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DYNAMICS OF THE THREE DIMENSIONAL VISCOUS PRIMITIVE EQUATIONS OF LARGE-SCALE MOIST ATMOSPHERE*

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Abstract. The objective of this paper is to study the long-time behavior of solutions for the three dimensional viscous primitive equations of large-scale moist atmosphere. Thanks to the presence of the strong coupling in the triple nonlinearity term and higher nonlinearity as well as the existence of corners of the domain under consideration, it is impossible to establish the existence of a more regular absorbing set. Therefore, it is very tricky to establish the existence of a global attractor in a more regular phase space. To overcome this difficulty, by obtaining the boundedness of the derivative of solutions in time, we prove the asymptotic compactness of the semigroup and establish the finite fractal dimension of the global attractor by using a smoothing property.

Keywords. Global attractor; Exponential attractor; Primitive equations; Smoothing property; Asymptotic a priori estimate.

AMS subject classifications. 35Q35; 35B40; 37C60.

1. Introduction

In this paper, we consider the long-time behavior of solutions for the following three dimensional viscous primitive equations of large-scale moist atmosphere in the pressure coordinate system (see [19, 23, 48, 53])

$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v+w\frac{\partial v}{\partial z}+\nabla\Phi+\frac{1}{Ro}fv^{\perp}+L_1v=0, \tag{1.1}$$
$$\frac{\partial \Phi}{\partial z}+\frac{bP}{p}(1+aq)T=0, \tag{1.2}$$
$$\nabla\cdot v+\frac{\partial w}{\partial z}=0, \tag{1.3}$$
$$\frac{\partial T}{\partial t}+v\cdot\nabla T+w\frac{\partial T}{\partial z}-\frac{bP}{p}(1+aq)w+L_2T=Q_1, \tag{1.4}$$
$$\frac{\partial q}{\partial t}+v\cdot\nabla q+w\frac{\partial q}{\partial z}+L_3q=Q_2 \tag{1.5}$$

in the domain

$$\Omega=M\times(0,1),$$

where M is a bounded domain in $\mathbb{R}^2$ with smooth boundary ∂M. The unknown functions for problem (1.1)-(1.5) are the horizontal velocity field $v=(v_1,v_2)$, the vertical velocity w in p-coordinate system, the mixing ratio of water vapor in the air q, the temperature T and the geopotential Φ. Here $v^{\perp}=(-v_2,v_1)$, $f=2\cos\theta_0$ is the Coriolis parameter, Ro is the Rossby number, P is an approximate value of pressure at the surface of the earth, p_0 represents the pressure of the upper atmosphere and $p_0>0$, the

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variable z satisfies $p=(P-p_0)z+p_0$ $(0<p_0\le p\le P)$, Q_1, Q_2 are given functions on Ω (here we don't consider the condensation of water vapor), a, b are positive constants and $a\approx 0.618$. The viscosity and the heat diffusion operators L_1, L_2 and L_3 are given by

$$L_1=-\frac{1}{Re_1}\Delta-\frac{1}{Re_2}\frac{\partial^2}{\partial z^2},$$

$$L_2=-\frac{1}{Rt_1}\Delta-\frac{1}{Rt_2}\frac{\partial^2}{\partial z^2},$$

$$L_3=-\frac{1}{Rt_3}\Delta-\frac{1}{Rt_4}\frac{\partial^2}{\partial z^2},$$

where Re_1, Re_2 are positive constants representing the horizontal and vertical Reynolds numbers, respectively, and Rt_1, Rt_3; Rt_2, Rt_4 are positive constants which stand for the horizontal and vertical eddy diffusivities, respectively. For the sake of simplicity, let $\nabla=(\partial_x,\partial_y)$ be the horizontal gradient operator and let $\Delta=\partial_x^2+\partial_y^2$ be the horizontal Laplacian. We observe that the above system is similar to the $3D$ Boussinesq system with the equation of vertical motion approximated by the hydrostatic balance.

Denote by Γ_u, Γ_b and Γ_l the upper, the bottom and the lateral boundaries of Ω, respectively. They are given by

$$\Gamma_u=\{(x,y,z)\in\overline{\Omega}:z=1\},$$

$$\Gamma_b=\{(x,y,z)\in\overline{\Omega}:z=0\},$$

$$\Gamma_l=\{(x,y,z)\in\overline{\Omega}:(x,y)\in\partial M,0\le z\le 1\}.$$

Equations (1.1)-(1.5) are subject to the following boundary conditions

$$\frac{\partial v}{\partial z}|_{\Gamma_u}=0,w|_{\Gamma_u}=0,\left(\frac{1}{Rt_2}\frac{\partial T}{\partial z}+\alpha T\right)|_{\Gamma_u}=0,\left(\frac{1}{Rt_4}\frac{\partial q}{\partial z}+\beta q\right)|_{\Gamma_u}=0, \tag{1.6}$$

$$\frac{\partial v}{\partial z}|_{\Gamma_b}=0,w|_{\Gamma_b}=0,\frac{\partial T}{\partial z}|_{\Gamma_b}=0,\frac{\partial q}{\partial z}|_{\Gamma_b}=0, \tag{1.7}$$

$$v\cdot\vec{n}|_{\Gamma_l}=0,\frac{\partial v}{\partial\vec{n}}\times\vec{n}|_{\Gamma_l}=0,\frac{\partial T}{\partial\vec{n}}|_{\Gamma_l}=0,\frac{\partial q}{\partial\vec{n}}|_{\Gamma_l}=0, \tag{1.8}$$

where $\vec{n}$ is the outward unit normal vector to Γ_l, α, β are positive constants.

In addition, we add the initial conditions to the system (1.1)-(1.8)

$$v(x,y,z,0)=v_0(x,y,z), \tag{1.9}$$

$$T(x,y,z,0)=T_0(x,y,z), \tag{1.10}$$

$$q(x,y,z,0)=q_0(x,y,z). \tag{1.11}$$

Large-scale dynamics of ocean and atmosphere is governed by the primitive equations which are derived from the Navier-Stokes equations with rotation coupled to thermodynamics and salinity diffusion-transport equations, which account for the buoyancy forces and stratification effects under the Boussinesq approximation. Moreover, due to the shallowness of the oceans and the atmosphere, i.e., the depth of the fluid layer is very small in comparison to the radius of the earth, the vertical large-scale motion in the oceans and the atmosphere is much smaller than the horizontal one, which in turn leads to modeling the vertical motion by the hydrostatic balance. As a result, one can obtain the system (1.1)-(1.3) which is known as the primitive equations for ocean and atmosphere dynamics (see [6, 42, 43, 48, 53]). We observe that one has to add the diffusion-transport equation of the salinity to the system (1.1)-(1.3) in the case of ocean dynamics, but we omitted it here in order to simplify our mathematical presentation. However, we emphasize that our results are equally valid when the salinity effects are taken into account.

In the past several decades, the primitive equations of the atmosphere, the ocean and the coupled atmosphere-ocean have been extensively studied from the mathematical point of view (see [3–7, 10, 14, 16, 19, 20, 22–24, 26, 26, 27, 27, 29–36, 38, 39, 41–43, 45–47, 49]). By introducing p-coordinate system and using some technical treatments, Lions, Temam and Wang in [42] obtained a new formulation for the primitive equations of large-scale dry atmosphere which is a little similar with Navier-Stokes equations of incompressible fluid, and they proved the existence of weak solutions for the primitive equations of the atmosphere. In [43], Lions, Temam and Wang introduced the primitive equations of large-scale ocean and proved the existence of weak solutions and the well-posedness of local-in-time strong solutions for the primitive equations of large-scale ocean, and estimated the dimension of the universal attractor. Based on the works of Lions, Temam and Wang in [42, 43], many authors continued to consider the well-posedness of solutions for the primitive equations of large-scale atmosphere (see [3, 4, 7, 15, 18, 19, 23, 30, 31, 45, 52, 57, 58]). However, the uniqueness of weak solutions and the global existence of strong solutions for the three dimensional primitive equations of large-scale ocean and atmosphere dynamics with any initial datum remain unresolved. Until 2007, Cao and Titi [6] decomposed the three dimensional primitive equations of large-scale ocean and atmosphere dynamics into two systems by using the idea of the decomposition of semigroup, one is similar with the two dimensional incompressible Navier-Stokes equations, the other is the reaction-convection-diffusion equations. As is known, the solutions of each system are fairly regular. Cao and Titi performed some a priori estimates about the solutions of each system by which they obtained some a priori estimates of strong solutions for the three dimensional primitive equations of large-scale ocean and atmosphere dynamics, which implies the well-posedness of strong solutions for the three dimensional primitive equations of large-scale ocean and atmosphere dynamics, they resolved the open question posed in [42, 43]. Meanwhile, the long-time behavior of solutions for the three dimensional primitive equations of large-scale ocean and atmosphere dynamics has been considered extensively (see [9, 14, 19, 20, 22, 23, 28, 29, 34–36, 40, 56, 61]). In particular, in [20], Guo and Huang obtained a weakly compact global attractor $\mathcal{A}$ for the primitive equations of large-scale atmosphere which captures all the trajectories. The existence of a global attractor in V for the primitive equations of large-scale atmosphere and ocean dynamics was proved by Ning Ju in [34] by using the Aubin-Lions compactness theorem under the assumption $Q \in L^2(\Omega)$. In [35, 36], the authors have proved the finite dimensional global attractor for the 3D viscous primitive equations by using the squeezing property. The regularity of the global attractor for the three dimensional viscous primitive equations was established in [55] by asymptotic a priori estimates. Recently, in [61], the authors have established the existence of a global attractor for the three dimensional primitive equations of large-scale moist atmosphere by using the Aubin-Lions compactness theorem under the assumption $Q_1, Q_2 \in L^2(\Omega)$. The existence of a pullback attractor in V for the process $\{U(t,\tau)\}_{t\geq\tau}$ associated with the three dimensional non-autonomous primitive equations of large-scale ocean and atmosphere dynamics was established in [40] by verifying the pullback $\mathcal{D}$-condition. In [46, 47], the author has proved the existence of a uniform attractor for the three dimensional non-autonomous primitive equations with oscillating external force and investigated its structure as well as the upper semi-continuous properties. The global well-posedness and long-time dynamics of solutions for the three-dimensional stochastic primitive equations of large-scale ocean was considered in [10, 17, 21, 28, 60]. As we know, the solutions of the stationary primitive equations of large-scale moist atmosphere are contained in the global attractor for the corresponding evolutionary primitive equations of large-scale moist atmosphere, thus, it is meaningful to consider the regularity of the global attractor for the three dimensional primitive equations of large-scale moist atmosphere.

Although the global attractor represents the first important step in the understanding of long-time behavior of dynamical systems generated by the three dimensional primitive equations of large-scale moist atmosphere, it may also present two essential drawbacks: On the one hand, the rate of attraction of the trajectories may be small and it is usually very difficult to estimate this rate in terms of the physical parameters of the problem. On the other hand, it is very sensitive to perturbations such that the global attractor can change drastically under very small perturbations of the initial dynamical system. These drawbacks obviously lead to essential difficulties in numerical simulations of global attractors and even make the global attractor unobservable in some sense. An alternative object describing the long-term dynamics is an inertial manifold, which is free from the above-mentioned drawbacks (see [13]). Unfortunately, its existence can be proved only under very restrictive spectral gap assumptions, which can be verified in few particular dynamical systems, mainly arising from one-dimensional parabolic equations (see [25]). In order to overcome this difficulty, an intermediate object has been introduced in [8, 11], an exponential attractor or inertial set. The exponential attractors contain the global attractor, are finite dimensional, and attract the trajectories exponentially fast. In contrast to the global attractor, an exponential attractor attracts the trajectories exponentially and will thus be more stable. Meanwhile, it also provides a way of proving that the global attractor has finite fractal dimension. Furthermore, in some situations, the global attractor can be very simple and thus fails to capture interesting transient behaviors. In such situations, an exponential attractor could be a more suitable object. Therefore, it is useful to explore the existence of an exponential attractor for the three dimensional primitive equations of large-scale moist atmosphere.

The main purpose of this paper is to study the long-time behavior of solutions for the three dimensional viscous primitive equations of large-scale moist atmosphere. In the next section, we reformulate problem (1.1)-(1.11) and give some notations used in the sequel. Section 3 is devoted to performing some a priori estimates of solutions of problem (2.4)-(2.12) to obtain the existence of absorbing sets in V and $(H^2(\Omega))^4\cap V$ of the semigroup generated by problem (2.4)-(2.12). In the last section, we prove the existence of a global attractor in $(H^2(\Omega))^4\cap V$ for problem (2.4)-(2.12) by an asymptotic a priori estimate and construct an exponential attractor by using the smoothing property of the semigroup generated by problem (2.4)-(2.12) by using the idea of [11–13, 50]. As a byproduct, we obtain that the fractal dimension of the global attractor for the semigroup, generated by problem (2.4)-(2.12), is finite, which is in consistent with the results in [35, 36].

Throughout this paper, let X be a Banach space endowed with the norm $\|\cdot\|_X$ and let $\|u\|_p$ be the $L^p(\Omega)$-norm of u for $1\leq p\leq\infty$, and let C be a generic positive constant.

