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EXISTENCE AND UNIQUENESS FOR A STATIONARY HYBRID QUANTUM HYDRODYNAMICAL MODEL WITH GENERAL PRESSURE FUNCTIONAL*

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Abstract. In this paper we generalize the results obtained in [F. Di Michele, M. Mei, B. Rubino, and R. Sampalmieri, Int. J. Numer. Anal. Model., 13:898–925, 2016], where a hybrid model for semiconductor devices has been presented. In particular we consider a more general pressure function, which allows us to account also for the isotropic case. General Dirichlet boundary conditions are also included. In this case we need a different and more restrictive subsonic condition which directly involves the first derivative of the quantum function $Q(x)$. The existence of solutions is obtained by regularizing the problem and performing a suitable *vanishing viscosity* limit. Also the zero-charge-space limit is discussed and our results are tested on a simple toy model.

Keywords. Hybrid quantum hydrodynamic model; isotropic pressure; stationary solutions; existence; uniqueness; classical limit.

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1. Introduction

In this paper the authors continue their research on hybrid quantum models for semiconductor devices as introduced in [6, 11, 12], and [13]. When one talks of *hybrid approach* we think of models having a part of the device described by using quantum equations (such as Schrödinger, quantum drift-diffusion (QDD) or quantum hydrodynamic (QHD)), and the other parts by classical models, for example hydrodynamical (HD) or drift diffusion (DD) equations. The main problem of this approach is to introduce proper transmission conditions that must be prescribed at the interface between classical and quantum zones of the device. This choice is somehow arbitrary since it cannot be based on experimental data.

The main advantage of our hybrid approach is that the solution of the hybrid problem is obtained as *limit solution* of a regularized sequence of problems. In this way we avoid the introduction of artificial interface conditions between classical and quantum regions.

Hybrid models for semiconductors have been of interest in the recent years since it is well known that the nano size of the latest semiconductor devices requires to take into account the quantum effect, and that this effect is localized in a small region of the device. Moreover, this approach allows the reduction of the computational costs simplifying the numerical simulation, without the massive use of parallel algorithms. One of the seminal papers in which the hybrid coupling between quantum and classical systems

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was studied is the paper of Ben Abdallah [2], where a suitable set of transmission conditions, linking classical Boltzmann equation and stationary Schrödinger equations, is discussed. Subsequently, in [5], a one-dimensional stationary Schrödinger drift-diffusion model, including collisions is analyzed. To link quantum zone and classical region, the authors prescribe the continuity of current density at the interface. In [3] Ben Abdallah, Méhats and Vauchelet introduced a hybrid drift-diffusion-Schrödinger-Poisson (DDSP) model, while the optimal parallelization strategy of numerical solutions of this model is performed in [22].

In [1, 16] and [17] the DDSP model is applied to study the electron transport in strongly confined structures, such as nanotubes; the continuity of the total current is assumed at the interface between classical and quantum domains. In [8], the hydrodynamic hybrid model is studied by prescribing the continuity of the charge density, while a small jump of the current density is accepted and justified from the physical point of view, by using scaling arguments. In [12] the authors define a generalized enthalpy function, which also contains a quantum term. This new quantity is assumed to be constant through the interfaces, allowing the information exchange between classical and quantum problems.

This short survey shows that many different strategies can be adopted to establish a physically reasonable set of interface conditions; usually the continuity of certain physical quantities is preserved at the expenses of others: the same concept of hybrid model introduces an error at the interface, due to the arbitrariness of this choice. Our approach proposes a hybrid model matching classical and quantum hydrodynamical equations, derived in [11], by introducing a modified form of the Bohm potential (more precisely, in comparison with the QHD equation, we introduce a new term in the quantum potential, namely $Q'\sqrt{n_x}/\sqrt{n}$. This make our model basically different from the standard QHD model). From the mathematical point of view, we notice that the performed localization of the quantum effects introduces a degeneracy in the working equations that makes the mathematical dealing more complex.

In the present paper the steady-state, one-dimensional, hybrid model, as derived in the above cited papers, is considered while introducing a general pressure function that allows to deal with the isotropic cases, while the isothermal case has been already discussed in [11]. The working system reads as

$$\begin{cases} 2n\varepsilon^2\left(Q(x)\dfrac{(\sqrt{n})_{xx}}{\sqrt{n}}+Q'(x)\dfrac{(\sqrt{n})_x}{\sqrt{n}}\right)_x-\left(p(n)+\dfrac{J^2}{n}\right)_x+nV_x=\dfrac{J}{\tau}, \\ \lambda^2V_{xx}=n-C(x), \end{cases} \tag{1.1}$$

here $x\in\Omega=[0,1]$.

As usual, the current density J is assumed to be a constant and the second equation in (1.1) is the Poisson equation, describing the evolution of the self-consistent electrical potential V. Moreover n is the charge density, τ and λ are strictly positive parameters for the scaled relaxation time and the scaled Debye length, respectively. Finally ε is the scaled Planck constant and will be considered sufficiently small in the sequel ($\varepsilon\ll 1$). We introduce $Q\in C^1(\Omega)$, with $0\le Q(x)\le 1$ which is the so-called quantum effect function.

The quantum function $Q(x)$ satisfying $Q(x)=0$ in some parts of $[0,1]$ and $Q(x)>0$ in the other parts can be designed in many ways. One of the typical examples for $Q(x)$ is a regularized step function such that $Q(x)>0$ around the endpoints $x=0$ and $x=1$ (but it can also be equal to zero in the middle part of $(0,1)$). Namely, we take $Q(x)$ as:

$$Q(x) = 0 \quad \text{for } x \in [x_1, x_2], \qquad \text{and } Q(x) > 0 \quad \text{for } x \in [0, x_1) \cup (x_2, 1], \tag{1.2}$$

with $0 < x_1 < x_2 < 1$ (a typical example is the one in the Figure 1.1).

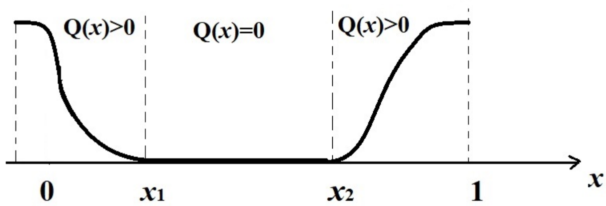


FIG. 1.1. *Function* $Q(x)$.

This problem has been studied in [18] in the non-hybrid case, namely for the standard QHD model (quantum hydrodynamic model), corresponding to $Q(x) \equiv 1$.

The doping profile is modelled by $C(x) \in L^2(\Omega)$, where $C(x) \geq C_0 > 0$ for all $x \in \Omega$.

The pressure function is assumed to be

$$p(n) = \frac{Tn^\gamma}{\gamma}, \quad \gamma > 1, \tag{1.3}$$

where T is the scaled electron temperature. The value $\gamma = 1$ corresponds to the isothermal case already discussed in [11], whereas for $\gamma > 1$ we are in the isotropic case and this is the case that will be considered in the present paper.

The stationary problem (1.1) is supplemented by the following boundary conditions

$$n(0) = n_0, \quad n(1) = n_1, \tag{1.4}$$

$$n_x(0) = n_x(1) = 0, \tag{1.5}$$

with n_0 and n_1 strictly positive. Concerning the potential V, we set

$$V(0) = V_0, \quad V(1) = V_1, \tag{1.6}$$

where

$$V_0 = V(0) = -2\varepsilon^2 Q(0) \frac{(\sqrt{n})_{xx}(0)}{\sqrt{n_0}} + \frac{T}{\gamma - 1} n_0^{\gamma-1} + \frac{J^2}{2n_0^2}. \tag{1.7}$$

Let us briefly discuss the boundary conditions (1.5) in the following:

REMARK 1.1. The boundary conditions (1.5) are the natural conditions for the standard QHD as in [15]. This allows the problem to be well posed. To solve the hybrid problem (1.1) we define a sequence of approximating problems obtained by regularizing $Q(x)$ as in (2.9). For each of these approximating problems we assume $Q_q(x) > 0$, then the same boundary conditions (1.5) are still necessary. When passing to the limit (see Theorem 2.3), assuming (1.2), the limit hybrid solution inherits them (see the proof on page 19). In the case of *classic boundary* ($Q(x) > 0$ for $x \in [x_1, x_2]$ and $Q(x) = 0$ for $x \in [0, x_1) \cup (x_2, 1]$), we are not able to prove the convergence $(w_q)_x \to w_x$ in

$x = 0$ and $x = 1$ (see Remark 2.5). Anyway, in Section 5, assuming (1.5), we perform numerical simulations also in this last case, observing that, since Q is strictly positive in a sub-interval of $[0, 1]$, there the problem is of the fourth order and then the single condition (1.4) is not enough to determine it.

Finally we remark that similar conditions have been used both for the classic HD model [7, 14, 20] and for the QHD model [15, 18, 19]. Basically these conditions are necessary to obtain a well-posed steady-state system (1.1).

Let us divide $(1.1)_1$ by n and integrate with respect to x. Then, in view of (1.6), we have

$$V(x) = -2\varepsilon^2 Q(x)\frac{(\sqrt{n})_{xx}}{\sqrt{n}} - 2\varepsilon^2 Q'(x)\frac{(\sqrt{n})_x}{\sqrt{n}} + \frac{J^2}{2n^2} + \frac{T}{\gamma-1}n^{\gamma-1} + \frac{J}{\tau}\int_0^x \frac{1}{n}ds, \quad (1.8)$$

and we further obtain, by using the boundary conditions (1.4) and (1.6), that

$$V_1 = V(1) = -2\varepsilon^2 Q(1)\frac{(\sqrt{n})_{xx}(1)}{\sqrt{n_1}} + \frac{T}{\gamma-1}n_1^{\gamma-1} + \frac{J^2}{2{n_1}^2} + \frac{J}{\tau}\int_0^1 \frac{1}{n}dx. \quad (1.9)$$

In this paper, we propose the constant J as a parameter and leave $V(1)$ to be a number automatically determined by (1.9).

Throughout the paper we consider the following boundary problem for the steady-state H-QHD model (1.1)

$$\begin{cases} 2\varepsilon^2\left(Q(x)\frac{(\sqrt{n})_{xx}}{\sqrt{n}} + Q'(x)\frac{(\sqrt{n})_x}{\sqrt{n}}\right)_x - \left(\frac{T}{\gamma-1}n^{\gamma-1} + \frac{J^2}{2n^2}\right)_x = -V_x + \frac{J}{\tau n}, \\ \lambda^2 V_{xx} = n - C(x), \\ n(0) = n_0, \quad n(1) = n_1, \\ n_x(0) = n_x(1) = 0, \quad V(0) = V_0, \quad J = J_0. \end{cases} \quad (1.10)$$

We just remark that in [11] a less general set of boundary conditions and pressure functional have been used, namely $\gamma = 1$ and $n_0 = n_1 = 1$. The problem considered in this paper is therefore more general, although the assumptions we need in order to prove the existence of a regular solution are more strict. Many other papers, related to hybrid model for semiconductor devices, are available in the literature. In particular, see [1, 2, 5, 8–10, 16, 21, 22] and references therein.

In the last section, for the sake of completeness, we present some numerical simulations obtained by means of the Fortran routine COLNEW, from the package SCILAB. [4].

2. Main results

Starting from $(1.10)_1$ differentiated with respect to x, in view of the Poisson equation and observing that $p'(n) = Tn^{\gamma-1}$, we obtain the following fourth-order differential equation for the electron density n

$$\begin{cases} 2\varepsilon^2\left(Q(x)\frac{(\sqrt{n})_{xx}}{\sqrt{n}} + Q'(x)\frac{(\sqrt{n})_x}{\sqrt{n}}\right)_{xx} - \left(\left(Tn^{\gamma-1} - \frac{J^2}{n^2}\right)\frac{n_x}{n}\right)_x \\ \qquad + \frac{1}{\lambda^2}(n - C(x)) = -\frac{J}{\tau n^2}n_x, \\ \lambda^2 V_{xx} = n - C(x), \\ n(0) = n_0, \quad n(1) = n_1, \quad n_x(0) = n_x(1) = 0, \\ V(0) = V_0, \quad J = J_0. \end{cases} \quad (2.1)$$

It is well known that the flow is subsonic when

$$\text{velocity of the flow} := \frac{|J|}{n} < \sqrt{p'(n)}. \tag{2.2}$$

Dividing by n the square of (2.2), we get

$$\frac{p'(n)}{n} - \frac{J^2}{n^3} > 0, \ \ i.e.\ T > \frac{J^2}{n^{\gamma+1}}, \ \text{for the } n \text{ under consideration}, \tag{2.3}$$

which ensures the uniform ellipticity of the term $\left(\left(\frac{p'(n)}{n} - \frac{J^2}{n^3}\right) n_x\right)_x$.

By using (2.2) or equivalently (2.3), we could obtain a lower bound for the charge density

$$\text{classical subsonic condition:} \ \ n > \left(\frac{J^2}{T}\right)^{\frac{1}{\gamma+1}} =: n_{\star\star}. \tag{2.4}$$

As it will be shown in the proofs in the sequel, in our case more strict subsonic-type conditions are required, such as

$$Tn^{(\gamma-1)} - \frac{J^2}{n^2} > 2\varepsilon^2 (Q'(x))^2. \tag{2.5}$$

We have to verify that there exists a $n_\star > 0$ such that, for all $n > n_\star$, the inequality (2.5) holds for all $\gamma > 1$. Obviously the boundary terms need to satisfy (2.5) and

$$n_0, n_1 > n_\star. \tag{2.6}$$

For the doping profile $C(x)$, we require:

$$C_0 := \min_{x\in[0,1]} C(x) > n_\star. \tag{2.7}$$

Remark 2.1. We observe that when $Q = 1$ and $Q' = 0$, that is in the pure quantum case, we recover (2.4), as expected.

