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APPROXIMATIONS OF THE STOCHASTIC 3D NAVIER-STOKES EQUATIONS WITH DAMPING*

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Abstract. The stochastic three-dimensional Navier-Stokes equation with damping is considered in this paper. We show that solutions of three-dimensional stochastic Navier-Stokes equation with damping driven by Brownian motion can be approximated by three-dimensional stochastic Navier-Stokes equation with damping driven by pure jump noise/random kicks on the spaces $D([0,T],V)$ and $D([0,T],H)$ for $3<\beta<5$ with any $\alpha>0$ and $\alpha\geq\frac{1}{4}$ as $\beta=3$.

Keywords. Stochastic Navier-Stokes equation; Approximations; Weak convergence.

AMS subject classifications. 35Q30; 76D05; 57Q55; 60H15.

1. Introduction

In this paper, we consider the following stochastic three-dimensional Navier-Stokes equations with damping:

$$\begin{cases} du-\nu\Delta udt+(u\cdot\nabla)udt+\alpha|u|^{\beta-1}udt+\nabla pdt=\sum_{i=1}^{m}\sigma^i(u)dW^i(t),\\ \nabla\cdot u=0,\\ u(x,0)=h, \end{cases} \tag{1.1}$$

where $\mathbb{T}^3\subset\mathbb{R}^3$ is a periodic domain, $u=(u_1,u_2,u_3)$ is the velocity, p is the pressure, $\beta\geq 1$ and $\alpha>0$ denote constants, and $t\geq 0$. $W=(W^1(t),\cdots,W^m(t))$ denotes a $m-$dimensional standard Brownian motion. The given function h is the initial velocity, and the constant $\nu>0$ represents the viscosity coefficient of the flow. For simplicity, we set $\nu=1$.

The deterministic three-dimensional Navier-Stokes equations with damping have been investigated in [19]. The existence and regularity of solutions for three-dimensional Navier-Stokes equations with damping were proved in [4]. Moreover, Cai and Jiu have obtained the global weak solutions for $\beta\geq 1$ and the global strong solutions for $\frac{7}{2}\leq\beta\leq 5$. By using the Fourier splitting method, Jiang has proved the asymptotic behavior of strong solutions for the three-dimensional Navier-Stokes equations with damping for $\beta\geq 3$ as $\alpha=1$ in [14].

Recently, in various papers, stochastic three-dimensional Navier-Stokes equations with damping were proved in [9, 10, 15, 16]. Rӧckner and Zhang have proved the existence and uniqueness of solutions for stochastic tamed three-dimensional Navier-Stokes equations, moreover, they obtained the ergodicity by using asymptotic strong Feller property in [12, 22]. The large deviation principle for the stochastic tamed 3D Navier-Stokes equations was obtained by using weak convergence method in [21]. By using monotonicity method, Rӧckner and Zhang have obtained the existence and uniqueness of strong solutions for stochastic 3D tamed Navier-Stokes equations. Meanwhile,

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they obtained the small-time large deviation principle in [20]. Dong and Zhang have proved the existence and uniqueness of strong solution for the stochastic 3D tamed Navier-Stokes equations driven by multiplicative Lévy noise by using the Galerkin approximation method. Moreover, they obtained the large deviation principles by using weak convergence method in [7]. For the well-posedness for stochastic partial differential equations, we refer to [3, 5, 11, 17, 18].

To obtain the approximations for the stochastic three-dimensional Navier-Stokes equation with damping, we overcome the main difficulty that lies in dealing with the nonlinear terms $B(u,u)=P((u\cdot\nabla)u)$ and $g(u)=\alpha P(|u|^{\beta-1}u)$. We get the approximations for $3<\beta<5$ with any $\alpha>0$ and $\alpha\geq\frac{1}{4}$ as $\beta=3$. Inspired by [13], we have $4\nu\alpha\geq 1$. But if $\nu=1$, by using a similar method, we have a similar result. Meanwhile, we can improve the result for the global well-posedness of a 3D MHD in porous media by using a similar method as in [25]. C represents a nonnegative constant which may have different values from line to line.

This paper is organized as follows. In Section 2, we recall some fundamental concepts and symbols. In Section 3, we will show the approximations on the spaces $D([0,T],V)$ and $D([0,T],H)$ for $3<\beta<5$ with any $\alpha>0$ and $\alpha\geq\frac{1}{4}$ as $\beta=3$.

2. Preliminaries

Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a filtered probability space, where $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ is a filtration. $\nu^i(dx), i=1,2,\cdots,m$ are $\sigma-$finite measures on the measurable space $(\mathbb{R}_0,\mathcal{B}(\mathbb{R}_0))$, where $\mathbb{R}_0=\mathbb{R}\backslash\{0\}$. Assume that $N^i, i=1,\cdots,m$ are mutually independent $\mathcal{F}_t-$Poisson random measures on $[0,T]\times\mathbb{R}_0$ with intensity measure $dt\times\nu^i(dz)$, respectively. For $U\in\mathcal{B}(\mathbb{R}_0)$ with $\nu^i(U)<\infty$, we will denote by $\tilde{N}^i((0,t]\times U)=N^i((0,t]\times U)-t\nu^i(U)$ the compensated Poisson random measure on $[0,T]\times\Omega\times\mathbb{R}_0$. Let $\mathbb{T}^3=[0,L]^3$ with $\int_{\mathbb{T}^3}u(x)dx=0$. Let $D([0,T],V)$ be the space of all càdlàg paths from $[0,T]$ into V equipped with the Skorohod topology. Assume that $\dot{C}_p^\infty(\mathbb{T}^3;\mathbb{R}^3)$ is the space of all infinitely differentiable functions such that $\int_{\mathbb{T}^3}u(x)dx=0$ and $u(x+Lw_i)=u(x)$, for any $x\in\mathbb{R}^3$ and $i=1,2,3$, here, $\{w_1,w_2,w_3\}$ is the canonical basis of $\mathbb{R}^3$.