2. New formulation and functional setting

2.1. New formulation. Integrating Equation (1.3) in the z direction, we obtain

$$w(x,y,z,t)=w(x,y,0,t)-\int_0^z \nabla\cdot v(x,y,\zeta,t)\,d\zeta.$$

Employing $w(x,y,0,t)=w(x,y,1,t)=0$ (see(1.6) and (1.7)), we find

$$w(x,y,z,t)=-\int_0^z \nabla\cdot v(x,y,\zeta,t)\,d\zeta \tag{2.1}$$

and

$$\int_0^1 \nabla\cdot v(x,y,\zeta,t)\,d\zeta=\nabla\cdot\int_0^1 v(x,y,\zeta,t)\,d\zeta=0. \tag{2.2}$$

Integrating Equation (1.2) with respect to z, we obtain

$$\Phi(x,y,z,t)=\Phi_s(x,y,t)-\int_0^z \frac{bP}{p(\zeta)}(1+aq(x,y,\zeta,t))T(x,y,\zeta,t)\,d\zeta, \tag{2.3}$$

where $\Phi_s(x,y,t)$ is a free function to be determined.

We infer from (2.1), (2.3) the following new formulation for problem (1.1)-(1.11)

$$\begin{aligned}
&\frac{\partial v}{\partial t}+(v\cdot\nabla)v-\Big(\int_0^z \nabla\cdot v(x,y,\zeta,t)\,d\zeta\Big)\frac{\partial v}{\partial z}+\nabla\Phi_s(x,y,t)+\frac{1}{Ro}fv^{\perp}+L_1v\\
&\quad-\int_0^z \frac{bP}{p(\zeta)}\nabla[(1+aq(x,y,\zeta,t))T(x,y,\zeta,t)]\,d\zeta=0, && (2.4)\\
&\frac{\partial T}{\partial t}+v\cdot\nabla T-\Big(\int_0^z \nabla\cdot v(x,y,\zeta,t)\,d\zeta\Big)\frac{\partial T}{\partial z}+L_2T\\
&\quad+\frac{bP}{p}(1+aq)\Big(\int_0^z \nabla\cdot v(x,y,\zeta,t)\,d\zeta\Big)=Q_1, && (2.5)\\
&\frac{\partial q}{\partial t}+v\cdot\nabla q-\Big(\int_0^z \nabla\cdot v(x,y,\zeta,t)\,d\zeta\Big)\frac{\partial q}{\partial z}+L_3q=Q_2 && (2.6)
\end{aligned}$$

with the following boundary conditions

$$\begin{aligned}
&\frac{\partial v}{\partial z}|_{\Gamma_u}=0,\ \frac{\partial v}{\partial z}|_{\Gamma_b}=0,\ v\cdot\vec{n}|_{\Gamma_l}=0,\ \frac{\partial v}{\partial\vec{n}}\times\vec{n}|_{\Gamma_l}=0, && (2.7)\\
&\Big(\frac{1}{Rt_2}\frac{\partial T}{\partial z}+\alpha T\Big)|_{\Gamma_u}=0,\ \frac{\partial T}{\partial z}|_{\Gamma_b}=0,\ \frac{\partial T}{\partial\vec{n}}|_{\Gamma_l}=0, && (2.8)\\
&\Big(\frac{1}{Rt_4}\frac{\partial q}{\partial z}+\beta q\Big)|_{\Gamma_u}=0,\ \frac{\partial q}{\partial z}|_{\Gamma_b}=0,\ \frac{\partial q}{\partial\vec{n}}|_{\Gamma_l}=0 && (2.9)
\end{aligned}$$

and the initial data

$$\begin{aligned}
v(x,y,z,0)&=v_0(x,y,z), && (2.10)\\
T(x,y,z,0)&=T_0(x,y,z), && (2.11)\\
q(x,y,z,0)&=q_0(x,y,z). && (2.12)
\end{aligned}$$

Denote by

$$\bar{v}(x,y)=\int_0^1 v(x,y,\zeta)\,d\zeta$$

and

$$\tilde{v}=v-\bar{v}.$$

Taking the average of (2.4) and combining Green's formula with the boundary conditions (2.7), we obtain

$$\begin{aligned}&\frac{\partial\bar{v}}{\partial t}+(\bar{v}\cdot\nabla)\bar{v}+\overline{(\tilde{v}\cdot\nabla)\tilde{v}+(\nabla\cdot\tilde{v})\tilde{v}}+\nabla\Phi_s(x,y,t)-\frac{1}{Re_1}\Delta\bar{v}\\&\quad+\frac{1}{Ro}f\bar{v}^{\perp}-\overline{\int_0^z\frac{bP}{p(\zeta)}\nabla[(1+aq(x,y,\zeta,t))T(x,y,\zeta,t)]\,d\zeta}=0,\end{aligned}\tag{2.13}$$

which is subject to the boundary conditions

$$\nabla\cdot\bar{v}=0,\bar{v}\cdot\vec{n}|_{\Gamma_l}=0,\ \frac{\partial\bar{v}}{\partial\vec{n}}\times\vec{n}|_{\Gamma_l}=0.\tag{2.14}$$

Substracting (2.13) from (2.4), we have

$$\begin{aligned}&\frac{\partial\tilde{v}}{\partial t}+(\tilde{v}\cdot\nabla)\tilde{v}-\Big(\int_{-h}^z\nabla\cdot\tilde{v}(x,y,\zeta,t)\,d\zeta\Big)\frac{\partial\tilde{v}}{\partial z}+(\tilde{v}\cdot\nabla)\bar{v}+(\bar{v}\cdot\nabla)\tilde{v}+\frac{1}{Ro}f\tilde{v}^{\perp}+L_1\tilde{v}\\&\quad-\int_0^z\frac{bP}{p(\zeta)}\nabla[(1+aq(x,y,\zeta,t))T(x,y,\zeta,t)]\,d\zeta-\overline{(\tilde{v}\cdot\nabla)\tilde{v}+(\nabla\cdot\tilde{v})\tilde{v}}\\&\quad+\overline{\int_0^z\frac{bP}{p(\zeta)}\nabla[(1+aq(x,y,\zeta,t))T(x,y,\zeta,t)]\,d\zeta}=0,\end{aligned}\tag{2.15}$$

which is supplemented with the boundary conditions

$$\frac{\partial\tilde{v}}{\partial z}|_{\Gamma_u}=0,\frac{\partial\tilde{v}}{\partial z}|_{\Gamma_b}=0,\tilde{v}\cdot\vec{n}|_{\Gamma_l}=0,\frac{\partial\tilde{v}}{\partial\vec{n}}\times\vec{n}|_{\Gamma_l}=0.\tag{2.16}$$

2.2. Functional spaces and some lemmas. To study problem (2.4)-(2.12), we introduce some function spaces. Let

$$\begin{aligned}\mathcal{V}_1&=\Big\{v\in(C^\infty(\bar{\Omega}))^2:\frac{\partial v}{\partial z}|_{\Gamma_u}=0,\frac{\partial v}{\partial z}|_{\Gamma_b}=0,v\cdot\vec{n}|_{\Gamma_l}=0,\frac{\partial v}{\partial\vec{n}}\times\vec{n}|_{\Gamma_l}=0,\\&\qquad\int_0^1\nabla\cdot v(x,y,\zeta)\,d\zeta=0\Big\},\\\mathcal{V}_2&=\Big\{T\in C^\infty(\bar{\Omega}):\Big(\frac{1}{Rt_2}\frac{\partial T}{\partial z}+\alpha T\Big)|_{\Gamma_u}=0,\frac{\partial T}{\partial z}|_{\Gamma_b}=0,\frac{\partial T}{\partial\vec{n}}|_{\Gamma_l}=0\Big\},\\\mathcal{V}_3&=\Big\{q\in C^\infty(\bar{\Omega}):\Big(\frac{1}{Rt_4}\frac{\partial q}{\partial z}+\beta q\Big)|_{\Gamma_u}=0,\frac{\partial q}{\partial z}|_{\Gamma_b}=0,\frac{\partial q}{\partial\vec{n}}|_{\Gamma_l}=0\Big\}.\end{aligned}$$

Denote the closure of $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ by V_1, V_2, V_3 with respect to the following norms, respectively, given by

$$\|v\|^2=\frac{1}{Re_1}\int_\Omega|\nabla v|^2\,dxdydz+\frac{1}{Re_2}\int_\Omega|\partial_z v|^2\,dxdydz,$$

$$\|T\|^2=\frac{1}{Rt_1}\int_\Omega|\nabla T|^2\,dxdydz+\frac{1}{Rt_2}\int_\Omega|\partial_z T|^2\,dxdydz+\alpha\int_M|T(z=1)|^2\,dxdy,$$
$$\|q\|^2=\frac{1}{Rt_3}\int_\Omega|\nabla q|^2\,dxdydz+\frac{1}{Rt_4}\int_\Omega|\partial_z q|^2\,dxdydz+\beta\int_M|q(z=1)|^2\,dxdy,$$
$$\|(v,T,q)\|_V^2=\|v\|^2+\|T\|^2+\|q\|^2,$$
$$\|(v,T,q)\|_H^2=\|v\|^2_{L^2(\Omega)}+\|T\|^2_{L^2(\Omega)}+\|q\|^2_{L^2(\Omega)}$$

for any $v\in\mathcal{V}_1$, $T\in\mathcal{V}_2$, $q\in\mathcal{V}_3$, and let $H_1=$ the closure of $\mathcal{V}_1$ with respect to the norm in $(L^2(\Omega))^2$, $V=V_1\times V_2\times V_3$, $H=H_1\times L^2(\Omega)\times L^2(\Omega)$.

3. Some a priori estimates of strong solutions

3.1. The well-posedness of strong solutions. We start with the following general existence and uniqueness of solutions for problem (2.4)-(2.12) which can be obtained by the methods used in [6,34,37,51]. Here we only state it.

THEOREM 3.1. *Assume that $Q_1\in L^2(\Omega)$ and $Q_2\in L^2(\Omega)$. Then for each $(v_0,T_0,q_0)\in V$, there exists a unique strong solution $(v,T,q)\in C(\mathbb{R}^+;V)$ for problem (2.4)-(2.12), which depends continuously on the initial data in V.*

By Theorem 3.1, we can define the operator semigroup $\{S(t)\}_{t\geq 0}$ in V as

$$S(.):\mathbb{R}^+\times V\to V,$$

which is (V,V)-continuous.

3.2. Some a priori estimates of strong solutions. In this subsection, we carry out some a priori estimates of strong solutions for problem (2.4)-(2.12), which imply the existence of absorbing sets for the semigroup $\{S(t)\}_{t\geq 0}$ associated with problem (2.4)-(2.12).

3.2.1. $L^2(\Omega)$ estimates of q. Taking the inner product of Equation (2.6) with q in $L^2(\Omega)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|q\|_2^2+\|q\|^2=\int_\Omega Q_2 q\,dxdydz. \tag{3.1}$$

Thanks to

$$\|q\|_2^2\leq 2\|q(z=1)\|^2_{L^2(M)}+2\|\partial_z q\|_2^2,$$

we find

$$\frac{\|q\|_2^2}{2Rt_4+\frac{2}{\beta}}\leq\frac{1}{Rt_4}\int_\Omega|\partial_z q|^2\,dxdydz+\beta\int_M|q(z=1)|^2\,dxdy. \tag{3.2}$$

It follows from inequalities (3.1)-(3.2) that

$$\frac{d}{dt}\|q\|_2^2+\|q\|^2\leq(2Rt_4+\frac{2}{\beta})\|Q_2\|_2^2.$$

Using inequality (3.2) again, we obtain

$$\frac{d}{dt}\|q\|_2^2+\frac{\|q\|_2^2}{2Rt_4+\frac{2}{\beta}}\leq(2Rt_4+\frac{2}{\beta})\|Q_2\|_2^2.$$

We infer from the classical Gronwall inequality that

$$\|q\|_2^2 \le \|q_0\|_2^2 \exp(\frac{-t}{2Rt_4+\frac{2}{\beta}}) + (2Rt_4+\frac{2}{\beta})^2\|Q_2\|_2^2,$$

which implies that

$$\|q\|_2^2 + \int_t^{t+1} \|q(\tau)\|^2 \, d\tau \le \rho_1 \tag{3.3}$$

for any $t \ge T_1$. For brevity, we omit writing out these bounds explicitly here and we also omit writing out other similar bounds in our future discussion for all other uniform a priori estimates.

3.2.2. $(L^2(\Omega))^3$ estimates of (v,T). Multiplying Equation (2.4) by v and integrating over Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \|v\|^2 \\ &= \int_\Omega \left(\int_0^z \frac{bP}{p(\zeta)} \nabla[(1+aq(x,y,\zeta,t))T(x,y,\zeta,t)]\, d\zeta \right) \cdot v \, dxdydz. \end{aligned} \tag{3.4}$$

Taking the inner product of Equation (2.5) with T in $L^2(\Omega)$, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|T\|_2^2 + \|T\|^2 \\ &= \int_\Omega Q_1 T \, dxdydz - \int_\Omega \left(\frac{bP}{p}(1+aq)(\int_0^z \nabla\cdot v(x,y,\zeta,t)\, d\zeta) \right) T \, dxdydz. \end{aligned} \tag{3.5}$$

Integrating by parts and combining equality (2.2) with boundary conditions (2.7), we obtain

$$\begin{aligned} &\int_\Omega \left(\int_0^z \frac{bP}{p(\zeta)} \nabla[(1+aq(x,y,\zeta,t))T(x,y,\zeta,t)]\, d\zeta \right) \cdot v \, dxdydz \\ &= -\int_\Omega \left(\int_0^z \frac{bP}{p(\zeta)} (1+aq(x,y,\zeta,t))T(x,y,\zeta,t)\, d\zeta \right) (\nabla\cdot v) \, dxdydz \\ &= \int_\Omega \left(\frac{bP}{p}(1+aq)(\int_0^z \nabla\cdot v(x,y,\zeta,t)\, d\zeta) \right) T \, dxdydz. \end{aligned} \tag{3.6}$$

It follows from equalities (3.4)-(3.6) and Hölder's inequality that

$$\frac{1}{2}\frac{d}{dt}(\|v\|_2^2 + \|T\|_2^2) + \|v\|^2 + \|T\|^2 \le \|Q_1\|_2 \|T\|_2. \tag{3.7}$$

Notice that

$$\frac{\|v\|_2^2}{C_M} + \frac{\|T\|_2^2}{2Rt_2+\frac{2}{\alpha}} \le \|T\|^2 + \|v\|^2. \tag{3.8}$$

Therefore, we deduce from inequalities (3.7)-(3.8), Hölder's inequality and Young's inequality that

$$\frac{d}{dt}(\|v\|_2^2 + \|T\|_2^2) + \frac{\|v\|_2^2}{C_M} + \frac{\|T\|_2^2}{2Rt_2+\frac{2}{\alpha}} \le (2Rt_2+\frac{2}{\alpha})\|Q_1\|_2^2,$$

which implies that

$$\|v\|_2^2+\|T\|_2^2+\int_t^{t+1}\|v(\tau)\|^2+\|T(\tau)\|^2\,d\tau\le\rho_2 \tag{3.9}$$

for any $t\ge T_2\ge T_1$.