As extensively discussed in [11] the fourth-order elliptic Equation (2.1) is locally degenerate because $Q(x) = 0$ in the classical part of the domain. This increases the difficulties in solving problem (2.1). Therefore we introduce a sequence of strictly positive functions $Q_q(x) \geq q > 0$, constructed such that $Q_q(x) \to Q(x)$ when $q \to 0$, and we prove that $(n_q, V_q)(x)$ is the unique solution of the relative approximating problem. The solution $(n, V)(x)$ of the really hybrid problem is then obtained taking the hybrid limit $q \to 0$ of $(n_q, V_q)(x)$.

Setting $w = \sqrt{n}$, (2.1) reduces to

$$\begin{cases} 2\varepsilon^2 \left(Q(x)\dfrac{w_{xx}}{w} + Q'(x)\dfrac{w_x}{w}\right)_{xx} - 2\left(\left(Tw^{2(\gamma-1)} - \dfrac{J^2}{w^4}\right)\dfrac{w_x}{w}\right)_x \\ \qquad + \dfrac{1}{\lambda^2}(w^2 - C(x)) = -\dfrac{2J}{\tau w^3} w_x, \\ \lambda^2 V_{xx} = w^2 - C(x), \\ w(0) = w_0 \ \ w(1) = w_1, \ \ w_x(0) = w_x(1) = 0, \\ V(0) = V_0, \ \ J = J_0, \end{cases} \tag{2.8}$$

where $w_0 = \sqrt{n_0}$ and $w_1 = \sqrt{n_1}$.

We now introduce the modified QHD equations (H-Q_qHD), obtained by replacing $Q(x)$ with $Q_q(x)$:

$$\begin{cases} 2\varepsilon^2 \left(Q_q \dfrac{(w_q)_{xx}}{w_q} + Q_q' \dfrac{(w_q)_x}{w_q} \right)_{xx} - 2 \left(\left(T w_q^{2(\gamma-1)} - \dfrac{J^2}{w_q^4} \right) \dfrac{(w_q)_x}{w_q} \right)_x \\ \qquad + \dfrac{1}{\lambda^2} (w_q^2 - C(x)) = -\dfrac{2J}{\tau w_q^3} (w_q)_x, \\ \lambda^2 (V_q)_{xx} = w_q^2 - C(x), \\ w_q(0) = w_0, \quad w_q(1) = w_1, \quad (w_q)_x(0) = (w_q)_x(1) = 0, \\ V_q(0) = V_0, \quad J = J_0. \end{cases} \tag{2.9}$$

Here we assume $Q_q'(0) = Q_q'(1) = 0$.

The subsonic-type condition (2.5) becomes

$$T w_q^{2(\gamma-1)} - \frac{J^2}{w_q^4} > 2\varepsilon^2 (Q_q'(x))^2. \tag{2.10}$$

Now we look for a unique solution to (2.9).

THEOREM 2.1 (Existence of solutions for the modified H-Q_qHD problem). *Assume (2.6), (2.7) and (2.10), and that $Q_q(x)$ is a positive, smooth and bounded function, defined on Ω such that*

$$0 < q \le Q_q(x) \le 1, \quad \alpha = \max(|Q_q'|_\infty, |Q_q''|_\infty) < \infty \quad \textit{for all} \quad x \in \Omega, \tag{2.11}$$

and

$$\varepsilon^2 \max_{x \in \Omega} \frac{|Q_q'(x)|^2}{Q_q(x)} < \frac{1}{6} \left(T \underline{w}^{2(\gamma-1)} - \frac{J^2}{\underline{w}^4} \right), \tag{2.12}$$

where

$$\underline{w} = \sqrt{\underline{n}} \quad \textit{and} \quad \underline{n} := \min\{n_0, n_1, C_0\}. \tag{2.13}$$

Then the solution to (2.9) exists and $(n_q, V_q) \in H^4(\Omega) \times H^2(\Omega)$.

REMARK 2.2. The term on the right-hand side in (2.12) must be positive, namely $\underline{w}^2 > \left(\frac{J^2}{T} \right)^{\frac{1}{\gamma+1}}$. Such a condition is verified as a consequence of (2.10). Moreover we observe that (2.12) implies (2.10), but we prefer to state both for clarity.

REMARK 2.3. Conditions (2.10) and (2.12) essentially mean that the quantity $\dfrac{(Q_q'(x))^2}{Q_q(x)}$ is bounded and this prevents that real step functions can be assumed as quantum functions, although a Heaviside function could be a reasonable choice to model, from a mathematical point of view, the interfaces between the classical and the quantum regions. Anyway we observe that, from a physical point of view, the doping process in heterojunctions does not allow a sharp jump (in the mathematical meaning of the term) and a transient region certainly exists even if it is not easy to evaluate its scale.

We assume that (2.12) holds also when $q \to 0$.

THEOREM 2.2 (Uniqueness for the modified H-Q_qHD problem). *Assume* (2.10), (2.11) *and* (2.12). *Then there exists* J *and* ε *sufficiently small and* γ *sufficiently close to* 1, *such that the boundary value problem* (2.9) *admits a unique solution.*

We introduce now the space:

$$W(\Omega)=\{w \in H^1(\Omega) : \sqrt{Q}w_{xx} \in L^2(\Omega),\ w(0) = w_0,\ w(1) = w_1,\ w_x(0) = w_x(1) = 0\}.$$

As we mentioned before, when $Q(x) = 0$ in some part of the domain Ω, the H-QHD system (2.8) becomes regionally degenerate in the fourth-order ellipticity, then it cannot possess smooth solutions. Therefore, we introduce the definition of its weak solution as follows.

DEFINITION 2.1. $(w, V)(x) \in W(\Omega) \times L^2(\Omega)$ *is said to be a weak solution of* (2.8), *if it holds that*

$$\begin{aligned} &2\varepsilon^2 \int_0^1 \left(Q(x)\frac{w_{xx}}{w} + Q'(x)\frac{w_x}{w}\right)\phi_{xx}dx + 2\int_0^1 \left(\left(Tw^{2(\gamma-1)} - \frac{J^2}{w^4}\right)\frac{w_x}{w}\right)\phi_x dx \\ &\quad + \int_0^1 \frac{1}{\lambda^2}(w^2 - C(x))\phi dx + \int_0^1 \frac{J}{\tau w^2}\phi_x dx = 0, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \int_0^1 V\phi\, dx = &- 2\varepsilon^2 \int_0^1 Q(x)\frac{w_{xx}}{w}\phi\, dx - 2\varepsilon^2 \int_0^1 Q'(x)\frac{w_x}{w}\phi\, dx \\ &+ \int_0^1 \frac{J^2}{2w^4}\phi\, dx + \frac{T}{\gamma-1}\int_0^1 w^{2(\gamma-1)}\phi\, dx \\ &+ \frac{J}{\tau}\int_0^1 \left(\int_0^x \frac{1}{w^2(s)}\, ds\right)\phi\, dx, \end{aligned} \tag{2.15}$$

for any $\phi \in C_0^\infty(\Omega)$.

REMARK 2.4. To our purpose it is enough to ask $w \in H^1(\Omega) \cap \{\sqrt{Q}w_{xx} \in L^2(\Omega)\}$, in order to obtain Theorem 2.3, as proved in Section 4.

We recall the approximation assumptions concerning the hybrid quantum effect function $0 \le Q(x) \le 1$: Let $\{Q_q(x)\}$ be a q-dependent sequence satisfying, for all $q \in \mathbb{R}_+$,

$$\begin{cases} Q_q \to Q, \quad Q'_q \to Q' \quad \text{uniformly in} \quad \Omega, \quad \text{for} \quad q \to 0, \\ \|Q'_q\|_{L^2} \le \bar{K}, \text{ uniformly in } q, \\ \varepsilon^2 \max\limits_{x\in\Omega} \dfrac{|Q'_q(x)|^2}{Q_q(x)} < \dfrac{1}{6}\left(T\underline{n}^{(\gamma-1)} - \dfrac{J^2}{\underline{n}^2}\right), \end{cases} \tag{2.16}$$

for all $x \in \Omega$, where $\underline{n} > n_\star$.

Now we present the existence result for the solutions to the weak hybrid quantum hydrodynamic Equation (2.14).

THEOREM 2.3 (Hybrid limits and existence of H-QHD solution). *Let* $\{Q_q(x)\}$ *be an approximation satisfying* (2.16), *and* $(w_q, V_q)(x)$ *be the solution to the system* (2.9). *Let* $Q \in C^1(\Omega)$ *be the quantum effect function limit of the sequence* $\{Q_q(x)\}$. *Assume* *the subsonic conditions* (2.6) *and* (2.7)*. Then there exists a pair of functions* $(w, V)(x)$ *such that the following convergence results can be established*

$$\begin{cases} w_q \rightharpoonup w & \text{in } H^1(\Omega), \\ w_q \to w & \text{in } C^0(\Omega), \\ \sqrt{Q_q} w_{q,xx} \rightharpoonup \sqrt{Q} w_{xx} & \text{in } L^2(\Omega), \\ V_q \rightharpoonup V & \text{in } L^2(\Omega), \end{cases} \quad \text{as } q \to 0. \tag{2.17}$$

Moreover, when $Q(x)$ *satisfies* (1.2)*, then the pair* $(w, V)(x)$ *is the weak solution to the H-QHD system* (2.8)*.*

REMARK 2.5. In the case of *classic boundary*, opposite to (1.2), when $Q(x)$ satisfies:

$$Q(x) = 0 \text{ for } x \in [0, x_1] \cup [x_2, 1], \quad \text{and } Q(x) > 0 \text{ for } [x_1, x_2],$$

we cannot establish the results stated by Theorem 2.3, since the weak limit of w_q as $q \to 0$ cannot be proved as the weak solution to the H-QHD system (2.8) due to some technical difficulty in verifying the boundary conditions $w_x(0) = w_x(1) = 0$. This problem is also highlighted by the oscillations that appear at the extrema of Ω, in the case of classic boundary, as can be seen in the simulations (see Figures 6.2 and 6.1). The origin of these oscillations is not clear and it should be further investigated in the future, although they could also arise from the intrinsic instability of the problem under consideration.

In Section 5 we discuss the zero-space-charge for both the regularized and fully hybrid equations. Here we just mention, as the main result of the section, the result concerning the zero-space-charge limits for the hybrid case, that is

THEOREM 2.4. *Assume* (2.12) *and let* $Q(x) \in C^2(\Omega)$ *such that* $0 \leq Q(x) \leq 1$*. Let* $C(x) \geq \underline{n}$ *be a given function which verifies*

$$C(0) = n_0,\ C(1) = n_1,\ C_x(0) = C_x(1) = 0. \tag{2.18}$$

If (w_λ, V_λ) *is the solution to the problem* (2.8) *then, for* $\lambda \to 0$*, one has*

$$\begin{aligned} w_\lambda(x) \rightharpoonup w := \sqrt{C(x)} \quad & \text{in} \quad H^1(\Omega) \\ w_\lambda(x) \to w := \sqrt{C(x)} \quad & \text{in} \quad C^0(\bar{\Omega}) \\ V_\lambda(x) \rightharpoonup \tilde{V}(x) \quad & \text{in} \quad L^2(\Omega), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} \tilde{V}(x) = & -2\varepsilon^2 \left(Q(x) \frac{\sqrt{C}_{xx}}{\sqrt{C}} + Q'(x) \frac{\sqrt{C}_x}{\sqrt{C}} \right) \\ & + \frac{J^2}{2C^2} + \frac{T}{\gamma - 1} C^{\gamma - 1} + \frac{J}{\tau} \int_0^x \frac{ds}{C(s)}. \end{aligned} \tag{2.20}$$

3. Existence and uniqueness of H-Q$_q$HD solution with a general pressure functional

Following the same approach proposed in [11], we first construct an approximating sequence of solutions (w_q, V_q) to the quantum hydrodynamic model H$_q$QHD (2.9).

Now the working space is:

$$\tilde{W} = \{ w \in H^2 \,:\, w(0) = w_0,\ w(1) = w_1,\ w_x(0) = w_x(1) = 0\},$$

since $Q_q > q > 0$.

One of the main problems usually related to the solution of the quantum systems is to prove that the charge density n is strictly positive. In order to overcome this difficulty in [15] and in [18] the authors introduce a truncated problem in terms of a new variable $u = \ln n$. Here we adopt a different approach, namely, we first prove that the solution is strictly positive then we get the a priori estimates.

LEMMA 3.1 (Strict positivity of the solution to (2.9)). *Assume that $Q_q(x)$ satisfies (2.11) and (2.12) and that the subsonic conditions (2.10), (2.6), (2.7) hold. If $w_q \in H^2(\Omega)$ is the solution to the problem (2.9), then*

$$w_q \geq \sqrt{\underline{n}} = \underline{\omega} \qquad \textit{for all} \quad x \in \Omega \tag{3.1}$$

where $\underline{n} = \min\{n_0, n_1, C_0\}$.

