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SELF-SIMILAR SOLUTIONS OF THE SPHERICALLY SYMMETRIC EULER EQUATIONS FOR GENERAL EQUATIONS OF STATE*

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Abstract. The study of spherically symmetric motion is important for the theory of explosion waves. In this paper, we construct rigorously self-similar solutions to the Riemann problem of the spherically symmetric Euler equations for general equations of state. We use the assumption of self-similarity to reduce the spherically symmetric Euler equations to a system of nonlinear ordinary differential equations, from which we obtain detailed structures of solutions besides their existence.

Keywords. Compressible Euler equations; van der Waals gas; spherical symmetry; self-similar solution.

AMS subject classifications. 35L65; 35L60; 35L67.

1. Introduction

The 3D isentropic Euler equations have the form

$$\begin{cases}\rho_t+(\rho u_1)_{x_1}+(\rho u_2)_{x_2}+(\rho u_3)_{x_3}=0,\\(\rho u_1)_t+(\rho u_1^2+p)_{x_1}+(\rho u_1u_2)_{x_2}+(\rho u_1u_3)_{x_3}=0,\\(\rho u_2)_t+(\rho u_1u_2)_{x_1}+(\rho u_2^2+p)_{x_2}+(\rho u_2u_3)_{x_3}=0,\\(\rho u_3)_t+(\rho u_1u_3)_{x_1}+(\rho u_2u_3)_{x_2}+(\rho u_3^2+p)_{x_3}=0,\end{cases}\tag{1.1}$$

where ρ is the density, (u_1,u_2,u_3) is the velocity. System (1.1) is closed through the equation of state $p=p(\tau)$, where $\tau=1/\rho$ is the specific volume.

The global existence of solution to the Cauchy problem for system (1.1) is still a complicated open problem. Thus it has been beneficial to consider some special problems. In this paper, we consider system (1.1) with the Riemann initial data

$$\big(\rho,u_1,u_2,u_3\big)(0,x_1,x_2,x_3)=\big(\rho_0,u_0\sin\varphi\cos\theta,u_0\sin\varphi\sin\theta,u_0\cos\varphi\big),\tag{1.2}$$

where $(x_1,x_2,x_3)=(x\sin\varphi\cos\theta,x\sin\varphi\sin\theta,x\cos\varphi)$, $x=\sqrt{x_1^2+x_2^2+x_3^2}$ is the radial variable, $\varphi\in[0,\pi]$, $\theta\in[0,2\pi)$, and ρ_0 and u_0 are two constants.

The problem (1.1), (1.2) allows us to look for a spherically symmetric solution, i.e.,

$$\rho=\rho(x,t),\quad u_1=u(x,t)\sin\varphi\cos\theta,\quad u_2=u(x,t)\sin\varphi\sin\theta,\quad u_3=u(x,t)\cos\varphi.$$

We can then reduce system (1.1) to

$$\begin{cases}\rho_t+(\rho u)_x+\dfrac{2\rho u}{x}=0,\\(\rho u)_x+(\rho u^2+p)_x+\dfrac{2\rho u^2}{x}=0.\end{cases}\tag{1.3}$$

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Then the problem (1.1), (1.2) can be reduced to a Riemann initial-boundary value problem for (1.3) with the initial and boundary conditions

$$(u,\rho)(x,0)=(u_0,\rho_0), \quad (\rho u)(0,t)=0. \tag{1.4}$$

The problem (1.3), (1.4) allows us to look for self-similar solutions that depend only on the self-similar variable $\xi = x/t$.

The self-similar solutions for (1.3) were first studied by Guderley, Taylor, et al.; see [4] and the survey paper [8]. Taylor [15] used the assumption of self-similarity to reduce the spherically symmetric Euler equations for polytropic gases to a system of nonlinear autonomous ordinary differential equations and solved the "spherical piston" problem. Zhang and Zheng [17] constructed several 2D self-similar radially symmetric solutions with swirl for polytropic gases. Hu [7] constructed 2D self-similar axisymmetric solutions for the Euler equations for a two-constant equation of state. For more general existence of weak solutions of (1.3), we refer the reader to [1–3, 5, 10, 12, 13].

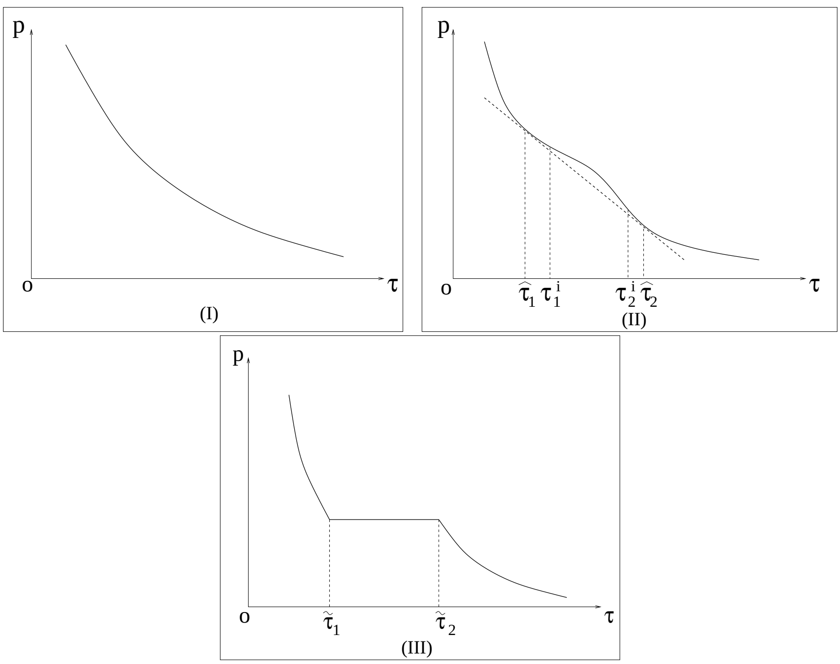


FIG. 1.1. *Equations of state.*

In this paper, we study the problem (1.3), (1.4) for the following three types of equations of state $p = p(\tau)$:

I. $p'(\tau) < 0$ and $p''(\tau) > 0$ as $\tau > 0$; see Figure 1.1(I).

II. $p'(\tau) < 0$ as $\tau > 0$; there exist two inflection points τ_1^i and τ_2^i where $\tau_1^i < \tau_2^i$, such that $p''(\tau) > 0$ as $\tau \in (0, \tau_1^i) \cup (\tau_2^i, +\infty)$ and $p''(\tau) < 0$ as $\tau \in (\tau_1^i, \tau_2^i)$; see Figure 1.1(II).

III. $p(\tau)$ is continuous on $(0, +\infty)$; $p'(\tau) < 0$ and $p''(\tau) > 0$ as $\tau \in (0, \tilde{\tau}_1) \cup (\tilde{\tau}_2, +\infty)$; $p(\tau)$ is constant on $[\tilde{\tau}_1, \tilde{\tau}_2]$; see Figure 1.1(III).

These three types of equations of state can be referred for instance to the van der Waals equation of state $p = \frac{A}{(\tau-1)^\gamma} - \frac{1}{\tau^2}$, where A is a constant corresponding to the entropy, γ

is a constant between 1 and 5/3. The third type of equation of state may be seen as a van der Waals equation of state complemented with Maxwell's equal areas law and may be used as a simple model of phase transition; see [6, 14, 16] and the references cited therein. For the van der Waals equation of state, τ needs to be greater than 1. However, we do not add this restriction for the equations of state considered in the present paper.

We make the following assumptions for these equations of state:

(A1) There exists a $\nu > 0$ such that $\lim\limits_{\rho\to 0} \frac{1}{\rho^\nu}\frac{\mathrm{d}p}{\mathrm{d}\rho} = 0$.

(A2) For equation of state II, we assume that there exist $\hat{\tau}_1$ and $\hat{\tau}_2$ where $\hat{\tau}_1 < \tau_1^i < \tau_2^i < \hat{\tau}_2$, such that

$$\frac{p(\hat{\tau}_1) - p(\hat{\tau}_2)}{\hat{\tau}_1 - \hat{\tau}_2} = p'(\hat{\tau}_1) = p'(\hat{\tau}_2).$$

(A3) For equation of state III, we assume $\lim\limits_{\tau\to\tilde{\tau}_1^-} p'(\tau) < p'(\tau_c)$, where $\tau_c > \tilde{\tau}_2$ is determined by $\frac{p(\tau_c) - p(\tilde{\tau}_1)}{\tau_c - \tilde{\tau}_1} = p'(\tau_c)$.

We also assume that $p''(\tau)$ is continuous for equations of state I and II and is piecewise continuous for equation of state III. The main result of the paper can be stated as the following theorem.

THEOREM 1.1. *For equations of state I–III, the Riemann initial-boundary value problem (1.3), (1.4) has a solution for any data* (u_0, ρ_0).

We use the assumption of self-similarity to reduce the spherically symmetric Euler Equations (1.3) to a system of nonlinear ordinary differential equations, from which we obtain detailed structures of solutions of (1.3), (1.4) besides their existence. There are many differences between our results and the previous results for polytropic gases. Firstly, system (1.3) cannot, by self-similar transformation, be reduced to an autonomous system of ordinary differential equations for general equations of state, so that the method in [4, 18, 19] does not work here. Secondly, the solution for (1.3), (1.4) for polytropic gases is continuous as $u_0 > 0$, whereas the solution for nonconvex equations of state may be discontinuous as $u_0 > 0$. Finally, the solution for (1.3), (1.4) for polytropic gases contains only one shock as $u_0 < 0$, whereas the solution for nonconvex equations of state may contain two or even more shocks as $u_0 < 0$.

2. Preliminaries

2.1. System of ordinary differential equations. By a self-similar transformation, system (1.3) can be written as

$$\begin{cases} -\xi\dfrac{\mathrm{d}\rho}{\mathrm{d}\xi} + \dfrac{\mathrm{d}(\rho u)}{\mathrm{d}\xi} + \dfrac{2\rho u}{\xi} = 0, \\ -\xi\dfrac{\mathrm{d}u}{\mathrm{d}\xi} + u\dfrac{\mathrm{d}u}{\mathrm{d}\xi} + \dfrac{1}{\rho}\dfrac{\mathrm{d}p}{\mathrm{d}\xi} = 0. \end{cases} \tag{2.1}$$

Thus, we have

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}\xi} = -\dfrac{2p'(\rho)u}{\xi\left[p'(\rho) - (u-\xi)^2\right]}, \\ \dfrac{\mathrm{d}\rho}{\mathrm{d}\xi} = \dfrac{2\rho u(u-\xi)}{\xi\left[p'(\rho) - (u-\xi)^2\right]}. \end{cases} \tag{2.2}$$

In what follows, we shall use $p'(\rho)$ and $p'(\tau)$ to denote $\frac{\mathrm{d}p}{\mathrm{d}\rho}$ and $\frac{\mathrm{d}p}{\mathrm{d}\tau}$, respectively.

Let $s=1/\xi$. Then, system (2.2) can be changed into

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}s}=\dfrac{2p'(\rho)us}{s^2p'(\rho)-(1-us)^2}, \\ \dfrac{\mathrm{d}\rho}{\mathrm{d}s}=\dfrac{2\rho u(1-us)}{s^2p'(\rho)-(1-us)^2}. \end{cases} \tag{2.3}$$

The initial condition $(u,\rho)(x,0)=(u_0,\rho_0)$ can be changed into

$$(u,\rho)\,|_{s=0}=(u_0,\rho_0). \tag{2.4}$$

Since $p''(\tau)$ is assumed to be continuous, the initial value problem (2.3), (2.4) is a classically well-posed problem which has a unique local solution for any (u_0,ρ_0). Throughout the paper, we denote by $(u_1,\rho_1)(s)$ the (local) classical solution to the initial value problem (2.3), (2.4).

In view of the denominators of the right-hand sides of (2.3), we define

$$h(\rho_1(s),s):=\frac{1}{s}-\sqrt{p'\big(\rho_1(s)\big)}. \tag{2.5}$$

By computations, we have the following properties:

- if $u_1(s)<h(\rho_1(s),s)$ then $s^2p'(\rho_1)-(1-u_1s)^2<0$;
- if $u_1(s)=h(\rho_1(s),s)$ then $s^2p'(\rho_1)-(1-u_1s)^2=0$;
- if $h(\rho_1(s),s)<u_1(s)<\frac{1}{s}+\sqrt{p'\big(\rho_1(s)\big)}$ then $s^2p'(\rho_1)-(1-u_1s)^2>0$.

2.2. Shock waves. It is known that a weak solution (u,ρ) to (1.3) satisfies the Rankine-Hugoniot condition across any discontinuity at (x,t):

$$\frac{\rho_1u_1-\rho_2u_2}{\rho_1-\rho_2}=\frac{\rho_1u_1^2+p_1-\rho_2u_2^2-p_2}{\rho_1u_1-\rho_2u_2}=\sigma, \tag{2.6}$$

where $(u_1,\rho_1)=(u,\rho)(x+0,t)$, $(u_2,\rho_2)=(u,\rho)(x-0,t)$, and σ is the speed of the discontinuity. For any (u_*,ρ_*), we let the shock set through (u_*,ρ_*) be the set of points (u,ρ) satisfying the Rankine-Hugoniot condition

$$\frac{\rho_*u_*-\rho u}{\rho_*-\rho}=\frac{\rho_*u_*^2+p_*-\rho u^2-p}{\rho_*u_*-\rho u}=\sigma(u_*,\rho_*;u,\rho).$$

We need to use the entropy condition (E) given by Liu [11].