3.2.3. $L^6(\Omega)$ estimates of q. Multiplying Equation (2.6) by $|q|^4q$ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{6}\frac{d}{dt}\|q\|_6^6+\frac{5}{9}\||q|^3\|^2\le&\||q|^3\|_{\frac{10}{3}}^{\frac{5}{3}}\|Q_2\|_2\\ \le&C\|Q_2\|_2\||q|^3\|_2^{\frac{2}{3}}\||q|^3\|\\ =&C\|Q_2\|_2\|q\|_6^2\||q|^3\|.\end{aligned}$$

Using Young's inequality, we obtain

$$\frac{d}{dt}\|q\|_6^2\le C\|Q_2\|_2^2.$$

Therefore, we infer from the uniform Gronwall inequality and inequality (3.3) that

$$\|q\|_6^2+\int_t^{t+1}\||q(\tau)|^3\|^2\,d\tau\le\rho_3 \tag{3.10}$$

for any $t\ge T_2+1$.

3.2.4. $L^6(\Omega)$ estimates of T. Taking the inner product of Equation (2.5) with $|T|^4T$ in $L^2(\Omega)$, we deduce

$$\begin{aligned}&\frac{1}{6}\frac{d}{dt}\|T\|_6^6+\frac{5}{9}\||T|^3\|^2\\ &\le\||T|^3\|_{\frac{10}{3}}^{\frac{5}{3}}\|Q_1\|_2+\int_\Omega\frac{bP}{p}(1+aq)(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta)|T|^4T\,dxdydz\\ &\le C\|Q_1\|_2\||T|^3\|_2^{\frac{2}{3}}\||T|^3\|+C\|\nabla v\|_2\||T|^3\|_2^{\frac{2}{3}}\||T|^3\|+I_1,\end{aligned} \tag{3.11}$$

where

$$I_1=\int_\Omega\frac{abP}{p}q(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta)|T|^4T\,dxdydz.$$

Now, we estimate I_1 as follows.

$$\begin{aligned}I_1\le&C\int_0^1\|q\|_{L^6(M)}\left\|\int_0^1|\nabla v|(x,y,\zeta,t)\,d\zeta\right\|_{L^2(M)}\||T|^5\|_{L^3(M)}\,dz\\ \le&C\left\|\int_0^1|\nabla v|(x,y,\zeta,t)\,d\zeta\right\|_{L^2(M)}\int_0^1\|q\|_{L^6(M)}\||T|^3\|_{L^2(M)}^{\frac{2}{3}}\||T|^3\|_{H^1(M)}\,dz\\ \le&C\left(\int_0^1\|\nabla v\|_{L^2(M)}\,d\zeta\right)\|q\|_6\||T|^3\|_2^{\frac{2}{3}}\||T|^3\|\\ \le&C\|\nabla v\|_2\|q\|_6\||T|^3\|_2^{\frac{2}{3}}\||T|^3\|.\end{aligned} \tag{3.12}$$

We deduce from inequalities (3.11)-(3.12) that

$$\frac{d}{dt}\|T\|_6^2 \le C\|Q_1\|_2^2 + C\|\nabla v\|_2^2 + C\|q\|_6^2\|\nabla v\|_2^2.$$

Combining the uniform Gronwall inequality with inequalities (3.9), (3.10), we obtain

$$\|T\|_6^2 + \int_t^{t+1} \||T(\tau)|^3\|^2 \, d\tau \le \rho_4 \tag{3.13}$$

for any $t \ge T_2 + 2$.

3.2.5. $(L^6(\Omega))^2$ estimates of $\tilde{v}$. Multiplying Equation (2.15) by $|\tilde{v}|^4\tilde{v}$ and integrating over Ω, we deduce

$$\begin{aligned}
&\frac{1}{6}\frac{d}{dt}\|\tilde{v}\|_6^6 + \frac{1}{Re_1}\int_\Omega |\nabla \tilde{v}|^2|\tilde{v}|^4 \, dxdydz + \frac{1}{Re_2}\int_\Omega |\partial_z \tilde{v}|^2|\tilde{v}|^4 \, dxdydz + \frac{4}{9}\||\tilde{v}|^3\|^2 \\
\le & C\int_\Omega |\bar{v}||\nabla \tilde{v}||\tilde{v}|^5 \, dxdydz + C\int_M \Big(\int_0^1 |\tilde{v}|^2 \, dz\Big)\Big(\int_0^1 |\nabla \tilde{v}||\tilde{v}|^4 \, dz\Big) \, dxdy + I_2,
\end{aligned} \tag{3.14}$$

where

$$I_2 = \int_\Omega \left(\int_0^z \frac{bP}{p(\zeta)}[(1+aq)T] \, d\zeta - \int_0^1 \int_0^\eta \frac{bP}{p(\zeta)}[(1+aq)T] \, d\zeta \, d\eta\right)(\nabla\cdot|\tilde{v}|^4\tilde{v}) \, dxdydz.$$

In the following, we estimate I_2 by using Hölder's inequality.

$$\begin{aligned}
I_2 \le & C\left\|\int_0^z \frac{bP}{p(\zeta)}[(1+aq)T] \, d\zeta\right\|_6 \||\nabla \tilde{v}||\tilde{v}|^2\|_2\|\tilde{v}\|_6^2 \\
\le & C\|T\|_6\||\nabla \tilde{v}||\tilde{v}|^2\|_2\|\tilde{v}\|_6^2 + C\left\|\int_0^1 |qT| \, d\zeta\right\|_{L^6(M)} \||\nabla \tilde{v}||\tilde{v}|^2\|_2\|\tilde{v}\|_6^2.
\end{aligned}$$

Due to

$$\begin{aligned}
\left\|\int_0^1 |qT| \, d\zeta\right\|_{L^6(M)}^6 = & \int_M \left|\int_0^1 |qT| \, dz\right|^6 dxdy \\
\le & \int_M \Big(\int_0^1 |q|^2 \, dz\Big)^3\Big(\int_0^1 |T|^2 \, dz\Big)^3 dxdy \\
\le & \left(\int_M \Big(\int_0^1 |q|^2 \, dz\Big)^6 dxdy\right)^{\frac{1}{2}} \left(\int_M \Big(\int_0^1 |T|^2 \, dz\Big)^6 dxdy\right)^{\frac{1}{2}} \\
\le & \left(\int_0^1 \Big(\int_M |q|^{12} \, dxdy\Big)^{\frac{1}{6}} dz\right)^3 \left(\int_0^1 \Big(\int_M |T|^{12} \, dxdy\Big)^{\frac{1}{6}} dz\right)^3 \\
\le & C\left(\int_0^1 \|q\|_{L^6(M)}\|q\|_{H^1(M)} \, dz\right)^3 \left(\int_0^1 \|T\|_{L^6(M)}\|T\|_{H^1(M)} \, dz\right)^3 \\
\le & C\|q\|_6^3\|q\|^3\|T\|_6^3\|T\|^3,
\end{aligned}$$

which implies that

$$I_2 \le C\|T\|_6\|\nabla \tilde{v}||\tilde{v}|^2\|_2\|\tilde{v}\|_6^2 + C\|q\|_6^{\frac{1}{2}}\|q\|^{\frac{1}{2}}\|T\|_6^{\frac{1}{2}}\|T\|^{\frac{1}{2}}\||\nabla \tilde{v}||\tilde{v}|^2\|_2\|\tilde{v}\|_6^2. \tag{3.15}$$

It follows from Hölder's inequality that

$$\int_\Omega |\bar{v}||\nabla\tilde{v}||\tilde{v}|^5\,dxdydz \le \int_M |\bar{v}|(\int_0^1 |\nabla\tilde{v}|^2|\tilde{v}|^4\,dz)^{\frac{1}{2}}(\int_0^1 |\tilde{v}|^6\,dz)^{\frac{1}{2}}\,dxdy$$
$$\le \|\bar{v}\|_{L^4(M)}\||\nabla\tilde{v}||\tilde{v}|^2\|_2(\int_0^1(\int_M |\tilde{v}|^{12}\,dxdy)^{\frac{1}{2}}\,dz)^{\frac{1}{2}}. \tag{3.16}$$

Thanks to

$$\int_M |\tilde{v}|^{12}\,dxdy = \int_M ||\tilde{v}|^3|^4\,dxdy$$
$$\le C\int_M |\tilde{v}|^6\,dxdy\int_M |\nabla|\tilde{v}|^3|^2\,dxdy,$$

we obtain

$$(\int_0^1(\int_M |\tilde{v}|^{12}\,dxdy)^{\frac{1}{2}}\,dz)^{\frac{1}{2}} \le C(\int_\Omega |\tilde{v}|^6\,dxdydz)^{\frac{1}{4}}(\int_\Omega |\nabla|\tilde{v}|^3|^2\,dxdydz)^{\frac{1}{4}}. \tag{3.17}$$

Therefore,we deduce from inequalities (3.16)-(3.17) that

$$\int_\Omega |\bar{v}||\nabla\tilde{v}||\tilde{v}|^5\,dxdydz$$
$$\le C\|\tilde{v}\|_6^{\frac{3}{2}}\|v\|_2^{\frac{1}{2}}\|\nabla v\|_2^{\frac{1}{2}}(\int_\Omega |\nabla|\tilde{v}|^3|^2\,dxdydz)^{\frac{1}{4}}(\int_\Omega |\nabla\tilde{v}|^2|\tilde{v}|^4\,dxdydz)^{\frac{1}{2}}. \tag{3.18}$$

Repeating a similar process as above, we deduce

$$\int_M(\int_0^1 |\tilde{v}|^2\,dz)(\int_0^1 |\nabla\tilde{v}||\tilde{v}|^4\,dz)\,dxdy \le C\||\nabla\tilde{v}||\tilde{v}|^2\|_2\|\tilde{v}\|_6^3\|\tilde{v}\|_{H^1(\Omega)}. \tag{3.19}$$

We infer from inequalities (3.14)-(3.15), (3.18)-(3.19) that

$$\frac{d}{dt}\|\tilde{v}\|_6^6 + \frac{2}{Re_1}\int_\Omega |\nabla\tilde{v}|^2|\tilde{v}|^4\,dxdydz + \frac{2}{Re_2}\int_\Omega |\partial_z\tilde{v}|^2|\tilde{v}|^4\,dxdydz + 2\||\tilde{v}|^3\|^2$$
$$\le C(\|v\|_2^2\|\nabla v\|_2^2 + \|\tilde{v}\|_{H^1(\Omega)}^2)\|\tilde{v}\|_6^6 + C\|T\|_6^2\|\tilde{v}\|_6^4 + C\|q\|_6\|q\|\|T\|_6\|T\|\|\tilde{v}\|_6^4.$$

Therefore, it follows from inequalities (3.3), (3.9), (3.10) and inequality (3.13) that

$$\|\tilde{v}\|_6^2 + \int_t^{t+1}\int_\Omega |\nabla\tilde{v}|^2|\tilde{v}|^4\,dxdydz\,d\tau \le \rho_5 \tag{3.20}$$

for any $t \ge T_2 + 3$.

