values are smooth enough, i.e. $\rho_0\in C^1$, $u_0,v_0,n_0\in C^2$, and $0<c_0^{-1}\leq\rho_0\leq c_0$ and $u_0=\frac{m_0}{\rho_0}$, $v_0=\frac{l_0}{\rho_0}$, and then construct the Galerkin approximation of ρ, u, v and n.

Step 1. Recall that

$$
\phi_j(x)=\sin(j\pi x),\quad j=1,2,...
$$

is an orthogonal base of $L^2(0,1)$. For any positive integer k, set

$$
\mathcal{X}_k=\text{span}\{\phi_1,\phi_2,\cdots\phi_k\}.
$$

and

$$
u_0^k=\sum_{j=0}^k\bar{c}_j^k\phi_j(x),\quad v_0^k=\sum_{j=0}^k\bar{d}_j^k\phi_j(x),
$$

for some constants

$$
\bar{c}_j^k=\int_0^1u_0\phi_j,\quad \bar{d}_j^k=\int_0^1v_0\phi_j.
$$

Then $(u_0^k,v_0^k)\to(u_0,v_0)$ in C^2 as $k\to\infty$. Let

$$
u_k=\sum_{j=0}^kc_j^k(t)\phi_j(x),\quad v_k=\sum_{j=0}^kd_j^k(t)\phi_j(x)
$$

be the finite-dimensional approximation of u, and v, and we want to solve the approximation system:

$$
\begin{cases}
(\rho_k)_t+(\rho_ku_k)_x=0,\\
\rho_k(u_k)_t+\rho_ku_k(u_k)_x+\left(\rho_k^\gamma\right)_x=J_k^1-(n_k)_{xx}(n_k)_x,\\
\rho_k(v_k)_t+\rho_ku_k(v_k)_x=J_k^2,\\
\gamma_1\left(\dot{n}_k-\frac{1}{2}(v_k)_x\right)-\gamma_2\left((u_k)_x\cos n_k\sin n_k+\frac{1}{2}(v_k)_x(1-2\cos^2n_k)\right)=(n_k)_{xx}.
\end{cases}
\tag{4.1}
$$

Here J_k^1, J_k^2 have the same form as J^1, J^2, but with u,v replaced by u_k, v_k. For this system, we consider the following initial and boundary values

$$(\rho_k, u_k, v_k, n_k)(x,0)=(\rho_0, u_0^k, v_0^k, n_0)(x), \tag{4.2}$$

$$u_k(0,t)=v_k(0,t)=u_k(1,t)=v_k(1,t)=0, \quad (n_k)_x(0,t)=(n_k)_x(1,t)=0. \tag{4.3}$$

Step 2. The first step is to solve ρ_k and n_k by assuming $u_k,v_k\in C^0(0,T;C^2)$ for a fixed k. To this end, we rewrite the equations of ρ_k and n_k in the Lagrange coordinate system.

Without loss of generality, in this section, we assume that

$$\int_0^1 \rho_0(x)\,dx=1. \tag{4.4}$$

For any $T>0$, we introduce the Lagrangian coordinate $(X,\tau)\in(0,1)\times[0,T)$ by

$$X(x,t)=\int_0^x \rho_k(y,t)\,dy, \quad \tau(x,t)=t.$$

If $\rho_k(x,t)\in C^1((0,1)\times[0,T))$ is positive and $\int_0^1\rho_k(x,t)\,dx=1$ for all $t\in[0,T)$, then the map $(x,t)\to(X,\tau):(0,1)\times(0,T)\to(0,1)\times(0,T)$ is a C^1-bijection such that $X(0,t)=0$, $X(1,t)=1$. By the chain rule, we have

$$\frac{\partial}{\partial t}=-\rho_k u_k\frac{\partial}{\partial X}+\frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x}=\rho_k\frac{\partial}{\partial X}.$$

The equation of ρ_k can be rewritten as

$$(\rho_k)_\tau+\rho_k^2(u_k)_X=0, \tag{4.5}$$

along with the initial condition

$$\rho_k(X,0)=\rho_0. \tag{4.6}$$

Suppose $u_k\in C^0(0,T;C^2)$ with $\|u_k\|_{C^0(0,T;C^2)}\le M_0$. Then ρ_k can be solved explicitly by

$$\rho_k(X,\tau)=\frac{\rho_0(X)}{1+\rho_0(X)\int_0^\tau(u_k)_X(X,s)\,ds}. \tag{4.7}$$

Hence, for any $T\le\frac{1}{2c_0M_0}$, we have

$$\rho_k(X,\tau)\le\frac{\rho_0(X)}{1-\left|\rho_0(X)\int_0^\tau(u_k)_X(X,s)\,ds\right|}\le\frac{c_0}{1-c_0M_0T}\le 2c_0, \tag{4.8}$$

$$\rho_k(X,\tau)\ge\frac{\rho_0(X)}{1+\left|\rho_0(X)\int_0^\tau(u_k)_X(X,s)\,ds\right|}\ge\frac{c_0^{-1}}{1+c_0M_0T}\ge\frac{c_0^{-1}}{2}. \tag{4.9}$$

Similarly, since $\rho_0\in C^1$, $u_k\in C^0(0,T;C^2)$, we conclude that for sufficiently small $T(c_0,M_0)>0$,

$$\|\rho_k\|_{C^0(0,T;C^1)}+\|(\rho_k)_t\|_{C^0((0,1)\times(0,T))}\le M_1, \tag{4.10}$$

for some positive constant M_1.

Furthermore, suppose that ρ_k^1,ρ_k^2 are solutions of Equation (4.5) corresponding to $u_k^1,u_k^2\in C^0(0,T;C^2)$, with the same initial condition, we can conclude from (4.5) that

$$\left(\frac{1}{\rho_k^1}-\frac{1}{\rho_k^2}\right)_\tau=\left(u_k^1-u_k^2\right)_X.$$

Integrating with respect to τ, we obtain

$$\rho_k^1-\rho_k^2=\rho_k^1\rho_k^2\int_0^\tau\left(u_k^1-u_k^2\right)_X$$

which, combined with (4.10), implies that

$$\|\rho_k^1-\rho_k^2\|_{C^0(0,T;C^1)}\le C(M_1,T)T\|u_k^1-u_k^2\|_{C^0(0,T;C^2)}. \tag{4.11}$$

Step 3. Similarly, we can rewrite the equation of n in the Lagrange coordinate as

$$\begin{aligned}&\gamma_1\left((n_k)_\tau-\frac{1}{2}\rho_k(v_k)_X\right)-\rho_k\left(\rho_k(n_k)_X\right)_X\\&=\frac{\gamma_2}{2}\left(\rho_k(u_k)_X\sin(2n_k)+\rho_k(v_k)_X\cos(2n_k)\right).\end{aligned} \tag{4.12}$$

For this system, we consider the following initial and boundary values

$$n_k(X,0)=n_0(X), \tag{4.13}$$

$$(n_k)_X(0,\tau)=(n_k)_X(1,\tau)=0. \tag{4.14}$$

By the standard Schauder theory of parabolic equations (cf. [6]), we conclude that

$$\begin{aligned}&\|n_k\|_{C^1(0,T;C^2)}\\&\le C\|n_0\|_{C^2}+C\|\rho_k(v_k)_X\|_{C^0((0,1)\times(0,T))}+C\|\rho_k(u_k)_X\|_{C^0((0,1)\times(0,T))}\le M_2,\end{aligned} \tag{4.15}$$

for some positive constant M_2.