Proof. Let $(w_q - \sqrt{\underline{n}})^- := \min(0, w_q - \sqrt{\underline{n}})$. Since $w_q(0) = \sqrt{n_0} > \sqrt{\underline{n}}$ and $w_q(1) = \sqrt{n_1} > \sqrt{\underline{n}}$, then one has $(w_q - \sqrt{\underline{n}})^-|_{\partial\Omega} = 0$, and $(w_q - \sqrt{\underline{n}})^- \in H_0^1(\Omega) \cap H^2(\Omega)$. Now, let us consider the weak formulation of the problem $(2.9)_1$ by using the test function $(w_q - \sqrt{\underline{n}})^-$, obtaining

$$\begin{aligned}
&2\varepsilon^2 \int_0^1 Q_q \frac{((w_q - \sqrt{\underline{n}})^-)^2_{xx}}{w_q}\,dx + 2\int_0^1 \left(T w_q^{2(\gamma-1)} - \frac{J^2}{w_q^4}\right) \frac{((w_q - \sqrt{\underline{n}})^-)^2_x}{w_q}\,dx \\
&\quad + 2\varepsilon^2 \int_0^1 Q_q' \frac{((w_q - \sqrt{\underline{n}})^-)_x((w_q - \sqrt{\underline{n}})^-)_{xx}}{w_q}\,dx \\
&= -\frac{1}{\lambda^2}\int_0^1 (w_q^2 - (\sqrt{\underline{n}})^2)(w_q - \sqrt{\underline{n}})^-\,dx \\
&\quad + \frac{1}{\lambda^2}\int_0^1 (C(x) - \underline{n})(w_q - \sqrt{\underline{n}})^-\,dx - \int_0^1 \frac{J}{\tau w_q^2}((w_q - \sqrt{\underline{n}})^-)_x\,dx \\
&=: L_1 + L_2 + L_3.
\end{aligned} \tag{3.2}$$

One has

$$L_1 + L_2 \leq -\frac{1}{\lambda^2}\int_0^1 ((w_q - \sqrt{\underline{n}})^-)^2(w_q + \sqrt{\underline{n}})\,dx + \frac{1}{\lambda^2}\int_0^1 (C(x) - \underline{n})(w_q - \sqrt{\underline{n}})^-\,dx. \tag{3.3}$$

Notice that Ω can be written as a disjoint union of $\Omega^\pm$ and isolated points, where $\Omega^+ = \cup_i {\Omega_i}^+$, $\Omega^- = \cup_i {\Omega_i}^-$ and

$${\Omega_i}^+ = \{x \in \Omega \text{ such that } w_q \geq \sqrt{\underline{n}}\}, \quad {\Omega_j}^- = \{x \in \Omega \text{ such that } w_q < \sqrt{\underline{n}}\},$$

then we can write

$$\begin{aligned}
L_3 &= -\int_0^1 \frac{J}{\tau w_q^2}((w_q - \sqrt{\underline{n}})^-)_x\,dx \\
&= -\sum_i \int_{{\Omega_i}^+} \frac{J}{\tau w_q^2}((w_q - \sqrt{\underline{n}})^-)_x\,dx - \sum_j \int_{{\Omega_j}^-} \frac{J}{\tau w_q^2}((w_q - \sqrt{\underline{n}})^-)_x\,dx.
\end{aligned}$$

Clearly the first sum above is zero, therefore

$$L_3 = -\sum_j \int_{\Omega_j^-} \frac{J}{\tau w_q^2}((w_q - \sqrt{\underline{n}})^-)_x \, dx.$$

Consequently, we compute L_3 on each interval Ω_j^-. Without loss of generality we just consider a single interval $\Omega_j^- = (a_j, b_j)$, which is properly contained in the open interval $(0,1)$. Since we assume $w_q \in H^2(0,1)$, then w_q is a continuous function in $[a_j, b_j]$. This implies that $w_q(a_j) = w_q(b_j) = \sqrt{\underline{n}}$, and thus

$$L_3 = -\int_{a_j}^{b_j} \frac{J}{\tau w_q^2}(w_q)_x \, dx = \frac{J}{\tau w_q(b_j)} - \frac{J}{\tau w_q(a_j)} = 0. \tag{3.4}$$

Now, in view of (2.10) and (2.11), we show that the first three terms of the left-hand side of (3.2) can be seen as a strictly positive quadratic form which satisfies the following inequality

$$\begin{aligned} &\int_0^1 (\mathcal{A}_0 \frac{((w_q - \sqrt{\underline{n}})^-)^2_{xx}}{w_q} + \mathcal{B}_0 \frac{((w_q - \sqrt{\underline{n}})^-)_x((w_q - \sqrt{\underline{n}})^-)_{xx}}{w_q} \\ &\quad + \mathcal{C}_0 \frac{((w_q - \sqrt{\underline{n}})^-)^2_x}{w_q}) dx \\ &\geq c_1 \int_0^1 \frac{((w_q - \sqrt{\underline{n}})^-)^2_{xx}}{w_q} dx + c_2 \int_0^1 \frac{((w_q - \sqrt{\underline{n}})^-)^2_x}{w_q} dx \end{aligned} \tag{3.5}$$

provided $\mathcal{B}_0^2 - 4\mathcal{A}_0\mathcal{C}_0 < 0$, where
$\mathcal{A}_0 = 2\varepsilon^2 Q_q(x)$, $\mathcal{B}_0 = 2\varepsilon^2 Q'_q(x)$, $\mathcal{C}_0 = 2\left(T w_q^{2(\gamma-1)} - \frac{J^2}{w_q^4}\right)$
and c_1 and c_2 are positive constants ($\mathcal{C}_0 > 0$ by (2.10)).

Therefore the quadratic form above is strictly positive if

$$\mathcal{B}_0^2 - 4\mathcal{A}_0\mathcal{C}_0 = 4\varepsilon^2 \left[\varepsilon^2 (Q'_q(x))^2 - 4Q_q(x)\left(T w^{2(\gamma-1)} - \frac{J^2}{w_q^4}\right)\right] < 0,$$

that is given by (2.12). Then, considering (3.3), in view of (3.4), we get

$$\begin{aligned} &c_1 \int_0^1 \frac{((w_q - \sqrt{\underline{n}})^-)^2_{xx}}{w_q} dx + c_2 \int_0^1 \frac{((w_q - \sqrt{\underline{n}})^-)^2_x}{w_q} dx \\ &\quad + c_3 \int_0^1 ((w_q - \sqrt{\underline{n}})^-)^2 (w_q + \sqrt{\underline{n}}) \, dx \\ &\leq \frac{1}{\lambda^2} \int_0^1 (C(x) - \underline{n})(w_q - \sqrt{\underline{n}})^- \, dx, \end{aligned} \tag{3.6}$$

which implies $(w_q - \sqrt{\underline{n}})^- = 0$ for all x in $[0,1]$, namely, $w_q \geq \sqrt{\underline{n}} > 0$ for all $x \in [0,1]$, that is (3.1). ∎

REMARK 3.1. Now we can say that (2.12) holds for any $w_q > \underline{w}$.

LEMMA 3.2 (A priori estimates). *Let $w_q \in H^2(\Omega)$ be the solution to the problem (2.9). Assume the subsonic conditions (2.10), (2.6) and (2.7), and that $Q_q(x)$ satisfies (2.11) and (2.12). Then*

$$\|w_q\|_{L^\infty(\Omega)} \leq w_M, \tag{3.7}$$

and

$$\underline{c_1}\int_0^1 (w_q)_{xx}^2\,dx + \underline{c_2}\int_0^1 (w_q)_x^2\,dx \le K, \tag{3.8}$$

where $w_M \ge \sqrt{\underline{n}}$, $\underline{c}_1$ depends on q, and $\underline{c}_2$, $K>0$ are $q-$independent constants.

Proof. Let $w_D \in C^\infty$ be a strictly positive function such that $w_D > \sqrt{\underline{n}}$, which verifies the boundary conditions of (2.9), and multiply $(2.9)_1$ by $(w_q - w_D) \in (H^2 \cap H_0^1)(\Omega)$. After integration on the whole domain, we have

$$\begin{aligned}
&2\varepsilon^2\int_0^1 Q_q\frac{(w_q)_{xx}^2}{w_q}\,dx + 2\int_0^1\left(Tw_q^{2(\gamma-1)} - \frac{J^2}{w_q^4}\right)\frac{(w_q)_x^2}{w_q}\,dx\\
&\quad + 2\varepsilon^2\int_0^1 Q_q'\frac{(w_q)_x(w_q)_{xx}}{w_q}\,dx\\
=&2\varepsilon^2\int_0^1 Q_q\frac{(w_D)_{xx}(w_q)_{xx}}{w_q}\,dx + 2\int_0^1\left(Tw_q^{2(\gamma-1)} - \frac{J^2}{w_q^4}\right)\frac{(w_q)_x(w_D)_x}{w_q}\,dx\\
&\quad + 2\varepsilon^2\int_0^1 Q_q'\frac{(w_q)_x(w_D)_{xx}}{w_q}\,dx\\
&\quad - \frac{1}{\lambda^2}\int_0^1 (w_q^2 - w_D^2)(w_q - w_D)\,dx\\
&\quad + \frac{1}{\lambda^2}\int_0^1 (C(x) - w_D^2)(w_q - w_D)\,dx + \int_0^1\left(\frac{J}{\tau w_q^2}\right)_x (w_q - w_D)\,dx\\
=:&I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{3.9}$$

We start estimating the first three terms on the right-hand side, using the Young's inequality and recalling that $0 < Q_q \le 1$ and $0 < \varepsilon \ll 1$

$$\begin{aligned}
I_1 + I_2 + I_3 \le&\varepsilon^2\int_0^1 Q_q^2\frac{(w_q)_{xx}^2}{w_q}\,dx + \varepsilon^2\int_0^1\frac{(w_D)_{xx}^2}{w_q}\,dx\\
&+ \int_0^1\left(Tw_q^{2(\gamma-1)} - \frac{J^2}{w_q^4}\right)\frac{(w_q)_x^2}{w_q}\,dx\\
&+ \int_0^1\left(Tw_q^{2(\gamma-1)} - \frac{J^2}{w_q^4}\right)\frac{(w_D)_x^2}{w_q}\,dx\\
&+ \varepsilon^2\int_0^1 (Q'_q)^2\frac{(w_{qx})^2}{w_q}\,dx + \varepsilon^2\int_0^1\frac{(w_{Dxx})^2}{w_q}\,dx\\
\le&\varepsilon^2\int_0^1 Q_q\frac{(w_q)_{xx}^2}{w_q}\,dx + \int_0^1\left(Tw_q^{2(\gamma-1)} - \frac{J^2}{w_q^4}\right)\frac{(w_q)_x^2}{w_q}\,dx\\
&+ \varepsilon^2\int_0^1 (Q'_q)^2\frac{w_{qx}^2}{w_q}\,dx\\
&+ \int_0^1\left(Tw_q^{2(\gamma-1)} - \frac{J^2}{w_q^4}\right)\frac{(w_D)_x^2}{w_q}\,dx + \int_0^1\frac{(w_D)_{xx}^2}{w_q}\,dx,\\
\le&\varepsilon^2\int_0^1 Q_q\frac{(w_q)_{xx}^2}{w_q}\,dx + \int_0^1\left(Tw_q^{2(\gamma-1)} - \frac{J^2}{w_q^4} + \varepsilon^2(Q'_q)^2\right)\frac{(w_q)_x^2}{w_q}\,dx\\
&+ \frac{\|w_{Dx}\|_\infty^2}{\sqrt{\underline{n}}}\int_0^1 Tw_q^{2|\gamma-1|}\,dx + \frac{\|w_{Dxx}\|_\infty^2}{\sqrt{\underline{n}}}.
\end{aligned} \tag{3.10}$$

Simple calculations, in view of the Cauchy inequality, give the following estimate for I_4 and I_5

$$
\begin{aligned}
I_4 + I_5 \leq &- \frac{1}{\lambda^2}\int_0^1 (w_q - w_D)^2 (w_q + w_D)\, dx + \frac{1}{2\lambda^2\sqrt{\underline{n}}}\int_0^1 (C(x) - w_D^2)^2\, dx \\
&+ \frac{1}{2\lambda^2}\sqrt{\underline{n}}\int_0^1 (w_q - w_D)^2\, dx \\
\leq &- \frac{1}{\lambda^2}\int_0^1 (w_q - w_D)^2 \left(w_q + w_D - \frac{\sqrt{\underline{n}}}{2}\right) dx \\
&+ \frac{1}{2\lambda^2\sqrt{\underline{n}}}\int_0^1 (C(x) - w_D^2)^2\, dx. \qquad (3.11)
\end{aligned}
$$

Concerning the term I_6, we have

$$
\begin{aligned}
I_6 &= - \int_0^1 \frac{J}{\tau w_q^2}(w_q - w_D)_x\, dx \\
&= - \int_0^1 \frac{J}{\tau w_q^2}(w_q)_x\, dx + \int_0^1 \frac{J}{\tau w_q^2} w_{Dx}\, dx \\
&\leq \frac{J}{\tau}\left|\frac{1}{w_1} - \frac{1}{w_0}\right| + J\frac{\|w_{Dx}\|_\infty}{\tau \underline{n}}. \qquad (3.12)
\end{aligned}
$$

From (3.9) and the previous estimates, we finally obtain

$$
\begin{aligned}
&\varepsilon^2 \int_0^1 Q_q \frac{(w_q)_{xx}^2}{w_q}\, dx + 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_q)_x (w_q)_{xx}}{w_q}\, dx \\
&+ \int_0^1 \left(T w_q^{2(\gamma-1)} - \frac{J^2}{w_q^4} - \varepsilon^2 (Q_q')^2\right) \frac{(w_q)_x^2}{w_q}\, dx \\
&+ \frac{1}{\lambda^2}\int_0^1 (w_q - w_D)^2 \left(w_q + w_D - \frac{\sqrt{\underline{n}}}{2}\right) dx \\
\leq &\frac{1}{2\lambda^2\sqrt{\underline{n}}}\int_0^1 (C(x) - w_D^2)^2\, dx \\
&+ \frac{\|w_{Dx}\|_\infty^2}{\sqrt{\underline{n}}} T w_M^{2|\gamma-1|} + \frac{\|w_{Dxx}\|_\infty^2}{\sqrt{\underline{n}}} + \frac{J}{\tau}\left|\frac{1}{w_1} - \frac{1}{w_0}\right| + J\frac{\|w_{Dx}\|_\infty}{\tau \underline{n}}, \qquad (3.13)
\end{aligned}
$$

where the fourth term on the left-hand side is strictly positive since $w_D, w_q > \sqrt{\underline{n}}$.