3.2.6. $(H^1(M))^2$ estimates of $\bar{v}$. Taking the inner product of Equation (2.13) with $-\Delta\bar{v}$ in $L^2(\Omega)$ and combining the boundary conditions (2.14), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\bar{v}\|_{L^2(M)}^2 + \frac{1}{Re_1}\int_M |\Delta\bar{v}|^2\,dxdy$$
$$\le C\int_M |\bar{v}||\nabla\bar{v}||\Delta\bar{v}|\,dxdy + C\int_M(\int_0^1 |\nabla\tilde{v}||\tilde{v}|\,dz)|\Delta\bar{v}|\,dxdy, \tag{3.21}$$

where we have used the following equalities

$$\int_M \nabla\Phi_s(x,y,t)\cdot\Delta\bar{v}\,dxdy=0,$$
$$\frac{1}{Ro}\int_M f\bar{v}^{\perp}\cdot\Delta\bar{v}\,dxdy=0,$$
$$\int_M \overline{\int_0^z \frac{bP}{p(\zeta)}\nabla[(1+aq(x,y,\zeta,t))T(x,y,\zeta,t)]\,d\zeta}\cdot\Delta\bar{v}\,dxdy=0.$$

In the following, we give the estimates of each term of the right-hand side of inequality (3.21).

$$\begin{aligned}\int_M |\bar{v}||\nabla\bar{v}||\Delta\bar{v}|\,dxdy&\le C\|\bar{v}\|_{L^4(M)}\|\nabla\bar{v}\|_{L^4(M)}\|\Delta\bar{v}\|_{L^2(M)}\\&\le C\|\bar{v}\|^{\frac{1}{2}}_{L^2(M)}\|\nabla\bar{v}\|_{L^2(M)}\|\Delta\bar{v}\|^{\frac{3}{2}}_{L^2(M)},\end{aligned}\tag{3.22}$$

$$\int_M\Big(\int_0^1|\tilde{v}||\nabla\tilde{v}|\,dz\Big)|\Delta\bar{v}|\,dxdy\le C\||\nabla\tilde{v}||\tilde{v}|^2\|_2^{\frac{1}{2}}\|\nabla\tilde{v}\|_2^{\frac{1}{2}}\|\Delta\bar{v}\|.\tag{3.23}$$

It follows from inequalities (3.21)-(3.23) that

$$\begin{aligned}&\frac{d}{dt}\|\nabla\bar{v}\|^2_{L^2(M)}+\frac{1}{Re_1}\int_M|\Delta\bar{v}|^2\,dxdy\\&\le C\|\bar{v}\|^2_{L^2(M)}\|\nabla\bar{v}\|^4_{L^2(M)}+C\||\nabla\tilde{v}||\tilde{v}|^2\|_2^2+C\|\nabla\tilde{v}\|_2^2.\end{aligned}$$

In view of inequalities (3.9), (3.20) and the uniform Gronwall inequality, we obtain

$$\|\nabla\bar{v}\|^2_{L^2(M)}\le\rho_6\tag{3.24}$$

for any $t\ge T_2+4$.

3.2.7. $(L^2(\Omega))^2$ estimates of v_z. Denoted by $u=v_z$. It is clear that u satisfies the following equation obtained by differentiating Equation (2.4) with respect to z:

$$\begin{aligned}&\frac{\partial u}{\partial t}+L_1u+(v\cdot\nabla)u-\Big(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta\Big)\frac{\partial u}{\partial z}+(u\cdot\nabla)v\\&\quad-(\nabla\cdot v)u+\frac{1}{Ro}fu^{\perp}-\frac{bP}{p}\nabla[(1+aq)T]=0\end{aligned}\tag{3.25}$$

subject to the boundary conditions

$$u|_{\Gamma_u}=0,u|_{\Gamma_b}=0,u\cdot\vec{n}|_{\Gamma_l}=0,\frac{\partial u}{\partial\vec{n}}\times\vec{n}|_{\Gamma_l}=0.\tag{3.26}$$

Multiplying Equation (3.25) by u and integrating over Ω, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|_2^2+\|u\|^2\\&=-\int_\Omega[(u\cdot\nabla)v-(\nabla\cdot v)u-\frac{bP}{p}\nabla((1+aq)T)]\cdot u\,dxdydz\\&\le C\int_\Omega|v||u||\nabla u|\,dxdydz+C\int_\Omega|T||\nabla u|\,dxdydz+C\int_\Omega|T||q||\nabla u|\,dxdydz.\end{aligned}\tag{3.27}$$

Next, we estimate the right-hand side of inequality (3.27) term by term.

$$\int_\Omega |T||\nabla u|\,dxdydz \le \|T\|_2\|\nabla u\|_2, \tag{3.28}$$

$$\begin{aligned}\int_\Omega |v||u||\nabla u|\,dxdydz &\le \|v\|_6\|u\|_3\|\nabla u\|_2\\ &\le C\|v\|_6\|u\|_2^{\frac{1}{2}}\|u\|^{\frac{3}{2}},\end{aligned} \tag{3.29}$$

$$\begin{aligned}\int_\Omega |T||q||\nabla u|\,dxdydz &\le \|T\|_3\|q\|_6\|\nabla u\|_2\\ &\le C\|T\|_2^{\frac{1}{2}}\|T\|^{\frac{1}{2}}\|q\|_6\|\nabla u\|_2.\end{aligned} \tag{3.30}$$

It follows from inequalities (3.27)-(3.30) that

$$\frac{d}{dt}\|u\|_2^2+\|u\|^2 \le C\|v\|_6^4\|u\|_2^2+C\|T\|_2^2+C\|q\|_6^4\|T\|^2.$$

It is shown in reference [34] that

$$\|v\|_6 \le C\|v\|_2+C\|\nabla \bar{v}\|_2+\|\tilde{v}\|_6,$$

which implies that

$$\|v\|_6^2 \le \rho_7 \tag{3.31}$$

for any $t \ge T_2+4$.

Thanks to the uniform Gronwall inequality, inequality (3.9) and inequality (3.31), we obtain

$$\|\partial_z v\|_2^2+\int_t^{t+1}\|\partial_z v(\tau)\|^2\,d\tau \le \rho_8 \tag{3.32}$$

for any $t \ge T_2+5$.

3.2.8. $(L^2(\Omega))^2$ estimates of (T_z,q_z). Taking the inner product of equation (2.6) with $-\frac{\partial^2 q}{\partial z^2}$ in $L^2(\Omega)$ and combining the boundary conditions (2.2), (2.9), we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|q_z\|^2+Rt_4\beta\|q\|^2_{L^2(\Gamma_u)})+\frac{1}{Rt_3}\|\nabla q_z\|_2^2+\frac{1}{Rt_4}\|\partial_z q_z\|_2^2+\frac{\beta Rt_4}{Rt_3}\|\nabla q\|^2_{L^2(\Gamma_u)}\\ &=-\int_\Omega Q_2\partial_z q_z+\int_\Omega\left[v\cdot\nabla q-\left(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta\right)q_z\right]\frac{\partial^2 q}{\partial z^2}\,dxdydz\\ &\le\|Q_2\|_2\|\partial_z q_z\|_2-Rt_4\beta\int_{\Gamma_u}(v\cdot\nabla q)q-\int_\Omega[v_z\cdot\nabla q-(\nabla\cdot v)q_z]\,q_z\,dxdydz\\ &\le\|Q_2\|_2\|\partial_z q_z\|_2+\frac{Rt_4\beta}{2}\int_{\Gamma_u}(\nabla\cdot v)|q|^2+C\|\nabla v_z\|_2\|q\|_6\|q_z\|_3\\ &\quad+C\|v\|_6\|q_z\|_3\|\nabla q_z\|_2+C\|v_z\|_3\|q\|_6\|\nabla q_z\|_2.\end{aligned} \tag{3.33}$$

Using Hölder's inequality, we have

$$\int_{\Gamma_u}(\nabla\cdot v)|q|^2\,dxdy=\int_M\left(\int_\eta^1\nabla\cdot v_\zeta(x,y,\zeta,t)\,d\zeta+\int_0^1\nabla\cdot v(x,y,\zeta,t)\,d\zeta\right)|q(z=1)|^2\,dxdy$$

$$\le C(\|\nabla v_z\|_2+\|\nabla v\|_2)\|q\|^2_{L^4(\Gamma_u)}$$
$$\le C(\|\nabla v_z\|_2+\|\nabla v\|_2)\|q\|_{L^2(\Gamma_u)}\|q\|^3_{L^6(\Gamma_u)}. \tag{3.34}$$

Multiplying Equation (2.5) by $-\frac{\partial^2 T}{\partial z^2}$ and integrating over Ω, and using the boundary conditions (2.2), (2.8), we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|T_z\|^2+Rt_2\alpha\|T\|^2_{L^2(\Gamma_u)})+\frac{1}{Rt_1}\|\nabla T_z\|_2^2+\frac{1}{Rt_2}\|\partial_z T_z\|_2^2+\frac{\alpha Rt_2}{Rt_1}\|\nabla T\|^2_{L^2(\Gamma_u)}\\
&=-\int_\Omega Q_1\partial_z T_z+\int_\Omega\left[v\cdot\nabla T-\left(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta\right)T_z\right]\frac{\partial^2 T}{\partial z^2}\,dxdydz\\
&\quad+\int_\Omega\frac{bP}{p}(1+aq)(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta)\frac{\partial^2 T}{\partial z^2}\,dxdydz\\
&\le\|Q_1\|_2\|\partial_z T_z\|_2-Rt_2\alpha\int_{\Gamma_u}(v\cdot\nabla T)T-\int_\Omega[v_z\cdot\nabla T-(\nabla\cdot v)T_z]T_z\,dxdydz\\
&\quad-\int_\Omega\frac{bP}{p}(1+aq)(\nabla\cdot v)T_z\,dxdydz-\int_\Omega\frac{abP}{p}q_z(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta)T_z\,dxdydz\\
&\quad+\int_\Omega\frac{bP(P-p_0)}{p^2}(1+aq)(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta)T_z\,dxdydz\\
&\le\|Q_1\|_2\|\partial_z T_z\|_2+\frac{Rt_2\alpha}{2}\int_{\Gamma_u}(\nabla\cdot v)|T|^2+C\|\nabla v_z\|_2\|T\|_6\|T_z\|_3\\
&\quad+C\|v_z\|_3\|T\|_6\|\nabla T_z\|_2+C\|q\|_6\|\nabla v\|_2\|T_z\|_3+C\|\nabla v\|_2\|T_z\|_2\\
&\quad+C\|v\|_6\|T_z\|_3\|\nabla T_z\|_2+C\|\nabla q_z\|_2\|v\|_6\|T_z\|_3+C\|\nabla T_z\|_2\|v\|_6\|q_z\|_3.
\end{aligned}\tag{3.35}$$

Similarly, we have

$$\int_{\Gamma_u}(\nabla\cdot v)|T|^2\,dxdy\le C(\|\nabla v_z\|_2+\|\nabla v\|_2)\|T\|_{L^2(\Gamma_u)}\|T\|^3_{L^6(\Gamma_u)}. \tag{3.36}$$

Therefore, by virtue of Young's inequality, the uniform Gronwall inequality and inequalities (3.33)-(3.36), we obtain

$$\begin{aligned}
&\|q_z\|^2+Rt_4\beta\|q\|^2_{L^2(\Gamma_u)}+\|T_z\|^2+Rt_2\alpha\|T\|^2_{L^2(\Gamma_u)}+\frac{1}{Rt_3}\int_t^{t+1}\|\nabla q_z\|_2^2\\
&\quad+\frac{1}{Rt_4}\int_t^{t+1}\|\partial_z q_z\|_2^2+\frac{1}{Rt_1}\int_t^{t+1}\|\nabla T_z\|_2^2+\frac{1}{Rt_2}\int_t^{t+1}\|\partial_z T_z\|_2^2\\
&\quad+\frac{\alpha Rt_2}{Rt_1}\int_t^{t+1}\|\nabla T\|^2_{L^2(\Gamma_u)}+\frac{\beta Rt_4}{Rt_3}\int_t^{t+1}\|\nabla q\|^2_{L^2(\Gamma_u)}\le\rho_9
\end{aligned}\tag{3.37}$$

for any $t\ge T_2+6$.

3.2.9. $L^2(\Omega)$ estimates of $(\nabla v,\nabla T,\nabla q)$. Taking the inner product of equation (2.4) with $-\Delta v$ in $L^2(\Omega)$ and combining the boundary condition (2.7), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2+\frac{1}{Re_1}\int_\Omega|\Delta v|^2\,dxdydz+\frac{1}{Re_2}\int_\Omega|\nabla\partial_z v|^2\,dxdydz\\
&\le C\int_M(\int_0^1|\nabla v|\,dz)(\int_0^1|\partial_z v||\Delta v|\,dz)\,dxdy+C\int_\Omega|v||\nabla v||\Delta v|\,dxdydz\\
&\quad-\int_\Omega(\int_0^z\frac{bP}{p(\zeta)}\nabla[(1+aq(x,y,\zeta,t))T(x,y,\zeta,t)]\,d\zeta)\cdot\Delta v\,dxdydz.
\end{aligned}\tag{3.38}$$

In the following, we estimate each term of the right-hand side of inequality (3.38).

$$\int_\Omega |v||\nabla v||\Delta v|\,dxdydz \le C\|v\|_6\|\nabla v\|_3\|\Delta v\|_2$$
$$\le C\|v\|_6\|\nabla v\|_2^{\frac{1}{2}}(\|\nabla\partial_z v\|_2+\|\Delta v\|_2)^{\frac{3}{2}}, \tag{3.39}$$

$$\int_M (\int_0^1 |\nabla v|\,dz)(\int_0^1 |\partial_z v||\Delta v|\,dz)\,dxdy \le C\|v_z\|_2^{\frac{1}{2}}\|\nabla v_z\|_2^{\frac{1}{2}}\|\nabla v\|_2^{\frac{1}{2}}\|\Delta v\|_2^{\frac{3}{2}} \tag{3.40}$$

and

$$\int_\Omega (\int_0^z \frac{bP}{p(\zeta)}\nabla[(1+aq(x,y,\zeta,t))T(x,y,\zeta,t)]\,d\zeta)\cdot\Delta v\,dxdydz$$
$$\le C\|\nabla T\|_2\|\Delta v\|_2+C\|q\|_6\|\nabla T\|_3\|\Delta v\|_2+C\|T\|_6\|\nabla q\|_3\|\Delta v\|_2. \tag{3.41}$$

Multiplying Equation (2.6) by $-\Delta q$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla q\|_2^2+\frac{1}{Rt_3}\int_\Omega |\Delta q|^2\,dxdydz+\frac{1}{Rt_4}\int_\Omega |\nabla\partial_z q|^2\,dxdydz+\beta\int_{\Gamma_u}|\nabla q|^2\,dxdy$$
$$\le C\int_M (\int_0^1 |\nabla v|\,dz)(\int_0^1 |\partial_z q||\Delta q|\,dz)\,dxdy+C\int_\Omega |v||\nabla q||\Delta q|\,dxdydz+\|Q_2\|_2\|\Delta q\|_2$$
$$\le C\|\nabla v\|_2^{\frac{1}{2}}\|\Delta v\|_2^{\frac{1}{2}}\|q_z\|_2^{\frac{1}{2}}\|\nabla q_z\|_2^{\frac{1}{2}}\|\Delta q\|_2+C\|v\|_6\|\nabla q\|_3\|\Delta q\|_2+\|Q_2\|_2\|\Delta q\|_2. \tag{3.42}$$