DEFINITION 2.1. *A discontinuity between two states (u_1,ρ_1) and (u_2,ρ_2) satisfies the entropy condition (E) if*

$$\sigma(u_1,\rho_1;u_2,\rho_2)\geq\sigma(u_1,\rho_1;u,\rho) \tag{2.7}$$

for all (u,ρ) on the shock set through (u_1,ρ_1) between (u_1,ρ_1) and (u_2,ρ_2). A shock which satisfies the entropy condition (E) will be called an admissible shock.

In this paper, we are only concerned with forward shock waves. So, we give a geometric interpretation of entropy condition (E) for forward shock waves.

LEMMA 2.1. *A forward shock between two states (u_1,τ_1) and (u_2,τ_2) satisfies the entropy condition (E) if and only if*

$$\sqrt{-\frac{p_2-p_1}{\tau_2-\tau_1}} \geq \sqrt{-\frac{p-p_1}{\tau-\tau_1}} \tag{2.8}$$

for all $\tau\in\big(\min\{\tau_1,\tau_2\},\max\{\tau_1,\tau_2\}\big)$. Here, "1" denotes the fluid in front of the shock, and "2" denotes the fluid behind the shock.

Proof. From the Rankine-Hugoniot conditions for forward shock waves we have

$$\begin{cases} \rho_1(u_1-\sigma)=\rho_2(u_2-\sigma)<0, \\ \rho_1(u_1-\sigma)^2+p_1=\rho_2(u_2-\sigma)^2+p_2. \end{cases} \tag{2.9}$$

From (2.9) we get

$$\frac{\rho_2^2(u_2-\sigma)^2}{\rho_1}+p_1=\rho_2(u_2-\sigma)^2+p_2,$$

and consequently

$$(u_2-\sigma)^2=\frac{p_2-p_1}{\rho_2-\rho_1}\cdot\frac{\rho_1}{\rho_2}=-\tau_2^2\frac{p_2-p_1}{\tau_2-\tau_1}.$$

Thus, we have

$$\sigma(u_1,\rho_1;u_2,\rho_2)=u_2+\tau_2\sqrt{-\frac{p_2-p_1}{\tau_2-\tau_1}}. \tag{2.10}$$

Similarly, we have

$$\sigma(u_1,\rho_1;u_2,\rho_2)=u_1+\tau_1\sqrt{-\frac{p_2-p_1}{\tau_2-\tau_1}}. \tag{2.11}$$

Thus, for all (u,ρ) on the forward shock set through (u_1,ρ_1), we have

$$\sigma(u_1,\rho_1;u,\rho)=u_1+\tau_1\sqrt{-\frac{p-p_1}{\tau-\tau_1}}. \tag{2.12}$$

Then by (2.7) we get this lemma. □

We define

$$\phi(\tau;u_1,\tau_1):=u_1+(\tau_1-\tau)\sqrt{-\frac{p-p_1}{\tau-\tau_1}}.$$

Then by (2.10), (2.11), and Lemma 2.1, we have the following corollaries about forward admissible shocks.

COROLLARY 2.1. *For equation of state I, the set $\mathcal{S}_c$ of the states which can be connected to (u_1,τ_1) by a forward admissible compression shock on the left is given by:*

$$\mathcal{S}_c=\{(u,\tau)\,|\,u=\phi(\tau;u_1,\tau_1),\tau<\tau_1\}.$$

COROLLARY 2.2. *For equation of state II, the set $\mathcal{S}_c$ of the states which can be connected to (u_1,τ_1) by a forward admissible compression shock on the left is given by:*

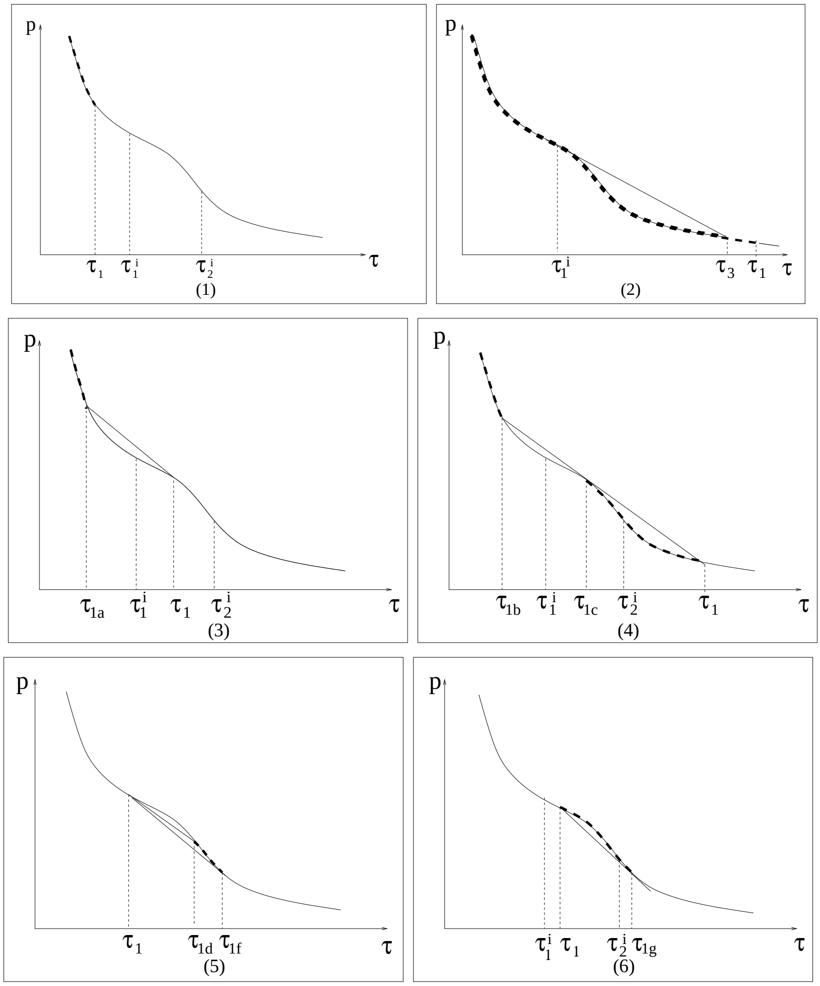


FIG. 2.1. *Admissible shocks for the equation of state II.*

- *If* $\tau_1 \in (0,\tau_1^i] \cup (\tau_3,+\infty)$, *then* $\mathcal{S}_c = \{(u,\tau)\,|\,u = \phi(\tau;u_1,\tau_1), \tau < \tau_1\}$, *where* τ_3 *is determined by*

$$\frac{p(\tau_3)-p(\tau_1^i)}{\tau_3-\tau_1^i} = p'(\tau_1^i);$$

  see Figure 2.1(1–2).
- *If* $\tau_1 \in (\tau_1^i,\tau_2^i]$, *then* $\mathcal{S}_c = \{(u,\tau)\,|\,u = \phi(\tau,u_1,\tau_1), \tau < \tau_{1a}\}$, *where* τ_{1a} *is determined by*

$$\frac{p(\tau_{1a})-p(\tau_1)}{\tau_{1a}-\tau_1} = p'(\tau_1) \quad \textit{and} \quad \tau_{1a} < \tau_1;$$

  see Figure 2.1(3).
- *If* $\tau_1 \in (\tau_2^i,\tau_3)$, *then* $\mathcal{S}_c = \{(u,\tau)\,|\,u = \phi(\tau;u_1,\tau_1), \tau < \tau_{1b}$ *and* $\tau_{1c} < \tau < \tau_1\}$, *where*

τ_{1b} and τ_{1c} are determined by

$$\frac{p(\tau_{1b})-p(\tau_1)}{\tau_{1b}-\tau_1}=\frac{p(\tau_{1c})-p(\tau_1)}{\tau_{1c}-\tau_1}=p'(\tau_{1c}) \quad \text{and} \quad \tau_{1b}<\tau_1^i<\tau_{1a}<\tau_2^i;$$

see Figure 2.1(4).

COROLLARY 2.3. *For equation of state II, the set $\mathcal{S}_r$ of the states which can be connected to (u_1,τ_1) by a forward admissible rarefaction shock on the left is given by:*

- *If $\tau_1\in(\hat{\tau}_1,\tau_1^i]$, then $\mathcal{S}_r=\{(u,\tau)\,|\,u=\phi(\tau;u_1,\tau_1),\tau_{1d}<\tau<\tau_{1f}\}$, where τ_{1d} and τ_{1f} are determined by*

$$\frac{p(\tau_{1d})-p(\tau_1)}{\tau_{1d}-\tau_1}=p'(\tau_1), \quad \frac{p(\tau_{1f})-p(\tau_1)}{\tau_{1f}-\tau_1}=p'(\tau_{1f}), \quad \text{and} \quad \tau_1^i<\tau_{1d}<\tau_{1f};$$

  see Figure 2.1(5).

- *If $\tau_1\in(\tau_1^i,\tau_2^i)$, then $\mathcal{S}_r=\{(u,\tau)\,|\,u=\phi(\tau;u_1,\tau_1),\tau_1<\tau<\tau_{1g}\}$, where τ_{1g} is determined by*

$$\frac{p(\tau_{1g})-p(\tau_1)}{\tau_{1g}-\tau_1}=p'(\tau_{1g}) \quad \text{and} \quad \tau_{1g}>\tau_2^i;$$

  see Figure 2.1(6).

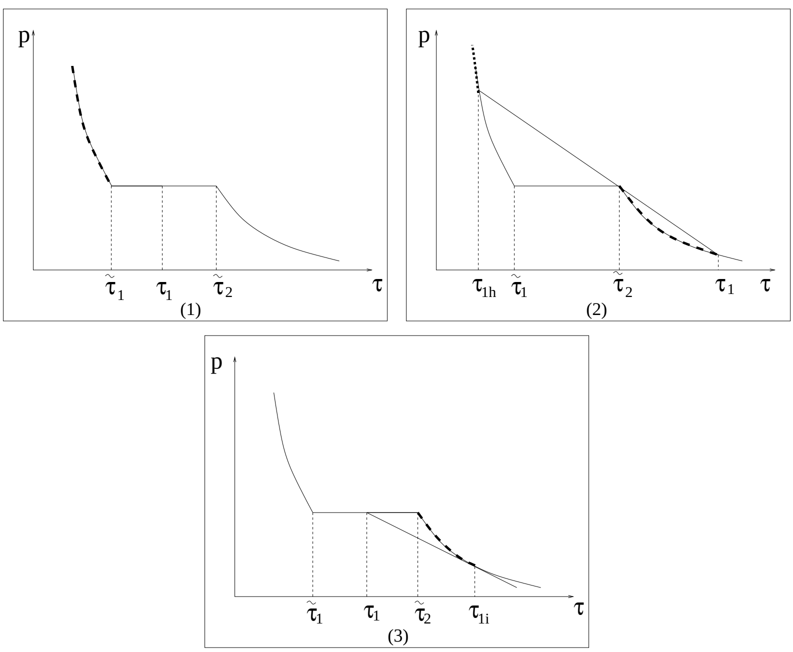


FIG. 2.2. *Admissible shocks for equation of state III.*

COROLLARY 2.4. *For equation of state III, the set $\mathcal{S}_c$ of the states which can be connected to (u_1,τ_1) by a forward admissible compression shock on the left is given by:*

- *If $\tau_1\in(0,\tilde{\tau}_2]$, then $\mathcal{S}_c=\{(u,\tau)\,|\,u=\phi(\tau;u_1,\tau_1),\tau<\min\{\tilde{\tau}_1,\tau_1\}\}$; see Figure 2.2(1).*

- *If* $\tau_1\in(\tilde\tau_2,+\infty)$*, then* $\mathcal{S}_c=\{(u,\tau)\,|\,u=\phi(\tau;u_1,\tau_1),0<\tau<\tau_{1h}$ *and* $\tilde\tau_2<\tau<\tau_1\}$*, where* τ_{1h} *is determined by*

$$\frac{p(\tau_{1h})-p(\tau_1)}{\tau_{1h}-\tau_1}=\frac{p(\tilde\tau_2)-p(\tau_1)}{\tilde\tau_2-\tau_1};$$

see Figure 2.2(2).

Corollary 2.5. *For equation of state III, if* $\tau_1\in(\tilde\tau_1,\tilde\tau_2)$*, then the set* $\mathcal{S}_r$ *of the states which can be connected to* (u_1,τ_1) *by a forward admissible rarefaction shock on the left is given by*

$$\mathcal{S}_r=\{(u,\tau)\,|\,u=\phi(\tau;u_1,\tau_1),\tilde\tau_2<\tau<\tau_{1i}\}$$

where τ_{1i} *is determined by*

$$\frac{p(\tau_{1i})-p(\tau_1)}{\tau_{1i}-\tau_1}=p'(\tau_{1i});$$

see Figure 2.2(3).