We close the proof showing that the first three terms of the left-hand side in (3.13) form a strictly positive quadratic form. This will imply that

$$
\int_0^1 \left(\mathcal{A}_1 \frac{(w_q)_{xx}^2}{w_q} + \mathcal{B}_1 \frac{(w_q)_x (w_q)_{xx}}{w_q} + \mathcal{C}_1 \frac{(w_q)_x^2}{w_q}\right) dx \geq c_1 \int_0^1 \frac{(w_q)_{xx}^2}{w_q} dx + c_2 \int_0^1 \frac{(w_q)_x^2}{w_q} dx \qquad (3.14)
$$

where

$$
\mathcal{A}_1 = \varepsilon^2 Q_q,\ \mathcal{B}_1 = 2\varepsilon^2 Q_q',\ \mathcal{C}_1 = \left(T w_q^{2(\gamma-1)} - \frac{J^2}{w_q^4} - \varepsilon^2 (Q_q')^2\right)
$$

and c_1 and c_2 are positive constants. The quadratic form above is strictly positive when $\mathcal{B}_1^2 - 4\mathcal{A}_1\mathcal{C}_1 < 0$, that is

$$\begin{aligned}&4\varepsilon^2\Big[\varepsilon^2(Q_q'(x))^2 - Q_q(x)\left(Tw_q^{2(\gamma-1)} - \frac{J^2}{w_q^4} - \varepsilon^2(Q_q'(x))^2\right)\Big]\\ = \;& 4\varepsilon^2 Q_q\Big[\varepsilon^2\frac{(Q_q'(x))^2}{Q_q}(1+Q_q) - \left(Tw_q^{2(\gamma-1)} - \frac{J^2}{w_q^4}\right)\Big] < 0\end{aligned} \tag{3.15}$$

that follows from (2.10) and (2.12). Now, to establish the uniform upper bound w_M for w_q, we apply the Implicit Function Theorem. Combining (3.14) with (3.13), we get

$$\begin{aligned}&c_1\int_0^1 \frac{(w_q)_{xx}^2}{w_q}\,dx + c_2\int_0^1\frac{(w_q)_x^2}{w_q}\,dx\\ \le&\frac{1}{2\lambda^2\sqrt{\underline{n}}}\int_0^1 (C(x)-w_D^2)^2dx + \frac{T}{\sqrt{\underline{n}}}w_M^{2|\gamma-1|}\|w_{Dx}\|_\infty^2\\ &+\frac{1}{\sqrt{\underline{n}}}\|w_{Dxx}\|_\infty^2 + \frac{J}{\tau}\left|\frac{1}{w_1}-\frac{1}{w_0}\right| + J\frac{\|w_{Dx}\|_\infty}{\tau\underline{n}}\\ =:&\bar{K}_0(\gamma, w_D, w_M, \underline{n}, w_0, w_1, C, \tau) =: K_0.\end{aligned} \tag{3.16}$$

From (3.16), since $\dfrac{(w_q)_x^2}{w_q} = 4[(\sqrt{w_q})_x]^2$, we get

$$\underline{c_2}\int_0^1 [(\sqrt{w_q})_x]^2\,dx \le K_0 \tag{3.17}$$

and (3.17) easily implies $\|\sqrt{w_q}\|_\infty \le K_1 + \sqrt{w_0}$, where $K_1 = \sqrt{\frac{K_0}{\underline{c_2}}}$.

As in [18], we can prove that the following equation in $(\gamma, \sqrt{w_M})$

$$\begin{aligned}&\sqrt{w_M}\\ =&\sqrt{\frac{\frac{1}{2\lambda^2\sqrt{\underline{n}}}\int_0^1[C(x)-w_D^2]^2dx + \frac{T}{\underline{w}}\sqrt{w_M}^{4|\gamma-1|}\|w_{Dx}\|_\infty^2 + \frac{1}{\underline{w}}\|w_{Dxx}\|_\infty^2 + \frac{J}{\tau}\left|\frac{1}{w_1}-\frac{1}{w_0}\right| + J\frac{\|w_{Dx}\|_\infty}{\tau\underline{w}^2}}{\underline{c_2}}}\\ &+\sqrt{w_0} \ge \sqrt{\underline{w}}\end{aligned} \tag{3.18}$$

has a solution

$$\begin{aligned}&(\gamma, \sqrt{w_M})\\ =&\left(1, \sqrt{\frac{\frac{1}{2\lambda^2}\int_0^1[C(x)-w_D^2]^2dx + \frac{T}{\underline{w}}\|w_{Dx}\|_\infty^2 + \frac{1}{\underline{w}}\|w_{Dxx}\|_\infty^2 + \frac{J}{\tau}\left|\frac{1}{w_1}-\frac{1}{w_0}\right| + J\frac{\|w_{Dx}\|_\infty}{\tau\underline{w}^2}}{\underline{c_2}}} + \sqrt{w_0}\right).\end{aligned} \tag{3.19}$$

Therefore, by the Implicit Function Theorem, there exists a $\gamma_0 > 0$ such that for $|\gamma - 1| < \gamma_0$, the Equation (3.18) admits a solution $(\gamma, \sqrt{w_M})$. This defines the bound w_M. Then (3.8) follows from (3.16), where $\underline{c_1}$ and $\underline{c_2}$ are positive constants, in view of (3.14). ∎

Lemma 3.3. *Under the assumption of Lemma 3.2, the variable u_q, defined as $u_q = 2\ln w_q$, verifies the following estimate*

$$\underline{c_3}\|(u_q)_{xx}\|_{L^2(\Omega)} + \underline{c_4}\|(u_q)_x\|_{L^2(\Omega)} \le K_0, \tag{3.20}$$

where $\underline{c_3}$ and $\underline{c_4}$ are strictly positive constants.

Proof. First of all we observe that u_q is bounded from below and from above:

$$2\ln\sqrt{\underline{n}} = u_m \leq u_q \leq u_M = 2\ln w_M.$$

Moreover, from the results listed in the previous lemma, it follows that

$$\|u_{q_x}\|^2 = 4\left\|\frac{w_{q_x}}{w_q}\right\|^2 \leq K \tag{3.21}$$

$$\|u_{q_{xx}}\|^2 \leq 8\left(\left\|\frac{w_{q_{xx}}}{w_q}\right\|^2 + \left\|\frac{w_{q_x}^2}{w_q^2}\right\|^2\right) \leq K \tag{3.22}$$

where K is a positive constant which does not depend on q. ∎

In order to apply the standard theory, we write $(2.9)_1$ in the new variable $u_q = 2\ln w_q$

$$\begin{aligned}\varepsilon^2\left(Q_q\left((u_q)_{xx} + \frac{(u_q)_x^2}{2}\right) + Q_q'(u_q)_x\right)_{xx} + (J^2 e^{-2u_q}(u_q)_x)_x \\ - \frac{T}{\gamma-1}(e^{u_q(\gamma-1)})_{xx} + \frac{e^{u_q} - C(x)}{\lambda^2} - \left(\frac{J}{\tau}e^{-u_q}\right)_x = 0.\end{aligned} \tag{3.23}$$

Equation (3.23) is coupled with the following conditions

$$u_q(0) = \ln n_0 = u_0, \quad u_q(1) = \ln n_1 = u_1, \quad (u_q)_x(0) = (u_q)_x(1) = 0. \tag{3.24}$$

Then the existence results for the H-Q_qHD system read as follows:

THEOREM 3.1 (Existence of H-Q_qHD solutions). *Assume inequality (2.10), then there exists at least one weak solution $u_q \in H^2(\Omega)$ to the boundary value problem (3.23)-(3.24).*

Proof. Here we just summarize the results of [15] and [18] and adapt them to our problem. In fact, since $Q_q(x) \geq q > 0$, Equation (3.23) is basically a QHD model and standard methods can be applied also in our case. We linearize Equation (3.23)

$$\begin{aligned}\varepsilon^2\left(Q_q\left((u_q)_{xx} + \frac{\sigma}{2}\nu_x^2\right) + Q_q'(u_q)_x\right)_{xx} + \sigma J^2\left(e^{-2\nu}\nu_x\right)_x \\ - T(e^{(\gamma-1)\nu}u_{q_x})_x + \frac{\sigma}{\lambda^2}\left(\frac{e^\nu - 1}{\nu}u_q + 1 - C\right) - \sigma\frac{J}{\tau}(e^{-\nu})_x = 0,\end{aligned} \tag{3.25}$$

where $\nu \in X = C^{0,1}(\Omega)$. The relative boundary conditions are

$$u_q(0) = \sigma\ln n_0 = \sigma u_0, \quad u_q(1) = \sigma\ln n_1 = \sigma u_1, \quad (u_q)_x(0) = (u_q)_x(1) = 0. \tag{3.26}$$

Obviously the solution of Equation (3.25) satisfies (3.26) for a given $\sigma \in [0,1]$. Let

$$\begin{aligned}a(u_q,\phi) = &\int_0^1 \left(\varepsilon^2\left(Q_q(u_q)_{xx} + Q_q'(u_q)_x\right)\phi_{xx} + T(e^{(\gamma-1)\nu}u_{q_x})\phi_x\right) dx \\ &+ \int_0^1 \frac{\sigma}{\lambda^2}\frac{e^\nu - 1}{\nu}u_q\phi\, dx.\end{aligned}$$

It is easy to check that $a(u_q, \phi)$ is a continuous and coercive bilinear form in $H^2(\Omega)$, for $\phi \in H^2(\Omega)$ and for each u_q. Moreover, the functional F, defined as

$$
\begin{aligned}
F(\phi) = &\int_0^1 \left(-Q_q \frac{\varepsilon^2\sigma}{2} \nu_x^2 \phi_{xx} + \sigma J^2 e^{-2\nu_k} \nu_x \phi_x - \frac{\sigma}{\lambda^2}(1-C)\phi \right) dx \\
&- \int_0^1 \left(\sigma \frac{J}{\tau} e^{-\nu} \phi_x \right) dx
\end{aligned}
$$

is linear and continuous in $H^2(\Omega)$ for $\phi \in H^2(\Omega)$. Therefore, for Lax Milgram Lemma the boundary value problem (3.25)-(3.26) admits a unique solution $u \in H^2(\Omega)$. In this way we define a continuous and compact fixed-point operator on $X \equiv H^2$ such that

$$
S : X \times [0,1] \to X, \ \ (\nu, \sigma) \to u_q \tag{3.27}
$$

with $S(\nu, 0) = 0$ for all $\nu \in X$. Moreover, we can show that there is a constant $c > 0$ that verifies

$$
\|u\|_X \le c, \tag{3.28}
$$

for all $(u_q, \sigma) \in X \times [0,1]$ satisfying $S(u_q, \sigma) = u_q$.

Indeed, for $\sigma = 1$ the inequality (3.28) is a direct consequence of the a priori estimates discussed in the previous lemma and in a similar way we obtain (3.28) for $0 < \sigma < 1$.

Therefore we can apply the Leray-Schauder fixed-point theorem to get the existence of a fixed point u_q. ∎

In the following theorem, using a standard approach, we prove the uniqueness of subsonic solution to (2.9), assuming J and ε small enough and γ close to 1.