Furthermore, suppose that n_k^1,n_k^2 are solutions of Equation (4.12) corresponding to $\rho_k^1,\rho_k^2\in C^1((0,1)\times(0,T))$ and $u_k^1,u_k^2\in C^0(0,T;C^2)$, subject to the same initial condition. Denote

$$\bar n_k=n_k^1-n_k^2,\quad \bar\rho_k=\rho_k^1-\rho_k^2,\quad \bar u_k=u_k^1-u_k^2.$$

Then from (4.12) we have that

$$\begin{aligned}&\gamma_1(\bar n_k)_\tau-(\rho_k^1)^2(\bar n_k)_{XX}\\&=\bar\rho_k(\rho_k^1+\rho_k^2)(n_k^2)_{XX}+\bar\rho_k(\rho_k^1)_X(n_k^1)_X+\rho_k^2(\bar\rho_k)_X(n_k^1)_X+\rho_k^2(\rho_k^2)_X(\bar n_k)_X\\&\quad+\frac{\gamma_1}{2}\left(\bar\rho_k(v_k^1)_X+\rho_k^2(\bar v_k^1)_X\right)\\&\quad-\frac{\gamma_2}{2}\left(\bar\rho_k(v_k^1)_X\cos(2n_k^1)+\rho_k^2(\bar v_k)_X\cos(2n_k^1)-2\rho_k^2(v_k^2)_X\sin(\bar n_k)\sin(n_k^1+n_k^2)\right)\\&\quad+\frac{\gamma_2}{2}\left(\bar\rho_k(u_k^1)_X\sin(2n_k^1)+\rho_k^2(\bar u_k)_X\sin(2n_k^1)+2\rho_k^2(u_k^2)_X\sin(\bar n_k)\cos(n_k^1+n_k^2)\right).\end{aligned}$$

By the standard $W_2^{2,1}$-estimate of parabolic equations (cf. [6]), we conclude that

$$\begin{aligned}&\|\bar n_k\|_{W_2^{2,1}([0,1]\times(0,T))}\\&\le C\|\bar\rho_k\|_{L^2(0,T;H^1)}+C\|\bar n_k\|_{L^2(0,T;L^2)}+C\|\bar v_k\|_{L^2(0,T;H^1)}+C\|\bar u_k\|_{L^2(0,T;H^1)}\\&\le CT^{\frac12}\|\bar\rho_k\|_{C^0(0,T;C^1)}+C\|\bar n_k\|_{L^2(0,T;L^2)}+CT^{\frac12}\|\bar v_k\|_{C^0(0,T;C^1)}+CT^{\frac12}\|\bar u_k\|_{C^0(0,T;C^1)}\\&\le CT^{\frac12}\|\bar u_k\|_{C^0(0,T;C^2)}+CT^{\frac12}\|\bar v_k\|_{C^0(0,T;C^1)}+C\|\bar n_k\|_{L^2(0,T;L^2)}.\end{aligned}$$

Since $\bar{n}_k(\tau,0)=0$, we obtain that

$$\|\bar{n}_k\|_{L^2(0,T;L^2)} \le CT\|\bar{n}_k\|_{W_2^{2,1}([0,1]\times(0,T))}.$$

If we choose $T>0$ small enough, we obtain

$$\|\bar{n}_k\|_{W_2^{2,1}([0,1]\times(0,T))} \le C(M_1,M_2,T)T^{\frac{1}{2}}\left(\|\bar{u}_k\|_{C^0(0,T;C^2)}+\|\bar{v}_k\|_{C^0(0,T;C^1)}\right). \tag{4.16}$$

Step 4. To obtain the estimates for u_k and v_k, first notice that the equation of u_k and v_k can be understood in the weak sense, i.e., for any $\phi(x)\in\mathcal{X}_k$ and $t\in[0,T]$, it holds

$$\begin{aligned}&\int_0^1 \rho_k u_k\phi-\int \rho_0 u_0^k\phi\\ =&\int_0^t\int_0^1 \mathcal{P}^1(\rho_k,u_k,v_k,n_k)\phi+(\alpha_2+\alpha_3)\int_0^t\int_0^1 \dot{n}\cos n\sin n\phi_x,\end{aligned} \tag{4.17}$$

$$\begin{aligned}&\int_0^1 \rho_k v_k\phi-\int \rho_0 v_0^k\phi\\ =&\int_0^t\int_0^1 \mathcal{P}^2(\rho_k,u_k,v_k,n_k)\phi-\int_0^t\int_0^1 \left(\alpha_2\dot{n}_k\cos^2 n_k-\alpha_3\dot{n}_k\sin^2 n_k\right)\phi_x,\end{aligned} \tag{4.18}$$

where

$$\begin{aligned}&\mathcal{P}^1(\rho_k,u_k,v_k,n_k)\\ =&(\alpha_0+\alpha_5+\alpha_6+\alpha_8)\left((u_k)_x\cos^2 n_k\right)_x+\alpha_1\left((u_k)_x\cos^4 n_k\right)_x+(\alpha_4+\alpha_7)(u_k)_{xx}\\ &+\alpha_0\left((v_k)_x\cos n_k\sin n_k\right)_x+\alpha_1\left((v_k)_x\cos^3 n_k\sin n_k\right)_x\\ &+\frac{1}{2}(\alpha_2+\alpha_3+\alpha_5+\alpha_6)\left((v_k)_x\cos n_k\sin n_k\right)_x-(\rho_k u_k u_k)_x-\left(\rho_k^\gamma\right)_x-(n_k)_{xx}(n_k)_x,\end{aligned}$$

and

$$\begin{aligned}&\mathcal{P}^2(\rho_k,u_k,v_k,n_k)\\ =&\alpha_1\left((v_k)_x\cos^2 n_k\sin^2 n_k\right)_x+\frac{1}{2}(-\alpha_2+\alpha_5)\left((v_k)_x\cos^2 n_k\right)_x\\ &+\frac{1}{2}(\alpha_3+\alpha_6)\left((v_k)_x\sin^2 n_k\right)_x+\frac{1}{2}\alpha_4(v_k)_{xx}\\ &+\alpha_1\left((u_k)_x\cos^3 n_k\sin n_k\right)_x+(\alpha_6+\alpha_8)\left((u_k)_x\cos n_k\sin n_k\right)_x-(\rho_k v_k v_k)_x.\end{aligned}$$

Similarly to the energy inequality (1.10), we can obtain the same form of energy estimates for the system (4.1) so that

$$\|u_k\|_{C^0(0,T;C^2)}+\|v_k\|_{C^0(0,T;C^2)} \le C\|u_k\|_{C^0(0,T;L^2)}+C\|v_k\|_{C^0(0,T;L^2)} \le M_3, \tag{4.19}$$

provided $\inf\limits_{(x,t)}\rho_k(x,t)>0$. Here we have used the fact that the dimension of $\mathcal{X}_k$ is finite.

To apply the contraction map theorem, we define the linear operator $\mathcal{N}[\rho_k]:\mathcal{X}_k\to\mathcal{X}_k^*$ by

$$\langle\mathcal{N}[\rho_k]\psi,\,\phi\rangle=\int_0^1 \rho_k\psi\phi,\ \ \psi,\phi\in\mathcal{X}_k.$$

It is easy to see that

$$\|\mathcal{N}[\rho_k]\|_{\mathcal{L}(\mathcal{X}_k,\mathcal{X}_k^*)} \le C(k)\|\rho_k\|_{L^1}.$$

If $\inf_x \rho_k > 0$, the operator $\mathcal{N}[\rho_k]$ is invertible and

$$\|\mathcal{N}^{-1}[\rho_k]\|_{\mathcal{L}(\mathcal{X}_k^*,\mathcal{X}_k)} \le \left(\inf_x \rho_k\right)^{-1}.$$

Furthermore, for any $\rho_k^i \in L^1$ and $\inf_x \rho_k^i > 0$, $i=1,2$, it is easy to see that

$$\mathcal{N}^{-1}[\rho_k^1] - \mathcal{N}^{-1}[\rho_k^2] = \mathcal{N}^{-1}[\rho_k^2]\left(\mathcal{N}[\rho_k^2] - \mathcal{N}[\rho_k^1]\right)\mathcal{N}^{-1}[\rho_k^1],$$

which implies that

$$\left\|\mathcal{N}^{-1}[\rho_k^1] - \mathcal{N}^{-1}[\rho_k^2]\right\|_{\mathcal{L}(\mathcal{X}_k^*,\mathcal{X}_k)} \le C\left\|\mathcal{N}[\rho_k^1] - \mathcal{N}[\rho_k^2]\right\|_{\mathcal{L}(\mathcal{X}_k,\mathcal{X}_k^*)} \le C\|\rho_k^1 - \rho_k^2\|_{L^1}. \tag{4.20}$$

Hence by the estimates (4.11), (4.16) and (4.20), we can apply the standard contraction map theorem to obtain the local existence of a unique solution $u_k, v_k \in C(0,T_k;\mathcal{X}_k)$ to (4.17) and (4.18) for some $T_k > 0$. Then by the Equations (4.5) and (4.12), we can solve for ρ_k, n_k, which provides a unique local solution to the approximated system (4.1) for any fixed k.

Step 5. In this step, we will establish a uniform estimate of the local solution until T_k in order to extend the solution beyond T_k to any time $T > 0$, which implies the existence of unique global solution of the system (4.1) for any fixed k. We first show the following uniform estimate for ρ_k

Claim: For any $x \in [0,1]$ and $t \in [0,T_k]$, it holds

$$\frac{1}{c_1 e^t} \le \rho_k(x,t) \le c_1 e^t \tag{4.21}$$

for some constant $c_1 > 0$.