Taking the inner product of equation (2.5) with $-\Delta T$ in $L^2(\Omega)$ and combining the boundary condition (2.8), we find

$$\frac{1}{2}\frac{d}{dt}\|\nabla T\|_2^2+\frac{1}{Rt_1}\int_\Omega |\Delta T|^2\,dxdydz+\frac{1}{Rt_2}\int_\Omega |\nabla\partial_z T|^2\,dxdydz+\alpha\int_{\Gamma_u}|\nabla T|^2\,dxdy$$
$$\le C\int_M (\int_0^1 |\nabla v|\,dz)(\int_0^1 |\partial_z T||\Delta T|\,dz)\,dxdy+C\int_\Omega |v||\nabla T||\Delta T|\,dxdydz+\|Q_1\|_2\|\Delta T\|_2$$
$$-\int_\Omega \frac{bP}{p}(1+aq)(\int_0^z \nabla\cdot v(x,y,\zeta,t)\,d\zeta)\Delta T\,dxdydz$$
$$\le C\|\nabla v\|_2^{\frac{1}{2}}\|\Delta v\|_2^{\frac{1}{2}}\|T_z\|_2^{\frac{1}{2}}\|\nabla T_z\|_2^{\frac{1}{2}}\|\Delta T\|_2+C\|v\|_6\|\nabla T\|_3\|\Delta T\|_2+\|Q_1\|_2\|\Delta T\|_2$$
$$+C\|\nabla v\|_2\|\Delta T\|_2+C\|q\|_6\|\nabla v\|_3\|\Delta T\|_2. \tag{3.43}$$

From the uniform Gronwall inequality, Young's inequality and inequalities (3.37)-(3.43), we obtain

$$\|\nabla v\|_2^2+\|\nabla q\|_2^2+\|\nabla T\|_2^2+\frac{1}{Rt_3}\int_t^{t+1}\|\Delta q\|_2^2+\frac{1}{Rt_4}\int_t^{t+1}\|\nabla\partial_z q\|_2^2$$
$$+\beta\int_t^{t+1}\|\nabla q\|_{L^2(\Gamma_u)}^2+\frac{1}{Rt_1}\int_t^{t+1}\|\Delta T\|_2^2+\frac{1}{Rt_2}\int_t^{t+1}\|\nabla\partial_z T\|_2^2$$
$$+\alpha\int_t^{t+1}\|\nabla T\|_{L^2(\Gamma_u)}^2+\frac{1}{Re_1}\int_t^{t+1}\|\Delta v\|_2^2+\frac{1}{Re_2}\int_t^{t+1}\|\nabla\partial_z v\|_2^2 \le \rho_{10} \tag{3.44}$$

for any $t \ge T_2+7$.

3.2.10. $(L^6(\Omega))^2$ estimates of v_z. Multiplying Equation (3.25) by $|u|^4u$ and integrating over Ω, and combining the boundary condition (3.26), we find

$$\begin{aligned}&\frac{1}{6}\frac{d}{dt}\|u\|_6^6+\frac{1}{Re_1}\int_\Omega|\nabla u|^2|u|^4dxdydz+\frac{1}{Re_2}\int_\Omega|\partial_z u|^2|u|^4dxdydz+\frac{4}{9}\||u|^3\|^2\\&=\int_\Omega(\nabla\cdot v)|u|^6dxdydz-\int_\Omega[(u\cdot\nabla)v]\cdot|u|^4udxdydz\\&+\int_\Omega\frac{bP}{p}\nabla[(1+aq)T]\cdot|u|^4udxdydz.\end{aligned}\tag{3.45}$$

Next, we estimate each term of the right-hand side of equality (3.45).

$$\begin{aligned}|\int_\Omega(\nabla\cdot v)|u|^6dxdydz|\le&C\int_\Omega|v||\nabla|u|^3||u|^3dxdydz\\\le&C\|v\|_6\||u|^3\|_3\|\nabla|u|^3\|_2\\\le&C\|v\|_6\|u\|_6^{\frac{3}{2}}(\|\nabla|u|^3\|_2+\|\partial_z|u|^3\|_2)^{\frac{3}{2}},\end{aligned}\tag{3.46}$$

$$\begin{aligned}&|\int_\Omega\frac{bP}{p}\nabla[(1+aq)T]\cdot|u|^4udxdydz|\\&\le\||u|^3\|_{\frac{10}{3}}^{\frac{5}{3}}\|\frac{bP}{p}\nabla[(1+aq)T]\|_2\\&\le C\|\frac{bP}{p}\nabla[(1+aq)T\|_2\||u|^3\|_2^{\frac{2}{3}}\||u|^3\|\\&\le C(\|\nabla T\|_2+\|\nabla T\|_3\|q\|_6+\|\nabla q\|_3\|T\|_6)\|u\|_6^2\||u|^3\|\end{aligned}\tag{3.47}$$

and

$$\begin{aligned}&|-\int_\Omega[(u\cdot\nabla)v]\cdot|u|^4udxdydz|\\&\le C\int_\Omega|u|^5|v||\nabla u|dxdydz+C\int_\Omega|u|^3|v||\nabla|u|^3|dxdydz\\&\le C\|v\|_6\||u|^3\|_3\|\nabla|u|^3\|_2+C\|v\|_6\||u|^3\|_3\||\nabla u||u|^2\|_2.\end{aligned}\tag{3.48}$$

Combining inequalities (3.45)-(3.48) with the uniform Gronwall inequality and Young's inequality, we have

$$\|\partial_z v\|_6^2\le\rho_{11}\tag{3.49}$$

for any $t\ge T_2+8$.

3.2.11. $(L^6(\Omega))^2$ estimates of (T_z,q_z). Denoted by $\theta=T_z$. It is clear that θ satisfies the following equation by differentiating Equation (2.5) with respect to z:

$$\begin{aligned}&\frac{\partial\theta}{\partial t}+L_2\theta+v\cdot\nabla\theta-(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta)\frac{\partial\theta}{\partial z}+\partial_z v\cdot\nabla T-(\nabla\cdot v)\theta\\&+\frac{bP}{p}(1+aq)(\nabla\cdot v)-\frac{bP(P-p_0)}{p^2}(1+aq)(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta)\\&+\frac{abP}{p}q_z(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta)=\partial_z Q_1\end{aligned}\tag{3.50}$$

supplemented with the boundary conditions

$$(\frac{1}{Rt_2}\theta+\alpha T)|_{\Gamma_u}=0,\theta|_{\Gamma_b}=0,\frac{\partial\theta}{\partial\vec{n}}|_{\Gamma_l}=0. \tag{3.51}$$

Taking the inner product of Equation (3.50) with $|\theta|^4\theta$ in $L^2(\Omega)$ and combining the boundary conditions (3.51), we know

$$\begin{aligned}&\frac{1}{6}\frac{d}{dt}\|\theta\|_6^6+\frac{5}{9Rt_1}\int_\Omega|\nabla|\theta|^3|^2dxdydz+\frac{5}{9Rt_2}\int_\Omega|\partial_z|\theta|^3|^2dxdydz+\alpha^5Rt_2^4\int_{\Gamma_u}\frac{\partial\theta}{\partial z}|T|^4T\,dxdy\\&=\int_\Omega(\nabla\cdot v)|\theta|^6dxdydz+\int_\Omega\partial_zQ_1|\theta|^4\theta\,dxdydz-\int_\Omega(v_z\cdot\nabla T)|\theta|^4\theta\,dxdydz\\&-\int_\Omega\frac{bP}{p}(1+aq)(\nabla\cdot v)|\theta|^4\theta-\int_\Omega\frac{abP}{p}q_z(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta)|\theta|^4\theta\\&+\int_\Omega\frac{bP(P-p_0)}{p^2}(1+aq)(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta)|\theta|^4\theta.\end{aligned} \tag{3.52}$$

In the following, we give the estimates of each term in the right-hand side of equality (3.52).

$$\begin{aligned}\left|\int_\Omega(\nabla\cdot v)|\theta|^6dxdydz\right|&\le C\int_\Omega|v||\nabla|\theta|^3||\theta|^3dxdydz\\&\le C\|v\|_6\||\theta|^3\|_3\|\nabla|\theta|^3\|_2\\&\le C\|v\|_6\||\theta|^3\|_2^{\frac{1}{2}}(\|\nabla|\theta|^3\|_2+\|\partial_z|\theta|^3\|_2)^{\frac{3}{2}},\end{aligned} \tag{3.53}$$

$$\begin{aligned}\left|\int_\Omega\partial_zQ_1|\theta|^4\theta\,dxdydz\right|&\le C\int_\Omega|\partial_zQ_1||\theta|^5dxdydz\\&\le C\|\partial_zQ_1\|_2\||\theta|^3\|_{\frac{10}{3}}^{\frac{5}{3}}\\&\le C\|Q_1\|_{H^1(\Omega)}\||\theta|^3\|_2^{\frac{2}{3}}\||\theta|^3\|_{H^1(\Omega)},\end{aligned} \tag{3.54}$$

$$\begin{aligned}\left|\int_\Omega(v_z\cdot\nabla T)|\theta|^4\theta\,dxdydz\right|&\le C\int_\Omega|v_z||\nabla T||\theta|^5dxdydz\\&\le C\|v_z\|_6\|\nabla T\|_3\||\theta|^3\|_{\frac{10}{3}}^{\frac{5}{3}}\\&\le C\|v_z\|_6\|\nabla T\|_2^{\frac{1}{2}}\|\nabla T\|_{H^1(\Omega)}^{\frac{1}{2}}\||\theta|^3\|_2^{\frac{2}{3}}\||\theta|^3\|_{H^1(\Omega)},\end{aligned} \tag{3.55}$$

$$\begin{aligned}\left|\int_\Omega\frac{bP}{p}(1+aq)(\nabla\cdot v)|\theta|^4\theta\,dxdydz\right|&\le C(\|\nabla v\|_2+\|q\|_6\|\nabla v\|_3)\||\theta|^3\|_{\frac{10}{3}}^{\frac{5}{3}}\\&\le C(\|\nabla v\|_2+\|q\|_6\|\nabla v\|_3)\||\theta|^3\|_2^{\frac{2}{3}}\||\theta|^3\|_{H^1(\Omega)},\end{aligned} \tag{3.56}$$

$$\begin{aligned}&\left|\int_\Omega\frac{abP}{p}q_z(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta)|\theta|^4\theta\,dxdydz\right|\\&\le C\|q_z\|_6\|\nabla v\|_3\||\theta|^3\|_{\frac{10}{3}}^{\frac{5}{3}}\end{aligned}$$

$$\leq C\|q_z\|_6\|\nabla v\|_3\||\theta|^3\|_2^{\frac{2}{3}}\||\theta|^3\|_{H^1(\Omega)}, \tag{3.57}$$

$$\left|\int_\Omega \frac{bP(P-p_0)}{p^2}(1+aq)(\int_0^z \nabla\cdot v(x,y,\zeta,t)d\zeta)|\theta|^4\theta\, dxdydz\right|$$
$$\leq C(\|\nabla v\|_2+\|q\|_6\|\nabla v\|_3)\||\theta|^3\|_2^{\frac{2}{3}}\||\theta|^3\|_{H^1(\Omega)}, \tag{3.58}$$

$$\begin{aligned}
&\alpha^5 Rt_2^4\int_{\Gamma_u}\frac{\partial\theta}{\partial z}|T|^4T\,dxdy\\
=&\alpha^5 Rt_2^5\int_{\Gamma_u}(\frac{\partial T}{\partial t}+v\cdot\nabla T-\frac{1}{Rt_1}\Delta T-Q_1)|T|^4T\,dxdy\\
=&\frac{\alpha^5 Rt_2^5}{6}\frac{d}{dt}\int_{\Gamma_u}|T|^6\,dxdy+\alpha^5 Rt_2^5\int_{\Gamma_u}(v\cdot\nabla T)|T|^4T\,dxdy\\
&+\frac{5\alpha^5 Rt_2^5}{9Rt_1}\int_{\Gamma_u}|\nabla|T|^3|^2\,dxdy-\alpha^5 Rt_2^5\int_{\Gamma_u}Q_1|T|^4T\,dxdy,
\end{aligned} \tag{3.59}$$

$$\begin{aligned}
&\left|\alpha^5 Rt_2^5\int_{\Gamma_u}(v\cdot\nabla T)|T|^4T\,dxdy-\alpha^5 Rt_2^5\int_{\Gamma_u}Q_1|T|^4T\,dxdy\right|\\
\leq& C\|v\|_{L^4(\Gamma_u)}\|\nabla|T|^3\|_{L^2(\Gamma_u)}\||T|^3\|_{L^4(\Gamma_u)}+C\|Q_1\|_{L^2(\Gamma_u)}\||T|^3\|^{\frac{5}{3}}_{L^{\frac{10}{3}}(\Gamma_u)}\\
\leq& C\|v\|_{H^1(\Omega)}\|\nabla|T|^3\|^{\frac{3}{2}}_{L^2(\Gamma_u)}\|T\|^{\frac{3}{2}}_{L^6(\Gamma_u)}+C\|Q_1\|_{H^1(\Omega)}\||T|^3\|_{L^2(\Gamma_u)}\||T|^3\|^{\frac{2}{3}}_{H^1(\Gamma_u)}.
\end{aligned} \tag{3.60}$$