THEOREM 3.2 (Uniqueness of H-Q$_q$HD solutions). *Under the assumptions* (2.10), (2.11) *and* (2.12) *there exist* $J > 0$ *and* $\varepsilon > 0$ *sufficiently small and* γ *sufficiently close to* 1, *such that the boundary value problem* (3.23)-(3.24) *admits unique solution.*

Proof. As usual, we will prove the theorem by contradiction, following basically the approach proposed in [15] and [18]. Let $u_q, v_q \in H^2(\Omega)$ be two solutions to (3.23)-(3.24). Now we consider the difference of the equations satisfied by u_q and v_q respectively

$$
\begin{aligned}
&\varepsilon^2 \left(Q_q (u_q - v_q)_{xx}\right)_{xx} + \varepsilon^2 \left(Q_q \left(\frac{(u_q)_x^2}{2} - \frac{(v_q)_x^2}{2} \right) \right)_{xx} + \varepsilon^2 (Q_q'(u_q - v_q)_x)_{xx} \\
&- \frac{J^2}{2} (e^{-2u_q} - e^{-2v_q})_{xx} - \frac{T}{(\gamma - 1)} (e^{u_q(\gamma-1)} - e^{v_q(\gamma-1)})_{xx} \\
&+ \frac{e^{u_q} - e^{v_q}}{\lambda^2} - \frac{J}{\tau} \left(e^{-u_q} - e^{-v_q}\right)_x = 0
\end{aligned} \tag{3.29}
$$

coupled with the following boundary conditions

$$
(u_q - v_q)(0) = (u_q - v_q)(1) = 0, \qquad (u_q - v_q)_x(0) = (u_q - v_q)_x(1) = 0. \tag{3.30}
$$

We multiply (3.29) by $(u_q - v_q) \in H_0^1(\Omega) \cap H^2(\Omega)$ and integrate it by parts on the whole domain

$$
\varepsilon^2 \int_0^1 Q_q(x)(u_q - v_q)_{xx}^2 \, dx + \frac{\varepsilon^2}{2} \int_0^1 Q_q(x)(u_q + v_q)_x (u_q - v_q)_x (u_q - v_q)_{xx} \, dx
$$

$$
\begin{aligned}
&-\frac{\varepsilon^2}{2}\int_0^1 Q_q''(x)(u_q-v_q)_x^2\,dx + T\int_0^1 e^{u_q(\gamma-1)}(u_q-v_q)_x^2\,dx\\
&+\frac{1}{\lambda^2}\int_0^1 (e^{u_q}-e^{v_q})(u_q-v_q)\,dx\\
=&-T\int_0^1 (e^{u_q(\gamma-1)}-e^{v_q(\gamma-1)})v_{q_x}(u_q-v_q)_x\,dx\\
&+J^2\int_0^1 e^{-2u_q}(u_q-v_q)_x^2\,dx + J^2\int_0^1 (e^{-2u_q}-e^{-2v_q})v_{q_x}(u_q-v_q)_x\,dx\\
&-\frac{J}{\tau}\int_0^1 (e^{-u_q}-e^{-v_q})(u_q-v_q)_x\,dx. \qquad (3.31)
\end{aligned}
$$

We observe that the term $\frac{1}{\lambda^2}\int_0^1 (e^{u_q}-e^{v_q})(u_q-v_q)\,dx$ is positive and, concerning the second term on the left-hand side, thanks to the apriori estimates (3.21), and recalling that $(u_q)_x(0)=(v_q)_x(0)=0$, we can find a constant K_2 such that $\|(v_q)_x\|_\infty$, $\|(u_q)_x\|_\infty \le K_2$, then we can write

$$
(u_q+v_q)_x^2 \le 2(\|(u_q)_x\|_\infty^2 + \|(v_q)_x\|_\infty^2) \le 4K_2^2,
$$

obtaining

$$
\begin{aligned}
&\frac{\varepsilon^2}{2}\int_0^1 Q_q(x)(u_q+v_q)_x(u_q-v_q)_x(u_q-v_q)_{xx}\,dx\\
\le&\frac{\varepsilon^2}{4}\int_0^1 Q_q(x)(u_q-v_q)_{xx}^2\,dx + K_2^2\varepsilon^2\int_0^1 (u_q-v_q)_x^2\,dx.
\end{aligned}
$$

Meanwhile, we estimate from below some terms on the left-hand side, and use Poincaré inequality on the right-hand side

$$
\begin{aligned}
&\frac{3\varepsilon^2}{4}\int_0^1 Q_q(x)(u_q-v_q)_{xx}^2\,dx - \varepsilon^2K_2^2\int_0^1 (u_q-v_q)_x^2\,dx\\
&-\frac{\alpha\varepsilon^2}{2}\int_0^1 (u_q-v_q)_x^2\,dx + Te^{-K_1(\gamma-1)}\int_0^1 (u_q-v_q)_x^2\,dx\\
\le&T(\gamma-1)e^{K_1(\gamma-1)}\|(v_q)_x\|_2\int_0^1 (u_q-v_q)_x^2\,dx + J^2e^{2K_1}\int_0^1 (u_q-v_q)_x^2\,dx\\
&+2J^2e^{2K_1}\|(v_q)_x\|_2\int_0^1 (u_q-v_q)_x^2\,dx + \frac{J}{\tau}e^{K_1}c_p\int_0^1 (u_q-v_q)_x^2\,dx. \qquad (3.32)
\end{aligned}
$$

Here c_p is the Poincaré constant, α has been defined in (2.11) and K_1 is the bound such that $\|v_q\|_\infty$, $\|u_q\|_\infty \le K_1$. From the apriori estimates we can find a constant K_3 such that $\|(v_q)_x\|_2$, $\|(u_q)_x\|_2 \le K_3$. Then we get

$$
\begin{aligned}
&\frac{3\varepsilon^2}{4}q\int_0^1 (u_q-v_q)_{xx}^2\,dx + \Big(Te^{-(\gamma-1)K_1} - \frac{\alpha\varepsilon^2}{2} - \varepsilon^2K_2^2 - K_3T(\gamma-1)e^{K_1(\gamma-1)} - J^2e^{2K_1}\\
&-2J^2e^{2K_1}K_3 - \frac{J}{\tau}e^{K_1}c_p\Big)\int_0^1 (u_q-v_q)_x^2\,dx \le 0. \qquad (3.33)
\end{aligned}
$$

This guarantees the uniqueness under the hypotheses of smallness of J, ε and $(\gamma-1)$ and concludes the proof. □

Proof. **(Proof of Theorem 2.1.)** Following [15] and using the regularity of the function $Q_q(x)$, it is not difficult to show that there exists a solution $u_q \in H^4(\Omega)$ to (3.23)-(3.24). Consequently, observing that $w_m^2 \le n_q = e^{u_q} \le w_M^2$, the boundary value problem (2.9) admits a unique solution $w_q \in H^4(\Omega)$.

Finally, $V_q(x) \in H^2(\Omega)$, thanks to the Poisson equation. This concludes the proof. ∎

Proof of Theorem 2.2. Theorem 3.2 immediately implies Theorem 2.2.

4. Hybrid limit

Finally, we consider the hybrid case represented by the system (2.8), corresponding to $0 \le Q(x) \le 1$. It means that the quantum effect function $Q(x)$ is equal to zero in the classical region and $Q(x) > 0$ in the quantum region.

Proof. **(Proof of Theorem 2.3.)** Let $Q(x) \in C^1[0,1]$ and $\{Q_q(x)\}$ be a suitable approximating sequence satisfying (2.16) and (w_q, V_q) be the solution to the approximating problem (2.9). Within this section, all the $q-$independent constants are indicated by $\bar{K}$ or $\bar{c}_i$. First of all we need to prove the following $q-$independent *a priori* estimates:

$$\|w_q\|_{H^1(\Omega)} \le \bar{K}, \quad \|\sqrt{Q_q}(w_q)_{xx}\|_{L^2(\Omega)} \le \bar{K}. \tag{4.1}$$

Proceding as in the proof of Lemma 3.2, we obtain the following inequality

$$\begin{aligned}
&\varepsilon^2 \int_0^1 Q_q \frac{(w_q)_{xx}^2}{w_q}\, dx + 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_q)_x (w_q)_{xx}}{w_q}\, dx \\
&+ \int_0^1 \left(T w_q^{2(\gamma-1)} - \frac{J^2}{w_q^4} - \varepsilon^2 (Q_q')^2 \right) \frac{(w_q)_x^2}{w_q}\, dx \\
&+ \frac{1}{\lambda^2} \int_0^1 (w_q - w_D)^2 \left(w_q + w_D - \frac{\sqrt{n}}{2} \right) dx \le \bar{K}.
\end{aligned} \tag{4.2}$$

Then, following [11], we rearrange the first three terms of the left-hand side in (4.2), obtaining

$$\begin{aligned}
&\int_0^1 \frac{1}{2}\varepsilon^2 Q_q \frac{{w_q}_{xx}^2}{w_q}\, dx + \int_0^1 \frac{2\varepsilon^2 Q_q'}{w_q} {w_q}_x {w_q}_{xx}\, dx \\
&\quad + \int_0^1 \left[\frac{1}{2} \left(T w_q^{2(\gamma-1)} - \frac{J^2}{w_q^4} - 2\varepsilon^2 (Q_q')^2 \right) \frac{{w_q}_x^2}{w_q} \right] dx \\
&\quad + \int_0^1 \left[\frac{1}{2}\varepsilon^2 Q_q \frac{{w_q}_{xx}^2}{w_q} + \frac{1}{2} \left(T w_q^{2\gamma-3} - \frac{J^2}{w_q^5} \right) {w_q}_x^2 \right] dx \\
&=: \int_0^1 (\mathcal{A}_2 {w_q}_{xx}^2 + \mathcal{B}_2 {w_q}_x {w_q}_{xx} + \mathcal{C}_2 {w_q}_x^2) dx \\
&\quad + \int_0^1 \left[\frac{1}{2}\varepsilon^2 Q_q \frac{{w_q}_{xx}^2}{w_q} + \frac{1}{2} \left(T w_q^{2\gamma-3} - \frac{J^2}{w_q^5} \right) {w_q}_x^2 \right] dx < \bar{K}.
\end{aligned} \tag{4.3}$$

The first integral on the right-hand side is positive since $\mathcal{B}_2^2 - 4\mathcal{A}_2\mathcal{C}_2 < 0$. Indeed, if $\mathcal{A}_2 = \frac{1}{2}\varepsilon^2 Q_q$, $\mathcal{B}_2 = 2\varepsilon^2 Q_q'$, $\mathcal{C}_2 = \frac{1}{2}\left(T w_q^{2(\gamma-1)} - \frac{J^2}{w_q^4} - 2\varepsilon^2 (Q')_q^2 \right)$, one has

$$\mathcal{B}_2^2 - 4\mathcal{A}_2\mathcal{C}_2 = 4\varepsilon^2 \left[\varepsilon^2 (Q_q'(x))^2 - \frac{1}{4} Q_q(x) \left(T w_q^{2(\gamma-1)} - \frac{J^2}{w_q^4} - 2\varepsilon^2 (Q_q'(x))^2 \right) \right]$$

$$=4\varepsilon^2\left[\varepsilon^2(Q_q'(x))^2(1+\frac{Q_q(x)}{2})-\frac{1}{4}Q_q(x)\left(Tw_q^{2(\gamma-1)}-\frac{J^2}{w_q^4}\right)\right]<0 \quad (4.4)$$

by (2.12).

Also, the second term on the right-hand side of (4.3) is positive by (2.10), then we have

$$\int_0^1\left[\frac{1}{2}\varepsilon^2 Q_q\frac{{w_q}_{xx}^2}{w_q}+\frac{1}{2}\left(Tw_q^{2\gamma-3}-\frac{J^2}{w_q^5}\right){w_q}_x^2\right]dx\le\bar{K}, \tag{4.5}$$

and from (2.10) we can find a positive constant $\bar{c_1}$ $q-$independent, such that

$$\left(Tw_q^{2(\gamma-1)}-\frac{J^2}{w_q^4}\right)>\bar{c_1}.$$

Then we obtain

$$\bar{c_1}\int_0^1[(\sqrt{w_q}-\sqrt{w_0})_x]^2\le\bar{K}, \tag{4.6}$$

and thus

$$\|w_q\|_{L^\infty(\Omega)}\le\bar{K}. \tag{4.7}$$

Using the uniform bounds for w_q and the assumption $0<Q_q(x)\le 1$, we can rewrite (4.5) as

$$\bar{c_2}\frac{\varepsilon^2}{2}\int_0^1 Q_q(x){w_q}_{xx}^2(x)dx+\bar{c_3}\int_0^1 {w_q}_x^2(x)\,dx\le\bar{K}, \tag{4.8}$$

which obviously implies (4.1), namely, w_q is uniformly bounded in $H^1(\Omega)$ and $\sqrt{Q_q}w_{qxx}$ is uniformly bounded in $L^2(\Omega)$. Therefore, there exists a $w(x)$ as the hybrid limit of the sequence w_q:

$$\begin{aligned}&w_q\rightharpoonup w \quad \text{in} \quad H^1(\Omega),\\ &\text{and by (2.16)},\\ &\sqrt{Q_q}w_{qxx}\rightharpoonup\sqrt{Q}w_{xx} \quad \text{in} \quad L^2(\Omega),\end{aligned} \tag{4.9}$$

for $q\to 0$.