Indeed, similar to the energy inequality (1.10), we can obtain the same form of energy estimate for system (4.1) so that

$$\begin{aligned} &\|(u_k)_x\|_{L^2(0,T_k;H^2)} + \|(v_k)_x\|_{L^2(0,T_k;H^2)} \\ \le& C\|(u_k)_x\|_{L^2((0,1)\times(0,T_k))} + C\|(v_k)_x\|_{L^2((0,1)\times(0,T_k))} \le M_4. \end{aligned} \tag{4.22}$$

By the first equation of (4.1), we can find $x_0(t) \in (0,1)$ such that

$$\rho_k(x_0(t),t) = \int_0^1 \rho_k = \int_0^1 \rho_0 = 1.$$

Then

$$\frac{1}{\rho_k(x,t)} = \frac{1}{\rho_k(x_0(t),t)} + \int_{x_0(t)}^x \left(\frac{1}{\rho_k}\right)_y \le 1 + \frac{1}{2}\left\|\frac{1}{\rho_k(x,t)}\right\|_{L^\infty} + \frac{1}{2}\int_0^1 \rho_k \left|\left(\frac{1}{\rho_k}\right)_x\right|^2$$

which implies

$$\left\|\frac{1}{\rho_k(x,t)}\right\|_{L^\infty} \le 2 + \int_0^1 \rho_k \left|\left(\frac{1}{\rho_k}\right)_x\right|^2. \tag{4.23}$$

By the first equation of (4.1), we have

$$\begin{aligned}&\frac{d}{dt}\int_0^1\rho_k\left|\left(\frac{1}{\rho_k}\right)_x\right|^2\\&=\int_0^1(\rho_k)_t\left|\left(\frac{1}{\rho_k}\right)_x\right|^2+2\int_0^1\rho_k\left(\frac{1}{\rho_k}\right)_x\left(\frac{1}{\rho_k}\right)_{xt}\\&=-\int_0^1(\rho_ku_k)_x\left|\left(\frac{1}{\rho_k}\right)_x\right|^2+2\int_0^1\rho_k\left(\frac{1}{\rho_k}\right)_x\left(\frac{(\rho_ku_k)_x}{\rho_k^2}\right)_x.\end{aligned}\tag{4.24}$$

The last term on the right-hand side can be computed by

$$\begin{aligned}2\int_0^1\rho_k\left(\frac{1}{\rho_k}\right)_x\left(\frac{(\rho_ku_k)_x}{\rho_k^2}\right)_x&=2\int_0^1\rho_k\left(\frac{1}{\rho_k}\right)_x\left[\left(\left(-\frac{1}{\rho_k}\right)_xu_k\right)_x+\left(\frac{(u_k)_x}{\rho_k}\right)_x\right]\\&=-\int_0^1\rho_ku_k\frac{\partial}{\partial x}\left|\left(\frac{1}{\rho_k}\right)_x\right|^2+2\int_0^1\left(\frac{1}{\rho_k}\right)_x(u_k)_{xx}.\end{aligned}\tag{4.25}$$

Combining (4.25) with (4.24), we conclude that

$$\frac{d}{dt}\int_0^1\rho_k\left|\left(\frac{1}{\rho_k}\right)_x\right|^2=2\int_0^1\left(\frac{1}{\rho_k}\right)_x(u_k)_{xx}.\tag{4.26}$$

The right-hand side can be estimated as follows

$$\begin{aligned}\left|\int_0^1\left(\frac{1}{\rho_k}\right)_x(u_k)_{xx}\right|&\le\int_0^1\rho_k^{\frac{1}{2}}\left|\left(\frac{1}{\rho_k}\right)_x\right|\rho_k^{-\frac{1}{2}}|(u_k)_{xx}|\\&\le\frac{1}{2}\int_0^1\rho_k\left|\left(\frac{1}{\rho_k}\right)_x\right|^2+\frac{1}{2}\left\|\frac{1}{\rho_k(x,t)}\right\|_{L^\infty}\int_0^1|(u_k)_{xx}|^2\\&\le\frac{1}{2}\left(1+\int_0^1|(u_k)_{xx}|^2\right)\int_0^1\rho_k\left|\left(\frac{1}{\rho_k}\right)_x\right|^2+\int_0^1|(u_k)_{xx}|^2.\end{aligned}$$

where we have used (4.23) in last inequality. Denote

$$\mathcal{Q}(\rho_k)=\int_0^1\rho_k\left|\left(\frac{1}{\rho_k}\right)_x\right|^2,\quad a(t)=1+\int_0^1|(u_k)_{xx}|^2.$$

Then by (4.25), we have

$$\frac{d}{dt}\mathcal{Q}(\rho_k)\le a(t)\mathcal{Q}(\rho_k)+\int_0^1|(u_k)_{xx}|^2$$

which is equivalent to

$$\mathcal{Q}(\rho_k)-\mathcal{Q}(\rho_0)\le2\int_0^t\int_0^1|(u_k)_{xx}|^2+\int_0^ta(t)\mathcal{Q}(\rho_k)\le2M_4+\int_0^ta(t)\mathcal{Q}(\rho_k),$$

where we have used (4.22) in last step. By the Gronwall inequality, we obtain

$$\mathcal{Q}(\rho_k)\le\big(\mathcal{Q}(\rho_0)+2M_4\big)\exp\left(\int_0^ta(s)\right)\le\big(\mathcal{Q}(\rho_0)+2M_4\big)\exp(t+M_4)\le Ce^t.\tag{4.27}$$

Combining (4.23) and (4.27) together, we can prove the left-hand side of (4.21).

Denote $\gamma=1+2\delta$ for some $\delta>0$. Then it holds

$$\|\rho_k^\delta\|_{L^\infty}\leq\int_0^1\rho_k^\delta+\delta\int_0^1\rho_k^{\delta-1}(\rho_k)_x\leq\left(\int_0^1\rho_k^\gamma\right)^{\frac{\delta}{\gamma}}+C\left(\int_0^1\rho_k^\gamma\right)^{\frac{1}{2}}(\mathcal{Q}(\rho_k))^{\frac{1}{2}}\leq Ce^t,$$

which completes the proof of the Claim.

By using the uniform estimate (4.21) for ρ_k and the energy inequality, we can show

$$\|u_k\|_{C^0(0,T_k;\mathcal{X}_k)}+\|v_k\|_{C^0(0,T_k;\mathcal{X}_k)}\leq C\|u_k\|_{C^0(0,T_k;L^2)}+C\|v_k\|_{C^0(0,T_k;L^2)}\leq M_5.$$

Therefore, we can extend the solution beyond T_k to any time $T>0$, which implies the existence of a unique smooth solution of the system (4.1) for any fixed k.

5. Existence of global weak solutions.

Step 1. Taking $k\to\infty$ in the approximated system (4.1), we may obtain the existence of a global weak solution with a smooth initial and boundary value and $\rho_0>\delta>0$. Since the limit process of this step is similar to the next step when $\delta\to 0$, we omit the details of this step.

Step 2. We first approximate the general initial and boundary data in Theorem 1.1 by smooth functions. We may extend n to $\tilde{n}_0\in H^1(\mathbb{R})$ such that $n_0=\tilde{n}_0$ on $(0,1)$, and obtain the smooth approximation of initial data by the standard mollification as follows

$$\rho_0^\delta=\eta_\delta*\hat{\rho}_0+\delta,\quad u_0^\delta=\frac{1}{\sqrt{\rho_0^\delta}}\eta_\delta*\left(\widehat{\frac{m_0}{\sqrt{\rho_0}}}\right),\quad v_0^\delta=\frac{1}{\sqrt{\rho_0^\delta}}\eta_\delta*\left(\widehat{\frac{l_0}{\sqrt{\rho_0}}}\right),\quad n_0^\delta=\frac{\eta_\delta*\tilde{n}_0}{|\eta_\delta*\tilde{n}_0|}$$

where, for small $\delta>0$, $\eta_\delta=\frac{1}{\delta}\eta\left(\frac{\cdot}{\delta}\right)$ is the standard mollifier, $\hat{f}$ is the zero extension of f from $(0,1)$ to $\mathbb{R}$. Therefore $\rho_0^\delta,u_0^\delta,v_0^\delta,n_0^\delta\in C^{2+\alpha}([0,1])$ for $0<\alpha<1$, and it holds

$$\rho_0^\delta\geq\delta>0,\quad \rho_0^\delta\to\rho_0 \text{ in } L^\gamma,\quad n_0^\delta\to n_0 \text{ in } H^1, \tag{5.1}$$

$$\sqrt{\rho_0^\delta}u_0^\delta\to\frac{m_0}{\sqrt{\rho_0}} \text{ in } L^2,\quad \sqrt{\rho_0^\delta}v_0^\delta\to\frac{l_0}{\sqrt{\rho_0}} \text{ in } L^2, \tag{5.2}$$

$$\rho_0^\delta u_0^\delta\to m_0 \text{ in } L^{\frac{2\gamma}{\gamma+1}},\quad \rho_0^\delta v_0^\delta\to l_0 \text{ in } L^{\frac{2\gamma}{\gamma+1}}, \tag{5.3}$$

as $\delta\to 0$.