Denoted by $\eta=q_z$. It is clear that η satisfies the following equation by differentiating Equation (2.6) with respect to z:

$$\frac{\partial\eta}{\partial t}+L_3\eta+v\cdot\nabla\eta-(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta)\frac{\partial\eta}{\partial z}+\partial_z v\cdot\nabla q-(\nabla\cdot v)\eta=\partial_z Q_2 \tag{3.61}$$

supplemented with the boundary conditions

$$(\frac{1}{Rt_4}\eta+\beta q)|_{\Gamma_u}=0,\eta|_{\Gamma_b}=0,\frac{\partial\eta}{\partial\vec{n}}|_{\Gamma_l}=0. \tag{3.62}$$

Similarly, we have the following inequality

$$\begin{aligned}
&\frac{d}{dt}(\|\eta\|_6^6+\beta^5Rt_4^5\|q\|^6_{L^6(\Gamma_u)})+\frac{2}{Rt_3}\int_\Omega|\nabla|\eta|^3|^2\,dxdydz+\frac{2}{Rt_4}\int_\Omega|\partial_z|\eta|^3|^2\,dxdydz\\
&\quad+\frac{2\beta^5Rt_4^5}{Rt_3}\int_{\Gamma_u}|\nabla|q|^3|^2\,d\tilde{x}\\
&\leq C\|v\|_6^4\|\eta\|_6^6+C\|Q_2\|^2_{H^1(\Omega)}\|\eta\|_6^4+C\|v_z\|_6^2\|\nabla q\|_2\|\nabla q\|_{H^1(\Omega)}\|\eta\|_6^4\\
&\quad+C\|v\|^4_{H^1(\Omega)}\|q\|^6_{L^6(\Gamma_u)}+C\|Q_2\|^{\frac{3}{2}}_{H^1(\Omega)}\|q\|^{\frac{9}{2}}_{L^6(\Gamma_u)}.
\end{aligned} \tag{3.63}$$

Employing the uniform Gronwall inequality and Young's inequality, using inequalities (3.50)-(3.63), yields

$$\|q_z\|_6^2+\|q\|^2_{L^6(\Gamma_u)}+\|T_z\|_6^2+\|T\|^2_{L^6(\Gamma_u)}\leq\rho_{12} \tag{3.64}$$

for any $t\geq T_2+9$.

3.2.12. *H* estimates of (v_t,T_t,q_t). Denoted by $\pi=v_t$, $\xi=T_t$, $\chi=q_t$. It is clear that π, ξ, χ satisfies the following equations by differentiating Equations (2.4)-(2.6) with respect to t, respectively.

$$\begin{aligned}&\frac{\partial\pi}{\partial t}+L_1\pi+(v\cdot\nabla)\pi-\left(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta\right)\frac{\partial\pi}{\partial z}+(\pi\cdot\nabla)v+\frac{1}{Ro}f\pi^{\perp}\\&\quad-\int_0^z\frac{abP}{p(\zeta)}\nabla[\chi(x,y,\zeta,t)T(x,y,\zeta,t)]d\zeta+\nabla\partial_t\Phi_s(x,y,t)-\left(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)d\zeta\right)\frac{\partial v}{\partial z}\\&\quad-\int_0^z\frac{bP}{p(\zeta)}\nabla[(1+aq(x,y,\zeta,t))\xi(x,y,\zeta,t)]d\zeta=0, \qquad (3.65)\end{aligned}$$

$$\begin{aligned}&\frac{\partial\xi}{\partial t}+L_2\xi+v\cdot\nabla\xi-\left(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta\right)\frac{\partial\xi}{\partial z}+\frac{abP}{p}\chi\left(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta\right)\\&\quad-\left(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)d\zeta\right)\frac{\partial T}{\partial z}+\pi\cdot\nabla T+\frac{bP}{p}(1+aq)\left(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)d\zeta\right)=0,\end{aligned} \qquad (3.66)$$

$$\begin{aligned}&\frac{\partial\chi}{\partial t}+L_3\chi+v\cdot\nabla\chi-\left(\int_0^z\nabla\cdot v(x,y,\zeta,t)d\zeta\right)\frac{\partial\chi}{\partial z}+\pi\cdot\nabla q\\&\quad-\left(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)d\zeta\right)\frac{\partial q}{\partial z}=0 \qquad (3.67)\end{aligned}$$

subject to the boundary conditions

$$\frac{\partial\pi}{\partial z}|_{\Gamma_u}=0,\frac{\partial\pi}{\partial z}|_{\Gamma_b}=0,\pi\cdot\vec{n}|_{\Gamma_l}=0,\frac{\partial\pi}{\partial\vec{n}}\times\vec{n}|_{\Gamma_l}=0, \qquad (3.68)$$

$$(\frac{1}{Rt_2}\frac{\partial\xi}{\partial z}+\alpha\xi)|_{\Gamma_u}=0,\frac{\partial\xi}{\partial z}|_{\Gamma_b}=0,\frac{\partial\xi}{\partial\vec{n}}|_{\Gamma_l}=0, \qquad (3.69)$$

$$(\frac{1}{Rt_4}\frac{\partial\chi}{\partial z}+\beta\chi)|_{\Gamma_u}=0,\frac{\partial\chi}{\partial z}|_{\Gamma_b}=0,\frac{\partial\chi}{\partial\vec{n}}|_{\Gamma_l}=0. \qquad (3.70)$$

Multiplying Equations (3.65), (3.66), (3.67) by π, ξ, χ respectively, integrating over Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\pi\|_2^2+\|\pi\|^2=&-\int_\Omega\left[(\pi\cdot\nabla)v-\left(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)d\zeta\right)\frac{\partial v}{\partial z}\right]\cdot\pi\,dxdydz\\&+\int_\Omega\int_0^z\frac{abP}{p(\zeta)}\nabla[\chi(x,y,\zeta,t)T(x,y,\zeta,t)]d\zeta\cdot\pi\,dxdydz\\&+\int_\Omega\int_0^z\frac{bP}{p(\zeta)}\nabla[(1+aq(x,y,\zeta,t))\xi(x,y,\zeta,t)]d\zeta\cdot\pi\,dxdydz, \qquad (3.71)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\xi\|_2^2+\|\xi\|^2=&-\int_\Omega(\pi\cdot\nabla T)\xi\,dxdydz+\int_\Omega\left(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)d\zeta\right)\frac{\partial T}{\partial z}\xi\,dxdydz\\&-\int_\Omega\frac{abP}{p}\chi\left(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta\right)\xi\,dxdydz\\&-\int_\Omega\frac{bP}{p}(1+aq)\left(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)\,d\zeta\right)\xi\,dxdydz \qquad (3.72)\end{aligned}$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\chi\|_2^2+\|\chi\|^2 \\
=&-\int_\Omega(\pi\cdot\nabla q)\chi\,dxdydz+\int_\Omega\left(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)d\zeta\right)\frac{\partial q}{\partial z}\chi\,dxdydz. \qquad (3.73)
\end{aligned}
$$

Next, we will estimate the right-hand side of equalities (3.71)-(3.73) term by term.

$$
\begin{aligned}
&\left|-\int_\Omega\left[(\pi\cdot\nabla)v-(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)d\zeta)\frac{\partial v}{\partial z}\right]\cdot\pi\,dxdydz\right| \\
\le& C\|\nabla\pi\|_2\|v_z\|_6\|\pi\|_3+C\|v\|_6\|\pi\|_3\|\nabla\pi\|_2, \qquad (3.74)
\end{aligned}
$$

$$
\begin{aligned}
&\left|-\int_\Omega(\pi\cdot\nabla T)\xi\,dxdydz+\int_\Omega\left(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)d\zeta\right)\frac{\partial T}{\partial z}\xi\,dxdydz\right| \\
\le& C\|\nabla\pi\|_2\|T\|_6\|\xi\|_3+C\|\nabla\xi\|_2\|T\|_6\|\pi\|_3+C\|\nabla\pi\|_2\|T_z\|_6\|\xi\|_3, \qquad (3.75)
\end{aligned}
$$

$$
\begin{aligned}
&\left|-\int_\Omega(\pi\cdot\nabla q)\chi\,dxdydz+\int_\Omega\left(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)d\zeta\right)\frac{\partial q}{\partial z}\chi\,dxdydz\right| \\
\le& C\|\nabla\pi\|_2\|q\|_6\|\chi\|_3+C\|\nabla\chi\|_2\|q\|_6\|\pi\|_3+C\|\nabla\pi\|_2\|q_z\|_6\|\chi\|_3, \qquad (3.76)
\end{aligned}
$$

$$
\left|\int_\Omega\int_0^z\frac{abP}{p(\zeta)}\nabla[\chi(x,y,\zeta,t)T(x,y,\zeta,t)]\,d\zeta\cdot\pi\,dxdydz\right|\le C\|\chi\|_3\|T\|_6\|\nabla\pi\|_2, \qquad (3.77)
$$

$$
\begin{aligned}
&\left|-\int_\Omega\frac{abP}{p}\chi\left(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta\right)\xi\,dxdydz\right| \\
\le& C\|\nabla\chi\|_2\|v\|_6\|\xi\|_3+C\|\nabla\xi\|_2\|v\|_6\|\chi\|_3 \qquad (3.78)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega\int_0^z\frac{bP}{p(\zeta)}\nabla[(1+aq(x,y,\zeta,t))\xi(x,y,\zeta,t)]\,d\zeta\cdot\pi\,dxdydz \\
&\quad-\int_\Omega\frac{bP}{p}(1+aq)\left(\int_0^z\nabla\cdot\pi(x,y,\zeta,t)\,d\zeta\right)\xi\,dxdydz=0. \qquad (3.79)
\end{aligned}
$$

We infer from inequalities (3.71)-(3.79) that

$$
\begin{aligned}
&\frac{d}{dt}(\|\xi\|_2^2+\|\pi\|_2^2+\|\chi\|_2^2)+(\|\xi\|^2+\|\pi\|^2+\|\chi\|^2) \\
\le&(\|v\|_6^4+\|v_z\|_6^4+\|T\|_6^4+\|T_z\|_6^4+\|q\|_6^4+\|q_z\|_6^4)(\|\xi\|_2^2+\|\pi\|_2^2+\|\chi\|_2^2). \qquad (3.80)
\end{aligned}
$$

Taking the inner product of Equation (2.6) with χ in $L^2(\Omega)$, we obtain

$$
\begin{aligned}
\|\chi\|_2^2=&-\int_\Omega\left[v\cdot\nabla q-\left(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta\right)\frac{\partial q}{\partial z}\right]\chi\,dxdydz \\
&-\int_\Omega L_3q\chi\,dxdydz+\int_\Omega Q_2\chi\,dxdydz \\
\le&\|Q_2\|_2\|\chi\|_2+C\|v\|_6\|\nabla q\|_3\|\chi\|_2+\|L_3q\|_2\|\chi\|_2+C\|\nabla v\|_3\|q_z\|_6\|\chi\|_2. \qquad (3.81)
\end{aligned}
$$

Similarly, we have

$$
\|\xi\|_2^2\le\|Q_1\|_2\|\xi\|_2+C\|v\|_6\|\nabla T\|_3\|\xi\|_2+\|\nabla v\|_3\|T_z\|_6\|\xi\|_2
$$

$$+\|L_2T\|_2\|\xi\|_2+C\|\nabla v\|_2\|\xi\|_2+C\|q\|_6\|\nabla v\|_3\|\xi\|_2 \tag{3.82}$$

and

$$\|\pi\|_2^2\le C\|v\|_6\|\nabla v\|_3\|\pi\|_2+C\|\nabla v\|_3\|v_z\|_6\|\pi\|_2+\|L_1v\|_2\|\pi\|_2+C\|v\|_2\|\pi\|_2$$
$$+C\|\nabla T\|_2\|\pi\|_2+C\|q\|_6\|\nabla T\|_3\|\pi\|_2+C\|T\|_6\|\nabla q\|_3\|\pi\|_2. \tag{3.83}$$

By the uniform Gronwall inequality and Young's inequality, we derive from inequalities (3.80)-(3.83) that

$$\|v_t\|_2^2+\|T_t\|_2^2+\|q_t\|_2^2\le\rho_{13} \tag{3.84}$$

for any $t\ge T_2+10$.

Moreover, we have

$$\int_t^{t+1}\|v_t\|^2\,d\tau+\int_t^{t+1}\|T_t\|^2\,d\tau+\int_t^{t+1}\|q_t\|^2\,d\tau\le\rho_{14} \tag{3.85}$$

for any $t\ge T_2+10$.