Here, the compression shocks are the admissible shocks across which the density is increasing; the rarefaction shocks are the admissible shocks across which the density is decreasing. Corollaries 2.1–2.5 are obvious, so we omit their proofs. We also refer the reader to [6,9] for more details.

3. Self-similar solutions for $u_0>0$

In this section we will construct the self-similar solutions of the problem (1.3), (1.4) for $u_0>0$.

3.1. Equation of state I.

We first study the properties of the local solution $(u_1,\rho_1)(s)$.

Lemma 3.1. *For any fixed* $S>0$*, if the initial value problem (2.3), (2.4) has a classical solution* $(u_1,\rho_1)(s)$ *in* $(0,S)$ *and* $s^2p'(\rho_1)-(1-u_1s)^2<0$ *for* $0<s<S$*, then we have*

$$u_1(s)>0,\quad \rho_1(s)>0,\quad \frac{\mathrm{d}u_1(s)}{\mathrm{d}s}<0,\quad \textit{and}\quad \frac{\mathrm{d}\rho_1(s)}{\mathrm{d}s}<0\quad \textit{for } 0<s<S.$$

Proof. Assume there exists a $0<s_*<S$ such that $u_1(s)>0$ for $0<s<s_*$ and $u_1(s_*)=0$. Then by (2.3) we have

$$\int_{u_0}^{0}\frac{1}{u}\,\mathrm{d}u=\int_0^{s_*}\frac{2p'(\rho_1(s))s}{s^2p'(\rho_1(s))-(1-u_1(s)s)^2}\,\mathrm{d}s,$$

which leads to a contradiction. Thus, we have $u_1(s)>0$ and $\frac{\mathrm{d}u_1(s)}{\mathrm{d}s}<0$ for $0<s<S$. Similarly, we can prove $\rho_1(s)>0$ as $0<s<S$. From $s^2p'(\rho_1)-(1-u_1s)^2<0$ and $u_1(s)<0$, we have $u_1(s)s<1$, consequently we have $\frac{\mathrm{d}\rho_1(s)}{\mathrm{d}s}<0$ for $0<s<S$. We then complete the proof of the lemma. ∎

Lemma 3.2. *For any fixed* $S>0$*, if the initial value problem (2.3), (2.4) has a classical solution* $(u_1,\rho_1)(s)$ *in* $(0,S)$ *and* $\rho_1(s)>0$ *and* $h(\rho_1(s),s)>0$ *for* $0<s<S$*, then* $0<u_1(s)<h(\rho_1(s),s)$ *for* $0<s<S$*.*

Proof. It is obvious that $0<u_1(s)<h(\rho_1(s),s)$ as s is sufficiently small. By a direct computation, we have

$$\frac{d(u_1-h)}{\mathrm{d}s}=\frac{2p'(\rho_1)u_1s}{s^2p'(\rho_1)-(1-u_1s)^2}+\frac{1}{s^2}+\frac{2p''(\rho_1)\rho_1u_1(1-u_1s)}{2\sqrt{p'(\rho_1)}\big(s^2p'(\rho_1)-(1-u_1s)^2\big)}. \tag{3.1}$$

Suppose that $s_0\in(0,S)$ is the "first" point such that $\rho_1(s_0)>0$, $h(\rho_1(s_0),s_0)>0$ and $u_1(s_0)=h(\rho_1(s_0),s_0)$. Then we have

$$\begin{aligned}
&p'(\rho_1)u_1s+\frac{p''(\rho_1)\rho_1u_1(1-u_1s)}{2\sqrt{p'(\rho_1)}}\\
&=\frac{u_1}{2\sqrt{-p'(\tau_1)}}\Big(-2\tau_1^2p'(\tau_1)s\sqrt{-p'(\tau_1)}+\tau_1^2p''(\tau_1)(1-u_1s)+2\tau_1p'(\tau_1)(1-u_1s)\Big)\\
&=\frac{u_1\tau_1^2p''(\tau_1)s\sqrt{p'(\rho_1)}}{2\sqrt{-p'(\tau_1)}}\ >\ 0
\end{aligned} \tag{3.2}$$

for $s=s_0$, where $\tau_1=1/\rho_1$. Here, we use $p'(\rho_1)=-\tau_1^2p'(\tau_1)$ and $p''(\rho_1)=2\tau_1^3p'(\tau_1)+\tau_1^4p''(\tau_1)$. From $0<u_1(s)<h(\rho_1(s),s)$ for $s<s_0$, we get $s^2p'(\rho_1)-(1-u_1s)^2<0$ for $s<s_0$. Hence, we have $\lim\limits_{s\to s_0^-}\frac{d(u_1-h)}{\mathrm{d}s}=-\infty$ which leads to a contradiction. We then complete the proof of the lemma. ∎

In what follows, we are going to show that there are the only following three cases for the local solution $(u_1,\rho_1)(s)$:

- There exists a $s_*>0$ such that $0<u_1(s)<h(\rho_1(s),s)$ for $s<s_*$ and $h(\rho_1(s_*),s_*)=u_1(s_*)=1/s_*$; see Figure 3.1(left).
- There exists a $s_*>0$ such that $0<u_1(s)<h(\rho_1(s),s)$ for $s<s_*$ and $u_1(s_*)=h(\rho_1(s_*),s_*)=0$; see Figure 3.2(left).
- $0<u_1(s)<h(\rho_1(s),s)$ for all $s>0$; see Figure 3.3(left).

LEMMA 3.3. *If $u_0>0$ is sufficiently large, then there exists a $s_*>0$ such that $0<u_1(s)<h(\rho_1(s),s)$ for $s<s_*$ and $h(\rho_1(s_*),s_*)=u_1(s_*)=1/s_*$.*

Proof. If $u_1(s)<h(\rho_1(s),s)$ then we have

$$u_1(s)s<1 \quad \text{and} \quad s^2p'(\rho_1(s))-(1-u_1(s)s)^2<0.$$

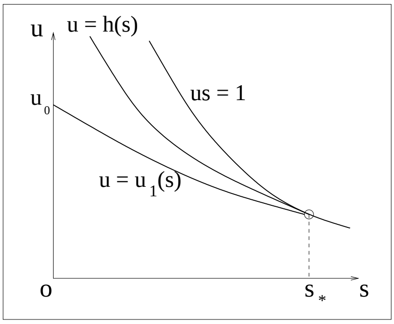
Consequently by (2.3) we have

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}u_1}=\frac{\rho_1(1-u_1s)}{p'(\rho_1)s}\ >\ \frac{\rho_1}{\sqrt{p'(\rho_1)}}.$$

Integrating this, we get

$$\int_0^{\rho_0}\frac{\sqrt{p'(\rho_1)}}{\rho_1}\,\mathrm{d}\rho_1\ \ge\ \int_{\rho_1(s)}^{\rho_0}\frac{\sqrt{p'(\rho_1)}}{\rho_1}\,\mathrm{d}\rho_1\ >\ \int_{u_1(s)}^{u_0}\mathrm{d}u_1. \tag{3.3}$$

Combining this with assumption (A1) and Lemmas 3.1 and 3.2, we know that when u_0 is sufficiently large, e.g. $u_0>\int_0^{\rho_0}\frac{\sqrt{p'(\rho)}}{\rho}\,\mathrm{d}\rho$, there exists a $s_*>0$ such that $\rho_1(s_*)=0$ and $u_1(s_*)=h(\rho_1(s_*),s_*)=1/s_*$. ∎



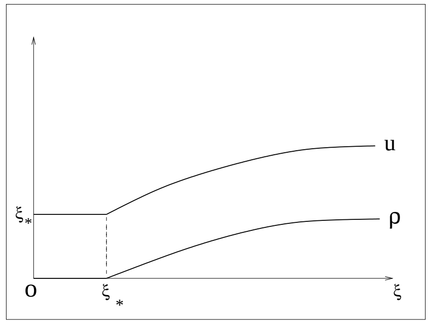


FIG. 3.1. *Continuous solution with a vacuum, where* $\xi = x/t$.

Therefore, the self-similar solution of the problem (1.3), (1.4) for this case has the form

$$(u,\rho)(x,t) = \begin{cases} (u_1,\rho_1)(s), & s < s_*, \\ \big(\xi_*,0\big), & s > s_*; \end{cases}$$

where $s = t/x$ and $\xi_* = 1/s_*$. This is a continuous solution with a growing vacuum region; see Figure 3.1(right).

LEMMA 3.4. *If $u_0 > 0$ is sufficiently small, then there exists a $s_* > 0$, such that $0 < u_1(s) < h(\rho_1(s),s)$ for $s < s_*$ and $u_1(s_*) = h(\rho_1(s_*),s_*) = 0$.*

Proof. Let $\varepsilon \in (0,\rho_0/2)$ be given such that

$$\max_{\rho\in[\rho_0-\varepsilon,\rho_0]} \sqrt{p'(\rho)} < \frac{3}{2}\sqrt{p'(\rho_0)}. \tag{3.4}$$

Let $s_0 = \frac{1}{2\sqrt{p'(\rho_0)}}$. It follows from (2.3) that if $u_0 > 0$ is sufficiently small then $\rho_1(s_0) > \rho_0 - \varepsilon$. Consequently, by (3.4) we have $h(\rho_1(s_0),s_0) > 0$.

From (2.3), we have

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}u_1} = \frac{\rho_1(1-u_1 s)}{p'(\rho_1)s} < \frac{\rho_1}{s_0 p'(\rho_1)} \quad \text{for } s > s_0.$$

Hence, we have

$$\int_{\rho_1(s)}^{\rho_0-\varepsilon} \frac{p'(\rho_1)}{\rho_1}\,\mathrm{d}\rho_1 < \int_{\rho_1(s)}^{\rho_1(s_0)} \frac{p'(\rho_1)}{\rho_1}\,\mathrm{d}\rho_1 < \frac{1}{s_0}\int_{u_1(s)}^{u_1(s_0)} \mathrm{d}u_1 < \frac{u_0}{s_0}. \tag{3.5}$$

Thus, when u_0 is sufficiently small there exists a $\rho_m > 0$ such that $\rho_1(s) > \rho_m$. Therefore, there must exist a $s_* > 0$ such that $h(\rho_1(s_*),s_*) = 0$. By Lemma 3.2 we also have $u_1(s_*) = 0$. We then complete the proof of this lemma. ∎

Therefore, the self-similar solution of the problem (1.3), (1.4) for this case has the form

$$(u,\rho)(x,t) = \begin{cases} (u_1,\rho_1)(s), & s < s_*, \\ \big(0,\rho_1(s_*)\big), & s > s_*; \end{cases}$$

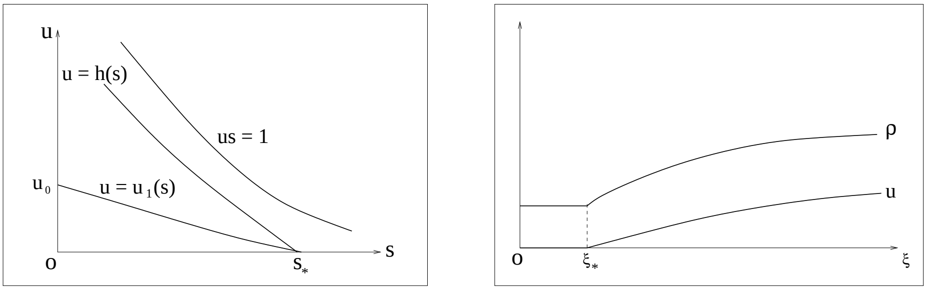


FIG. 3.2. *Continuous solution with a quiet constant state.*

where $s=t/x$. This is a continuous solution with a quiet constant state; see Figure 3.2(right).

LEMMA 3.5. *If the first case happens as $(u,\rho)(0)=(u^0,\rho^0)$, then there exists a sufficiently small $\varepsilon>0$ such that the first case will happen for $(u,\rho)(0)\in(u^0-\varepsilon,u^0+\varepsilon)\times\{\rho^0\}$.*

Proof. We denote by $(\bar u,\bar\rho)(s)$ the solution of system (2.3) with the initial data $(u,\rho)(0)=(u^0,\rho^0)$. Then there exists a $\bar s_*>0$ such that $\bar\rho(\bar s_*)=0$ and $\bar u(\bar s_*)=h(\bar\rho(\bar s_*),\bar s_*)=1/\bar s_*$.