First, we define the following spaces

$$
\begin{aligned}
\mathcal{V}&=\{u\in\dot{C}_p^\infty(\mathbb{T}^3,\mathbb{R}^3):\mathrm{div}u=0\},\\
H&=\text{ the closure of }\mathcal{V}\text{ in }L^2(\mathbb{T}^3):=L^2(\mathbb{T}^3,\mathbb{R}^3),\\
V&=\text{ the closure of }\mathcal{V}\text{ in }H^1(\mathbb{T}^3):=H^1(\mathbb{T}^3,\mathbb{R}^3).
\end{aligned}
$$

It is well known that H, V are separable Hilbert spaces and identify H and its dual H', we have $V\hookrightarrow H\hookrightarrow V'$ with dense and continuous injections, and $V\hookrightarrow H$ is compact. H and V are endowed, respectively, with the inner products

$$
\begin{aligned}
(u,v)&=\int_{\mathbb{T}^3}u\cdot v dx,\quad \forall u,v\in H,\\
((u,v))&=\sum_{i=1}^{3}\int_{\mathbb{T}^3}\nabla u_i\cdot\nabla v_i dx,\quad \forall u,v\in V,
\end{aligned}
$$

and norms $||\cdot||^2=(\cdot,\cdot)$, $||\nabla\cdot||^2=((\cdot,\cdot))=||\cdot||_V$. L^p-norm is defined by $||\cdot||_p$. We set $||\cdot||=||\cdot||_2$. H^3-norm is defined by $||\cdot||_{H^3}$.

Let P be the Helmholtz-Leray orthogonal projection of $L^2(\mathbb{T}^3)$ onto H. In periodic space, the Leray projector commutes with derivatives. $Au=-P\Delta u=-\Delta u$ is the Stokes operator defined by $\langle Au,v\rangle=((u,v))$. $B:V\times V\to V'$ is a bilinear operator defined by

$\langle B(u,v),w\rangle = b(u,v,w)$, $B(u) = B(u,u) = P((u\cdot\nabla)u)$, where

$$b(u,v,w) = \sum_{i,j=1}^{3}\int_{\mathbb{T}^3} u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

and $\langle\cdot,\cdot\rangle$ is the duality product between V and V'. For simplicity, we take $g(u) = \alpha P(|u|^{\beta-1}u)$. Assume that $\{e_i\}_{i=1}^{\infty}\subset H^3$ are an orthonormal basis of H composed of eigenfunctions of A with corresponding eigenvalues $0<\lambda_1\le\lambda_2\le\cdots\to\infty$ satisfying $Ae_i=\lambda_i e_i$, which is an orthogonal base in V. Let $l_k = \frac{e_k}{\sqrt{\lambda_k}}$, we have that $\mathbf{g}=\{l_k; k\in\mathbb{N}\}$ is an orthonormal basis of V.

Assumption 2.1. *For $\varepsilon>0$, assume that $\sigma^{i,\varepsilon}(\cdot), i=1,2,\cdots,m$ are measurable mappings from $V\times\mathbb{R}_0$ into V (resp. $H\times\mathbb{R}_0\to H$). Then there exist nonnegative constants C, ε_0 and k_1 is small enough such that for all $t\in[0,T]$, $p>2$ and $u,v\in H$,*

(A1) $\sup\limits_{\varepsilon\le\varepsilon_0}\sum\limits_{i=1}^{m}\int_{\mathbb{R}_0}||\sigma^{i,\varepsilon}(u,z)||^2\nu^i(dz)\le C(1+||u||^2)+k_1||\nabla u||^2$,

(A2) $\sup\limits_{\varepsilon\le\varepsilon_0}\sum\limits_{i=1}^{m}\int_{\mathbb{R}_0}||\sigma^{i,\varepsilon}(u,z)-\sigma^{i,\varepsilon}(v,z)||^2\nu^i(dz)\le C||u-v||^2$,

(A3) $\sup\limits_{\varepsilon\le\varepsilon_0}\sum\limits_{i=1}^{m}\int_{\mathbb{R}_0}||\sigma^{i,\varepsilon}(u,z)||^p\nu^i(dz)\le C(1+||u||^p)$,

and for $u,v\in V$,

(A4) $\sup\limits_{\varepsilon\le\varepsilon_0}\sum\limits_{i=1}^{m}\int_{\mathbb{R}_0}||\nabla\sigma^{i,\varepsilon}(u,z)||^2\nu^i(dz)\le C||\nabla u||^2$,

(A5) $\sup\limits_{\varepsilon\le\varepsilon_0}\sum\limits_{i=1}^{m}\int_{\mathbb{R}_0}||\nabla\sigma^{i,\varepsilon}(u,z)||^p\nu^i(dz)\le C||\nabla u||^p$,