3.2.13. V estimates of (v_t,T_t,q_t). Multiplying Equation (3.65) by $L_1\pi$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|\pi\|^2+\|L_1\pi\|_2^2$$
$$\le C\|v\|_6\|\nabla\pi\|_3\|L_1\pi\|_2+C\|\nabla v\|_2^{\frac12}\|\Delta v\|_2^{\frac12}\|\pi_z\|_2^{\frac12}\|\nabla\pi_z\|_2^{\frac12}\|L_1\pi\|_2+C\|\chi\|_6\|\nabla T\|_3\|L_1\pi\|_2$$
$$+C\|\pi\|_6\|\nabla v\|_3\|L_1\pi\|_2+C\|\nabla\pi\|_2^{\frac12}\|\Delta\pi\|_2^{\frac12}\|v_z\|_2^{\frac12}\|\nabla v_z\|_2^{\frac12}\|L_1\pi\|_2+C\|\nabla\chi\|_3\|T\|_6\|L_1\pi\|_2$$
$$+C\|\nabla\xi\|_2\|L_1\pi\|_2+C\|q\|_6\|\nabla\xi\|_3\|L_1\pi\|_2+C\|\xi\|_6\|\nabla q\|_3\|L_1\pi\|_2.$$

Similarly, we have the following inequalities

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2+\|L_2\xi\|_2^2$$
$$\le C\|v\|_6\|\nabla\xi\|_3\|L_2\xi\|_2+C\|\nabla v\|_2^{\frac12}\|\Delta v\|_2^{\frac12}\|\xi_z\|_2^{\frac12}\|\nabla\xi_z\|_2^{\frac12}\|L_2\xi\|_2+C\|\nabla\pi\|_2\|L_2\xi\|_2$$
$$+C\|\pi\|_6\|\nabla T\|_3\|L_2\xi\|_2+C\|\nabla\pi\|_2^{\frac12}\|\Delta\pi\|_2^{\frac12}\|T_z\|_2^{\frac12}\|\nabla T_z\|_2^{\frac12}\|L_2\xi\|_2$$
$$+C\|\nabla v\|_3\|\chi\|_6\|L_2\xi\|_2+C\|q\|_6\|\nabla\pi\|_3\|L_2\xi\|_2$$

and

$$\frac{1}{2}\frac{d}{dt}\|\chi\|^2+\|L_3\chi\|_2^2$$
$$\le C\|v\|_6\|\nabla\chi\|_3\|L_3\chi\|_2+C\|\nabla v\|_2^{\frac12}\|\Delta v\|_2^{\frac12}\|\chi_z\|_2^{\frac12}\|\nabla\chi_z\|_2^{\frac12}\|L_3\chi\|_2$$
$$+C\|\pi\|_6\|\nabla q\|_3\|L_3\chi\|_2+C\|\nabla\pi\|_2^{\frac12}\|\Delta\pi\|_2^{\frac12}\|q_z\|_2^{\frac12}\|\nabla q_z\|_2^{\frac12}\|L_3\chi\|_2.$$

Employing the uniform Gronwall inequality and Young's inequality, yield

$$\|v_t\|^2+\|T_t\|^2+\|q_t\|^2\le\rho_{15} \tag{3.86}$$

for any $t\ge T_2+11$.

3.2.14. $(H^2(\Omega))^4\cap V$ estimates of (v,T,q). Taking the inner product of Equation (2.4) with $L_1 v$ in $L^2(\Omega)$, we obtain

$$\begin{aligned}\|L_1v\|_2^2\le & C\|v\|_6\|\nabla v\|_3\|L_1v\|_2+C\|\nabla v\|_3\|v_z\|_6\|L_1v\|_2+C\|q\|_6\|\nabla T\|_3\|L_1v\|_2\\ &+C\|v\|_2\|L_1v\|_2+C\|\nabla T\|_2\|L_1v\|_2+\|v_t\|_2\|L_1v\|_2+\|\nabla q\|_3\|T\|_6\|L_1v\|_2.\end{aligned}$$

Similarly, we have the following inequalities

$$\begin{aligned}\|L_2T\|_2^2\le & C\|v\|_6\|\nabla T\|_3\|L_2T\|_2+C\|\nabla v\|_3\|T_z\|_6\|L_2T\|_2+C\|q\|_6\|\nabla v\|_3\|L_2T\|_2\\ &+C\|Q_1\|_2\|L_2T\|_2+C\|\nabla v\|_2\|L_2T\|_2+\|T_t\|_2\|L_2T\|_2\end{aligned}$$

and

$$\|L_3q\|_2^2\le C\|v\|_6\|\nabla q\|_3\|L_3q\|_2+C\|\nabla v\|_3\|q_z\|_6\|L_3q\|_2+C\|Q_2\|_2\|L_3q\|_2+\|q_t\|_2\|L_3q\|_2.$$

Therefore, we have

$$\|L_1v\|_2^2+\|L_2T\|_2^2+\|L_3q\|_2^2\le\rho_{16}$$

for any $t\ge T_2+10$, which implies that

$$\|(v,T,q)\|^2_{(H^2(\Omega))^4\cap V}\le\rho_{17} \tag{3.87}$$

for any $t\ge T_2+10$.

By virtue of inequalities (3.3), (3.9), (3.32), (3.37), (3.44), (3.87), we have

THEOREM 3.2. *Assume that $Q_1\in L^2(\Omega)$ and $Q_2\in L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t\ge0}$ associated with the initial-boundary problem* (2.4)-(2.12) *possesses an absorbing set in V. That is, there exists a positive constant $\mathcal{R}_1$ satisfying for any bounded subset B of V, there exists a positive time $\tau_1=\tau_{1,B}$ depending on the norm of B such that for any $t\ge\tau_1$, we have*

$$\|(v(t),T(t),q(t))\|_V=\|S(t)(v_0,T_0,q_0)\|_V\le\mathcal{R}_1.$$

THEOREM 3.3. *Assume that $Q_1\in H^1(\Omega)$ and $Q_2\in H^1(\Omega)$. Then the semigroup $\{S(t)\}_{t\ge0}$ associated with the initial-boundary problem* (2.4)-(2.12) *possesses an absorbing set in $(H^2(\Omega))^4\cap V$. That is, there exists a positive constant $\mathcal{R}_2$ satisfying for any bounded subset B of V, there exists a positive time $\tau_2=\tau_{2,B}$ depending on the norm of B such that for any $t\ge\tau_2$, we have*

$$\|(v(t),T(t),q(t))\|_{(H^2(\Omega))^4\cap V}=\|S(t)(v_0,T_0,q_0)\|_{(H^2(\Omega))^4\cap V}\le\mathcal{R}_2.$$

4. The existence of exponential attractor and global attractors

4.1. The existence of global attractors. The abstract theory of global attractor can be referred to [2,8,44,51,54,59]. In this section, we prove the existence of global attractors of the semigroup $\{S(t)\}_{t\ge0}$ generated by the initial-boundary problem (2.4)-(2.12).

Thanks to the compactness of $(H^2(\Omega))^4\cap V\subset V$, we have the following result.

THEOREM 4.1. *Assume that $Q_1\in H^1(\Omega)$ and $Q_2\in H^1(\Omega)$. Then the semigroup $\{S(t)\}_{t\ge0}$ corresponding to problem* (2.4)-(2.12) *has a global attractor $\mathcal{A}_V$ in V.*

REMARK 4.1. Under the assumptions that $Q_1 \in L^2(\Omega)$ and $Q_2 \in L^2(\Omega)$, the existence of a global attractor $\mathcal{A}_V$ in V of the semigroup $\{S(t)\}_{t\geq 0}$ associated with problem (2.4)-(2.12) can be also obtained by the Aubin-Lions compactness Lemma as in [34].

Next, we prove the asymptotical compactness of the semigroup $\{S(t)\}_{t\geq 0}$ generated by the initial-boundary problem (2.4)-(2.12).

THEOREM 4.2. *Assume that $Q_1 \in H^1(\Omega)$ and $Q_2 \in H^1(\Omega)$. Then the semigroup $\{S(t)\}_{t\geq 0}$ generated by problem (2.4)-(2.12) is asymptotically compact in $(H^2(\Omega))^4 \cap V$.*

Proof. Let B_0 be an absorbing set in $(H^2(\Omega))^4 \cap V$ of the semigroup $\{S(t)\}_{t\geq 0}$ generated by problem (2.4)-(2.12) obtained in Theorem 3.3. Then we need only to show that for any $\{(v_{0n}, T_{0n}, q_{0n})\}_{n=1}^{\infty} \subset B_0$ and $t_n \to \infty$, $\{(v_n(t_n), T_n(t_n), q_n(t_n))\}_{n=1}^{\infty}$ is pre-compact in $(H^2(\Omega))^4 \cap V$, where $(v_n(t_n), T_n(t_n), q_n(t_n)) = S(t_n)(v_{0n}, T_{0n}, q_{0n})$.

In fact, from Theorem 3.3, inequality (3.86) and the compactness of $(H^2(\Omega))^4 \cap V \subset (W^{1,3}(\Omega))^4 \cap V$, we know that $\{(v_n(t_n), T_n(t_n), q_n(t_n))\}_{n=1}^{\infty}$ and $\{(\frac{\partial v_n(t_n)}{\partial t}, \frac{\partial T_n(t_n)}{\partial t}, \frac{\partial q_n(t_n)}{\partial t})\}_{n=1}^{\infty}$ is pre-compact in $(W^{1,3}(\Omega))^4 \cap V$ and H, respectively. Without loss of generality, we assume that $\{(v_n(t_n), T_n(t_n), q_n(t_n))\}_{n=1}^{\infty}$ and $\{(\frac{\partial v_n(t_n)}{\partial t}, \frac{\partial T_n(t_n)}{\partial t}, \frac{\partial q_n(t_n)}{\partial t})\}_{n=1}^{\infty}$ is a Cauchy sequence in $(W^{1,3}(\Omega))^4 \cap V$ and H, respectively.

In the following, we will prove that $\{(v_n(t_n), T_n(t_n), q_n(t_n))\}_{n=0}^{\infty}$ is a Cauchy sequence in $(H^2(\Omega))^4 \cap V$.

Then, by simple calculations, we have

$$\begin{aligned}
&\|L_3 q_n(t_n) - L_3 q_m(t_m)\|_2^2 \\
\leq &\|\frac{\partial q_n(t_n)}{\partial t} - \frac{\partial q_m(t_m)}{\partial t}\|_2 \|L_3 q_n(t_n) - L_3 q_m(t_m)\|_2 \\
&+ \|v_n(t_n) - v_m(t_m)\|_3 \|\nabla q_n(t_n)\|_6 \|L_3 q_n(t_n) - L_3 q_m(t_m)\|_2 \\
&+ \|v_m(t_m)\|_6 \|\nabla q_n(t_n) - \nabla q_m(t_m)\|_3 \|L_3 q_n(t_n) - L_3 q_m(t_m)\|_2 \\
&+ \|\nabla v_n(t_n) - \nabla v_m(t_m)\|_3 \|\frac{\partial q_n(t_n)}{\partial z}\|_6 \|L_3 q_n(t_n) - L_3 q_m(t_m)\|_2 \\
&+ \|\nabla v_m(t_m)\|_6 \|\frac{\partial q_n(t_n)}{\partial z} - \frac{\partial q_m(t_m)}{\partial z}\|_3 \|L_3 q_n(t_n) - L_3 q_m(t_m)\|_2.
\end{aligned}$$

From Hölder inequality and Theorem 3.3, we deduce that $\{q_n(t_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $H^2(\Omega)$.

Similarly, we can also prove $\{(v_n(t_n), T_n(t_n))\}_{n=1}^{\infty}$ is a Cauchy sequence in $(H^2(\Omega))^3$. The proof of Theorem 4.2 is completed. ∎

Therefore, from Theorem 3.3 and Theorem 4.2, we immediately obtain the following result.

THEOREM 4.3. *Assume that $Q_1 \in H^1(\Omega)$ and $Q_2 \in H^1(\Omega)$. Then the semigroup $\{S(t)\}_{t\geq 0}$ associated with problem (2.4)-(2.12) has a global attractor $\mathcal{A}$ in $(H^2(\Omega))^4 \cap V$.*

4.2. The existence of an exponential attractor. In this subsection, inspired by the idea in [1, 11, 12, 50], we prove the existence of an exponential attractor in V for the three dimensional viscous primitive equations of large-scale moist atmosphere. The definition about exponential attractor can be referred to [1, 8, 11–13, 50].

In what follows, we first prove the smoothing property of the semigroup $\{S(t)\}_{t\geq 0}$ generated by problem (2.4)-(2.12).

THEOREM 4.4. *Assume that $Q_1 \in H^1(\Omega)$ and $Q_2 \in H^1(\Omega)$. Let $(v^i, \Phi_s^i, T^i, q^i)$ be the solution of problem (2.4)-(2.12) with the initial data (v_0^i, T_0^i, q_0^i), $i = 1, 2$. Then for any* *bounded subset $B\subset V$, there exists some time $\tau_3=\tau_3(B)$ such that the following estimate holds*

$$\begin{aligned}&\|(v^1(t),T^1(t),q^1(t))-(v^2(t),T^2(t),q^2(t))\|_V^2\\&\leq \varrho_1\frac{\bar t+1}{\bar t}e^{\varrho_2 t}\|(v_0^1,T_0^1,q_0^1)-(v_0^2,T_0^2,q_0^2)\|_H^2\end{aligned}\tag{4.1}$$

for any $(v_0^i,T_0^i,q_0^i)\in B$ and any $t\geq\tau_3$, where $\bar t=t-\tau_3$, ϱ_1 and ϱ_2 are positive constants which only depend on Ω, α, β, Rt_2, Rt_4, $\|Q_1\|_{H^1(\Omega)}$ and $\|Q_2\|_{H^1(\Omega)}$.