From assumption (A1), we can find a sufficiently small $\delta\in(0,1/\bar s_*)$ such that

$$\int_0^\delta \frac{\sqrt{p'(\rho)}}{\rho}\,\mathrm{d}\rho < \frac{1}{2\bar s_*}. \tag{3.6}$$

Since $\bar\rho(s)$ is continuous on $[0,\bar s_*]$, there exists a sufficiently small $\eta>0$ such that

$$\bar\rho(\bar s_*-\eta)<\frac{\delta}{2}. \tag{3.7}$$

When $\varepsilon>0$ is sufficiently small the solution $(u,\rho)(s)$ of system (2.3) with the initial data $(u,\rho)(0)\in(u^0-\varepsilon,u^0+\varepsilon)\times\{\rho^0\}$ satisfies

$$|\rho(\bar s_*-\eta)-\bar\rho(\bar s_*-\eta)|<\frac{\delta}{4} \quad\text{and}\quad |u(\bar s_*-\eta)-\bar u(\bar s_*-\eta)|<\frac{\delta}{4}. \tag{3.8}$$

It is similar to (3.3) that

$$\int_{\rho(s)}^{\rho(\bar s_*-\eta)} \frac{\sqrt{p'(\rho)}}{\rho}\,\mathrm{d}\rho > \int_{u(s)}^{u(\bar s_*-\eta)} \mathrm{d}u=u(\bar s_*-\eta)-u(s)$$

for $s>\bar s_*-\eta$. Combining this with (3.8), we get

$$\begin{aligned}\int_{\rho(s)}^{\delta} \frac{\sqrt{p'(\rho)}}{\rho}\,\mathrm{d}\rho &> \bar u(\bar s_*-\eta)-\frac{\delta}{4}-u(s)> \bar u(\bar s_*)-\frac{\delta}{4}-u(s)\\ &=\frac{1}{\bar s_*}-\frac{\delta}{4}-u(s) > \frac{3}{4\bar s_*}-u(s)\end{aligned}$$

for $s>\bar s_*-\eta$. Thus, by (3.6) and Lemmas 3.1 and 3.2 we know that there exists a s_* such that $u(s)<h(\rho(s),s)$ as $0<s<s_*$ and $u(s_*)=h(\rho(s_*),s_*)=1/s_*$. □

Lemma 3.6. *If the second case happens as* $(u,\rho)(0)=(u^0,\rho^0)$*, then there exists a sufficiently small* $\varepsilon>0$*, such that the second case will happen as* $(u,\rho)(0)\in(u^0-\varepsilon,u^0+\varepsilon)\times\{\rho^0\}$*.*

Proof. We denote by $(\bar{u},\bar{\rho})(s)$ the solution of system (2.3) with the initial data $(u,\rho)(0)=(u^0,\rho^0)$. Then there exists a $\bar{s}_*>0$ such that $\bar{u}(\bar{s}_*)=0$ and $\bar{\rho}(\bar{s}_*)=\rho_*>0$. Let

$$\mathcal{N}=\int_0^{\frac{\rho_*}{2}}\frac{p'(\rho)}{\rho}\,\mathrm{d}\rho.$$

There exists a sufficiently small $\eta<\frac{\bar{s}_*}{2}$ such that

$$0<\bar{u}(\bar{s}_*-\eta)<\frac{\mathcal{N}\bar{s}_*}{4}. \tag{3.9}$$

It is easy to see that if $\varepsilon>0$ is sufficiently small, then the solution $(u,\rho)(s)$ of system (2.3) with the initial data $(u,\rho)(0)\in(u^0-\varepsilon,u^0+\varepsilon)\times\{\rho^0\}$ satisfies

$$|\rho(\bar{s}_*-\eta)-\bar{\rho}(\bar{s}_*-\eta)|<\frac{\rho_*}{4}\quad\text{and}\quad|u(\bar{s}_*-\eta)-\bar{u}(\bar{s}_*-\eta)|<\frac{\mathcal{N}\bar{s}_*}{4}. \tag{3.10}$$

As in (3.5) we have

$$\int_{\rho(s)}^{\rho(\bar{s}_*-\eta)}\frac{p'(\rho)}{\rho}\,\mathrm{d}\rho\;<\;\frac{1}{\bar{s}_*-\eta}\int_{u(s)}^{u(\bar{s}_*-\eta)}\mathrm{d}u\quad\text{for } s>\bar{s}_*-\eta. \tag{3.11}$$

Combining this with (3.10) we get

$$\int_{\rho(s)}^{\frac{3\rho_*}{4}}\frac{p'(\rho)}{\rho}\,\mathrm{d}\rho\;<\;\frac{1}{\bar{s}_*-\eta}\int_{u(s)}^{\frac{\mathcal{N}\bar{s}_*}{2}}\mathrm{d}u\;<\;\mathcal{N},$$

since $\rho'(s)<0$. Thus, by (3.9) we know that there exists a $\rho_m>0$ such that $\rho(s)>\rho_m$ for $s>\bar{s}_*-\eta$. Consequently, there exists a $s_*>\bar{s}_*-\eta$ such that $h(\rho(s_*),s_*)=0$. ☐

Using Lemmas 3.5 and 3.6 and the argument of continuity, we know that for any given $\rho_0>0$, there exists a $u_c>0$, such that when $u_0=u_c$ the solution $(u_1,\rho_1)(s)$ of the initial value problem (2.3), (2.4) satisfies $0<u_1(s)<h(\rho_1(s),s)<1/s$ as $s>0$; see Figure 3.3(left). That is to say, the initial value problem (2.3), (2.4) has a global classical solution. Moreover, this solution satisfies $\lim\limits_{s\to+\infty}u_1(s)=\lim\limits_{s\to+\infty}\rho_1(s)=0$. In this case, the initial-boundary value problem (1.3), (1.4) has a self-similar smooth solution $(u,\rho)(x,t)=(u_1,\rho_1)(t/x)$; see Figure 3.3(right).

3.2. Equation of state II.

3.2.1. $\tau_0\geq\tau_2^i$**.** The discussion is similar to that of Section 3.1, since $\tau_1'(s)>0$ for $s>0$ and $p''(\tau)>0$ for $\tau>\tau_0$. So, we omit the details.

3.2.2. $\tau_1^i\leq\tau_0<\tau_2^i$**.** There are the following four cases:

- There exists a $s_*>0$ such that $0<u_1(s)<h(\rho_1(s),s)$ for $0<s<s_*$ and $h(\rho_1(s_*),s_*)=u_1(s_*)=\frac{1}{s_*}$.
- There exists a $s_*>0$ such that $0<u_1(s)<h(\rho_1(s),s)$ as $0<s<s_*$ and $u_1(s_*)=h(\rho_1(s_*),s_*)=0$.

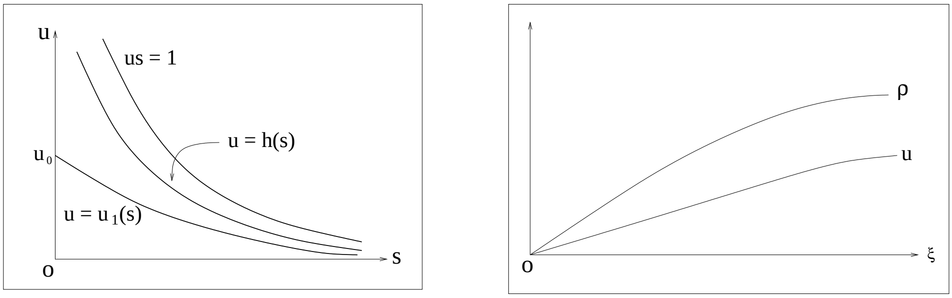


FIG. 3.3. *A global smooth self-similar solution.*

- $0 < u_1(s) < h(\rho_1(s), s)$ for all $s > 0$.
- There exists a $s_* > 0$ such that $0 < u_1(s) < h(\rho_1(s), s)$ for $0 < s < s_*$ and $0 < u_1(s_*) = h(\rho_1(s_*), s_*) < \frac{1}{s_*}$. (By Lemma 3.2, there holds $\tau_1(s_*) \in (\tau_0, \tau_2^i)$ in this case.)

It is easy to see that the first three cases can happen when τ_0 is sufficiently close to τ_2^i, and the discussions for these three cases are similar to that of Section 3.1.

In what follows, we are going to discuss the fourth case. We first show that the fourth case can happen at least in some cases. To confirm this, we consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}u} = \dfrac{s^2 p'(\rho) - (1-us)^2}{2p'(\rho)us}, \\ \dfrac{\mathrm{d}\rho}{\mathrm{d}u} = \dfrac{\rho(1-us)}{2p'(\rho)s}, \end{cases} \tag{3.12}$$

$$(s, \rho)\,|_{u=u_*} = (s_*, \rho_*), \tag{3.13}$$

where $u_* > 0$, $\rho_* > 0$, and $s_* > 0$ satisfy $s_*^2 p'(\rho_*) - (1 - u_* s_*)^2 = 0$, $u_* s_* < 1$ and $\tau_2^i < 1/\rho_* < \tau_2^i$.

LEMMA 3.7. *When $\delta > 0$ is sufficiently small, the initial value problem (3.12), (3.13) has a solution $(\hat{s}, \hat{\rho})(u)$ on $(u_*, u_* + \delta)$. Moreover, this solution satisfies $\frac{\mathrm{d}\hat{s}}{\mathrm{d}u} < 0$, $\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}u} < 0$, and $\frac{\mathrm{d}[\hat{s}^2 p'(\hat{\rho}) - (1-u\hat{s})^2]}{\mathrm{d}u} < 0$ for $u \in (u_*, u_* + \delta)$.*

Proof. It is easy to see that this initial value problem is a classically well-posed problem which has a unique local solution $(\hat{s}, \hat{\rho})(u)$.

By computation, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}u}\big(\hat{s}^2 p'(\hat{\rho}) - (1-u\hat{s})^2\big) \\ =\ & \big(2\hat{s}p'(\hat{\rho}) + 2u(1-u\hat{s})\big) \cdot \Big(\frac{\hat{s}^2 p'(\hat{\rho}) - (1-u\hat{s})^2}{2p'(\hat{\rho})u\hat{s}}\Big) + \hat{s}^2 p''(\hat{\rho})\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}u} + 2\hat{s}(1-u\hat{s}) \\ =\ & \big(2\hat{s}p'(\hat{\rho}) + 2u(1-u\hat{s})\big) \cdot \Big(\frac{\hat{s}^2 p'(\hat{\rho}) - (1-u\hat{s})^2}{2p'(\hat{\rho})u\hat{s}}\Big) + \frac{2\hat{s}\hat{\tau}^3 p''(\hat{\tau})}{p'(\hat{\rho})}(1-u\hat{s}) \ < \ 0 \end{aligned}$$

as $u = u_*$. Hence, we have $\hat{s}^2 p'(\hat{\rho}) - (1-u\hat{s})^2 < 0$ and $\frac{\mathrm{d}[\hat{s}^2 p'(\hat{\rho}) - (1-u\hat{s})^2]}{\mathrm{d}u} < 0$ as $u \in (u_*, u_* + \delta)$.

Moreover, in view of $s_*^2p'(\rho_*)-(1-u_*s_*)^2=0$ and $u_*s_*<1$, we have $1-u\hat{s}>0$ as $u\in(u_*,u_*+\delta)$. Consequently, we have $\frac{d\hat{s}}{du}<0$ and $\frac{d\hat{\rho}}{du}<0$ for $u\in(u_*,u_*+\delta)$. ∎

When $p''(\tau)<0$ and $p'(\tau)<0$ we have $p''(\rho)=2\tau^3p'(\tau)+\tau^4p''(\tau)<0$. Hence, in view of Lemma 3.7, there exists a $u^*>u_*$ such that $\hat{s}(u^*)=0$ and $\tau_1^i<\frac{1}{\hat{\rho}(u^*)}<\tau_2^i$, at least for some equations of state. Therefore, if we take $u_0=u^*$ and $\rho_0=\hat{\rho}(u^*)$ then the fourth case will happen.

We next construct the solution for the fourth case. From $s_*^2p'(\rho_1(s_*))-(1-u_1(s_*)s_*)^2=0$, we have $\lim\limits_{s\to s_*^-}\frac{du_1}{ds}=-\infty$ and $\lim\limits_{s\to s_*^-}\frac{d\rho_1}{ds}=-\infty$; see Figure 3.4(left). This implies that the problem (1.3), (1.4) does not have a global continuous solution. So, we need to look for a discontinuous solution.