(A6) $\sup\limits_{\varepsilon\le\varepsilon_0}\sum\limits_{i=1}^{m}\int_{\mathbb{R}_0}||\nabla(\sigma^{i,\varepsilon}(u,z)-\sigma^{i,\varepsilon}(v,z))||^2\nu^i(dz)\le C||\nabla(u-v)||^2$,

and for $\xi=(\xi^1,\xi^2,\xi^3)\in\mathbb{R}^3$ and $j=1,2,3$,

$$\sum_{i=1}^{m}|\sigma^i(\xi)|^2\le C|\xi|^2, \qquad \sum_{i=1}^{m}|\partial_{\xi^j}\sigma^i(\xi)|^2\le C.$$

Assumption 2.2. *For $\varepsilon>0$, assume that $\sigma^{i,\varepsilon}(\cdot)$ map from $H^2\times\mathbb{R}_0$ into H^2. Then there exist nonnegative constants C, ε_0 such that*

(B1) $\sup\limits_{\varepsilon\le\varepsilon_0}\sum\limits_{i=1}^{m}\int_{\mathbb{R}_0}||\Delta\sigma^{i,\varepsilon}(u,z)||^2\nu^i(dz)\le C||\Delta u||^2$.

We introduce the following conditions.

(H.1)

(i) For any $i\in\{1,\cdots,m\}$, any $M>0$,

$$\sup_{||u||\le M}\sup_{z\in\mathbb{R}_0}||\sigma^{i,\varepsilon}(u,z)||\to 0, \text{ as } \varepsilon\to 0. \tag{2.1}$$

(ii) For any $i\in\{1,\cdots,m\}$ and $k,j\in\mathbb{N}, u\in H$,

$$\int_{\mathbb{R}_0}(\sigma^{i,\varepsilon}(u,z),e_k)(\sigma^{i,\varepsilon}(u,z),e_j)\nu^i(dz)\to(\sigma^i(u),e_k)(\sigma^i(u),e_j), \text{ as } \varepsilon\to 0. \tag{2.2}$$

(H.2) For any $i\in\{1,\cdots,m\}$ and $u\in H$,

$$\int_{\mathbb{R}_0}||\sigma^{i,\varepsilon}(u,z)||^2\nu^i(dz)\to||\sigma^i(u)||^2,\quad \text{as} \quad \varepsilon\to 0. \tag{2.3}$$

(H.3)

(i) For any $i\in\{1,\cdots,m\}$, any $N>0$,

$$\sup_{||\nabla u||\leq N}\sup_{z\in\mathbb{R}_0}||\sigma^{i,\varepsilon}(u,z)||_V\to 0,\quad \text{as} \quad \varepsilon\to 0. \tag{2.4}$$

(ii) For any $i\in\{1,\cdots,m\}$ and $k,j\in\mathbb{N},u\in V$, $l_k,l_j\in\mathbf{g}$,

$$\int_{\mathbb{R}_0}((\sigma^{i,\varepsilon}(u,z),l_k))((\sigma^{i,\varepsilon}(u,z),l_j))\nu^i(dz)\to((\sigma^i(u),l_k))((\sigma^i(u),l_j)),\quad \text{as} \quad \varepsilon\to 0. \tag{2.5}$$

Finally, we write (1.1) as follows in the abstract form:

$$\begin{cases} du(t)=-[Au(t)+B(u(t),u(t))+g(u(t))]dt+\sum_{i=1}^m\sigma^i(u(t))dW^i(t),\\ u(0)=h. \end{cases} \tag{2.6}$$

3. Approximations of 3D SNSEs with damping by pure jump noise

In this section, let $\sigma^{i,\varepsilon}$ be given measurable maps. We introduce the following stochastic three-dimensional Navier-Stokes equations with damping driven by pure jump noise:

$$\begin{aligned} u^\varepsilon(t)=&h-\int_0^t Au^\varepsilon(s)ds-\int_0^t B(u^\varepsilon(s))ds-\int_0^t g(u^\varepsilon(s))ds\\ &+\sum_{i=1}^m\int_0^t\int_{\mathbb{R}_0}\sigma^{i,\varepsilon}(u^\varepsilon(s-),z)\tilde{N}^i(dzds). \end{aligned} \tag{3.1}$$

Inspired by [6,23], we have the following definition.

Definition 3.1. *A V-valued $(\mathcal{F}_t)$-adapted process $u^\varepsilon=(u^\varepsilon(t))_{t\geq 0}$ is said to be a solution to Equation (3.1) if*

(i) for any $T>0$, $u^\varepsilon\in D([0,T],V)\cap L^2([0,T]\times\Omega;dt\times\mathbb{P},H^2)$,

(ii) for any $t\geq 0$, (3.1) holds in H, $\mathbb{P}-a.s.$.

The following lemmas provide the existence and uniqueness of approximate solutions and uniform estimates.