Proof. For any bounded subset $B\subset V$, let $(v_0^i,T_0^i,q_0^i)\in B(i=1,2)$ and $(v,\Phi_s,T,q)=(v^1-v^2,\Phi_s^1-\Phi_s^2,T^1-T^2,q^1-q^2)$, then (v,Φ_s,T,q) satisfies the following equations

$$\begin{cases}\frac{\partial v}{\partial t}+(v^1\cdot\nabla)v-(\int_0^z\nabla\cdot v^1(x,y,\zeta,t)\,d\zeta)\frac{\partial v}{\partial z}+(v\cdot\nabla)v^2-(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta)\frac{\partial v^2}{\partial z}\\ \quad+\nabla\Phi_s(x,y,t)+\frac{1}{Ro}fv^{\perp}-\int_0^z\frac{bP}{p(\zeta)}\nabla[(1+aq^1(x,y,\zeta,t))T(x,y,\zeta,t)]\,d\zeta\\ \quad+L_1v-\int_0^z\frac{abP}{p(\zeta)}\nabla[q(x,y,\zeta,t)T^2(x,y,\zeta,t)]\,d\zeta=0,\\ \frac{\partial T}{\partial t}+v^1\cdot\nabla T-(\int_0^z\nabla\cdot v^1(x,y,\zeta,t)\,d\zeta)\frac{\partial T}{\partial z}+v\cdot\nabla T^2-(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta)\frac{\partial T^2}{\partial z}+L_2T\\ \quad+\frac{bP}{p}(1+aq^1)(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta)+\frac{abP}{p}q(\int_0^z\nabla\cdot v^2(x,y,\zeta,t)\,d\zeta)=0,\\ \frac{\partial q}{\partial t}+v^1\cdot\nabla q-(\int_0^z\nabla\cdot v^1(x,y,\zeta,t)\,d\zeta)\frac{\partial q}{\partial z}+v\cdot\nabla q^2-(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta)\frac{\partial q^2}{\partial z}+L_3q=0\end{cases}\tag{4.2}$$

with the following boundary conditions

$$\begin{cases}\frac{\partial v}{\partial z}|_{\Gamma_u}=0,\frac{\partial v}{\partial z}|_{\Gamma_b}=0,v\cdot\vec n|_{\Gamma_l}=0,\frac{\partial v}{\partial\vec n}\times\vec n|_{\Gamma_l}=0,\\ (\frac{1}{Rt_2}\frac{\partial T}{\partial z}+\alpha T)|_{\Gamma_u}=0,\frac{\partial T}{\partial z}|_{\Gamma_b}=0,\frac{\partial T}{\partial\vec n}|_{\Gamma_l}=0,\\ (\frac{1}{Rt_4}\frac{\partial q}{\partial z}+\beta q)|_{\Gamma_u}=0,\frac{\partial q}{\partial z}|_{\Gamma_b}=0,\frac{\partial q}{\partial\vec n}|_{\Gamma_l}=0\end{cases}\tag{4.3}$$

and the initial data

$$\begin{cases}v(x,y,z,0)=v_0^1(x,y,z)-v_0^2(x,y,z),\\ T(x,y,z,0)=T_0^1(x,y,z)-T_0^2(x,y,z),\\ q(x,y,z,0)=q_0^1(x,y,z)-q_0^2(x,y,z).\end{cases}\tag{4.4}$$

Multiplying the first equation of Equations (4.2) by v and integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|_2^2+\|v\|^2\\&=-\int_\Omega[(v\cdot\nabla)v^2]\cdot v\,dxdydz+\int_\Omega(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta)\frac{\partial v^2}{\partial z}\cdot v\,dxdydz\\&+\int_\Omega\int_0^z\frac{bP}{p(\zeta)}\nabla[(1+aq^1(x,y,\zeta,t))T(x,y,\zeta,t)]\,d\zeta\cdot v\,dxdydz\\&+\int_\Omega\int_0^z\frac{abP}{p(\zeta)}\nabla[q(x,y,\zeta,t)T^2(x,y,\zeta,t)]\,d\zeta\cdot v\,dxdydz.\end{aligned}\tag{4.5}$$

Taking the inner product of the second equation of Equations (4.2) with T in $L^2(\Omega)$ and combining the second equation of Equations (4.3), we get

$$\frac{1}{2}\frac{d}{dt}\|T\|_2^2+\|T\|^2=-\int_\Omega(v\cdot\nabla T^2)T\,dxdydz+\int_\Omega(\int_0^z\nabla\cdot v(x,y,\zeta,t)\,d\zeta)\frac{\partial T^2}{\partial z}T\,dxdydz$$

$$-\int_\Omega \frac{bP}{p}(1+aq^1)(\int_0^z \nabla\cdot v(x,y,\zeta,t)\,d\zeta)T\,dxdydz$$
$$-\int_\Omega \frac{abP}{p}q(\int_0^z \nabla\cdot v^2(x,y,\zeta,t)\,d\zeta)T\,dxdydz. \tag{4.6}$$

Multiplying the third equation of Equations (4.2) by q and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|q\|_2^2+\|q\|^2$$
$$=-\int_\Omega (v\cdot\nabla q^2)q\,dxdydz+\int_\Omega (\int_0^z \nabla\cdot v(x,y,\zeta,t)\,d\zeta)\frac{\partial q^2}{\partial z}q\,dxdydz. \tag{4.7}$$

From equalities (4.5)-(4.7) and Hölder's inequality, we deduce that

$$\frac{d}{dt}\|(v,T,q)\|_H^2+2\|(v,T,q)\|_V^2$$
$$\leq C\|v\|_3\|\nabla v^2\|_2\|v\|_6+C\|v_z^2\|_2^{\frac{1}{2}}\|\nabla v_z^2\|_2^{\frac{1}{2}}\|v\|_2^{\frac{1}{2}}\|\nabla v\|_2^{\frac{3}{2}}+C\|q\|_3\|\nabla v\|_2\|T^2\|_6$$
$$+C\|v\|_3\|\nabla T^2\|_2\|T\|_6+C\|T_z^2\|_2^{\frac{1}{2}}\|\nabla T_z^2\|_2^{\frac{1}{2}}\|T\|_2^{\frac{1}{2}}\|\nabla T\|_2^{\frac{1}{2}}\|\nabla v\|_2$$
$$+C\|q\|_6\|\nabla v^2\|_2\|T\|_3+C\|v\|_3\|\nabla q^2\|_2\|q\|_6+C\|q_z^2\|_2^{\frac{1}{2}}\|\nabla q_z^2\|_2^{\frac{1}{2}}\|q\|_2^{\frac{1}{2}}\|\nabla q\|_2^{\frac{1}{2}}\|\nabla v\|_2.$$

It follows from Theorem 3.3 and Young's inequality that there exists some time $\tau_3=\tau_3(B)>0$ such that

$$\frac{d}{dt}\|(v,T,q)\|_H^2+\|(v,T,q)\|_V^2\leq C\mathcal{R}_2^4\|(v,T,q)\|_H^2$$

for any $t\geq\tau_3$.

We infer from the classical Gronwall inequality that

$$\|(v(t),T(t),q(t))\|_H^2+\int_0^t \|(v(s),T(s),q(s))\|_V^2\,ds$$
$$\leq e^{C\mathcal{R}_2^4 t}\|(v(0),T(0),q(0))\|_H^2$$

for any $t\geq\tau_3$, where C is a positive constant.

Taking the inner product of the first equation of Equations (4.2) with L_1v in $L^2(\Omega)$ and combining the first equation of boundary conditions (4.3), we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2+\|L_1v\|_2^2$$
$$\leq\|v^1\|_6\|\nabla v\|_3\|L_1v\|_2+C\|\nabla v^1\|_2^{\frac{1}{2}}\|\Delta v^1\|_2^{\frac{1}{2}}\|v_z\|_2^{\frac{1}{2}}\|\nabla v_z\|_2^{\frac{1}{2}}\|L_1v\|_2$$
$$+\|v\|_3\|\nabla v^2\|_6\|L_1v\|_2+C\|\nabla v\|_2^{\frac{1}{2}}\|\Delta v\|_2^{\frac{1}{2}}\|v_z^2\|_2^{\frac{1}{2}}\|\nabla v_z^2\|_2^{\frac{1}{2}}\|L_1v\|_2+C\|v\|_2\|L_1v\|_2$$
$$+C\|\nabla T\|_2\|L_1v\|_2+C\|\nabla q^1\|_3\|T\|_6\|L_1v\|_2+C\|q^1\|_6\|\nabla T\|_3\|L_1v\|_2$$
$$+C\|\nabla q\|_3\|T^2\|_6\|L_1v\|_2+C\|q\|_6\|\nabla T^2\|_3\|L_1v\|_2. \tag{4.8}$$

Multiplying the third equation of Equations (4.2) by L_2T and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|T\|^2+\|L_2T\|_2^2\leq\|v^1\|_6\|\nabla T\|_3\|L_2T\|_2+C\|\nabla v^1\|_2^{\frac{1}{2}}\|\Delta v^1\|_2^{\frac{1}{2}}\|T_z\|_2^{\frac{1}{2}}\|\nabla T_z\|_2^{\frac{1}{2}}\|L_2T\|_2$$

$$+\|v\|_3\|\nabla T^2\|_6\|L_2T\|_2+C\|\nabla v\|_2^{\frac{1}{2}}\|\Delta v\|_2^{\frac{1}{2}}\|T_z^2\|_2^{\frac{1}{2}}\|\nabla T_z^2\|_2^{\frac{1}{2}}\|L_2T\|_2$$
$$+C\|\nabla v\|_2\|L_2T\|_2+C\|q^1\|_6\|\nabla v\|_3\|L_2T\|_2+C\|q\|_6\|\nabla v^2\|_3\|L_2T\|_2. \tag{4.9}$$

Taking the inner product of the third equation of Equations (4.2) with L_3q in $L^2(\Omega)$ and combining the third equation of boundary conditions (4.3), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|q\|^2+\|L_3q\|_2^2\\ \le&\|v^1\|_6\|\nabla q\|_3\|L_3q\|_2+C\|\nabla v^1\|_2^{\frac{1}{2}}\|\Delta v^1\|_2^{\frac{1}{2}}\|q_z\|_2^{\frac{1}{2}}\|\nabla q_z\|_2^{\frac{1}{2}}\|L_3q\|_2\\ &+\|v\|_3\|\nabla q^2\|_6\|L_3q\|_2+C\|\nabla v\|_2^{\frac{1}{2}}\|\Delta v\|_2^{\frac{1}{2}}\|q_z^2\|_2^{\frac{1}{2}}\|\nabla q_z^2\|_2^{\frac{1}{2}}\|L_3q\|_2.\end{aligned} \tag{4.10}$$

We deduce from inequalities (4.8)-(4.10), Theorem 3.3 and Young's inequality that

$$\frac{d}{dt}\|(v,T,q)\|_V^2+\|(L_1v,L_2T,L_3q)\|_H^2\le C(1+\mathcal{R}_2^4)\|(v,T,q)\|_V^2$$

for any $t\ge\tau_3$.

Multiplying now both sides of the above inequality by $\bar{t}=t-\tau_3$ and integrating the resulting relation over (τ_3,t), we obtain

$$\begin{aligned}\bar{t}\|(v(t),T(t),q(t))\|_V^2\le&C(1+\mathcal{R}_2^4)\int_{\tau_2}^t(s-\tau_2+1)\|(v(s),T(s),q(s))\|_V^2\,ds\\ \le&C(1+\mathcal{R}_2^4)(\bar{t}+1)\int_0^t\|(v(s),T(s),q(s))\|_V^2\,ds\\ \le&C(1+\mathcal{R}_2^4)(\bar{t}+1)e^{C\mathcal{R}_2^4t}\|(v(0),T(0),q(0))\|_H^2\end{aligned}$$

for any $t\ge\tau_3$. □

Next, we prove the time regularity of the semigroup $\{S(t)\}_{t\ge0}$ generated by problem (2.4)-(2.12). The proof is standard and we only state it here.

THEOREM 4.5. *Assume that $Q_1\in H^1(\Omega)$ and $Q_2\in H^1(\Omega)$. Then for any bounded subset $B\subset V$, there exists a positive constant ϱ_3 and a time $t^*=t^*(B)>0$ such that*

$$\|S(t)(v_0,T_0,q_0)-S(\tilde{t})(v_0,T_0,q_0)\|_V\le\varrho_3|t-\tilde{t}|^{\frac{1}{2}} \tag{4.11}$$

for any $t,\tilde{t}\ge t^*$ and any $(v_0,T_0,q_0)\in B$, where $S(t)(v_0,T_0,q_0)$ is the solution of problem (2.4)-(2.12) with initial data (v_0,T_0,q_0).

Finally, inspired by the idea in [1, 11, 12, 50], we can easily construct the existence of an exponential attractor for problem (2.4)-(2.12).

THEOREM 4.6. *Assume that $Q_1\in H^1(\Omega)$ and $Q_2\in H^1(\Omega)$. Let $\{S(t)\}_{t\ge0}$ be a semigroup generated by problem (2.4)-(2.12). Then the semigroup $\{S(t)\}_{t\ge0}$ possesses an exponential attractor $\mathcal{E}\subset V$, namely,*

(i) *$\mathcal{E}$ is compact and positively invariant with respect to $S(t)$, i.e.,*

$$S(t)\mathcal{E}\subset\mathcal{E}$$

for any $t\ge0$.

(ii) *The fractal dimension* $dim_F(\mathcal{E},V)$ *of* $\mathcal{E}$ *is finite.*

(iii) $\mathcal{E}$ *attracts exponentially any bounded subset* B *of* V*, that is, there exists a positive nondecreasing function* Q *and a constant* $\rho>0$ *such that*

$$dist_V(S(t)B,\mathcal{E}) \leq Q(\|B\|_V)e^{-\rho t}$$

for any $t\geq 0$*, where* $dist_V$ *denotes the non-symmetric Hausdorff distance between sets in* V *and* $\|B\|_V$ *stands for the size of* B *in* V*. Moveover, both* Q *and* ρ *can be explicitly calculated.*

REMARK 4.2. Thanks to $\mathcal{A}\subset\mathcal{E}$, we infer from Theorem 4.6 that the fractal dimension of the global attractor of problem (2.4)-(2.12) established in Theorem 4.1 is finite.

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