Since $s^2p'(\rho_1)-(1-u_1s)^2<0$ and $0<u_1s<1$ for $0<s<s_*$, we have

$$\frac{1}{s} > u_1(s)+\sqrt{p'(\rho_1(s))}=u_1(s)+\tau_1(s)\sqrt{-p'(\tau_1(s))} \quad \text{for } 0<s<s_*. \tag{3.14}$$

We first consider the possibility of the existence of a compression shock wave solution. By (3.14) and Corollary 2.2 we know that for any $0<s<s_*$, there exists an admissible forward compression shock with the speed $1/s$ and the frontside state $(u_1,\rho_1)(s)$. Moreover, the backside state $(u_2,\tau_2)(s)$ can be uniquely determined by

$$\begin{cases} \dfrac{1}{s}=u_1(s)+\tau_1(s)\sqrt{-\dfrac{p(\tau_2(s))-p(\tau_1(s))}{\tau_2(s)-\tau_1(s)}}, \\ u_2(s)=u_1(s)+(\tau_1(s)-\tau_2(s))\sqrt{-\dfrac{p(\tau_2(s))-p(\tau_1(s))}{\tau_2(s)-\tau_1(s)}}. \end{cases} \tag{3.15}$$

Since the shock is assumed to be a compression shock, i.e. $\tau_2(s)<\tau_1(s)$, we have $u_2(s)>u_1(s)>0$. By the entropy condition, we have

$$u_2(s)-\sqrt{p'(\rho_2(s))} < \frac{1}{s} < u_2(s)+\sqrt{p'(\rho_2(s))},$$

and consequently

$$\frac{1}{s}-\sqrt{p'(\rho_2(s))}<u_2(s)<\frac{1}{s}+\sqrt{p'(\rho_2(s))}. \tag{3.16}$$

We now assume there exists an admissible forward compression shock with the speed $1/s_1$ and the frontside state $(u_1,\rho_1)(s_1)$, where $s_1\in(0,s_*)$. Then we consider system (2.3) with the data

$$(u,\rho)\,|_{s=s_1} = (u_2,\rho_2)(s_1). \tag{3.17}$$

We have the following lemma:

LEMMA 3.8. *There exists a $s^*>s_1$ such that the solution $(u_3,\rho_3)(s)$ of the initial value problem (2.3), (3.17) satisfies*

$$\frac{1}{s}-\sqrt{p'(\rho_3(s))}<u_3(s)<\frac{1}{s}+\sqrt{p'(\rho_3(s))} \quad \textit{for} \;\; s_1<s<s^* \tag{3.18}$$

and $u(s^*)=\frac{1}{s^*}+\sqrt{p'(\rho(s^*))}>\frac{1}{s_*}$.

Proof. It is easy to see that if $\frac{1}{s}-\sqrt{p'(\rho_3(s))}<u_3(s)<\frac{1}{s}+\sqrt{p'(\rho_3(s))}$ then $s^2p'(\rho_3)-(1-u_3s)^2>0$. There are two situations: $u_2(s_1)s_1\geq 1$ and $u_2(s_1)s_1<1$.

If $u_2(s_1)s_1>1$, then we have $\mathrm{d}u_3/\mathrm{d}s>0$ and $\mathrm{d}\rho_3/\mathrm{d}s<0$ for $s>s_1$.

If $u_2(s_1)s_1<1$, then we have $\rho_3'(s_1)>0$. Using (3.1), (3.2), and the fact that $\rho_3'(s)>0$ for $u_3s<1$, we get $u_3(s)>1/s-\sqrt{p'(\rho_3(s))}$. Thus, there exists a $s_2>s_1$ such that $u_3(s_2)s_2=1$. Moreover, we have $\mathrm{d}u_3/\mathrm{d}s>0$ and $\mathrm{d}\rho_3/\mathrm{d}s<0$ for $s>s_2$.

If the curves $u=u_3(s)$ and $u=1/s+\sqrt{p'(\rho_3(s))}$ do not intersect with each other, then we must have

$$\lim_{s\to+\infty}\rho_3(s)=\rho_\infty>0 \quad\text{and}\quad \lim_{s\to+\infty}u_3(s)=u_\infty>0.$$

Thus, we have

$$\frac{\mathrm{d}u_3}{\mathrm{d}s}=\frac{2p'(\rho_3)u_3s}{s^2p'(\rho_3)-(1-u_3s)^2}>\frac{2u_3(s)}{s}>\frac{2u_2(s_1)}{s}$$

which leads to a contradiction. We then complete the proof of this lemma. ∎

Lemma 3.8 implies that the initial value problem (2.3), (3.17) does not have a solution on $(s_1,+\infty)$. If follows from (3.18) that $(u_3,\rho_3)(s)$ $(s_1<s<s^*)$ can not also be the frontside state of any admissible forward shock with the speed $1/s$. Therefore, the problem (1.3), (1.4) does not permit a compression shock wave solution in this situation. In what follows, we are going to look for a rarefaction shock wave solution.

For $\tau_1^i<\tau_1(s)<\tau_2^i$, we let $f(\tau_1(s))$ be defined such that

$$\frac{p(\tau_1)-p(f(\tau_1))}{\tau_1-f(\tau_1)}=p'(f(\tau_1)) \quad\text{and}\quad f(\tau_1)>\tau_2^i.$$

It can be seen that

$$-p'(\tau_1(s))<-p'(f(\tau_1(s))) \quad \text{for } 0<s<s_*. \tag{3.19}$$

Lemma 3.9. *There exists a $s_{**}\in(0,s_*)$ such that for any $s\in[s_{**},s_*]$, there exists an admissible forward rarefaction shock with the speed $1/s$ and the frontside state $(u_1,\rho_1)(s)$.*

Proof. According to Corollary 2.3, in order that $(u_1,\rho_1)(s)$ can be the frontside state of an admissible forward rarefaction shock with the speed $1/s$, there must hold

$$u_1(s)+\tau_1(s)\sqrt{-p'(\tau_1(s))}\leq\frac{1}{s}\leq u_1(s)+\tau_1(s)\sqrt{-p'(f(\tau_1(s)))}.$$

Since $u_1(s)<h(\rho_1(s),s)$ for $0<s<s_*$, we have

$$\frac{1}{s}\ >\ u_1(s)+\tau_1(s)\sqrt{-p'(\tau_1(s))} \quad \text{for } 0<s<s_*.$$

From $u_1(s_*)=h(\rho_1(s_*),s_*)>0$, $\tau_0<\tau_1(s_*)<\tau_2^i$, and (3.19), we have

$$\frac{1}{s_*}=u_1(s_*)+\tau_1(s_*)\sqrt{-p'(\tau_1(s_*))}\ <\ u_1(s_*)+\tau_1(s_*)\sqrt{-p'(f(\tau_1(s_*)))}. \tag{3.20}$$

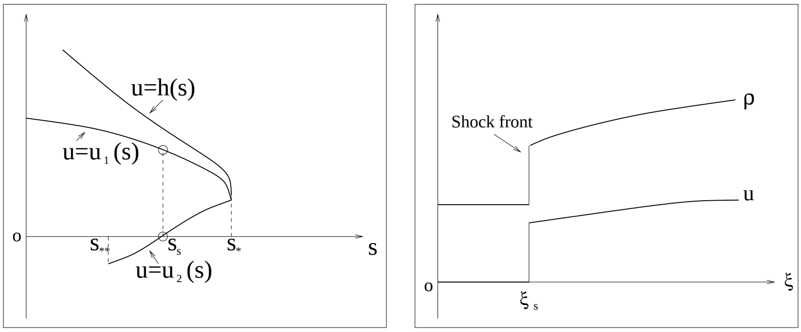


FIG. 3.4. *Discontinuous solution with a single rarefaction shock.*

Thus, there exists a $s_{**}\in(0,s_*)$ such that $1/s<u_1(s)+\tau_1(s)\sqrt{-p'(f(\tau_1(s)))}$ for $s_{**}<s<s_*$ and

$$\frac{1}{s_{**}}=u_1(s_{**})+\tau_1(s_{**})\sqrt{-p'(f(\tau_1(s_{**})))}. \tag{3.21}$$

We then complete the proof of this lemma. □

Let $(u_2,\tau_2)(s)$ $(s_{**}\le s<s_*)$ be determined by (3.15) and $\tau_2(s)>\tau_1(s)$. It is easy to see that $u_2(s_*)=u_1(s_*)>0$ and

$$\tau_2(s_{**})=f(\tau_1(s_{**}))>\tau_2^i. \tag{3.22}$$

If $u_2(s_{**})\le 0$, then there exists a $s_s\in[s_{**},s_*)$ such that $u_2(s_s)=0$; see Figure 3.4(left). In this case, the self-similar solution of the problem (1.3), (1.4) has the form:

$$(u,\rho)(x,t)=\begin{cases}(u_1,\rho_1)(s), & s<s_s,\\ (0,\rho_2(s_s)), & s>s_s,\end{cases}$$

where $s=t/x$; see Figure 3.4(right).

If $u_2(s)>0$ for $s\in[s_{**},s_*]$, then we consider system (3.12) with the initial data

$$(s,\rho)\,|_{u=u_2(s_{**})}=(s_{**},\rho_2(s_{**})). \tag{3.23}$$

LEMMA 3.10. *When $\delta>0$ is sufficiently small the initial value problem (3.12), (3.23) has a solution $(\bar s,\bar\rho)(u)$ on $(u_2(s_{**})-\delta,u_2(s_{**}))$. Moreover, this solution satisfies $\mathrm{d}\bar s/\mathrm{d}u<0$ and $\bar s^2p'(\bar\rho)-(1-u\bar s)^2<0$ for $u\in\big(u_2(s_{**})-\delta,u_2(s_{**})\big)$.*

Proof. It is easy to see that the initial value problem is a classically well-posed problem which has a unique local solution. From(2.10), (2.11), and (3.21) we have

$$\frac{1}{s_{**}}=u_2(s_{**})+\tau_2(s_{**})\sqrt{-p'(\tau_2(s_{**}))}. \tag{3.24}$$

Hence, we have $\bar s^2p'(\bar\rho)-(1-u\bar s)^2=0$ as $u=u_2(s_{**})$.

From (3.22) and (3.24), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}u}\big(\bar s^2p'(\bar\rho)-(1-u\bar s)^2\big)\\ &=\big(2\bar sp'(\bar\rho)+2u(1-u\bar s)\big)\Big(\frac{\bar s^2p'(\bar\rho)-(1-u\bar s)^2}{2p'(\bar\rho)u\bar s}\Big)+\frac{2\bar s\bar\tau^3p''(\bar\tau)}{p'(\bar\rho)}(1-u\bar s)>0\end{aligned}$$

for $u=u_2(s_{**})$.

Thus, when $\delta>0$ is sufficiently small we have

$$\bar{s}^2p'(\bar{\rho})-(1-u\bar{s})^2<0 \quad \text{as} \quad u\in\big(u_2(s_{**})-\delta,u_2(s_{**})\big).$$

We then complete the proof of this lemma. ∎

Let $u=\bar{u}_1(s)$ be the inverse function of $s=\bar{s}(u)$ and $\bar{\rho}_1(s)=\bar{\rho}(\bar{u}_1(s))$. It is obvious that $(\bar{u}_1,\bar{\rho}_1)(s)$ satisfies the system (2.3) in $\big(s_{**},\bar{s}(u_2(s_{**})-\delta)\big)$. Moreover, by Lemma 3.10 we also have

$$0\ <\ \bar{u}_1(s)\ <\ h(\bar{\rho}_1(s),s) \quad \text{and} \quad \bar{\tau}_1(s)>\tau_2^i$$

for $s\in\big(s_{**},\bar{s}(u_2(s_{**})-\delta)\big)$; see Figure 3.5. Thus, the discussion for $s>\bar{s}(u_2(s_{**})-\delta)$ is similar to that of Section 3.1. The structures of the solution can be illustrated in Figure 3.5.

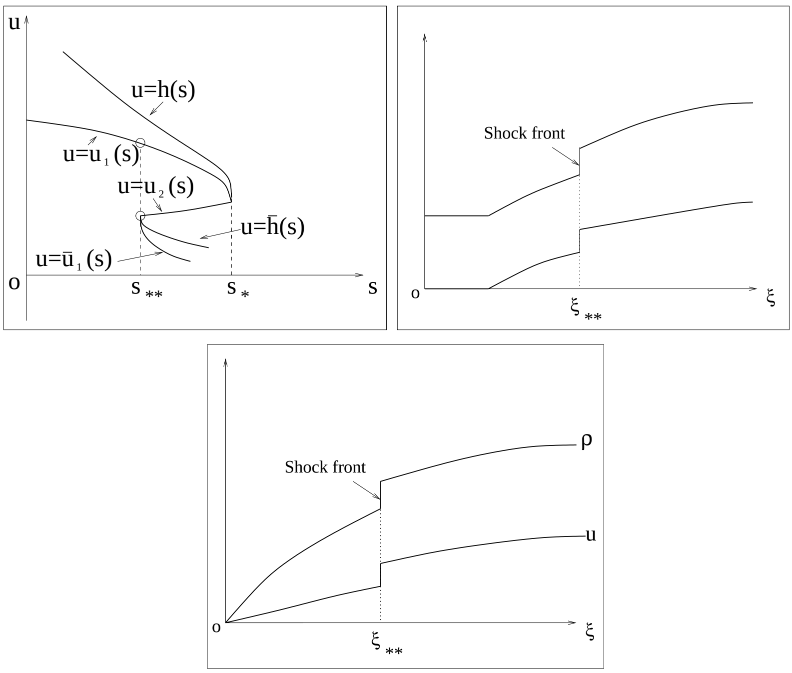


FIG. 3.5. *Discontinuous solutions with a single rarefaction shock.*

3.2.3. $\tau_0<\tau_1^i$. Like the case of $\tau_1^i\le\tau_0<\tau_2^i$, we also have the four cases. We only discuss the fourth case, i.e., there exists a $s_*>0$ such that $0<u_1(s)<h(\rho_1(s),s)$ for $0<s<s_*$ and $0<u_1(s_*)=h(\rho_1(s_*),s_*)<\frac{1}{s_*}$.