Let $(\rho_\delta,u_\delta,v_\delta,n_\delta)$ be a sequence of global weak solutions to

$$\begin{cases}(\rho_\delta)_t+(\rho_\delta u_\delta)_x=0,\quad \rho_\delta>0,\\ (\rho_\delta u\delta)_t+(\rho_\delta u_\delta^2)_x+\left(\rho_\delta^\gamma\right)_x=J_\delta^1-(n_\delta)_{xx}(n_\delta)_x,\\ (\rho_\delta v_\delta)_t+(\rho_\delta u_\delta v_\delta)_x=J_\delta^2,\\ \gamma_1\left(\dot{n}_\delta-\frac{1}{2}(v_\delta)_x\right)-\gamma_2\left((u_\delta)_x\cos n_\delta\sin n_\delta+\frac{1}{2}(v_\delta)_x(1-2\cos^2 n_\delta)\right)=(n_\delta)_{xx},\end{cases} \tag{5.4}$$

with the initial and boundary values

$$(\rho_\delta,u_\delta,v_\delta,n_\delta)(x,0)=(\rho_0^\delta,u_0^\delta,v_0^\delta,n_0^\delta)(x), \tag{5.5}$$

$$u_\delta(0,t)=v_\delta(0,t)=u_\delta(1,t)=v_\delta(1,t)=0,\quad (n_\delta)_x(0,t)=(n_\delta)_x(1,t)=0. \tag{5.6}$$

Here J_δ^1 and J_δ^2 have the same forms as J^1 and J^2, but with (u,v,n) replaced by $(u_\delta,v_\delta,n_\delta)$.

By Lemma 3.1–Lemma 3.3, we can find a subsequence $(\rho_\delta,u_\delta,v_\delta,n_\delta)$, still denoted as $(\rho_\delta,u_\delta,v_\delta,n_\delta)$, such that for any $T>0$, as $\delta\to 0$,

$$\rho_\delta \overset{*}{\rightharpoonup} \rho, \text{ in } L^\infty(0,T;L^\gamma), \qquad \rho_\delta \rightharpoonup \rho, \text{ in } L^{2\gamma}([0,1]\times[0,T]), \tag{5.7}$$

$$\rho_\delta^\gamma \rightharpoonup \overline{\rho^\gamma}, \text{ in } L^2([0,1]\times[0,T]), \tag{5.8}$$

$$u_\delta \rightharpoonup u, \text{ in } L^2(0,T;H_0^1), \quad v_\delta \rightharpoonup v, \text{ in } L^2(0,T;H_0^1), \tag{5.9}$$

$$n_\delta \overset{*}{\rightharpoonup} n, \text{ in } L^\infty([0,1]\times[0,T]), \quad (n_\delta)_x \overset{*}{\rightharpoonup} n_x, \text{ in } L^\infty(0,T;L^2), \tag{5.10}$$

$$(n_\delta)_t \rightharpoonup n_t, \text{ in } L^2([0,1]\times[0,T]), \quad (n_\delta)_{xx} \rightharpoonup n_{xx}, \text{ in } L^2([0,1]\times[0,T]). \tag{5.11}$$

Since $\rho_\delta>0$, for any nonnegative function $f\in C_0^\infty((0,1)\times(0,T))$ it holds that

$$\int_0^T\int_0^1 \rho f = \lim_{\delta\to 0}\int_0^T\int_0^1 \rho_\delta f \ge 0.$$

Since f is arbitrary, we conclude that $\rho\ge 0$ a.e. in $(0,1)\times(0,T)$.

We need to show the limit (ρ,u,v,n) is a solution to the system (4.1). We first state several compactness results that will be used in our proof.

LEMMA 5.1 ([23]). *Assume $X\subset E\subset Y$ are Banach spaces and $X\hookrightarrow\hookrightarrow E$ is compact. Then the following embeddings are compact*

$$\left\{f: f\in L^q(0,T;X), \frac{\partial f}{\partial t}\in L^1(0,T;Y)\right\}\hookrightarrow\hookrightarrow L^q(0,T;E), \text{ for any } 1\le q\le\infty,$$

$$\left\{f: f\in L^\infty(0,T;X), \frac{\partial f}{\partial t}\in L^r(0,T;Y)\right\}\hookrightarrow\hookrightarrow C([0,T];E), \text{ for any } 1<r<\infty.$$

LEMMA 5.2 ([7]). *Let $\bar O\subset\mathbb{R}^n$ be compact and X be a separable Banach space. Assume that $f_\delta:\bar O\to X^*$ is a sequence of measurable functions such that for any k*

$$\operatorname{esssup}_{\bar O}\|f_\delta\|_{X^*}\le N<\infty.$$

Moreover, the family of functions $\langle f_\delta,\Phi\rangle$ is equi-continuous for any Φ belonging to a dense subset of X. Then $f_\delta\in C(\bar O;X-w)$ for any k, i.e., for any $g\in X*$, $\langle f_\delta,g\rangle\in C(\bar O)$. Furthermore, there exists $f\in C(\bar O;X-w)$ such that (after taking possible subsequences)

$$f_\delta\to f, \quad \text{in } C(\bar O;X-w)$$

as $\delta\to 0$.

First observe that $\rho_\delta\in L^{2\gamma}([0,1]\times[0,T])$ and $u_\delta\in L^2(0,T;H_0^1)\subset L^2(0,T;L^\infty)$ imply

$$\rho_\delta u_\delta\in L^{\frac{2\gamma}{\gamma+1}}(0,T;L^{2\gamma}), \quad (\rho_\delta)_t=-(\rho_\delta u_\delta)_x\in L^{\frac{2\gamma}{\gamma+1}}(0,T;H^{-1}).$$

By Lemma 5.1 and Lemma 5.2, and $\frac{2\gamma}{\gamma+1}>1$, $\rho_\delta\in L^\infty(0,T;L^\gamma)$, $L^\gamma\hookrightarrow\hookrightarrow H^{-1}$, we conclude

$$\rho_\delta\to\rho, \text{ in } C(0,T;L^\gamma-\omega), \quad \rho_\delta\to\rho, \text{ in } C(0,T;H^{-1}), \tag{5.12}$$

where $f \in C(0,T;X-\omega)$ if for any $g \in X^*$, $\langle f(t),g\rangle \in C([0,T])$. Hence

$$\rho_\delta u_\delta \to \rho u, \text{ in } \mathcal{D}'((0,1)\times(0,T)), \quad \rho_\delta v_\delta \to \rho v, \text{ in } \mathcal{D}'((0,1)\times(0,T)), \tag{5.13}$$

and furthermore

$$\rho_t + (\rho u)_x = 0, \text{ in } \mathcal{D}'((0,1)\times(0,T)). \tag{5.14}$$

By (5.12), it also holds that

$$\rho(x,0) = \rho_0(x), \text{ weakly in } L^\gamma([0,1]). \tag{5.15}$$

By the fact $(n_\delta)_t \in L^2(0,T;L^2)$, (5.10) and (5.11), we can apply Lemma 5.1 to obtain

$$n_\delta \to n, \text{ in } C([0,1]\times[0,T]), \quad n_\delta \to n, \text{ in } L^2(0,T;C^1), \tag{5.16}$$

Combining with (5.9)-(5.11), we can show the limit n satisfies the following equation:

$$\gamma_1\left(\dot{n} - \frac{1}{2}v_x\right) - \gamma_2\left(u_x \cos n \sin n + \frac{1}{2}v_x(1-2\cos^2 n)\right) = n_{xx}. \tag{5.17}$$

By (5.16), it also holds that

$$n(x,0) = n_0(x), \text{ in } [0,1]. \tag{5.18}$$

By the fact $\sqrt{\rho_\delta} \in L^{2\gamma}([0,1]\times[0,T])$ and $\sqrt{\rho_\delta}u_\delta \in L^\infty(0,T;L^2)$, it holds

$$\rho_\delta u_\delta \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}).$$

Combining with (5.9), we have

$$\rho_\delta u_\delta^2 \rightharpoonup \rho u^2, \text{ in } L^2(0,T;L^{\frac{2\gamma}{\gamma+1}}). \tag{5.19}$$

By the second equation of system (5.4), we have

$$(\rho_\delta u_\delta)_t = -(\rho_\delta u_\delta^2)_x - \left(\rho_\delta^\gamma\right)_x + J_\delta^1 - (n_\delta)_{xx}(n_\delta)_x \in L^2(0,T;W^{-1,\frac{2\gamma}{\gamma+1}}),$$

where J_δ^1 has the same form as J^1, but with (u,v,n) replaced by $(u_\delta,v_\delta,n_\delta)$. By using Lemma 5.1 and Lemma 5.2, we conclude

$$\rho_\delta u_\delta \to \rho u, \text{ in } C(0,T;L^{\frac{2\gamma}{\gamma+1}}-\omega), \quad \rho_\delta u_\delta \to \rho u, \text{ in } C(0,T;H^{-1}). \tag{5.20}$$