Lemma 3.1. *Suppose that the Assumption 2.1 and $h\in H$ hold. Let u^ε denote the solution of Equation (3.1), we get*

$$\sup_{\varepsilon\leq\varepsilon_0}\mathbb{E}\{\sup_{0\leq t\leq T}||u^\varepsilon(t)||^2+\int_0^T||\nabla u^\varepsilon(s)||^2ds+\int_0^T||u^\varepsilon(s)||_{\beta+1}^{\beta+1}ds\}<\infty. \tag{3.2}$$

Proof. Applying Itô's formula to the process $||u^\varepsilon(t)||^2$, we get

$$\begin{aligned} ||u^\varepsilon(t)||^2=&||h||^2-2\int_0^t\langle Au^\varepsilon(s),u^\varepsilon(s)\rangle ds-2\int_0^t\langle g(u^\varepsilon(s)),u^\varepsilon(s)\rangle ds\\ &+2\int_0^t\int_{\mathbb{R}_0}(\sigma^\varepsilon(u^\varepsilon(s-),z),u^\varepsilon(s-))\widetilde{N}(dzds) \end{aligned}$$

$$+\int_0^t\int_{\mathbb{R}_0}||\sigma^\varepsilon(u^\varepsilon(s-),z)||^2N(dzds)$$
$$=||h||^2-2\int_0^t||\nabla u^\varepsilon(s)||^2ds-2\alpha\int_0^t||u^\varepsilon(s)||_{\beta+1}^{\beta+1}ds+I_1(t)+I_2(t). \tag{3.3}$$

For any $M>0$, we define the following stopping time

$$\tau_M^\varepsilon=T\wedge\inf\{t\geq0:||u^\varepsilon(t)||^2>M\}\wedge\inf\{t\geq0:\int_0^t||\nabla u^\varepsilon(s)||^2ds>M\}.$$

For $I_1(t)$, applying the Burkholder-Davis-Gundy inequality and the Hölder's inequality, we get

$$\begin{aligned}\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M^\varepsilon}|I_1(s)|\leq&2\mathbb{E}[\int_0^{t\wedge\tau_M^\varepsilon}\int_{\mathbb{R}_0}(\sigma^\varepsilon(u^\varepsilon(s-),z),u^\varepsilon(s-))^2\nu(dz)ds]^{\frac12}\\ \leq&C\mathbb{E}[\int_0^{t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||^2(1+||u^\varepsilon(s)||^2+k_1||\nabla u^\varepsilon(s)||^2)ds]^{\frac12}\\ \leq&C[\frac{1}{2C}\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||^2]^{\frac12}\\ &\times[2C\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}(1+||u^\varepsilon(s)||^2+k_1||\nabla u^\varepsilon(s)||^2)ds]^{\frac12}\\ \leq&\frac12\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||^2+2C^2t+2C^2\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||^2ds\\ &+2C^2k_1\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}||\nabla u^\varepsilon(s)||^2ds.\end{aligned} \tag{3.4}$$

For $I_2(t)$, then we get

$$\begin{aligned}\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M^\varepsilon}|I_2(s)|\leq&\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}\int_{\mathbb{R}_0}||\sigma^\varepsilon(u^\varepsilon(s-),z)||^2\nu(dz)ds\\ \leq&Ct+C\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||^2ds+Ck_1\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}||\nabla u^\varepsilon(s)||^2ds.\end{aligned} \tag{3.5}$$

Putting (3.4) and (3.5) into (3.3), we get

$$\begin{aligned}&\frac12\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||^2+2\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}||\nabla u^\varepsilon(s)||^2ds+2\alpha\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||_{\beta+1}^{\beta+1}ds\\ \leq&\mathbb{E}||h||^2+(2C^2+C)t+(2C^2+C)\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||^2ds+(2C^2+C)k_1\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}||\nabla u^\varepsilon(s)||^2ds.\end{aligned} \tag{3.6}$$

By choosing sufficiently small k_1 such that $(2C^2+C)k_1<1$, then we have

$$\begin{aligned}&\frac12\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||^2+\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}||\nabla u^\varepsilon(s)||^2ds+2\alpha\mathbb{E}\int_0^{t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||_{\beta+1}^{\beta+1}ds\\ \leq&\mathbb{E}||h||^2+(2C^2+C)t+(2C^2+C)\int_0^{t\wedge\tau_M^\varepsilon}\mathbb{E}\sup_{r\in[0,s]}||u^\varepsilon(r)||^2ds.\end{aligned} \tag{3.7}$$

By using Gronwall lemma, we have

$$\mathbb{E}(\sup_{0\le s\le t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||^2+\int_0^{t\wedge\tau_M^\varepsilon}||\nabla u^\varepsilon(s)||^2ds+\int_0^{t\wedge\tau_M^\varepsilon}||u^\varepsilon(s)||_{\beta+1}^{\beta+1}ds)\le C(\mathbb{E}||h||^2+1). \tag{3.8}$$

Recall that $\tau_M^\varepsilon\uparrow T$ as $M\to\infty$, and $\mathbb{P}\{\tau_M^\varepsilon<T\}=0$ as $M\to\infty$. By the Fatou lemma, we have

$$\sup_{\varepsilon\le\varepsilon_0}\mathbb{E}\{\sup_{0\le t\le T}||u^\varepsilon(t)||^2+\int_0^T||\nabla u^\varepsilon(s)||^2ds+\int_0^T||u^\varepsilon(s)||_{\beta+1}^{\beta+1}ds\}<\infty. \tag{3.9}$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *Suppose that the Assumption 2.1 and $h\in H$ hold. Let u^ε denote the solution of Equation (3.1), we get for any $p>1$,*

$$\sup_{\varepsilon\le\varepsilon_0}\mathbb{E}\{\sup_{0\le t\le T}||u^\varepsilon(t)||^{2p}+\int_0^T||u^\varepsilon(s)||^{2p-2}||\nabla u^\varepsilon(s)||^2ds + \int_0^T||u^\varepsilon(t)||^{2p-2}||u^\varepsilon(s)||_{\beta+1}^{\beta+1}ds\}<\infty. \tag{3.10}$$