If $\tau_0\ge\hat{\tau}_1$, then we have the same results as Lemmas 3.8 and 3.9, since $f(\tau_1(s))$ can be also defined for $s\in[\tau_0,s_*]$. Then the discussion will be similar to that of Section 3.2.2. We omit the details.

If $\tau_0<\hat{\tau}_1$, then we let $\hat{s}$ be the point such that $\tau_1(\hat{s})=\hat{\tau}_1$. Since $u_1(s)<h(\rho_1(s),s)$ as $0<s<s_*$, we have

$$\frac{1}{\hat{s}}>u_1(\hat{s})+\hat{\tau}_1\sqrt{-p'(\hat{\tau}_1)}=u_1(\hat{s})+\hat{\tau}_1\sqrt{-p'(\hat{\tau}_2)}=u_1(\hat{s})+\hat{\tau}_1\sqrt{-p'(f(\hat{\tau}_1))},$$

since $f(\hat{\tau}_1)=\hat{\tau}_2$. Then there exists a $s_{**}\in(\hat{s},s_*)$ such that

$$u_1(s)+\tau_1(s)\sqrt{-p'(\tau_1(s))}<\frac{1}{s}<u_1(s)+\tau_1(s)\sqrt{-p'(f(\tau_1(s)))}$$

for $s\in(s_{**},s_*)$ and $1/s_{**}=u_1(s_{**})+\tau_1(s_{**})\sqrt{-p'(f(\tau_1(s_{**})))}$. Then the discussion will be similar to that of Section 3.2.2. We omit the details.

3.3. Equation of state III. We first define

$$b_1:=\lim_{\rho\to\tilde{\rho}_1^+}\sqrt{p'(\rho)}\quad\text{and}\quad b_2:=\lim_{\rho\to\tilde{\rho}_2^-}\sqrt{p'(\rho)},\tag{3.25}$$

where $\tilde{\rho}_i=\frac{1}{\tilde{\tau}^i}$ $(i=1,2)$.

3.3.1. $\tau_0\geq\tilde{\tau}_2$**.** The discussion is similar to that of Section 3.1, since $\tau_1'(s)>0$ as $s>0$ and $p''(\tau)>0$ as $\tau>\tau_0$. We omit the details.

3.3.2. $\tilde{\tau}_1<\tau_0<\tilde{\tau}_2$**.** Let s_* be defined so that

$$\int_{\rho_0}^{\tilde{\rho}_2}\frac{1}{\rho}\mathrm{d}\rho=\int_0^{s_*}\frac{2u_0}{u_0s-1}\mathrm{d}s.$$

Hence, we have

$$u_1(s)=u_0,\quad \rho_1(s)=\rho_0\exp\Big(\int_0^s\frac{2u_0}{u_0s-1}\mathrm{d}s\Big),\quad 0<s<s_*.$$

If $u_0\leq 1/s_*-b_2$ then the discussion for $s\geq s_*$ is similar to that of Section 3.1, i.e., the problem (1.3), (1.4) has a continuous solution.

In what follows, we are going to discuss the case of $u_0>1/s_*-b_2$. Like the fourth case of Section 3.2.2, the problem does not have a compression shock wave solution in this situation. So, we look for a rarefaction shock wave solution.

For $\tilde{\tau}_1\leq\tau_1\leq\tilde{\tau}_2$, we let $g(\tau_1)$ be determined by

$$\frac{p(\tau_1)-p(g(\tau_1))}{\tau_1-g(\tau_1)}=p'(g(\tau_1))\quad\text{and}\quad g(\tau_1)>\tilde{\tau}_2.\tag{3.26}$$

LEMMA 3.11. *There exists a $s_{**}\in(0,s_*)$ such that for any $s\in[s_{**},s_*]$, there exists an admissible forward rarefaction shock with the speed $1/s$ and the frontside state $(u_1,\rho_1)(s)$.*

Proof. According to Corollary 2.5, in order that $(u_1,\rho_1)(s)$ can be the frontside state of an admissible forward rarefaction shock with the speed $1/s$, there needs

$$0<\frac{1}{s}\leq u_1(s)+\tau_1(s)\sqrt{-p'(g(\tau_1(s)))}.$$

It follows from $u_0>1/s_*-b_2$ that

$$\frac{1}{s_*}\;<\;u_0+b_2=u_0+\tau_1(s_*)\sqrt{-p'(g(\tau_1(s_*)))},$$

since $g(\tau_1(s_*))=\tau_1(s_*)=\tilde{\tau}_2$. Therefore, there exists a $s_{**}\in(0,s_*)$ such that $1/s<u_0+\tau_1(s)\sqrt{-p'(g(\tau_1(s)))}$ as $s\in(s_{**},s_*)$ and $1/s_{**}=u_0+\tau_1(s_{**})\sqrt{-p'(g(\tau_1(s_{**})))}$. We then complete the proof of this lemma. ∎

Let $(u_2,\tau_2)(s)$ $(s_{**}\leq s<s_*)$ be determined by (3.15) and $\tau_2(s)>\tau_1(s)$. It is easy to see that $u_2(s_*)=u_0>0$. Then, the discussion will be similar to the fourth case of Section 3.2.2. We omit the details.

3.3.3. $\tau_0 < \tilde{\tau}_1$. We have the following two cases.

- There exists a $s_* > 0$ such that $u_1(s) < h(\rho_1(s), s)$ for $s < s_*$ and $u_1(s_*) = h(\rho_1(s_*), s_*) = 0$ and $\rho_1(s_*) < \tilde{\rho}_1$.
- There exists a $s_1 > 0$ such that $0 < u_1(s_1) < h(\rho_1(s), s) < 1/s$ for $0 < s < s_1$ and $\rho_1(s_1) = \tilde{\rho}_1$.

The structure of the solution for the first case can be illustrated by Figure 3.2. We only need to discuss the second case.

Let s_* be determined by

$$\int_{\tilde{\rho}_1}^{\tilde{\rho}_2} \frac{1}{\rho}\, \mathrm{d}\rho = \int_{s_1}^{s_*} \frac{2u_1(s_1)}{u_1(s_1)s - 1} \mathrm{d}s.$$

Hence, we have

$$u_1(s) = u_1(s_1), \quad \rho(s) = \tilde{\rho}_1 \exp\Big(\int_{s_1}^{s} \frac{2u_1(s_1)}{u_1(s_1)s - 1} \mathrm{d}s\Big), \quad \text{and} \quad u_1(s) < \frac{1}{s} = h(\rho_1(s), s)$$

for $s_1 < s < s_*$.

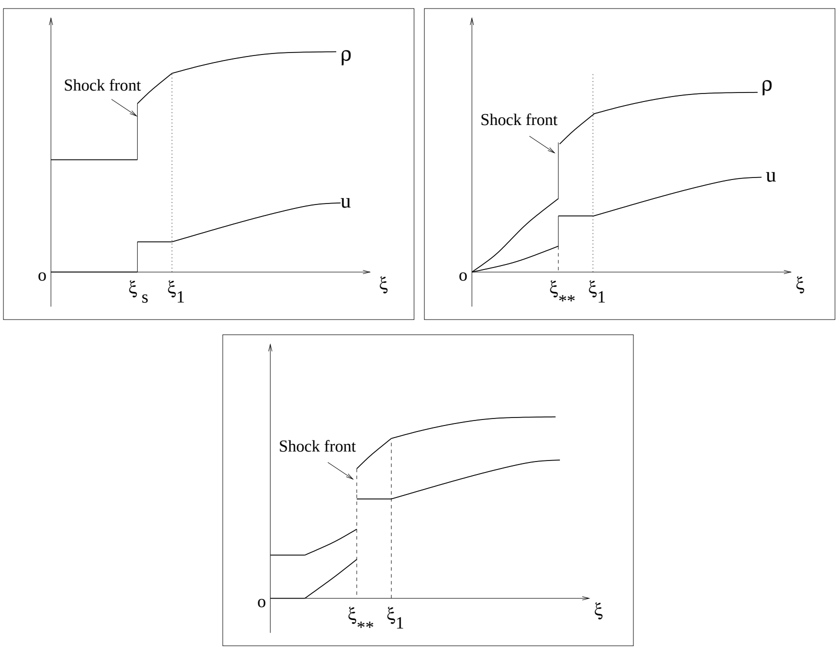


FIG. 3.6. *Discontinuous solutions with a single rarefaction shock.*

If $u_1(s_1) \le 1/s_* - b_2$, then the discussion for $s > s_*$ is similar to that of Section 3.1. In what follows, we are going to discuss the case of $u_1(s_1) > 1/s_* - b_2$. We look for a rarefaction shock wave solution.

LEMMA 3.12. *There exists a $s_{**} \in (s_1, s_*)$ such that for any $s \in [s_{**}, s_*]$, there exists an admissible forward rarefaction shock with the speed $1/s$ and the frontside state $(u_1, \rho_1)(s)$.*

Proof. Since $u_1(s)<h(\rho_1(s),s)$ for $0<s<s_*$, we have

$$\frac{1}{s}>u_1(s)+\tau_1(s)\sqrt{-p'(\tau_1(s))} \quad \text{as} \quad 0<s<s_*.$$

From assumption (A3) we also have

$$\frac{1}{s_1}>u_1(s_1)+b_1>u_1(s_1)+\tau_1(s_1)\sqrt{-p'(g(\tau_1(s_1)))}. \tag{3.27}$$

It follows from $u_1(s_1)>1/s_*-b_2$ that

$$\frac{1}{s_*} < u_1(s_1)+b_2=u_1+\tau_1(s_*)\sqrt{-p'(g(\tau_1(s_*)))}. \tag{3.28}$$

Combining (3.27) and (3.28), there exists a $s_{**}\in(s_1,s_*)$ such that $1/s<u_1(s)+\tau_1(s)\sqrt{-p'(g(\tau_1(s)))}$ as $s\in(s_{**},s_*)$ and $1/s_{**}=u_1(s_{**})+\tau_1(s_{**})\sqrt{-p'(g(\tau_1(s_{**})))}$. Thus by Corollary 2.5 we complete the proof of the lemma. ∎

Hence, the discussion for this case will be similar to the fourth case of Section 3.2.2. The wave structures of the solution can be illustrated in Figure 3.6.

4. Self-similar solutions for $u_0<0$

In this section, we will construct the self-similar solutions of the problem (1.3), (1.4) for $u_0<0$.

4.1. Equation of state I.

From $u_0<0$ we have

$$\frac{du_1}{ds}>0 \quad \text{and} \quad \frac{d\rho_1}{ds}>0. \tag{4.1}$$

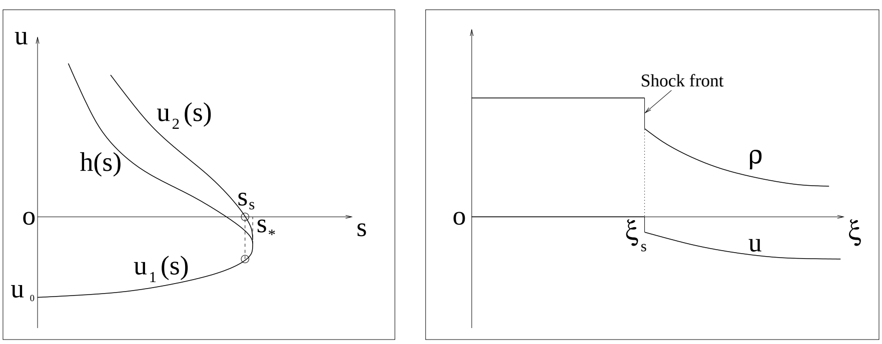


Fig. 4.1. *Discontinuous solution with a single compression shock.*

Lemma 4.1. *For any $u_0<0$, there exists a $s_*>0$ such that $u_1(s)<h(\rho_1(s),s)$ for $0<s<s_*$ and $u_1(s_*)=h(\rho_1(s_*),s_*)<0$; see Figure 4.1(left).*

Proof. The proof of this lemma proceeds in two steps.

Step 1. We first claim that the integral curves $u=u_1(s)$ and $u=h(\rho_1(s),s)$ can not intersect with each other at the $s-$axis.