Since $H^1(\Omega)\hookrightarrow C^0(\Omega)$, we further have

$$w_q\to w \quad \text{in} \quad C^0(\Omega), \quad \text{for} \quad q\to 0. \tag{4.10}$$

Particularly, when $Q(x)$ satisfies (1.2), namely, $Q(x)>0$ for $x\in[0,x_1)\cup(x_2,1]$, and $Q(x)=0$ for $x\in[x_1,x_2]$, then there exists a small number $0<\delta_*\ll 1$,

$$0<\delta_*<\min\{x_1,1-x_2\},$$

such that

$$Q_q(x),\ Q(x)\ge\frac{1}{2}Q(0), \quad \text{for } x\in[0,\delta_*],$$

and

$$Q_q(x),\ Q(x) \geq \frac{1}{2}Q(1), \quad \text{for } x \in [1-\delta_*, 1].$$

Thus, (4.9) implies that

$$w_q \rightharpoonup w \quad \text{in} \quad H^2([0,\delta_*]) \quad \text{and in } H^2([1-\delta_*, 1]).$$

This, along with the Sobolev embeddings
$H^2([0,\delta_*]) \hookrightarrow C^1([0,\delta_*])$ and $H^2([1-\delta_*,1]) \hookrightarrow C^1([1-\delta_*,1])$, guarantees

$$w_q \to w \quad \text{in} \quad C^1([0,\delta_*] \cup [1-\delta_*, 1]), \quad \text{for} \quad q \to 0.$$

Namely, we prove that the limit function $w(x)$ satisfies the following boundary conditions

$$w(0) = w_0, \quad w(1) = w_1, \quad w_x(0) = w_x(1) = 0.$$

Now we prove that w, with $Q(x)$ satisfying (1.2), is the weak solution of (2.8), that is w satisfies (2.14). Let us consider Equation (2.9). Multiplying (2.9) by ϕ, where $\phi \in C_0^\infty(\Omega)$ is any given test function, and integrating by parts, we have

$$\begin{aligned} 2\varepsilon^2 \int_0^1 \left(Q_q(x)\frac{w_{q_{xx}}}{w_q} + Q_q'(x)\frac{w_{q_x}}{w_q}\right)\phi_{xx}dx + 2\int_0^1 (Tw^{2(\gamma-1)}\frac{w_{q_x}}{w_q}\phi_x dx \\ - 2\int_0^1 \left(\frac{J^2}{{w_q}^4}\right)\frac{w_{q_x}}{w_q}\phi_x dx + \int_0^1 \frac{{w_q}^2 - C}{\lambda^2}\phi dx + \int_0^1 \left(\frac{J}{\tau {w_q}^2}\right)\phi_x dx = 0. \end{aligned} \tag{4.11}$$

Recalling (4.1) and that $w_q > \sqrt{\underline{n}} > 0$ (the subsonic condition), in view of (2.16), the weak form (4.11) converges in L^2, for $q \to 0$, to the weak form of the limit problem. Namely,

$$\begin{aligned} 2\varepsilon^2 \int_0^1 \left(Q(x)\frac{w_{xx}}{w} + Q'(x)\frac{w_x}{w}\right)\phi_{xx}dx + 2\int_0^1 Tw^{2(\gamma-1)}\frac{w_x}{w}\phi_x dx \\ - 2\int_0^1 \left(\frac{J^2}{w^4}\right)\frac{w_x}{w}\phi_x dx + \int_0^1 \frac{w^2 - C}{\lambda^2}\phi dx + \int_0^1 \left(\frac{J}{\tau w^2}\right)\phi_x dx = 0. \end{aligned} \tag{4.12}$$

Thus, we have proved that w is the weak solution of (2.8).

Now we consider the expression for the electric potential $V_q(x)$, obtained by integrating (1.1) with respect to x and using (1.4):

$$\begin{aligned} V_q(x) = &- 2\varepsilon^2 Q_q(x)\frac{w_{q_{xx}}}{w_q} - 2\varepsilon^2 Q_q'(x)\frac{w_{q_x}}{w_q} + \frac{J^2}{2{w_q}^4} + \frac{T}{\gamma-1}{w_q}^{2(\gamma-1)} \\ &+ \frac{J}{\tau}\int_0^x \frac{1}{{w_q}^2}\,dx. \end{aligned} \tag{4.13}$$

By (2.16) and the uniform estimates (4.1) and (3.1), one has that $\|V_q\|_{L^2} \leq \bar{K}$. Therefore, there exists V such that

$$V_q \rightharpoonup V \quad \text{in} \quad L^2(\Omega). \tag{4.14}$$

Now, we have to prove that the limit V is the weak solution of the hybrid problem. To this end, we multiply (4.13) by $\phi \in C_0^\infty(\Omega)$ and integrate it in Ω:

$$\begin{aligned}\int_0^1 V_q\phi\, dx = &- 2\varepsilon^2 \int_0^1 Q_q(x)\frac{w_{q_{xx}}}{w_q}\phi\, dx - 2\varepsilon^2 \int_0^1 Q_q'(x)\frac{w_{q_x}}{w_q}\phi\, dx\\ &+ \int_0^1 \frac{J^2}{2{w_q}^4}\phi\, dx + \int_0^1 \frac{T}{\gamma-1}{w_q}^{2(\gamma-1)}\phi\, dx\\ &+ \frac{J}{\tau}\int_0^1 \Big(\int_0^x \frac{1}{{w_q}^2(s)}\, ds\Big)\,\phi\, dx. \end{aligned}\tag{4.15}$$

Due to the uniform estimate in (4.1) and to the properties of $\{Q_q\}$, for $q \to 0$, we have

$$\begin{aligned}\int_0^1 V\phi\, dx = &- 2\varepsilon^2 \int_0^1 Q(x)\frac{w_{xx}}{w}\phi\, dx - 2\varepsilon^2 \int_0^1 Q'(x)\frac{w_x}{w}\phi\, dx\\ &+ \int_0^1 \frac{J^2}{2w^4}\phi\, dx + \int_0^1 \frac{T}{\gamma-1}w^{2(\gamma-1)}\phi\, dx\\ &+ \frac{J}{\tau}\int_0^1 \Big(\int_0^x \frac{1}{w^2(s)}\, ds\Big)\,\phi\, dx. \end{aligned}\tag{4.16}$$

Thus, we prove $V_q \rightharpoonup V$ in L^2 and the limit potential V verifies the Poisson equation in the weak sense. From (4.14) and $n_q = w_q^2$, we obtain (2.17). The proof of Theorem 2.3 is complete. ∎

5. Zero-space-charge limit for the hybrid model

Before proving Theorem 2.4, for the sake of completeness, we briefly discuss, in the spirit of [15], the limit ($\lambda \to 0$) for the Q_qHD model (2.9).

THEOREM 5.1 (Zero-space-charge limits for the H-Q_qHD problem). *Let $C(x) \in C^2(\Omega)$ as in Theorem 2.4, and $Q(x) \in C^2(\Omega)$ with $0 < q \le Q_q(x) \le 1$, verifying (2.12). If $(w_{q,\lambda}, V_{q,\lambda})$ is the solution to (2.9) then, for $\lambda \to 0$, one has*

$$\begin{aligned} w_{q,\lambda}(x) \rightharpoonup w_q := \sqrt{C(x)} \quad &in \quad H^1(\Omega)\\ w_{q,\lambda}(x) \to w_q := \sqrt{C(x)} \quad &in \quad C^0(\bar\Omega)\\ V_{q,\lambda}(x) \rightharpoonup V_q(x) \quad &in \quad L^2(\Omega), \end{aligned}\tag{5.1}$$

where

$$\begin{aligned} V_q(x) = &- 2\varepsilon^2\left(Q_q(x)\frac{\sqrt{C}_{xx}}{\sqrt{C}} + Q_q'(x)\frac{\sqrt{C}_x}{\sqrt{C}}\right)\\ &+ \frac{J^2}{2C^2} + \frac{T}{\gamma-1}C^{\gamma-1} + \frac{J}{\tau}\int_0^x \frac{ds}{C(s)}. \end{aligned}\tag{5.2}$$

Proof. Before proving the theorem above we just remark that, in order to obtain a set of $\lambda-$independent estimates, we need to consider a special doping profile function. As an example we can consider a straight line connecting the boundary values of the charge density n_0 and n_1. Under this assumption, proceeding as in Lemma 3.2, it is not difficult to show that there exists a constant $\tilde K$, independent of λ, such that $\|(w_{q,\lambda})\|_\infty \le \tilde K$, $\|(w_{q,\lambda})\|_{H^1} \le \tilde K$ and $\|V_{q,\lambda}\|_{L^2} \le \tilde K$. The last inequality directly implies the last limit in (5.1). Moreover, multiplying equation $(2.9)_1$, by the test function $(w_{q,\lambda} - \sqrt{C}) \in H^2(\Omega) \cap H_0^1(\Omega)$, after integration by part, we obtain

$$\begin{aligned}
&2\varepsilon^2 \int_0^1 Q_q \frac{(w_{q,\lambda})_{xx}^2}{w_{q,\lambda}}\,dx + 2\int_0^1 \left(Tw_{q,\lambda}^{2(\gamma-1)} - \frac{J^2}{(w_{q,\lambda})^4}\right)\frac{(w_{q,\lambda})_x^2}{w_{q,\lambda}}\,dx\\
&\quad + 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_{q,\lambda})_x (w_{q,\lambda})_{xx}}{w_{q,\lambda}}\,dx + \frac{1}{\lambda^2}\int_0^1 (w_{q,\lambda} - \sqrt{C})^2 (w_{q,\lambda} + \sqrt{C})\,dx\\
=&2\varepsilon^2 \int_0^1 Q_q \frac{(w_{q,\lambda})_{xx}}{w_{q,\lambda}} \sqrt{C}_{xx}\,dx + 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_{q,\lambda})_x}{w_{q,\lambda}}\sqrt{C}_{xx}\,dx\\
&\quad + 2\int_0^1 \left(T{w_{q,\lambda}}^{2(\gamma-1)} - \frac{J^2}{(w_{q,\lambda})^4}\right)\frac{(w_{q,\lambda})_x}{w_{q,\lambda}}\sqrt{C}_x\,dx\\
&\quad - \int_0^1 \frac{J}{\tau (w_{q,\lambda})^2}(w_{q,\lambda})_x\,dx + \int_0^1 \frac{J}{\tau (w_{q,\lambda})^2}\sqrt{C}_x\,dx\\
=:&Y_1 + Y_2 + Y_3 + Y_4 + Y_5.
\end{aligned} \tag{5.3}$$

By Young's inequality we get

$$\begin{aligned}
Y_1 &\le \varepsilon^2 \int_0^1 Q_q(x) \frac{(w_{q,\lambda})_{xx}^2}{w_{q,\lambda}}\,dx + \varepsilon^2 \int_0^1 Q_q(x)\frac{\sqrt{C}_{xx}^2}{w_{q,\lambda}}\,dx\\
&\le \varepsilon^2 \int_0^1 Q_q(x) \frac{(w_{q,\lambda})_{xx}^2}{w_{q,\lambda}}\,dx + \int_0^1 \frac{\sqrt{C}_{xx}^2}{\underline{w}}\,dx,
\end{aligned}$$

where $\underline{w} = \min\{w_{q,\lambda}\}$. Using the subsonic condition (2.12) one has

$$0 < M_1 \le \left(T{w_{q,\lambda}}^{2(\gamma-1)} - \frac{J^2}{(w_{q,\lambda})^4}\right) \le M_2$$

for some positive constants M_1 and M_2 and then

$$Y_3 \le \int_0^1 \left(T{w_{q,\lambda}}^{2(\gamma-1)} - \frac{J^2}{(w_{q,\lambda})^4}\right)\frac{(w_{q,\lambda})_x^2}{w_{q,\lambda}}\,dx + M_2\int_0^1 \frac{\sqrt{C}_x^2}{\underline{w}}\,dx.$$

Similarly, we can estimate

$$\begin{aligned}
Y_2 &\le \varepsilon^2 \int_0^1 [Q_q'(x)]^2 \frac{(w_{q,\lambda})_x^2}{w_{q,\lambda}}\,dx + \varepsilon^2 \int_0^1 \frac{[\sqrt{C}_{xx}]^2}{w_{q,\lambda}}\,dx,\\
Y_4 &= \frac{J}{\tau}\left|\frac{1}{w_1} - \frac{1}{w_0}\right|,\\
Y_5 &\le \int_0^1 \frac{J^2}{2\tau^2 (w_{q,\lambda})^4}\,dx + \int_0^1 \frac{[\sqrt{C}_x]^2}{2}\,dx \le \frac{J^2}{2\tau^2 \underline{w}^4} + \int_0^1 \frac{[\sqrt{C}_x]^2}{2}\,dx.
\end{aligned}$$

Substituting the above estimates on Y_i $(i = 1, \cdots, 5)$ into (5.3), we have

$$\begin{aligned}
&\varepsilon^2 \int_0^1 Q_q \frac{(w_{q,\lambda})_{xx}^2}{w_{q,\lambda}}\,dx + \int_0^1 \left[T(w_{q,\lambda})^{2(\gamma-1)} - \frac{J^2}{(w_{q,\lambda})^4} - \varepsilon^2 (Q_q')^2\right]\frac{(w_{q,\lambda})_x^2}{w_{q,\lambda}}\,dx\\
&+ 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_{q,\lambda})_x (w_{q,\lambda})_{xx}}{w_{q,\lambda}}\,dx + \frac{1}{\lambda^2}\int_0^1 ((w_{q,\lambda}) - \sqrt{C})^2 ((w_{q,\lambda}) + \sqrt{C})\,dx \le \tilde{K}.
\end{aligned} \tag{5.4}$$

Introducing, as usual, a strictly positive quadratic form, we obtain

$$\int_0^1 \mathcal{A}_3 \frac{(w_{q,\lambda})_{xx}^2}{w_{q,\lambda}}\,dx + \int_0^1 \mathcal{B}_3 \frac{(w_{q,\lambda})_x (w_{q,\lambda})_{xx}}{w_{q,\lambda}}\,dx + \int_0^1 \mathcal{C}_3 \frac{(w_{q,\lambda})_x^2}{w_{q,\lambda}}\,dx \le \tilde{K}, \tag{5.5}$$

where, in order to guarantee its strict positivity, we require

$$\mathcal{B}_3^2 - 4\mathcal{A}_3\mathcal{C}_3 < 0, \tag{5.6}$$

with

$$\begin{aligned}
\mathcal{A}_3 &= \varepsilon^2 Q_q(x),\\
\mathcal{B}_3 &= 2\varepsilon^2 Q_q'(x),\\
\mathcal{C}_3 &= \left[T(w_{q,\lambda})^{2(\gamma-1)} - \frac{J^2}{(w_{q,\lambda})^4} - \varepsilon^2 (Q_q')^2\right].
\end{aligned}$$