Combining with (5.9), we conclude that

$$\rho_\delta u_\delta^2 \to \rho u^2, \text{ in } \mathcal{D}'((0,1)\times(0,T)). \tag{5.21}$$

Therefore

$$(\rho u)_t + (\rho u^2)_x + \left(\overline{\rho^\gamma}\right)_x = J^1 - n_{xx}n_x, \text{ in } \mathcal{D}'((0,1)\times(0,T)). \tag{5.22}$$

By (5.20), it holds that

$$\rho u(x,0) = m_0(x), \text{ weakly in } L^{\frac{2\gamma}{\gamma+1}}([0,1]). \tag{5.23}$$

Similarly, we can also prove that

$$(\rho v)_t + (\rho u v)_x = J^2, \text{ in } \mathcal{D}'((0,1)\times(0,T)), \tag{5.24}$$

$$\rho v(x,0) = n_0(x), \text{ weakly in } L^{\frac{2\gamma}{\gamma+1}}([0,1]). \tag{5.25}$$

By (5.21), for some $t\in(0,T)$ and small $\epsilon>0$, it holds

$$\frac{1}{\epsilon}\int_t^{t+\epsilon}\int_0^1 \rho u^2 = \frac{1}{\epsilon}\int_t^{t+\epsilon}\lim_{\delta\to 0}\int_0^1 \rho_\delta u_\delta^2 \le \frac{1}{\epsilon}\int_t^{t+\epsilon}\overline{\lim_{\delta\to 0}}\int_0^1 \rho_\delta u_\delta^2.$$

Sending $\epsilon\to 0^+$ and using the Lebesgue Differentiation Theorem, we obtain

$$\int_0^1 \rho u^2 \le \overline{\lim_{\delta\to 0}}\int_0^1 \rho_\delta u_\delta^2,$$

for a.e. $t\in(0,T)$. Combining this limit with the lower semicontinuity, we can prove that the energy inequality is valid.

The only thing left is to show $\overline{\rho^\gamma} = \rho^\gamma$. To this end, we denote

$$A(n) = (A_{ij}(n))_{2\times 2}$$

where the elements of A_{ij} are given as follows

$$\begin{aligned}
A_{11}(n) &= (\alpha_0+\alpha_5+\alpha_6+\alpha_8)\cos^2 n + \alpha_1\cos^4 n + (\alpha_4+\alpha_7),\\
A_{12}(n) &= \alpha_0\cos n\sin n + \alpha_1\cos^3 n\sin n + \frac{1}{2}(\alpha_2+\alpha_3+\alpha_5+\alpha_6)\cos n\sin n,\\
A_{21}(n) &= \alpha_1\cos^3 n\sin n + (\alpha_6+\alpha_8)\cos n\sin n,\\
A_{22}(n) &= \alpha_1\cos^2 n\sin^2 n + \frac{1}{2}(-\alpha_2+\alpha_5)\cos^2 n + \frac{1}{2}(\alpha_3+\alpha_6)\sin^2 n + \frac{1}{2}\alpha_4.
\end{aligned}$$

By the relations (1.6), direct computations imply that there exist two positive constants $\lambda,\Lambda<\infty$ such that for any $\mathbf{y}\in\mathbb{R}^2$

$$\lambda|\mathbf{y}|^2 \le \mathbf{y}^T A(n)\mathbf{y} \le \Lambda|\mathbf{y}|^2. \tag{5.26}$$

In fact

$$\begin{aligned}
&\mathbf{y}^T A(n)\mathbf{y}\\
=&A_{11}(n)y_1^2 + (A_{12}(n)+A_{21}(n))y_1y_2 + A_{22}(n)y_2^2\\
=&\left[(\alpha_0+\alpha_5+\alpha_6+\alpha_8)\cos^2 n + \alpha_1\cos^4 n + (\alpha_4+\alpha_7)\right]y_1^2\\
&+\left[(\alpha_0+\alpha_6+\alpha_8)\cos n\sin n + 2\alpha_1\cos^3 n\sin n + \frac{1}{2}(\alpha_2+\alpha_3+\alpha_5+\alpha_6)\cos n\sin n\right]y_1y_2\\
&+\left[\alpha_1\cos^2 n\sin^2 n + \frac{1}{2}(-\alpha_2+\alpha_5)\cos^2 n + \frac{1}{2}(\alpha_3+\alpha_6)\sin^2 n + \frac{1}{2}\alpha_4\right]y_2^2\\
=&\frac{1}{4}\left(\frac{\gamma_2}{\sqrt{\gamma_1}}y_1\sin(2n) + \frac{1}{\sqrt{\gamma_1}}(\gamma_1-\gamma_2\cos(2n))y_2\right)^2\\
&+\frac{1}{4}\left(-\alpha_1-\frac{\gamma_2^2}{\gamma_1}\right)y_1^2 + (\alpha_4+\alpha_7)y_1^2 + \frac{1}{4}\left(2\alpha_4+\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1}\right)y_2^2\\
&\frac{1}{4}\left(\alpha_1+\frac{\gamma_2^2}{\gamma_1}\right)(y_1\cos(2n)+y_2\sin(2n))^2\\
&+(\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)\left[\left(y_1\cos n + \frac{1}{2}y_2\sin n\right)^2 - \frac{1}{4}y_2^2\sin^2 n\right].
\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{y}^T A(n)\mathbf{y} \ge & \frac{1}{4}\left(-\alpha_1 - \frac{\gamma_2^2}{\gamma_1}\right) y_1^2 + (\alpha_4+\alpha_7)y_1^2 + \frac{1}{4}\left(2\alpha_4+\alpha_5+\alpha_6 - \frac{\gamma_2^2}{\gamma_1}\right) y_2^2 \\ & - \frac{1}{4}(\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8) y_2^2 \sin^2 n.\end{aligned}$$

If we take

$$\lambda = \min\left\{(\alpha_4+\alpha_7) - \frac{1}{4}\left(\alpha_1 + \frac{\gamma_2^2}{\gamma_1}\right),\ \left(2\alpha_4+\alpha_5+\alpha_6 - \frac{\gamma_2^2}{\gamma_1}\right) - (\alpha_0+\alpha_1+\alpha_5+\alpha_6+\alpha_8)\right\},$$

then by the relation (1.6), we know that $\lambda > 0$ and we have shown the estimate (5.26).

By the definition of $A(n)$, we see that the matrix-valued function $A(\cdot) \in C^\infty$. By the estimate (5.26), the inverse matrix function A^{-1} exists and

$$\frac{d}{dn}\big(A^{-1}(n)\big) = A^{-1}\frac{d}{dn}\big(A(n)\big)A^{-1}.$$

The equations for $\mathbf{u} = (u,v)^T$ can be written as

$$\rho \mathbf{u}_t + \rho u \mathbf{u}_x + \mathbf{P}_x = \big(A(n)\mathbf{u}_x\big)_x + (B_1(n))_x - B_2(n) \tag{5.27}$$

where

$$\begin{aligned}&\mathbf{P} = (\overline{\rho^\gamma}, 0)^T,\\ &B_1(n) = \big((\alpha_2+\alpha_3)\dot{n}\cos n \sin n,\ \alpha_2 \dot{n}\cos^2 n - \alpha_3 \dot{n} \sin^2 n\big)^T,\\ &B_2(n) = \big(n_{xx} n_x, 0\big)^T.\end{aligned}$$

Similarly, we can rewrite the equations for $\mathbf{u}_\delta = (u_\delta, v_\delta)^T$, $\mathbf{P}_\delta = (\rho_\delta^\gamma, 0)^T$ in the similar form

$$\rho_\delta(\mathbf{u}_\delta)_t + \rho_\delta u_\delta (\mathbf{u}_\delta)_x + (\mathbf{P}_\delta)_x = \big(A(n_\delta)(\mathbf{u}_\delta)_x\big)_x + (B_1(n_\delta))_x - B_2(n_\delta). \tag{5.28}$$

Denote

$$\mathcal{H} = \mathbf{u}_x - A^{-1}(n)\mathbf{P}, \quad \mathcal{H}_\delta = (\mathbf{u}_\delta)_x - A^{-1}(n_\delta)\mathbf{P}_\delta.$$

We have the following lemma.