We argue by contradiction. Suppose that there is a $s_1>0$ such that $u_1(s)<0<h(\rho_1(s),s)$ for $s<s_1$ and $u_1(s_1)=h(\rho_1(s_1),s_1)=0$. Then, we have $s_1\sqrt{p'(\rho_1(s_1))}=1$. We consider the following system of ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}r}=2p'(\rho)us, \\ \dfrac{\mathrm{d}s}{\mathrm{d}r}=s^2p'(\rho)-(1-us)^2, \\ \dfrac{\mathrm{d}\rho}{\mathrm{d}r}=2\rho u(1-us). \end{cases} \tag{4.2}$$

At the point $(u,s,\rho)=(0,s_1,\rho_1(s_1))$, we find the linear part of the right-hand side of (4.2) is given by $M(u,s-s_1,\rho-\rho_1(s_1))^T$ where

$$M=\begin{pmatrix} 2\sqrt{p'(\rho_1(s_1))} & 0 & 0 \\ \dfrac{2}{\sqrt{p'(\rho_1(s_1))}} & 2\sqrt{p'(\rho_1(s_1))} & \dfrac{p''(\rho_1(s_1))}{p'(\rho_1(s_1))} \\ 2\rho_1(s_1) & 0 & 0 \end{pmatrix}.$$

Since

$$\begin{aligned} &\frac{p''(\rho_1(s_1))}{p'(\rho_1(s_1))}\cdot\frac{\rho_1(s_1)}{\sqrt{p'(\rho_1(s_1))}}+\frac{2}{\sqrt{p'(\rho_1(s_1))}} \\ =&\frac{1}{(p'(\rho_1(s_1)))^{3/2}}\Big(\rho_1(s_1)p''(\rho_1(s_1))+2p'(\rho_1(s_1))\Big)=\frac{\tau_1^3(s_1)p''(\tau_1(s_1))}{(p'(\rho_1(s_1)))^{3/2}}\;>\;0, \end{aligned}$$

we have that along the integral curves of (4.2), $\frac{\mathrm{d}s}{\mathrm{d}u}\to-\infty$ for $(u,s,\rho)\to(0,s_1,\rho_1(s_1))$; see Figure 4.2. This leads to a contradiction. Thus the integral curves $u=u_1(s)$ and $u=h(\rho_1(s),s)$ can not intersect with each other at the $s-$axis.

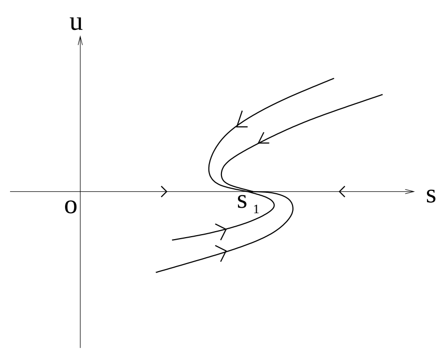


FIG. 4.2. *Integration curves of (4.2).*

Step 2. Let $m=\inf\limits_{\rho\in[\rho_0,+\infty)}\sqrt{p'(\rho)}$. Suppose that the curves $u=u_1(s)$ and $u=h(\rho_1(s),s)$ do not intersect with each other. Then by (4.1) we have

$$\lim_{s\to+\infty}u_1(s)=u_\infty<-m.$$

By (2.3), we have

$$\frac{\mathrm{d}u_1}{\mathrm{d}s} = \frac{p'(\rho_1)u_1 s}{s^2 p'(\rho_1) - (1-u_1 s)^2} > \frac{m^3 s}{(1-u_0 s)^2}.$$

Hence, we have

$$u_\infty - u_0 \; > \; u_1(s) - u_0 \; > \; \int_0^s \frac{m^3 s}{(1-u_0 s)^2}\mathrm{d}s \quad \text{for} \quad s>0.$$

This leads to a contradiction. We then complete the proof of this lemma. ◻

Lemma 4.1 implies that if $u_0<0$ then the problem (1.3), (1.4) does not have a global continuous solution. So, we need to look for a shock wave solution.

LEMMA 4.2. *For any $s\in(0,s_*)$, there exists an admissible compression shock with the speed $1/s$ and the frontside state $(u_1,\rho_1)(s)$.*

Proof. This lemma can be proved by the fact that $1/s > u_1(s)+\sqrt{p'(\rho_1(s))}$ for $0<s<s_*$. ◻

Let the backside state of the shock $(u_2,\rho_2)(s)$ $(0<s\le s_*)$ be determined by (3.15). It is easy to see that

$$u_2(s_*) = u_1(s_*) < 0 \quad \text{and} \quad \lim_{s\to 0} u_2(s) = +\infty.$$

Therefore, there exists a $s_s\in(0,s_*)$ such that $u_2(s_s)=0$. Hence, the self-similar solution of the problem (1.3), (1.4) has the form

$$(u,\rho)(x,t) = \begin{cases} (u_1,\rho_1)(s), & s<s_s, \\ (0,\rho_2(s_s)), & s>s_s, \end{cases}$$

where $s=t/x$; see Figure 4.1(right).

4.2. Equation of state II.

4.2.1. $\tau_0 \le \tau_1^i$. The discussion is similar to that of Section 4.1, since $\tau_1'(s)<0$ for $s>0$ and $p''(\tau)>0$ for $\tau<\tau_0$. We omit the details.

4.2.2. $\tau_1^i<\tau_0<\tau_2^i$. There are the following two cases:

- There exists a $s_*>0$ such that $u_1(s)<h(\rho_1(s),s)$ for $0<s<s_*$ and $u_1(s_*)=h(\rho_1(s_*),s_*)<0$.
- There exists a $s_*>0$ such that $u_1(s)<h(\rho_1(s),s)$ for $0<s<s_*$ and $u_1(s_*)=h(\rho_1(s_*),s_*)=0$. (Remark: In view of Lemma 4.1 we have $\tau_1(s_*)\in(\tau_1^i,\tau_2^i)$ in this case.)

The discussion for the first case is similar to that of Section 4.1, since $1/s > u_1(s)+\sqrt{p'(\rho_1(s))}$ for $0<s<s_*$.

The solution for the second case has the form

$$(u,\rho)(x,t) = \begin{cases} (u_1,\rho_1)(s), & s<s_*, \\ (0,\rho_1(s_*)), & s>s_*. \end{cases} \tag{4.3}$$

4.2.3. $\tau_0>\tau_2^i$. There are the following two cases.

- There exists a $s_*>0$, such that $u_1(s)<0<h_1(\rho_1(s),s)$ for $s<s_*$ and $u_1(s_*)=h(\rho_1(s_*),s_*)=0$ and $\tau_1(s_*)\in(\tau_1^i,\tau_2^i)$.
- There exists a $s_*>0$, such that $u_1(s)<h_1(\rho_1(s),s)$ for $s<s_*$ and $u_1(s_*)=h(\rho_1(s_*),s_*)<0$ and $\tau_1(s_*)\in[\tau_2^i,\tau_0)\cup(0,\tau_1^i]$.

REMARK 4.1. In view of (3.1) and (3.2), it is impossible to have a $s_*>0$ such that $u_1(s)<h_1(\rho_1(s),s)$ for $s<s_*$ and $u_1(s_*)=h(\rho_1(s_*),s_*)<0$ and $\tau_1(s_*)\in(\tau_1^i,\tau_2^i)$.

The solution for the first case is of the form (4.3). In what follows, we are going to discuss the second case. For $\tau_2^i<\tau_1<\tau_3$, we let $\psi(\tau_1)$ be defined such that

$$\frac{p(\tau_1)-p(\psi(\tau_1))}{\tau_1-\psi(\tau_1)}=p'(\psi(\tau_1)) \quad \text{and} \quad \tau_1^i<\psi(\tau_1)<\tau_2^i. \tag{4.4}$$

Here, τ_3 is defined in Corollary 2.2.

We first consider the case of $\tau_2^i<\tau_1(s_*)<\tau_0<\tau_3$. Let

$$\hat{\xi}(s)=u_1(s)+\tau_1(s)\sqrt{-p'(\psi(\tau_1(s)))} \quad \text{and} \quad F(s)=\frac{1}{s}-\hat{\xi}(s). \tag{4.5}$$

Then we have

$$\lim_{s\to 0}F(s)=+\infty \tag{4.6}$$

and

$$\begin{aligned} F(s_*)&=\frac{1}{s_*}-u_1(s_*)-\tau_1(s_*)\sqrt{-p'(\psi(\tau_1(s_*)))}\\ &=\frac{1}{s_*}-u_1(s_*)-\tau_1(s_*)\sqrt{-p'(\tau_1(s_*))}+\tau_1(s_*)\Big(\sqrt{-p'(\tau_1(s_*))}-\sqrt{-p'(\psi(\tau_1(s_*)))}\Big)\\ &=\tau_1(s_*)\Big(\sqrt{-p'(\tau_1(s_*))}-\sqrt{-p'(\psi(\tau_1(s_*)))}\Big)<0. \end{aligned} \tag{4.7}$$

Since $1/s > u_1(s)+\sqrt{p'(\rho_1(s))}$ and $\tau_2^i<\tau_1(s)<\tau_0$ for $0<s<s_*$, for any $s\in(0,s_*)$ there exists an admissible compression shock with the speed $1/s$ and the frontside state $(u_1,\rho_1)(s)$. Let the backside state of the shock $(u_2,\rho_2)(s)$ $(0<s\le s_*)$ be determined by (3.15). We have

$$u_2(s_*)=u_1(s_*)<0 \quad \text{and} \quad \lim_{s\to 0}u_2(s)=+\infty. \tag{4.8}$$

From Corollary 2.2 we know that if $F(s)=0$ then (3.15) has two solutions $(u_2^+,\tau_2^+)(s)$ and $(u_2^-,\tau_2^-)(s)$, where $u_2^+(s)>u_2^-(s)$ and $\tau_2^+(s)<\tau_1^i<\tau_2^-(s)<\tau_2^i$. So, by (4.6) and (4.7) we can see that $u_2(s)$ is piecewise continuous on $(0,s_*)$. Hence, we can not determine whether or not $u_2(s)$ has a zero point in $(0,s_*)$.

If there exists a $s_s\in(0,s_*)$ such that $u_2(s_s)=0$ then the problem (1.3), (1.4) admits a discontinuous solution with a single shock with the speed $1/s_s$; see Figure 4.1(right).

If $u_2(s)\neq 0$ for all $s\in(0,s_*)$, then by (4.8) there must exist a $s^*\in(0,s_*)$ such that

$$F(s^*)=0, \quad u_2^+(s^*)>0, \quad u_2^-(s^*)<0, \quad \text{and} \quad \tau_1^i<\tau_2^-(s^*)<\tau_2^i.$$

Then we consider system (3.12) with the data

$$(s,\rho)\,|_{u=u_2^-(s^*)}=(s_*,\rho_2^-(s^*)). \tag{4.9}$$

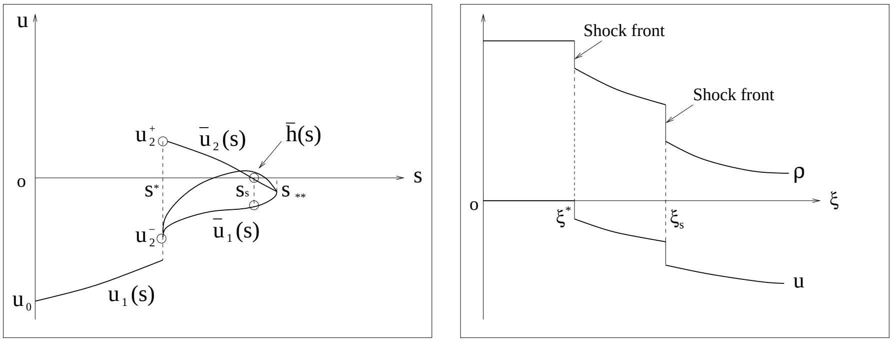


FIG. 4.3. *Discontinuous solution with two compression shocks.*

LEMMA 4.3. *When $\delta>0$ is sufficiently small, the initial value problem (3.12), (4.9) has a solution $(\bar{s},\bar{\rho})(u)$ on $(u_2^-(s^*),u_2^-(s^*)+\delta)$. Moreover, this solution satisfies $\frac{\mathrm{d}s}{\mathrm{d}u}>0$ and $s^2p'(\rho)-(1-us)^2<0$ in $(u_2^-(s^*),u_2^-(s^*)+\delta)$.*

Proof. The proof is similar to that of Lemma 3.10, we omit the details. ∎

Let $u=\bar{u}_1(s)$ be the inverse function of $s=\bar{s}(u)$ and $\bar{\rho}_1(s)=\bar{\rho}(\bar{u}_1(s))$. It is obvious that $(\bar{u}_1,\bar{\rho}_1)(s)$ satisfies (2.3) in $\big(s^*,\bar{s}(u_2^-(s^*)+\delta)\big)$. Moreover, by Lemma 3.10 we also have

$$0\;<\;\bar{u}_1(s)\;<\;h(\bar{\rho}_1(s),s)\quad\text{and}\quad\bar{\tau}_1(s)<\tau_2^i$$

for $s\in\big(s^*,\bar{s}(u_2^-(s^*)+\delta)\big)$.

When $s>s^*$ there are two cases: (a) there exists a $s_{**}>s^*$ such that $\bar{u}_1(s)<h(\bar{\rho}_1(s),s)$ for $s^*<s<s_{**}$ and $\bar{u}_1(s_{**})=h(\bar{\rho}_1(s_{**}),s_{**})=0$; (b) there exists a $s_{**}>s^*$ such that $\bar{u}_1(s)<h(\bar{\rho}_1(s),s)$ for $s^*<s<s_{**}$ and $\bar{u}_1(s_{**})=h(\bar{\rho}_1(s_{**}),s_{**})<0$.