The inequality (5.6) can be easily verified from (2.10) and (2.12). Indeed we have

$$\begin{aligned}
\mathcal{B}_3^2 - 4\mathcal{A}_3\mathcal{C}_3 &= 4\varepsilon^4 Q_q'(x)^2 - 4\varepsilon^2 Q_q(x)\left(T(w_{q,\lambda})^{2(\gamma-1)} - \frac{J^2}{(w_{q,\lambda})^4} - \varepsilon^2 (Q_q')(x)^2\right)\\
&= 4\varepsilon^2\left[\varepsilon^2 (Q_q'(x))^2 (1+Q_q(x)) - Q_q(x)\left(T(w_{q,\lambda})^{2(\gamma-1)} - \frac{J^2}{(w_{q,\lambda})^4}\right)\right]\\
&< 0.
\end{aligned} \tag{5.7}$$

Equation (5.4) clearly implies

$$\int_0^1 ((w_{q,\lambda}) - \sqrt{C})^2((w_{q,\lambda}) + \sqrt{C}) \le \lambda^2 \tilde{K}.$$

This guarantees the existence of subsequence $(w_{q,\lambda})$ (not relabeled) which converges to $\sqrt{C(x)}$ as in $(5.1)_1$ and $(5.1)_2$. To complete the proof, we multiply (4.13) by $\phi \in C_0^\infty(\Omega)$ used as a test function

$$\begin{aligned}
\int_0^1 V_{q\lambda}\phi\,dx = &- 2\varepsilon^2 \int_0^1 Q_q \frac{(w_{q,\lambda})_{xx}}{w_{q,\lambda}}\phi\,dx - 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_{q,\lambda})_x}{w_{q,\lambda}}\phi\,dx\\
&+ \frac{1}{2}\int_0^1 \frac{J^2}{(w_{q,\lambda})^4}\phi\,dx + \frac{T}{\gamma-1}\int_0^1 {w_{q,\lambda}}^{2(\gamma-1)}\phi\,dx\\
&+ \frac{J}{\tau}\int_0^1 \Big(\int_0^x \frac{1}{(w_{q,\lambda})^2}\,ds\Big)\phi\,dx,
\end{aligned} \tag{5.8}$$

which implies, for $\lambda \to 0$,

$$\begin{aligned}
\int_0^1 V_q\phi\,dx = &- 2\varepsilon^2 \int_0^1 Q_q \frac{\sqrt{C}_{xx}}{\sqrt{C}}\phi\,dx - 2\varepsilon^2 \int_0^1 Q_q' \frac{\sqrt{C}_x}{\sqrt{C}}\phi\,dx\\
&+ \int_0^1 \frac{J^2}{2C^2}\phi\,dx + \frac{T}{\gamma-1}\int_0^1 C^{(\gamma-1)}\phi\,dx\\
&+ \frac{J}{\tau}\int_0^1 \Big(\int_0^x \frac{1}{C}\,ds\Big)\phi\,dx
\end{aligned} \tag{5.9}$$

then also (5.2) is verified. This concludes the proof. □

Now we proceed with the proof of Theorem 2.4.

Proof. **(Proof of Theorem 2.4.)** We consider the smooth solution $(w_{q,\lambda}, V_{q,\lambda})(x)$ to (2.9) that verifies the following estimates

$$\|w_{q,\lambda}\|_{H^1} \leq \tilde{K}, \quad \|\sqrt{Q_q}(w_{q,\lambda})_{xx}\|_{L^2} \leq \tilde{K}, \quad \|V_{q,\lambda}\|_{L^2} \leq \tilde{K},$$

independently of q and λ. Moreover, as in (5.3), from the proof of Theorem 5.1, we have

$$\begin{aligned}
&\varepsilon^2 \int_0^1 Q_q \frac{(w_{q,\lambda})_{xx}^2}{w_{q,\lambda}}\, dx \\
&+ \int_0^1 \left[T(w_{q,\lambda})^{2(\gamma-1)} - \frac{J^2}{(w_{q,\lambda})^4} - \varepsilon^2 (Q_q')^2 \right] \frac{(w_{q,\lambda})_x^2}{w_{q,\lambda}}\, dx \\
&+ 2\varepsilon^2 \int_0^1 Q_q' \frac{(w_{q,\lambda})_x (w_{q,\lambda})_{xx}}{w_{q,\lambda}}\, dx + \frac{1}{\lambda^2} \int_0^1 ((w_{q,\lambda}) - \sqrt{C})^2 ((w_{q,\lambda}) + \sqrt{C})\, dx \leq \tilde{K},
\end{aligned} \tag{5.10}$$

that, in particular, implies

$$\int_0^1 (w_{q,\lambda} - \sqrt{C})^2 (w_{q,\lambda} + \sqrt{C})\, dx \leq \lambda^2 \tilde{K}.$$

Taking $q \to 0$, in (5.10), we get

$$\begin{aligned}
&\varepsilon^2 \int_0^1 Q \frac{(w_\lambda)_{xx}^2}{w_\lambda}\, dx \\
&+ \int_0^1 \left[T(w_\lambda)^{2(\gamma-1)} - \frac{J^2}{(w_\lambda)^4} - \varepsilon^2 (Q')^2 \right] \frac{(w_\lambda)_x^2}{w_\lambda}\, dx \\
&+ 2\varepsilon^2 \int_0^1 Q' \frac{(w_\lambda)_x (w_\lambda)_{xx}}{w_\lambda}\, dx + \frac{1}{\lambda^2} \int_0^1 ((w_\lambda) - \sqrt{C})^2 ((w_\lambda) + \sqrt{C})\, dx \leq \tilde{K}.
\end{aligned} \tag{5.11}$$

The inequality (5.11) can be rewritten as

$$\begin{aligned}
&\frac{\varepsilon^2}{2} \int_0^1 Q \frac{(w_\lambda)_{xx}^2}{w_\lambda}\, dx \\
&+ \frac{1}{2} \int_0^1 \left[T(w_\lambda)^{2(\gamma-1)} - \frac{J^2}{(w_\lambda)^4} - \varepsilon^2 (Q')^2 \right] \frac{(w_\lambda)_x^2}{w_\lambda}\, dx \\
&+ \int_0^1 \left[\mathcal{A}_4 \frac{[(w_\lambda)_{xx}]^2}{w_\lambda} + \mathcal{B}_4 \frac{(w_\lambda)_x (w_\lambda)_{xx}}{w_\lambda} + \mathcal{C}_4 \frac{[(w_\lambda)_x]^2}{w_\lambda} \right] dx \\
&+ \frac{1}{\lambda^2} \int_0^1 ((w_\lambda) - \sqrt{C})^2 ((w_\lambda) + \sqrt{C})\, dx \leq \tilde{K},
\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}
\mathcal{A}_4 &:= \frac{\varepsilon^2}{2} Q, \\
\mathcal{B}_4 &:= 2\varepsilon^2 Q', \\
\mathcal{C}_4 &:= \frac{1}{2} \left(T(w_\lambda)^{2(\gamma-1)} - \frac{J^2}{(w_\lambda)^4} - \varepsilon^2 (Q')^2 \right).
\end{aligned}$$

One has

$$\mathcal{B}_4^2 - 4\mathcal{A}_4\mathcal{C}_4 < 0,$$

by (2.12), which is assumed to hold also in the limit $q \to 0$ (see Remark 3.1), indeed

$$\mathcal{B}_4^2 - 4\mathcal{A}_4\mathcal{C}_4 = 4\varepsilon^2\left(\varepsilon^2(Q')^2(1+\frac{Q}{4}) - \frac{Q}{4}(T(w_\lambda)^{2(\gamma-1)} - \frac{J^2}{(w_\lambda)^4})\right).$$

So that

$$\int_0^1 \left[\mathcal{A}_4\frac{[(w_\lambda)_{xx}]^2}{w_\lambda} + \mathcal{B}_4\frac{(w_\lambda)_x(w_\lambda)_{xx}}{w_\lambda} + \mathcal{C}_4\frac{[(w_\lambda)_x]^2}{w_\lambda}\right] dx > 0.$$

Finally from (5.12), we further have

$$\begin{aligned}\frac{\varepsilon^2}{2}\int_0^1 Q\frac{(w_\lambda)_{xx}^2}{w_\lambda}\,dx + \frac{1}{2}\int_0^1 \left[T(w_\lambda)^{2(\gamma-1)} - \frac{J^2}{(w_\lambda)^4} - \varepsilon^2(Q')^2\right]\frac{(w_\lambda)_x^2}{w_\lambda}\,dx \\ + \frac{1}{\lambda^2}\int_0^1((w_\lambda)-\sqrt{C})^2((w_\lambda)+\sqrt{C})\,dx \le \tilde{K},\end{aligned} \tag{5.13}$$

which implies, together with (3.1), that

$$\|w_\lambda\|_{H^1} \le \tilde{K} \text{ and } \int_0^1 (w_\lambda - \sqrt{C})^2(w_\lambda + \sqrt{C})\,dx \le \lambda^2\tilde{K}.$$

Then we obtain

$$w_\lambda \rightharpoonup \sqrt{C} \text{ in } H^1 \text{ as } \lambda \to 0,$$

and

$$w_\lambda \to \sqrt{C} \text{ in } C^0 \text{ as } \lambda \to 0.$$

In the same way, from (4.15) and (4.16), we can prove

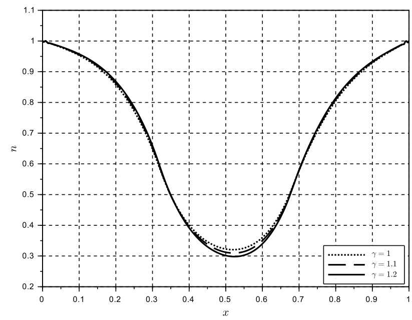
$$V_\lambda \rightharpoonup \tilde{V} \text{ in } L^2(\Omega) \text{ as } \lambda \to 0.$$

This completes the proof. □

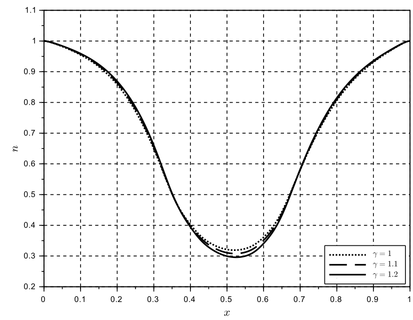
REMARK 5.1. We observe that (2.10) assures the necessary positiveness of the terms $\mathcal{C}_i$, $i = 0, ..., 4$ in the previous proofs.

6. Numerical simulations

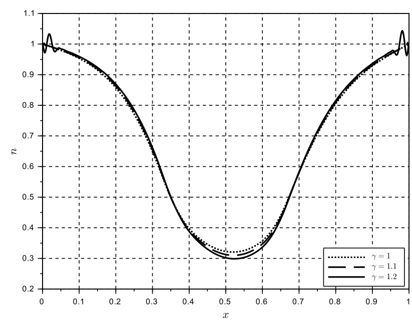
In this section we test numerically the model introduced in the first part of the paper, in order to evaluate the effects of the nonlinear pressure function. We just recall that, compared to quantum hydrodynamic model, the quantum hybrid model (1.1) is characterized by a more general Bohm potential, which allows the localization of the quantum effects in a given region of the device. As observed in the theoretical part, the hybrid problem is degenerate because of the hybrid term $Q(x)$. Through this section, q is strictly positive, but we consider different values of q to simulate the limit $q \to 0$. The problem, although not degenerate, is stiff, and this requires an accurate numerical treatment, especially close to the boundary.