Proof. Applying Itô's formula to the process $||u^\varepsilon(t)||^{2p}$, we deduce

$$\begin{aligned}||u^\varepsilon(t)||^{2p}=&||h||^{2p}-2p\int_0^t||u^\varepsilon(s)||^{2p-2}||\nabla u^\varepsilon(s)||^2ds-2p\alpha\int_0^t||u^\varepsilon(s)||^{2p-2}||u^\varepsilon(s)||_{\beta+1}^{\beta+1}ds\\&+2p\int_0^t\int_{\mathbb{R}_0}||u^\varepsilon(s)||^{2p-2}(\sigma^\varepsilon(u^\varepsilon(s-),z),u^\varepsilon(s-))\widetilde{N}(dzds)\\&+\int_0^t\int_{\mathbb{R}_0}(||\sigma^\varepsilon(u^\varepsilon(s-),z)+u^\varepsilon(s-)||^{2p}-||u^\varepsilon(s-)||^{2p}\\&-2p||u^\varepsilon(s)||^{2p-2}(\sigma^\varepsilon(u^\varepsilon(s-),z),u^\varepsilon(s-)))N(dzds).\end{aligned} \tag{3.11}$$

Taking the supremum over the interval $[0,t]$ on the equality (3.11), we deduce

$$\begin{aligned}&\sup_{s\in[0,t]}||u^\varepsilon(s)||^{2p}+2p\int_0^t||u^\varepsilon(s)||^{2p-2}||\nabla u^\varepsilon(s)||^2ds\\&\quad+2p\alpha\int_0^t||u^\varepsilon(s)||^{2p-2}||u^\varepsilon(s)||_{\beta+1}^{\beta+1}ds\\&\le||h||^{2p}+2p\sup_{s\in[0,t]}\left|\int_0^s\int_{\mathbb{R}_0}||u^\varepsilon(r)||^{2p-2}(\sigma^\varepsilon(u^\varepsilon(r-),z),u^\varepsilon(r-))\widetilde{N}(dzdr)\right|\\&\quad+\sup_{s\in[0,t]}|\int_0^s\int_{\mathbb{R}_0}(||\sigma^\varepsilon(u^\varepsilon(r-),z)+u^\varepsilon(r-)||^{2p}-||u^\varepsilon(r-)||^{2p}\\&\quad-2p||u^\varepsilon(r)||^{2p-2}(\sigma^\varepsilon(u^\varepsilon(r-),z),u^\varepsilon(r-)))N(dzdr)|\\&=||h||^{2p}+I_3(t)+I_4(t).\end{aligned} \tag{3.12}$$

By using the Burkholder-Davis-Gundy inequality and Young's inequality, we deduce for $\varepsilon_1>0$

$$\mathbb{E}I_3(t)\le 2pC\mathbb{E}\left(\int_0^t||u^\varepsilon(s)||^{4p-2}(1+||u^\varepsilon(s)||^2+k_1||\nabla u^\varepsilon(s)||^2)ds\right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\leq pC\varepsilon_1\mathbb{E}\sup_{s\in[0,t]}||u^\varepsilon(s)||^{2p}+\frac{pC^2T}{\varepsilon_1}+\frac{2pC^2T}{\varepsilon_1}\int_0^t\mathbb{E}\sup_{r\in[0,s]}||u^\varepsilon(r)||^{2p}ds\\
&\quad+\frac{pCk_1}{\varepsilon_1}\mathbb{E}\int_0^t||u^\varepsilon(s)||^{2p-2}||\nabla u^\varepsilon(s)||^2ds. \qquad (3.13)
\end{aligned}
$$

Applying the Taylor formula, we deduce

$$
\big|||x+h||^{2p}-||x||^{2p}-2p||x+h||^{2p-2}(x,h)\big|\leq C_p(||x||^{2p-2}||h||^2+||h||^{2p}),
$$

for any $x,h\in H$. Then we get

$$
\begin{aligned}
\mathbb{E}I_4(t)\leq&\mathbb{E}\int_0^t\int_{\mathbb{R}_0}|(||u^\varepsilon(s-)+\sigma^\varepsilon(u^\varepsilon(s-),z)||^{2p}-||u^\varepsilon(s-)||^{2p}\\
&-2p||u^\varepsilon(s-)||^{2p-2}(\sigma^\varepsilon(u^\varepsilon(s-),z),u^\varepsilon(s-)))|N(dzds)\\
\leq&C_p\mathbb{E}\int_0^t\int_{\mathbb{R}_0}(||u^\varepsilon(s-)||^{2p-2}||\sigma^\varepsilon(u^\varepsilon(s-),z)||^2+||\sigma^\varepsilon(u^\varepsilon(s-),z)||^{2p})\nu(dz)ds\\
\leq&C\mathbb{E}\int_0^t(1+||u^\varepsilon(s)||^{2p}+k_1||u^\varepsilon(s)||^{2(p-1)}||\nabla u^\varepsilon(s)||^2)ds\\
\leq&CT+C\mathbb{E}\int_0^t\sup_{r\in[0,s]}||u^\varepsilon(r)||^{2p}ds+Ck_1\mathbb{E}\int_0^t||u^\varepsilon(s)||^{2(p-1)}||\nabla u^\varepsilon(s)||^2ds. \qquad (3.14)
\end{aligned}
$$