LEMMA 5.3. *As $\delta \to 0$, it holds*

$$\rho_\delta \mathcal{H}_\delta \to \rho \mathcal{H}, \ \textit{in } \mathcal{D}'((0,1)\times(0,T)). \tag{5.29}$$

Proof. The main difficulty of the proof arises from $\rho u \notin L^2$. To overcome it, we need to mollify the density ρ by $\langle\hat{\rho}\rangle_\sigma = \eta_\sigma * \hat{\rho}$, where $\eta_\sigma = \frac{1}{\sigma}\sigma\left(\frac{\cdot}{\sigma}\right)$ is the standard mollifier, $\hat{f}$ is the zero extension of f from $(0,1)$ to $\mathbb{R}$. By Lemma 3.3 in [7], the zero-extension of $\hat{\rho}$ still satisfies the same equation

$$(\hat{\rho})_t + (\hat{\rho}\hat{u})_x = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)). \tag{5.30}$$

Denote $\tau^\sigma = (\langle\hat{\rho}\rangle_\sigma \hat{u})_x - \langle(\hat{\rho}\hat{u})_x\rangle_\sigma$. By Lemma 2.3 in [19], we know that $\tau^\sigma \in L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}\times(0,T)$, and as $\sigma \to 0$

$$\tau^\sigma \to 0, \quad \text{in } L^1(\mathbb{R}\times(0,T)). \tag{5.31}$$

Taking the standard mollifier as the test function, we obtain

$$(\langle\hat{\rho}\rangle_\sigma)_t+(\langle\hat{\rho}\rangle_\sigma\hat{u})_x=\tau^\sigma,\quad \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)). \tag{5.32}$$

Similarly, it also holds for the approximate solutions

$$(\langle\hat{\rho}_\delta\rangle_\sigma)_t+(\langle\hat{\rho}_\delta\rangle_\sigma\hat{u}_\delta)_x=\tau_\delta^\sigma,\quad \text{in } \mathcal{D}'(\mathbb{R}\times(0,T)), \tag{5.33}$$

where τ_δ^σ has the same form as τ^σ, but with ρ,u replaced by ρ_δ,u_δ. We also know that, for any $\delta>0$, $\tau_\delta^\sigma\in L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}\times(0,T))$, and as $\sigma\to0$

$$\tau_\delta^\sigma\to0,\quad \text{in } L^1(\mathbb{R}\times(0,T)). \tag{5.34}$$

Multiplying the Equation (5.28) by $\varphi\phi A^{-1}(n_\delta)\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma$ from left for any $\varphi\in C_0^\infty(0,T)$ and $\phi\in C_0^\infty(0,1)$, and integrating by parts, we obtain

$$\begin{aligned}
&\int_0^T\int_0^1\varphi\phi\mathcal{H}_\delta\langle\hat{\rho}_\delta\rangle_\sigma\\
=&\int_0^T\int_0^1\varphi'\phi\rho_\delta A^{-1}(n_\delta)\mathbf{u}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma+\int_0^T\int_0^1\varphi\phi\rho_\delta A^{-1}(n_\delta)\mathbf{u}_\delta\left(\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma\right)_t\\
&+\int_0^T\int_0^1\varphi\phi\rho_\delta\big(A^{-1}(n_\delta)\big)_t\mathbf{u}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma+\int_0^T\int_0^1\varphi\phi'\rho_\delta u_\delta A^{-1}(n_\delta)\mathbf{u}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma\\
&+\int_0^T\int_0^1\varphi\phi\rho_\delta\langle\hat{\rho}_\delta\rangle_\sigma u_\delta A^{-1}(n_\delta)\mathbf{u}_\delta+\int_0^T\int_0^1\varphi\phi\rho_\delta u_\delta\big(A^{-1}(n_\delta)\big)_x\mathbf{u}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma\\
&+\int_0^T\int_0^1\varphi\phi A^{-1}(n_\delta)(B_1(n_\delta))_x\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma-\int_0^T\int_0^1\varphi\phi A^{-1}(n_\delta)B_2(n_\delta)\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma\\
&-\int_0^T\int_0^1\varphi\phi'\mathcal{H}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma-\int_0^T\int_0^1\varphi\phi A^{-1}(n_\delta)\big(A(n_\delta)\big)_x\mathcal{H}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma.
\end{aligned}$$

The Equation (5.33) implies

$$\frac{\partial}{\partial t}\left(\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma\right)=-\langle\hat{\rho}_\delta\rangle_\sigma\hat{u}_\delta+\tau_\delta^\sigma.$$

Using this fact, we have

$$\begin{aligned}
&\int_0^T\int_0^1\varphi\phi\mathcal{H}_\delta\langle\hat{\rho}_\delta\rangle_\sigma\\
=&\int_0^T\int_0^1\varphi'\phi\rho_\delta A^{-1}(n_\delta)\mathbf{u}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma+\int_0^T\int_0^1\varphi\phi\rho_\delta A^{-1}(n_\delta)\mathbf{u}_\delta\int_0^x\tau_\delta^\sigma\\
&+\int_0^T\int_0^1\varphi\phi\rho_\delta\big(A^{-1}(n_\delta)\big)_t\mathbf{u}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma+\int_0^T\int_0^1\varphi\phi'\rho_\delta u_\delta A^{-1}(n_\delta)\mathbf{u}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma\\
&+\int_0^T\int_0^1\varphi\phi\rho_\delta u_\delta\big(A^{-1}(n_\delta)\big)_x\mathbf{u}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma\\
&+\int_0^T\int_0^1\varphi\phi A^{-1}(n_\delta)(B_1(n_\delta))_x\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma-\int_0^T\int_0^1\varphi\phi A^{-1}(n_\delta)B_2(n_\delta)\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma\\
&-\int_0^T\int_0^1\varphi\phi'\mathcal{H}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma-\int_0^T\int_0^1\varphi\phi A^{-1}(n_\delta)\big(A(n_\delta)\big)_x\mathcal{H}_\delta\int_0^x\langle\hat{\rho}_\delta\rangle_\sigma.
\end{aligned}$$

By the Lebesgue Dominated Convergence theorem and (5.34), we may take the limit $\sigma \to 0$ and get

$$\begin{aligned}
&\int_0^T\int_0^1 \varphi\phi\mathcal{H}_\delta\rho_\delta \\
=&\int_0^T\int_0^1 \varphi'\phi\rho_\delta A^{-1}(n_\delta)\mathbf{u}_\delta\int_0^x\rho_\delta+\int_0^T\int_0^1\varphi\phi\rho_\delta\big(A^{-1}(n_\delta)\big)_t\mathbf{u}_\delta\int_0^x\rho_\delta \\
&+\int_0^T\int_0^1 \varphi\phi'\rho_\delta u_\delta A^{-1}(n_\delta)\mathbf{u}_\delta\int_0^x\rho_\delta+\int_0^T\int_0^1\varphi\phi\rho_\delta u_\delta\big(A^{-1}(n_\delta)\big)_x\mathbf{u}_\delta\int_0^x\rho_\delta \\
&+\int_0^T\int_0^1 \varphi\phi A^{-1}(n_\delta)(B_1(n_\delta))_x\int_0^x\rho_\delta-\int_0^T\int_0^1\varphi\phi A^{-1}(n_\delta)B_2(n_\delta)\int_0^x\rho_\delta \\
&-\int_0^T\int_0^1\varphi\phi'\mathcal{H}_\delta\int_0^x\rho_\delta-\int_0^T\int_0^1\varphi\phi A^{-1}(n_\delta)\big(A(n_\delta)\big)_x\mathcal{H}_\delta\int_0^x\rho_\delta.
\end{aligned}\tag{5.35}$$

By the definition of $B_2(n_\delta)$ and integration by parts, we obtain

$$\begin{aligned}
&-\int_0^T\int_0^1\varphi\phi A^{-1}(n_\delta)B_2(n_\delta)\int_0^x\rho_\delta \\
=&\frac12\int_0^T\int_0^1\varphi\phi' A^{-1}(n_\delta)\big(|(n_\delta)_x|^2,0\big)^T\int_0^x\rho_\delta \\
&+\frac12\int_0^T\int_0^1\varphi\phi\big(A^{-1}(n_\delta)\big)_x\big(|(n_\delta)_x|^2,0\big)^T\int_0^x\rho_\delta \\
&+\frac12\int_0^T\int_0^1\varphi\phi\rho_\delta A^{-1}(n_\delta)\big(|(n_\delta)_x|^2,0\big)^T.
\end{aligned}\tag{5.36}$$

By the definition of $B_1(n_\delta)$, we obtain

$$\begin{aligned}
&\int_0^T\int_0^1\varphi\phi A^{-1}(n_\delta)\big(B_1(n_\delta)\big)_x\int_0^x\rho_\delta \\
=&\int_0^T\int_0^1\varphi\phi\big(A^{-1}(n_\delta)B_1(n_\delta)\big)_x\int_0^x\rho_\delta-\int_0^T\int_0^1\varphi\phi\big(A^{-1}(n_\delta)\big)_x B_1(n_\delta)\int_0^x\rho_\delta.
\end{aligned}$$