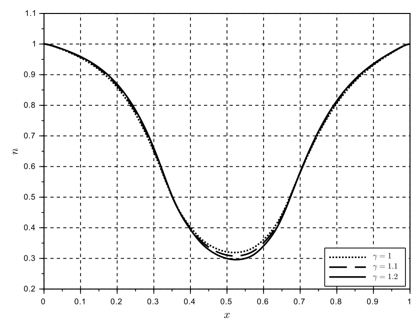
(a) Case 1, $q = 0.001$



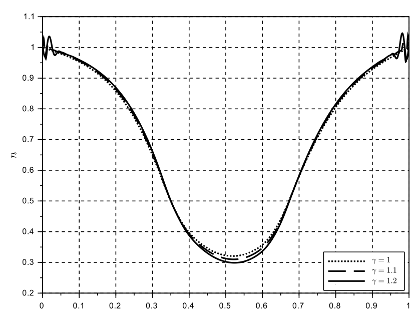
(b) Case 2, $q = 0.001$



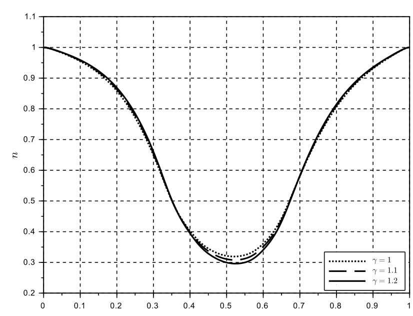
(c) Case 1, $q = 0.0001$



(d) Case 2, $q = 0.0001$



(e) Case 1, $q = 0.00001$

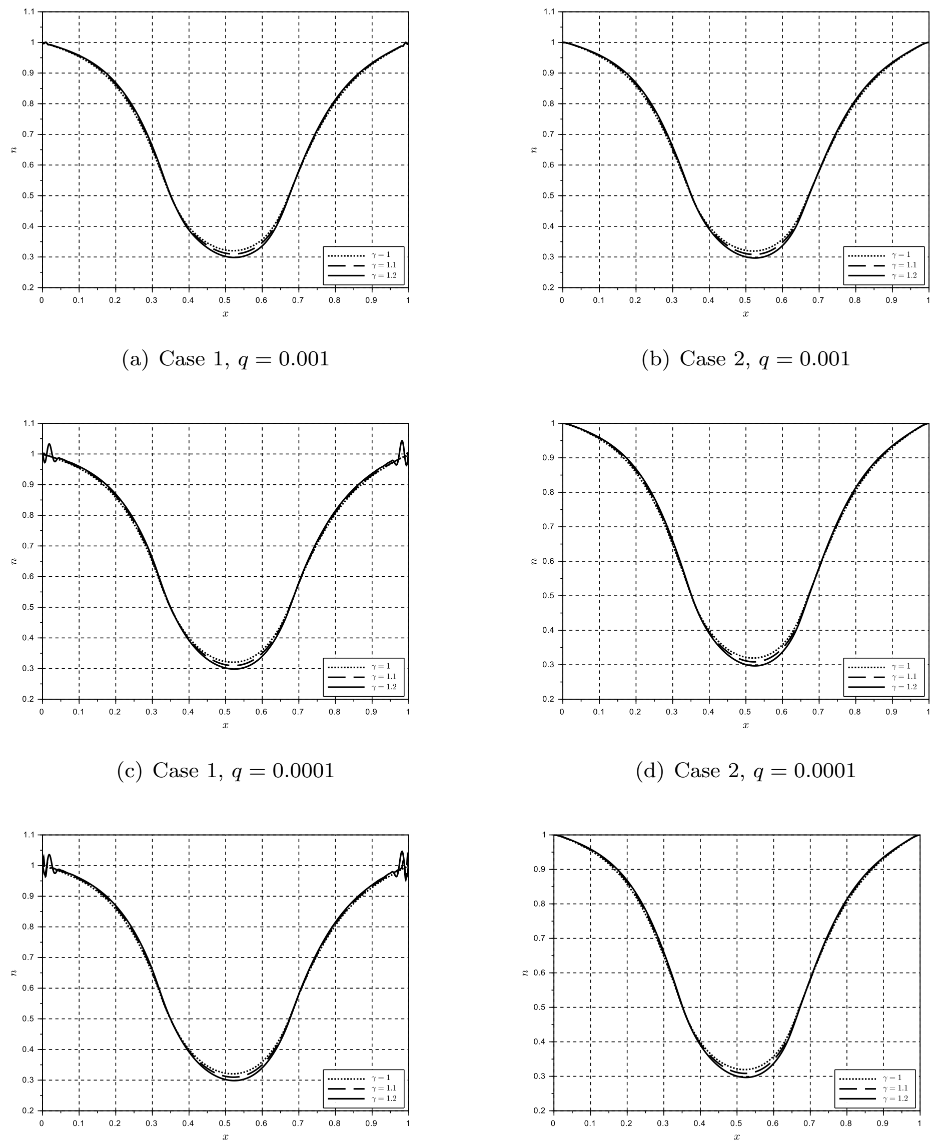


(f) Case 2, $q = 0.00001$

FIG. 6.1. *Behaviour of the charge density for a hybrid model having quantum region in the middle part of the domain (Case 1) and in the external part (Case 2) assuming $y_1 = 2/5$ and $y_2 = 3/5$. Three different values of the parameters γ and q are considered.*

Here we just perform some simple numerical tests using COLNEW (see [4]), a SCILAB function for boundary value problems. This tool was not specifically developed to solve the hybrid problem, but it has been widely used to simulate a simple device using the QHD equation and the hybrid QHD equation (see for example [11,15]) and it provides reasonably enough results.

Although in the theoretical part of paper we have proved the existence of a weak solution to the hybrid problem, assuming the boundaries are quantum, in this numerical



(a) Case 1, $q = 0.001$

(b) Case 2, $q = 0.001$

(c) Case 1, $q = 0.0001$

(d) Case 2, $q = 0.0001$

(e) Case 1, $q = 0.00001$

(f) Case 2, $q = 0.00001$

FIG. 6.2. *Behaviour of the charge density for a hybrid model having quantum region in the middle part of the domain (Case 1) and in the external part (Case 2) assuming* $y_1 = 1/3$ *and* $y_2 = 2/3$. *Three different values of the parameters* γ *and* q *are considered.*

section we also consider the classic boundary case.

We can observe that, assuming a smaller value of q, the oscillations detectable near the extrema (see Figures 6.1, 6.2, Case 1), in the classic boundary case, seem to reflect the technical difficulty in verifying the boundary conditions $w_x(0) = w_x(1) = 0$ for the limit hybrid solution, (see Remark 2.5).

Currently, we can not say if the oscillations depend on the inadequacy of the code or on the lack of convergence of $(w_q)_x \to w_x$ in $x = 0, 1$, when $q \to 0$.

We use, as a test device, a $n^+|n|n^+$ transistor on the domain $[0, 1]$. The doping profile $\bar{C}(x)$ modelling our device is such that: $\bar{C}(x) = C_m$, $0 < C_m < 1$ if $x \in [x_1, x_2]$ and $\bar{C}(x) = 1$ if $x \in [0, x_1) \cup (x_2, 1]$, with $0 < x_1 < x_2 < 1$. In order to simplify the numerical approach, we need to regularize $\bar{C}(x)$. We choose $x_1 = 1/3$ and $x_2 = 2/3$, setting

$$\bar{C}(x) \approx C(x) = 1 - (0.5 - C_m/2)(\tanh(100(x - 1/3)) + \tanh(100(x - 2/3)))$$

where $C_m = 0.02$.

Without loss of generality, we divide the device into a classical and a quantum part, and we consider two cases as follows

$$\text{Case 1} \begin{cases} \text{Quantum Region} & \forall x \in [y_1, y_2] \\ \text{Classical Region} & \forall x \in [0, y_1) \text{ and } x \in (y_2, 1], \end{cases} \tag{6.1}$$

$$\text{Case 2} \begin{cases} \text{Quantum Region} & \forall x \in [0, y_1) \text{ and } x \in (y_2, 1], \\ \text{Classical Region} & \forall x \in [y_1, y_2], \end{cases} \tag{6.2}$$

where $x_1 \leq y_1 < y_2 \leq x_2$. We only observe that we have not run simulations on Case 1 in previous papers, dealing with this line of research.

The approximating quantum functions $\{Q_q(x)\}$ are respectively

$$\text{Case 1} \quad Q_q(x) = \left(q - \frac{1}{2}\right)(\tanh(h(x - y_2)) - \tanh(h(x - y_1))) + q \tag{6.3}$$

and

$$\text{Case 2} \quad Q_q(x) = \left(\frac{1 - q}{2}\right)(\tanh(h(x - y_2)) - \tanh(h(x - y_1))) + 1, \tag{6.4}$$

where we fix $h = 40$, $x \in [0, 1]$ and $q = \min\{Q_q\} = 0.001, 0.0001, 0.00001$, to simulate the limit $q \to 0$.

τ	scaled relaxation time	0.125
λ	scaled Debye length	0.1
ε	scaled Plank constant	0.01
J	scaled current density	0.1
T	scaled temperature	1

TABLE 6.1. *Values of the scaled parameters used in the simulations.*

The boundary value problem we solve numerically is the following

$$\begin{aligned} &\varepsilon^2 \left(Q_q \left(u_{xx} + \frac{u_x^2}{2} \right) + Q_q' u_x \right)_{xx} + (J^2 e^{-2u} u_x)_x \\ &\quad - \frac{T}{\gamma - 1} (e^{(\gamma-1)u})_{xx} + \frac{e^u - C(x)}{\lambda^2} - \left(\frac{J}{\tau} e^{-u} \right)_x = 0, \\ &u(0) = u(1) = 0 \qquad u_x(0) = u_x(1) = 0 \end{aligned}$$

where, as usual, $u = \ln n$. The value of the physical parameters used in our tests are shown in Table 6.1.

Case 1	γ	n_{min}	s
$q = 0.001$			
	1	0.32067	0.056134
$y_1 = 1/3,\ y_2 = 2/3$	1.1	0.30963	0.036519
	1.2	0.29814	0.017763
	1	0.32047	0.056234
$y_1 = 2/5,\ y_2 = 3/5$	1.1	0.30976	0.036551
	1.2	0.29849	0.017848
$q = 0.0001$			
	1	0.32059	0.055704
$y_1 = 1/3,\ y_2 = 2/3$	1.1	0.30967	0.036105
	1.2	0.29843	0.017403
	1	0.32078	0.055725
$y_1 = 2/5,\ y_2 = 3/5$	1.1	0.31000	0.036159
	1.2	0.29875	0.017473
$q = 0.00001$			
	1	0.32058	0.055662
$y_1 = 1/3,\ y_2 = 2/3$	1.1	0.30965	0.036059
	1.2	0.29844	0.017363
	1	0.32076	0.055680
$y_1 = 2/5,\ y_2 = 3/5$	1.1	0.30993	0.036105
	1.2	0.29876	0.017432

TABLE 6.2. *Case 1. Table of the minimum values of the charge density and of the parameter s obtained for three different values of γ and q. Please notice that n_{min} decreases as γ increases.*

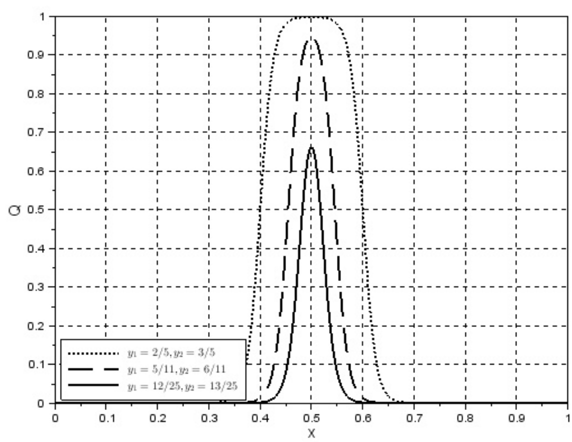


FIG. 6.3. *Quantum function (6.3) for different values of (y_1, y_2).*

In the figures below we show the behaviour of the charge density for the approximated quantum functions (6.3) and (6.4) at different values of the parameter γ, that is $\gamma = 1, 1.1, 1.2$ (which allow to verify condition (2.12)) and for different values of q. We just remark that (2.12) is not a numerical constraint, but comes out from the theoretical analysis of the problem. We have checked that, also for large values of γ, the

Case 2	γ	n_{min}	s
$q = 0.001$			
	1	0.31947	0.055900
$y_1 = 1/3,\ y_2 = 2/3$	1.1	0.308415	0.036213
	1.2	0.29676	0.017353
	1	0.31917	0.055898
$y_1 = 2/5,\ y_2 = 3/5$	1.1	0.30792	0.036160
	1.2	0.29597	0.017216
$q = 0.0001$			
	1	0.31947	0.055559
$y_1 = 1/3,\ y_2 = 2/3$	1.1	0.30841	0.035872
	1.2	0.296752	0.017013
	1	0.31917	0.055552
$y_1 = 2/5,\ y_2 = 3/5$	1.1	0.30791	0.058143
	1.2	0.29598	0.016870
$q = 0.00001$			
	1	0.31947	0.055525
$y_1 = 1/3,\ y_2 = 2/3$	1.1	0.30841	0.035838
	1.2	0.29675	0.016978
	1	0.31917	0.055495
$y_1 = 2/5,\ y_2 = 3/5$	1.1	0.30791	0.035757
	1.2	0.29598	0.016813

TABLE 6.3. *Case 2. Table of the minimum values of the charge density and of the parameter s obtained for three different values of γ and q. Please notice that n_{min} decreases as γ increases.*

numerical results are still reasonable. Moreover, two different values of y_1 and y_2 have been considered, namely $(1/3, 2/3)$ and $(2/5, 3/5)$.

A not negligible difference in the behaviour of the charge density between case (6.3) and case (6.4) can be observed close to the boundaries. In particular large oscillations of the solution at the boundaries can be observed in the Case 1. The amplitude of the oscillations increases as q decreases and as γ increases. The origin of this phenomena is not clear, it may be due to the intrinsic structure of the problem as well as due to the incapability of the numerical code to carefully describe the stiff problem on the boundaries. The problem of existence of solutions for the hybrid problem when the boundaries behave classically remains, therefore, an open problem. No significant differences in the behaviour of n throughout the domain can be observed. The position of the interval (y_1, y_2) does not play any relevant role. Probably the two couple (y_1, y_2) considered in this section are too close to each other to see remarkable differences. On the other hand, for $|y_2 - y_1|$ small enough and for the value of h we have fixed, the quantum function is not able to reach the value 1 (see Figure 6.3) and then the hybrid nature of our equation is lost. We observe that, for values of h big enough, the quantum function reaches the value 1 also for smaller intervals $|y_2 - y_1|$, but condition (2.12) is not necessarily verified.

In order to check that condition (2.12) can be verified, we introduce the parameter

s as follows

$$s = \frac{1}{6}\Big(T\underline{n}^{(\gamma-1)} - \frac{J^2}{\underline{n}^2}\Big) - \varepsilon^2 \max_{x\in\Omega} \frac{(Q_q'(x))^2}{Q_q(x)}.$$

Clearly, we expect $s > 0$.

The minimum values reached by n are shown in Tables 6.2 (Case 1)-6.3 (Case 2) together with the corresponding values of s assuming $q = 0.001$, $q = 0.0001$ and $q = 0.00001$. We remark that the constraint $n > \underline{n} = C_m = 0.02$ is verified in all cases and the value of n decreases as γ increases, in both cases.

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