Putting (3.13)-(3.14) into (3.12), then we get

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}||u^\varepsilon(s)||^{2p}+2p\mathbb{E}\int_0^t||u^\varepsilon(s)||^{2p-2}||\nabla u^\varepsilon(s)||^2ds\\
&\quad+2p\alpha\mathbb{E}\int_0^t||u^\varepsilon(s)||^{2p-2}||u^\varepsilon(s)||_{\beta+1}^{\beta+1}ds\\
&\leq\mathbb{E}||h||^{2p}+pC\varepsilon_1\mathbb{E}\sup_{s\in[0,t]}||u^\varepsilon(s)||^{2p}+\frac{pC^2T}{\varepsilon_1}+CT\\
&\quad+(\frac{2pC^2T}{\varepsilon_1}+C)\int_0^t\mathbb{E}\sup_{r\in[0,s]}||u^\varepsilon(r)||^{2p}ds\\
&\quad+Ck_1(\frac{p}{\varepsilon_1}+1)\mathbb{E}\int_0^t||u^\varepsilon(s)||^{2p-2}||\nabla u^\varepsilon(s)||^2ds. \qquad (3.15)
\end{aligned}
$$

Choosing k_1 sufficiently small such that $Ck_1(\frac{p}{\varepsilon_1}+1)\leq p$ and $\varepsilon_1=\frac{1}{2pC}$, we apply Gronwall's lemma to obtain

$$
\begin{aligned}
&\mathbb{E}(\sup_{t\in[0,T]}||u^\varepsilon(t)||^{2p}+\int_0^T||u^\varepsilon(s)||^{2p-2}||\nabla u^\varepsilon(s)||^2ds+\int_0^T||u^\varepsilon(s)||^{2p-2}||u^\varepsilon(s)||_{\beta+1}^{\beta+1}ds)\\
&\leq C(\mathbb{E}||h||^{2p}+1). \qquad (3.16)
\end{aligned}
$$

This completes the proof of Lemma 3.2. ∎

LEMMA 3.3. *Suppose that the Assumption 2.1 and $h\in V$ hold and $\beta>3$ with any $\alpha>0$ and $\alpha\geq\frac{1}{4}$ as $\beta=3$, then there exists a positive constant $C(T)$ such that*

$$
\sup_{\varepsilon\leq\varepsilon_0}\mathbb{E}(\sup_{0\leq t\leq T}||\nabla u^\varepsilon(t)||^2+\int_0^T||\Delta u^\varepsilon(s)||^2ds+\int_0^T\int_{\mathbb{T}^3}|u^\varepsilon|^{\beta-1}|\nabla u^\varepsilon|^2dxds
$$

$$+\int_0^T ||\nabla|u^\varepsilon|^{\frac{\beta+1}{2}}||^2 ds) < \infty. \tag{3.17}$$

Proof. We apply Itô's formula to $||\nabla u^\varepsilon(t)||^2$ for $t\in[0,T]$,

$$\begin{aligned}||\nabla u^\varepsilon(t)||^2 =& ||\nabla h||^2 - 2\int_0^t ||\Delta u^\varepsilon(s)||^2 ds - 2\alpha\int_0^t\int_{\mathbb{T}^3} |u^\varepsilon|^{\beta-1}|\nabla u^\varepsilon|^2 dxds \\ &- \frac{8\alpha(\beta-1)}{(\beta+1)^2}\int_0^t ||\nabla|u^\varepsilon|^{\frac{\beta+1}{2}}||^2 ds + I_5(t) + I_6(t) + I_7(t),\end{aligned} \tag{3.18}$$

where

$$I_5(t) = 2\int_0^t (B(u^\varepsilon(s)), \Delta u^\varepsilon(s)) ds, \tag{3.19}$$

$$I_6(t) = 2\int_0^t\int_{\mathbb{R}_0} ((\sigma^\varepsilon(u^\varepsilon(s-),z), u^\varepsilon(s-)))\tilde{N}(dzds), \tag{3.20}$$

$$I_7(t) = \int_0^t\int_{\mathbb{R}_0} ||\nabla\sigma^\varepsilon(u^\varepsilon(s-),z)||^2 N(dzds). \tag{3.21}$$

For I_5, inspired by [15], since

$$0 < \frac{2}{\beta-1} < 1, \qquad \beta > 3, \tag{3.22}$$

using Young's inequality and (3.22) to estimate $I_5(t)$, we have for $\varepsilon_2 > 0$

$$\begin{aligned}I_5(t) \le& \int_0^t ||\Delta u^\varepsilon||^2 ds + \int_0^t\int_{\mathbb{T}^3} |u^\varepsilon|^2 |\nabla u^\varepsilon|^{\frac{4}{\beta-1}} |\nabla u^\varepsilon|^{2-\frac{4}{\beta-1}} dxds \\ \le& \int_0^t ||\Delta u^\varepsilon||^2 ds + \varepsilon_2\int_0^t\int_{\mathbb{T}^3} |\nabla u^\varepsilon|^2 |u^\varepsilon|^{\beta-1} dxds + C_{\varepsilon_2}\int_0^t ||\nabla u^\varepsilon||^2 ds.\end{aligned} \tag{3.23}$$