It is not hard to see that there is a vector function $\mathcal{F}(n_\delta)$ (smooth in n_δ) such that

$$A^{-1}(n_\delta)B_1(n_\delta)=\mathcal{F}_t(n_\delta)+u_\delta\mathcal{F}_x(n_\delta).$$

Then

$$\begin{aligned}
&\int_0^T\int_0^1\varphi\phi A^{-1}(n_\delta)\big(B_1(n_\delta)\big)_x\int_0^x\rho_\delta \\
=&-\int_0^T\int_0^1\varphi'\phi\mathcal{F}_x(n_\delta)\int_0^x\rho_\delta-\int_0^T\int_0^1\varphi\phi' u_\delta\mathcal{F}_x(n_\delta)\int_0^x\rho_\delta \\
&-\int_0^T\int_0^1\varphi\phi\big(A^{-1}(n_\delta)\big)_x B_1(n_\delta)\int_0^x\rho_\delta.
\end{aligned}\tag{5.37}$$

To estimate the second term on right side of (5.35), we use $\varphi\phi n$ as the test function for the first equation of (5.4) to obtain

$$\int_0^T\int_0^1\varphi\phi\rho_\delta(n_\delta)_t=-\int_0^T\int_0^1\varphi'\phi\rho_\delta n_\delta-\int_0^T\int_0^1\varphi\rho_\delta u_\delta(n_\delta\phi)_x.$$

Similarly, it holds

$$\int_0^T\int_0^1 \varphi\phi\rho n_t = -\int_0^T\int_0^1 \varphi'\phi\rho n - \int_0^T\int_0^1 \varphi\rho u(n\phi)_x.$$

Taking the difference, and using (5.7), (5.13) and (5.16), we have

$$\rho_\delta(n_\delta)_t \to \rho n_t, \text{ in } \mathcal{D}'((0,1)\times(0,T)). \tag{5.38}$$

Furthermore, since

$$\int_0^x \rho_\delta \in L^\infty(0,T;W^{1,\gamma}), \quad \frac{\partial}{\partial t}\left(\int_0^x \rho_\delta\right) = -\rho_\delta u_\delta \in L^\infty\left(0,T;L^{\frac{2\gamma}{\gamma+1}}\right)$$

we obtain by Lemma 5.1 and (5.7)

$$\int_0^x \rho_\delta \to \int_0^x \rho, \quad \text{in } C([0,1]\times[0,T]), \quad \text{as } \delta\to 0. \tag{5.39}$$

Now, we are ready to take limit in (5.35). Letting $\delta\to 0$ in (5.35) (5.36) and (5.37), and using the facts (5.39), (5.38), (5.7)-(5.9), (5.13), (5.16) and (5.21), we obtain

$$\begin{aligned}
&\lim_{\delta\to 0}\int_0^T\int_0^1 \varphi\phi\mathcal{H}_\delta\rho_\delta\\
=&\int_0^T\int_0^1 \varphi'\phi\rho A^{-1}(n)\mathbf{u}\int_0^x\rho + \int_0^T\int_0^1 \varphi\phi\rho\left(A^{-1}(n)\right)_t\mathbf{u}\int_0^x\rho\\
&+\int_0^T\int_0^1 \varphi\phi'\rho u A^{-1}(n)\mathbf{u}\int_0^x\rho + \int_0^T\int_0^1 \varphi\phi\rho u\left(A^{-1}(n)\right)_x\mathbf{u}\int_0^x\rho\\
&+\int_0^T\int_0^1 \varphi\phi A^{-1}(n)(B_1(n))_x\int_0^x\rho - \int_0^T\int_0^1 \varphi\phi A^{-1}(n)B_2(n)\int_0^x\rho\\
&-\int_0^T\int_0^1 \varphi\phi'\mathcal{H}\int_0^x\rho - \int_0^T\int_0^1 \varphi\phi A(n)\left(A^{-1}(n)\right)_x\mathcal{H}\int_0^x\rho.
\end{aligned}\tag{5.40}$$

We may go through the same arguments for ρ and u, and show that right side of (5.40) is exactly

$$\int_0^T\int_0^1 \varphi\phi\mathcal{H}\rho,$$

which completes the proof of the lemma. ∎

We also need the following result.

Lemma 5.4 ([7]). *Let $\bar{O}\subset\mathbb{R}^n$ be a measurable set and $f_k\in L^1(O;\mathbb{R}^N)$ for $k\in\mathbb{Z}_+$ such that*

$$f_k \rightharpoonup f, \quad in \ \ L^1(O;\mathbb{R}^N).$$

Let $\Phi:\mathbb{R}^N\to(-\infty,\infty]$ be a lower semi-continuous convex function such that $\Phi(f_k)\in L^1(O)$ for any k and

$$\Phi(f_k) \rightharpoonup \overline{\Phi(f)}, \quad in \ \ L^1(O).$$

Then

$$\Phi(f) \le \overline{\Phi(f)}, \quad a.e. \ in \ \ O.$$

Moreover, if Φ *is strictly convex on an open convex set* $U \subset \mathbb{R}^N$ *and*

$$\Phi(f) = \overline{\Phi(f)}, \quad a.e. \ in \ \ O,$$

then

$$f_k \to f, \quad for \ \ a.e. \ y \in \{y \in O \,|\, f(y) \in U\}.$$

The proof of Theorem 1.1 will be completed by the following lemma.

LEMMA 5.5. *As* $\delta \to 0$*, it holds*

$$\lim_{\delta \to 0} \int_0^T \int_0^1 \rho_\delta \log(\rho_\delta) = \int_0^T \int_0^1 \rho \log \rho. \tag{5.41}$$

Proof. By Proposition 4.2 in [8], if $\rho \in L^2((0,1)\times(0,T))$, $u \in L^2(0,T;H_0^1)$ solves the equation

$$\rho_t + (\rho u)_x = 0, \quad \text{in } \mathcal{D}'((0,1)\times(0,T))$$

then

$$(b(\rho))_t + (b(\rho)u)_x + (b'(\rho)\rho - b(\rho))u_x = 0, \quad \text{in } \mathcal{D}'((0,1)\times(0,T)) \tag{5.42}$$

for any $b \in C^1(\mathbb{R})$ such that $b'(x) \equiv 0$ for all large enough $x \in \mathbb{R}$.

For any positive integers j, K, we may take a family of functions $b_K^j \in C^1(\mathbb{R})$ with

$$b_K^j(x) = \begin{cases} \left(x + \frac{1}{j}\right) \log\left(x + \frac{1}{j}\right), & \text{if } 0 \le x \le K, \\ \left(K + 1 + \frac{1}{j}\right) \log\left(K + 1 + \frac{1}{j}\right), & \text{if } x \ge K+1. \end{cases}$$

Since $\rho \in L^\infty(0,T;L^\gamma)$, we have $\rho < \infty$ a.e. in $(0,1)\times(0,T)$. This implies that $b_K^j(\rho) \to (\rho + \frac{1}{j})\log(\rho + \frac{1}{j})$ a.e. in $(0,1)\times(0,T)$ as $K \to \infty$. Hence, by using the Lebesgue Dominated Convergence theorem, we conclude

$$\left(\left(\rho + \frac{1}{j}\right)\log\left(\rho + \frac{1}{j}\right)\right)_t + \left(\left(\rho + \frac{1}{j}\right)\log\left(\rho + \frac{1}{j}\right)u\right)_x + \left(\rho - \frac{1}{j}\log\left(\rho + \frac{1}{j}\right)\right)u_x = 0,$$

in $\mathcal{D}'((0,1)\times(0,T))$.