The solution for case (a) has the form

$$(u,\rho)(x,t)=\begin{cases}(u_1,\rho_1)(s), & s<s^*,\\ (\bar{u}_1,\bar{\rho}_1)(s), & s^*<s<s_{**},\\ (0,\bar{\rho}_1(s_{**})), & s>s_{**}.\end{cases}$$

For case (b), since $1/s>\bar{u}_1(s)+\sqrt{p'(\bar{\rho}_1(s))}$ and $\bar{\tau}_1(s)<\tau_2^i$ for $s^*<s<s_{**}$, for any $s\in(s^*,s_{**})$ there exists an admissible compression shock with the speed $1/s$ and the frontside state $(\bar{u}_1,\bar{\rho}_1)(s)$. Let the backside state of the shock $(\bar{u}_2,\bar{\rho}_2)(s)$ ($s^*<s\le s_{**}$) be determined by (3.15). Then we have $\bar{u}_2(s_{**})=\bar{u}_1(s_{**})<0$. Thus by $\bar{u}_2(s^*)=u_2^+(s^*)>0$ we know that there exists a $s_s\in(s^*,s_{**})$ such that $\bar{u}_2(s_s)=0$. Hence, the problem (1.3), (1.4) admits a discontinuous solution with two compression shocks. The solution has the form

$$(u,\rho)(x,t)=\begin{cases}(u_1,\rho_1)(s), & s<s^*,\\ (\bar{u}_1,\bar{\rho}_1)(s), & s^*<s<s_s,\\ (0,\bar{\rho}_2(s_s)), & s>s_s;\end{cases}$$

see Figure 4.3.

We next discuss the case for $\tau_1(s_*)\in\{\tau_2^i\}\cup(0,\tau_1^i]$ or $\tau_0>\tau_3$. If $F(s)\ge0$ for $\tau_2^i<$

$\tau_1(s) \le \tau_3$, then

$$u_2(s) := \begin{cases} u_2^+(s), \ F(s) = 0; \\ u_2(s), \ \text{otherwise} \end{cases}$$

is a continuous function on $(0, s_*]$. Moreover, $u_2(s_*) = u_1(s_*) < 0$ and $\lim_{s\to 0} u_2(s) = +\infty$. Hence, there exists a $s_s \in (0, s^*)$ such that $u_2(s_s) = 0$, and consequently the problem (1.3), (1.4) admits a discontinuous solution with a single compression shock; see Figure 4.1(right). If $F(s)$ is not nonnegative for $\tau_2^i < \tau_1(s) \le \tau_3$, then the discussion will be similar to the previous discussions.

4.3. Equation of state III.

4.3.1. $\tau_0 \le \tilde{\tau}_1$. The discussion is similar to that of Section 4.1, since $\tau_1'(s) < 0$ for $s > 0$ and $p''(\tau) > 0$ for $\tau < \tau_0$.

4.3.2. $\tilde{\tau}_1 < \tau_0 \le \tilde{\tau}_2$. Let s_* be determined by

$$\int_{\rho_0}^{\tilde{\rho}_1} \frac{1}{\rho} \mathrm{d}\rho = \int_0^{s_*} \frac{2u_0}{u_0 s - 1} \mathrm{d}s.$$

Hence, we have

$$u_1(s) = u_0, \quad \rho_1(s) = \rho_0 \exp\Big(\int_0^s \frac{2u_0}{u_0 s - 1} \mathrm{d}s \Big), \quad 0 < s < s_*.$$

We then have the following two cases: (1) $u_0 < 1/s_* - b_1$; (2) $u_0 \ge 1/s_* - b_1$.

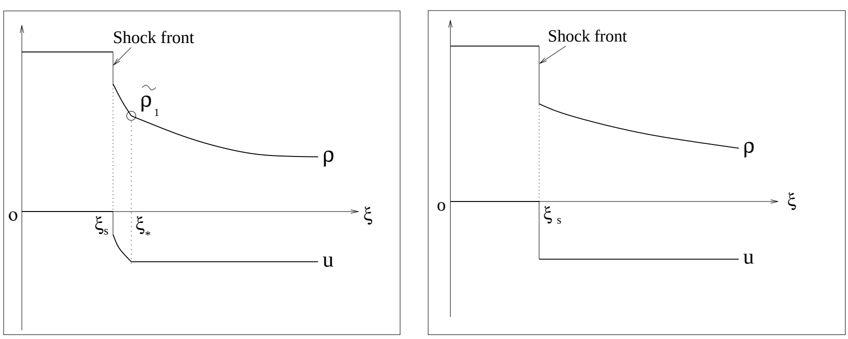


FIG. 4.4. *Solutions with a single compression shock. Left: $u_0 < 1/s_* - b_1$; right: $u_0 \ge 1/s_* - b_1$.*

If $u_0 < 1/s_* - b_1$ then we consider the system (2.3) with the initial data

$$(u, \rho)(s_*) = (u_0, \tilde{\rho}_1). \tag{4.10}$$

Like Lemma 4.1, there exists a $s^* > s_*$ such that the solution $(u_1, \rho_1)(s)$ of the problem (2.3), (4.10) satisfies $u_1(s) < h(\rho_1(s), s)$ for $s_* < s < s^*$ and $u_1(s^*) = h(\rho_1(s^*), s^*) < 0$. Moreover, for any $s \in (0, s^*)$ there exists an admissible forward compression shock with the speed $1/s$ and the frontside state $(u_1, \rho_1)(s)$. The backside state of the shock $(u_2, \rho_2)(s)$ can be determined by (3.15). It is easy to see that $u_2(s^*) = u_1(s^*) < 0$ and $\lim_{s\to 0} u_2(s) = +\infty$. Hence, there exists a $s_s \in (0, s^*)$ such that $u_2(s_s) = 0$, and consequently the problem (1.3), (1.4) admits a discontinuous solution with a single compression shock; see Figure 4.4.

Next, we discuss the case of $u_0 \geq 1/s_* - b_1$. By Corollary 2.5, we know that for any $s \in (0, s^*)$, there exists an admissible forward compression shock with the speed $1/s$ and the frontside state $(u_1, \rho_1)(s)$. The backside state of the shock $(u_2, \rho_2)(s)$ can be determined by (3.15). It is obvious that $\lim_{s\to 0} u_2(s) \to +\infty$. Using $u_0 > 1/s_* - b_1$, we also have $\lim_{s\to s_*} \tau_2(s) = \tilde{\tau}_1$ and $\lim_{s\to s_*} u_2(s) = u_0 < 0$. Thus, there must exist a $s_s \in (0, s_*)$ such that $u_2(s_s) = 0$. The solution for this case can be illustrated by Figure 4.4(right).

4.3.3. $\tau_0 > \tilde{\tau}_2$**.** We have the following three cases:

- There exists a $s_* > 0$, such that $u_1(s) < h(\rho_1(s), s)$ for $s < s_*$ and $u_1(s_*) = h(\rho_1(s_*), s_*) < 0$ and $\tau_1(s_*) \in [\tilde{\tau}_2, \tau_0)$.
- There exists a $s_* > 0$, such that $\tau_1(s_*) = \tilde{\tau}_1$, $u_1(s_*) \geq \frac{1}{s_*} - b_1$, and $u_1(s) < h(\rho_1(s), s)$ for $s < s_*$.
- There exists a $s_* > 0$, such that $u_1(s_*) = h(\rho_1(s_*), s_*) < 0$, $\tau_1(s_*) < \tilde{\tau}_1$, and $u_1(s) < h(\rho_1(s), s)$ for $s < s_*$. (Remark: We have $u_1(s_*) < \frac{1}{s_*} - b_1$ in this case.)

We now discuss the first case. For $\tau_1 > \tilde{\tau}_2$, we let $\kappa(\tau_1)$ be defined such that

$$\kappa(\tau_1) = \frac{p(\tilde{\tau}_2) - p(\tau_1)}{\tilde{\tau}_2 - \tau_1}. \tag{4.11}$$

Then we have $-\kappa(\tau_1) > -p'(\tau_1)$, since $\tau_1 > \tilde{\tau}_2$.

Let

$$\hat{\xi}(s) = u_1(s) + \tau_1(s)\sqrt{-\kappa(\tau_1(s))}, \quad F(s) = \frac{1}{s} - \hat{\xi}(s).$$

Then we have

$$\lim_{s\to 0} F(s) = +\infty \tag{4.12}$$

and

$$\begin{aligned} F(s_*) &= \frac{1}{s_*} - u_1(s_*) - \tau_1(s_*)\sqrt{-\kappa(\tau_1(s_*))} \\ &= \frac{1}{s_*} - u_1(s_*) - \tau_1(s_*)\sqrt{-p'(\tau_1(s_*))} + \tau_1(s_*)\left(\sqrt{-p'(\tau_1(s_*))} - \sqrt{-\kappa(\tau_1(s_*))}\right) \\ &= \tau_1(s_*)\left(\sqrt{-p'(\tau_1(s_*))} - \sqrt{-\kappa(\tau_1(s_*))}\right) < 0. \end{aligned} \tag{4.13}$$

Since $1/s > u_1(s) + \sqrt{p'(\rho_1(s))}$ and $\tilde{\tau}_2 < \tau_1(s) < \tau_0$ as $0 < s < s_*$, there exists an admissible compression shock with the speed $1/s$ and the frontside state $(u_1, \rho_1)(s)$ for any $s \in (0, s_*)$. The backside state of the shock $(u_2, \rho_2)(s)$ $(0 < s \leq s_*)$ can be determined by (3.15). Moreover, we have $u_2(s_*) = u_1(s_*) < 0$ and $\lim_{s\to 0} u_2(s) = +\infty$. However, by (4.12) and (4.13) we know that $u_2(s)$ is not continuous in $(0, s_*)$. Since, if $F(s) = 0$ then (3.15) has two solutions $(u_2^+, \rho_2^+)(s)$ and $(u_2^-, \rho_2^-)(s)$, where $u_2^+(s) > u_2^-(s)$ and $\rho_2^+(s) > \rho_2^-(s)$.

If there exists a $s_s \in (0, s_*)$ such that $u_2(s_s) = 0$, then the problem (1.3), (1.4) admits a discontinuous solution with a single shock.

If $u_2(s) \neq 0$ for all $s \in (0, s_*)$, then there must exist a $s^* \in (0, s_*)$ such that $u_2^+(s^*) > 0$, $u_2^-(s^*) < 0$, and $\tau_2^-(s^*) = \tilde{\tau}_2$. Then the discussion for $s > s^*$ is similar to that of Section

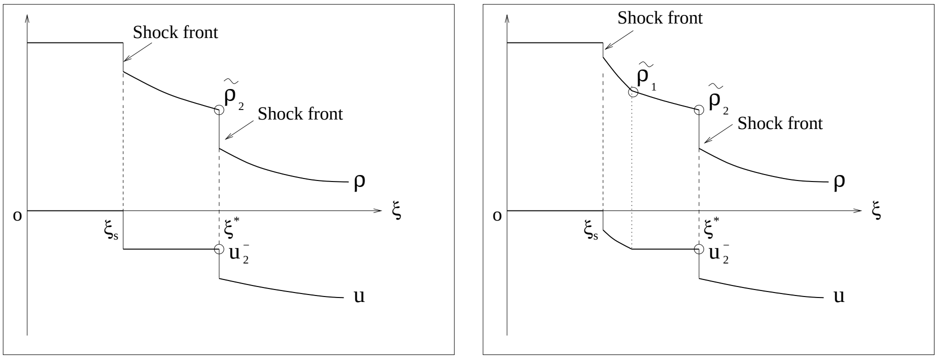


FIG. 4.5. *Solutions with two compression shocks for* $u_0 < 0$ *and* $\tau_0 > \tilde{\tau}_2$.

4.3.2. The problem has a discontinuous solution with two compression shocks; see Figure 4.5.

We next discuss the second case. Similarly, for any $s \in (0, s_*)$, there exists an admissible compression shock with the speed $1/s$ and the frontside state $(u_1, \rho_1)(s)$. The backside state of the shock $(u_2, \rho_2)(s)$ $(0 < s \leq s_*)$ can be determined by (3.15). Moreover, by $u_1(s_*) \geq \frac{1}{s_*} - b_1$ we have $\lim_{s \to s_*} u_2(s) = u_1(s_*) < 0$. Let s_1 be the point such that $\tau_1(s_1) = \tilde{\tau}_2$. Then the discussion can be divided into the following two cases: (1) $F(s) \geq 0$ for $s \in (0, s_1)$; (2) $F(s)$ is not nonnegative in $(0, s_1)$.

If $F(s) \geq 0$ for $s \in (0, s_1)$, we redefine $u_2(s) = \begin{cases} u_2(s), & F(s) > 0; \\ u_2^+(s), & F(s) = 0 \end{cases}$ for $0 < s < s_1$. Then $u_2(s)$ is a continuous function on $(0, s_*)$. Thus, there exists a $s_s \in (0, s_*)$ such that $u_2(s) = 0$ and consequently, the problem (1.3), (1.4) admits a discontinuous solution with a single compressible shock. If $F(s)$ is not nonnegative on $(0, s_1)$ then the discussion will be similar to the first case.

Actually, the discussion for the third case is similar to that of the second case. We omit the details.

We then complete the proof of Theorem 1.1.

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