Taking the supremum and expectation over the interval $[0,t]$ on both sides of the equality (3.18), we estimate the last two items,

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t} |I_6(s)| \le& 2\mathbb{E}[\int_0^t\int_{\mathbb{R}_0} ((\sigma^\varepsilon(u^\varepsilon(s-),z), u^\varepsilon(s-)))^2 \nu(dz)ds]^{\frac{1}{2}} \\ \le& \frac{1}{2}\mathbb{E}\sup_{0\le s\le t} ||\nabla u^\varepsilon(s)||^2 + C\mathbb{E}\int_0^t ||\nabla u^\varepsilon(s)||^2 ds,\end{aligned} \tag{3.24}$$

similarly, we have

$$\mathbb{E}\sup_{0\le s\le t} |I_7(s)| \le C\mathbb{E}\int_0^t ||\nabla u^\varepsilon(s)||^2 ds. \tag{3.25}$$

Putting (3.23)-(3.25) into (3.18), then we have

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\sup_{0\le s\le t} ||\nabla u^\varepsilon(s)||^2 + \mathbb{E}\int_0^t ||\Delta u^\varepsilon(s)||^2 ds + (2\alpha - \varepsilon_2)\mathbb{E}\int_0^t\int_{\mathbb{T}^3} |u^\varepsilon|^{\beta-1}|\nabla u^\varepsilon|^2 dxds \\ &+ \frac{8\alpha(\beta-1)}{(\beta+1)^2}\mathbb{E}\int_0^t ||\nabla|u^\varepsilon|^{\frac{\beta+1}{2}}||^2 ds \le \mathbb{E}||\nabla h||^2 + (2C + C_{\varepsilon_2})\int_0^t \mathbb{E}\sup_{r\in[0,s]} ||\nabla u^\varepsilon(r)||^2 ds.\end{aligned} \tag{3.26}$$

Choosing ε_2 sufficiently small and applying Gronwall's lemma, we have

$$\mathbb{E}\sup_{0\le s\le t}||\nabla u^\varepsilon(s)||^2+\mathbb{E}\int_0^t||\Delta u^\varepsilon(s)||^2ds+\mathbb{E}\int_0^t\int_{\mathbb{T}^3}|u^\varepsilon|^{\beta-1}|\nabla u^\varepsilon|^2dxds$$
$$+\mathbb{E}\int_0^t||\nabla|u^\varepsilon|^{\frac{\beta+1}{2}}||^2ds\le C(t)(\mathbb{E}||\nabla h||^2+1). \tag{3.27}$$

By Theorem 4.1 in [13], we have $1-\frac{1}{2\theta}\ge 0$ and $\alpha-\frac{\theta}{2}\ge 0$ for any $\theta>0$. Then we can get the above estimate easily for $\alpha\ge\frac{1}{4}$ as $\beta=3$. This completes the proof of Lemma 3.3. □

LEMMA 3.4. *Suppose that the Assumption 2.1, $p>1$ and $h\in V$ hold and $\beta>3$ with any $\alpha>0$ and $\alpha\ge\frac{1}{4}$ as $\beta=3$, then there exists a positive constant $C(T)$ such that*

$$\sup_{\varepsilon\le\varepsilon_0}\mathbb{E}(\sup_{0\le t\le T}||\nabla u^\varepsilon(t)||^{2p}+\int_0^T||\nabla u^\varepsilon(s)||^{2p-2}||\Delta u^\varepsilon(s)||^2ds$$
$$+\int_0^T||\nabla u^\varepsilon(s)||^{2p-2}\int_{\mathbb{T}^3}|u^\varepsilon|^{\beta-1}|\nabla u^\varepsilon|^2dxds$$
$$+\int_0^T||\nabla u^\varepsilon(s)||^{2p-2}||\nabla|u^\varepsilon|^{\frac{\beta+1}{2}}||^2ds)<\infty. \tag{3.28}$$

Proof. We apply Itô's formula to $||\nabla u^\varepsilon(t)||^{2p}$ for $t\in[0,T]$,

$$||\nabla u^\varepsilon(t)||^{2p}=||\nabla h||^{2p}-2p\int_0^t||\nabla u^\varepsilon(s)||^{2p-2}||\Delta u^\varepsilon(s)||^2ds$$
$$-2p\alpha\int_0^t||\nabla u^\varepsilon||^{2p-2}\int_{\mathbb{T}^3}|u^\varepsilon|^{\beta-1}|\nabla u^\varepsilon|^2dxds$$
$$-\frac{8p\alpha(\beta-1)}{(\beta+1)^2}\int_0^t||\nabla u^\varepsilon||^{2p-2}||\nabla|u^\varepsilon|^{\frac{\beta+1}{2}}||^2ds+I_8(t)+I_9(t)+I_{10}(t), \tag{3.29}$$

where

$$I_8(t)=2p\int_0^t||\nabla u^\varepsilon(s)||^{2p-2}(B(u^\varepsilon(s)),\Delta u^\varepsilon(s))ds, \tag{3.30}$$

$$I_9(t)=2p\int_0^t\int_{\mathbb{R}_0}||\nabla u^\varepsilon(s)||^{2p-2}((\sigma^\varepsilon(u^\varepsilon(s-),z),u^\varepsilon(s-)))\tilde{N}(dzds), \tag{3.31}$$

$$I_{10}(t)=\int_0^t\int_{\mathbb{R}_0}(||\nabla(u^\varepsilon(s-)+\sigma^\varepsilon(u^\varepsilon(s-),z))||^{2p}$$
$$-||\nabla u^\varepsilon(s-)||^{2p}-2p||\nabla u^\varepsilon(s-)||^{2(p-1)}((u^\varepsilon(s-),\sigma^\varepsilon(u^\varepsilon(s-),z))))N(dzds). \tag{3.32}$$