It is easy to see that $\left(\rho + \frac{1}{j}\right)\log\left(\rho + \frac{1}{j}\right) \in L^2((0,1)\times(0,T))$ since $\rho \in L^{2\gamma}((0,1)\times(0,T))$. By Lemma 3.3 in [7], the zero-extension of ρ outside $(0,1)$ satisfies the same equation. By the mollification, the integration by parts and the limiting process, we may take the test function to be the constant 1 so that

$$\begin{aligned} \int_0^T \int_0^1 \rho u_x = & \int_0^1 \left(\rho_0 + \frac{1}{j}\right)\log\left(\rho_0 + \frac{1}{j}\right) - \int_0^1 \left(\rho + \frac{1}{j}\right)\log\left(\rho + \frac{1}{j}\right)(T) \\ & + \frac{1}{j}\int_0^T \int_0^1 u_x \log\left(\rho + \frac{1}{j}\right). \end{aligned} \tag{5.43}$$

Similar estimates are valid for approximated solutions ρ_δ, u_δ. More precisely, we have

$$(\rho_\delta \log(\rho_\delta))_t + (\rho_\delta \log(\rho_\delta) u_\delta)_x + \rho_\delta (u_\delta)_x = 0, \tag{5.44}$$

in $\mathcal{D}'((0,1)\times(0,T))$, and

$$\int_0^T \int_0^1 \rho_\delta (u_\delta)_x = \int_0^1 \rho_0^\delta \log(\rho_0^\delta) - \int_0^1 \rho_\delta \log(\rho_\delta)(T). \tag{5.45}$$

Since $\rho_\delta \in L^\infty(0,T;L^\gamma)$, we have

$$\rho^\delta \log(\rho^\delta) \in L^\infty(0,T;L^{\tilde{\gamma}})$$

for $1<\tilde{\gamma}<\gamma$. By the Equation (5.44), we obtain

$$(\rho_\delta \log(\rho_\delta))_t \in L^{\frac{2\gamma}{\gamma+1}}(0,T;W^{-1,\frac{2\gamma}{\gamma+1}}).$$

By Lemma 5.2, we conclude as $\delta\to 0$

$$\rho^\delta \log(\rho^\delta) \to \overline{\rho\log(\rho)}, \quad \text{in } C([0,T];L^{\tilde{\gamma}}-\omega).$$

This implies

$$\lim_{\delta\to 0} \int_0^1 \rho^\delta \log(\rho^\delta)(T) = \int_0^1 \overline{\rho\log(\rho)}(T). \tag{5.46}$$

Since the function $x\log(x)$ is convex for any $x>0$, Lemma 5.4 implies that

$$\rho\log(\rho) \le \overline{\rho\log(\rho)}, \quad \text{a.e. in } (0,1)\times(0,T). \tag{5.47}$$

Subtracting (5.43) by (5.45) and sending $\delta\to 0$, we have

$$\begin{aligned}
&\int_0^1 \overline{\rho\log(\rho)}(T) - \int_0^1 \left(\rho+\frac{1}{j}\right)\log\left(\rho+\frac{1}{j}\right)(T)\\
=&\int_0^1 \rho_0\log(\rho_0) - \int_0^1 \left(\rho_0+\frac{1}{j}\right)\log\left(\rho_0+\frac{1}{j}\right)\\
&+\int_0^T\int_0^1 \rho(u)_x - \lim_{\delta\to 0}\int_0^T\int_0^1 \rho_\delta(u_\delta)_x - \frac{1}{j}\int_0^T\int_0^1 u_x\log\left(\rho+\frac{1}{j}\right).
\end{aligned} \tag{5.48}$$

The first two terms of right-hand side can be estimated as follows

$$\begin{aligned}
&\int_0^T\int_0^1 \rho(u)_x - \lim_{\delta\to 0}\int_0^T\int_0^1 \rho_\delta(u_\delta)_x\\
=&\int_0^T\int_0^1 \rho(u)_x - \lim_{\delta\to 0}\int_0^T\int_0^1 \rho_\delta \mathcal{H}_\delta^1 - \lim_{\delta\to 0}\int_0^T\int_0^1 A_{11}^{-1}(n_\delta)\rho_\delta^{\gamma+1}\\
=&\int_0^T\int_0^1 \rho(u)_x - \int_0^T\int_0^1 \rho\mathcal{H}^1 - \lim_{\delta\to 0}\int_0^T\int_0^1 A_{11}^{-1}(n_\delta)\rho_\delta^{\gamma+1}\\
=&\int_0^T\int_0^1 \rho A_{11}^{-1}(n)\overline{\rho^\gamma} - \lim_{\delta\to 0}\int_0^T\int_0^1 A_{11}^{-1}(n)\rho_\delta^{\gamma+1}
\end{aligned}$$

$$
\begin{aligned}
&-\lim_{\delta\to 0}\int_0^T\int_0^1 \left(A_{11}^{-1}(n_\delta)-A_{11}^{-1}(n)\right)\rho_\delta^{\gamma+1}\\
&=\lim_{\delta\to 0}\int_0^T\int_0^1 A_{11}^{-1}(n)\left(\rho\overline{\rho^\gamma}-\rho_\delta^{\gamma+1}\right),
\end{aligned}
\tag{5.49}
$$

where we have used Lemma 5.3 in the second equality, and (5.16), $\gamma>1$, and (3.20) in the last step. Here $\mathcal{H}^1$ is the first element of $\mathcal{H}$, and $A_{11}^{-1}(\cdot)$ is the (1,1) element of inverse matrix $A^{-1}(\cdot)$. By the estimate (5.26) and the property of 2×2 matrices, $A_{11}^{-1}(\cdot)>0$.

Since $\rho,\rho_\delta\geq 0$, it is not hard to verify that

$$
(\rho-\rho_\delta)^{\gamma+1}=(\rho-\rho_\delta)^\gamma(\rho-\rho_\delta)\leq(\rho^\gamma-\rho_\delta^\gamma)(\rho-\rho_\delta).
$$

Thus

$$
\begin{aligned}
&\overline{\lim_{\delta\to 0}}\int_0^T\int_0^1 A_{11}^{-1}(n)(\rho-\rho_\delta)^{\gamma+1}\\
&\leq\lim_{\delta\to 0}\int_0^T\int_0^1 A_{11}^{-1}(n)(\rho^\gamma-\rho_\delta^\gamma)(\rho-\rho_\delta)\\
&=\lim_{\delta\to 0}\int_0^T\int_0^1 A_{11}^{-1}(n)\left(\rho^{\gamma+1}-\rho^\gamma\rho_\delta-\rho_\delta^\gamma\rho+\rho_\delta^{\gamma+1}\right)\\
&=\lim_{\delta\to 0}\int_0^T\int_0^1 A_{11}^{-1}(n)\left(\rho_\delta^{\gamma+1}-\rho\overline{\rho^\gamma}\right)+\lim_{\delta\to 0}\int_0^T\int_0^1 A_{11}^{-1}(n)\left(\rho^{\gamma+1}-\rho^\gamma\rho_\delta-\rho_\delta^\gamma\rho+\rho\overline{\rho^\gamma}\right)\\
&=\lim_{\delta\to 0}\int_0^T\int_0^1 A_{11}^{-1}(n)\left(\rho_\delta^{\gamma+1}-\rho\overline{\rho^\gamma}\right).
\end{aligned}
\tag{5.50}
$$

Substituting (5.50) into (5.49), we have

$$
\int_0^T\int_0^1 \rho(u)_x-\lim_{\delta\to 0}\int_0^T\int_0^1\rho_\delta(u_\delta)_x\leq 0.
$$

Combing this inequality with (5.48), we conclude that

$$
\begin{aligned}
&\int_0^1\overline{\rho\log(\rho)}(T)-\int_0^1\left(\rho+\frac{1}{j}\right)\log\left(\rho+\frac{1}{j}\right)(T)\\
&\leq\int_0^1\rho_0\log(\rho_0)-\int_0^1\left(\rho_0+\frac{1}{j}\right)\log\left(\rho_0+\frac{1}{j}\right)-\frac{1}{j}\int_0^T\int_0^1 u_x\log\left(\rho+\frac{1}{j}\right).
\end{aligned}
$$

Sending $j\to\infty$, we obtain that

$$
\int_0^1\overline{\rho\log(\rho)}(T)-\int_0^1\rho\log(\rho)(T)\leq 0.
$$

This and (5.47) imply that $\overline{\rho\log(\rho)}=\rho\log(\rho)$, combined with (5.46), implies (5.41).

Combining Lemma 5.5 with Lemma 5.4, and using the strict convexity of $\rho\log\rho$ for $\rho\geq 0$, we know that

$$
\rho_\delta\to\rho,\quad \text{a.e. in } (0,1)\times(0,T).
$$

It follows from the Egorov theorem that for any $\epsilon > 0$, there is $I_\epsilon \subset (0,1) \times (0,T)$ such that $|((0,1) \times (0,T)) \setminus I_\epsilon| < \epsilon$ and

$$\sup_{(x,t) \in I_\epsilon} |\rho_\delta(x,t) - \rho(x,t)| \to 0.$$

Since ρ_δ is uniformly bounded in $L^{2\gamma}$, we can estimate

$$\int_0^T \int_0^1 |\rho_\delta - \rho|^\gamma \le \sup_{(x,t) \in I_\epsilon} |\rho_\delta(x,t) - \rho(x,t)||I_\epsilon| + C|((0,1) \times (0,T)) \setminus I_\epsilon|^{\frac{1}{2}} \|\rho_\delta - \rho\|_{L^{2\gamma}}^\gamma$$
$$\to 0, \quad \text{as} \quad \delta \to 0.$$

This implies that $\overline{\rho^\gamma} = \rho^\gamma$ in $(0,1) \times (0,T)$. This completes the proof of Lemma 5.5. □

Acknowledgements. The first author is supported in part by the National Natural Science Foundation of China grants 11601342, 11871345, and 61872429. The third author is partially supported by NSF DMS 1764417.

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