For $I_8(t)$, inspired by [15], since

$$0<\frac{2}{\beta-1}<1,\qquad \beta>3, \tag{3.33}$$

using Young's inequality and (3.33) to estimate $I_8(t)$, we have for $\varepsilon_3>0$

$$I_8(t)\le p\int_0^t||\nabla u^\varepsilon||^{2(p-1)}||\Delta u^\varepsilon||^2ds+p\int_0^t||\nabla u^\varepsilon||^{2(p-1)}\int_{\mathbb{T}^3}|u^\varepsilon|^2|\nabla u^\varepsilon|^{\frac{4}{\beta-1}}|\nabla u^\varepsilon|^{2-\frac{4}{\beta-1}}dxds$$

$$\leq p\int_0^t ||\nabla u^\varepsilon(s)||^{2(p-1)}||\Delta u^\varepsilon||^2 ds+\varepsilon_3\int_0^t ||\nabla u^\varepsilon(s)||^{2(p-1)}\int_{\mathbb{T}^3}|\nabla u^\varepsilon|^2|u^\varepsilon|^{\beta-1}dxds$$
$$+C_{\varepsilon_3}\int_0^t ||\nabla u^\varepsilon(s)||^{2p}ds. \tag{3.34}$$

For $I_9(t)$, applying the Burkholder-Davis-Gundy inequality and Young's inequality, we get

$$\mathbb{E}\sup_{0\leq s\leq t} I_9(s)\leq 2pC\mathbb{E}[\int_0^t\int_{\mathbb{R}_0}||\nabla u^\varepsilon(s-)||^{4(p-1)}||\nabla u^\varepsilon(s-)||^2||\nabla\sigma^\varepsilon(u^\varepsilon(s-),z)||^2\nu(dz)ds]^{\frac{1}{2}}$$
$$\leq\frac{1}{2}\mathbb{E}\sup_{0\leq s\leq t}||\nabla u^\varepsilon(s)||^{2p}+C\mathbb{E}\int_0^t||\nabla u^\varepsilon(s)||^{2p}ds. \tag{3.35}$$

By the Taylor formula, we get

$$\big|||\nabla(x+h)||^{2p}-||\nabla x||^{2p}-2p||\nabla x||^{2p-2}((x,h))\big|\leq C(||\nabla x||^{2p-2}||\nabla h||^2+||\nabla h||^{2p}), \tag{3.36}$$

for any $x,h\in V$. For $I_{10}(t)$, we get

$$\mathbb{E}\sup_{0\leq s\leq t} I_{10}(s)$$
$$\leq C_p\mathbb{E}\int_0^t\int_{\mathbb{R}_0}(||\nabla u^\varepsilon(s-)||^{2p-2}||\nabla\sigma^\varepsilon(u^\varepsilon(s-),z)||^2+||\nabla\sigma^\varepsilon(u^\varepsilon(s-),z)||^{2p})\nu(dz)ds$$
$$\leq C\int_0^t||\nabla u^\varepsilon(s)||^{2p}ds. \tag{3.37}$$

Putting (3.34)-(3.37) into (3.29), we have

$$\frac{1}{2}\mathbb{E}\sup_{0\leq s\leq t}||\nabla u^\varepsilon(s)||^{2p}+p\mathbb{E}\int_0^t||\nabla u^\varepsilon(s)||^{2p-2}||\Delta u^\varepsilon(s)||^2ds$$
$$+(2p\alpha-\varepsilon_3)\mathbb{E}\int_0^t||\nabla u^\varepsilon||^{2p-2}\int_{\mathbb{T}^3}|u^\varepsilon|^{\beta-1}|\nabla u^\varepsilon|^2dxds$$
$$+\frac{8p\alpha(\beta-1)}{(\beta+1)^2}\mathbb{E}\int_0^t||\nabla u^\varepsilon||^{2p-2}||\nabla|u^\varepsilon|^{\frac{\beta+1}{2}}||^2ds$$
$$\leq\mathbb{E}||\nabla h||^{2p}+(2C+C_{\varepsilon_3})\mathbb{E}\int_0^t||\nabla u^\varepsilon(s)||^{2p}ds. \tag{3.38}$$

Choosing sufficiently small ε_3 such that $2p\alpha-\varepsilon_3\geq p\alpha$, applying Gronwall's lemma, we have

$$\mathbb{E}\sup_{0\leq s\leq t}||\nabla u^\varepsilon(s)||^{2p}+\mathbb{E}\int_0^t||\nabla u^\varepsilon(s)||^{2p-2}||\Delta u^\varepsilon(s)||^2ds$$
$$+\mathbb{E}\int_0^t||\nabla u^\varepsilon||^{2p-2}\int_{\mathbb{T}^3}|u^\varepsilon|^{\beta-1}|\nabla u^\varepsilon|^2dxds$$
$$+\mathbb{E}\int_0^t||\nabla u^\varepsilon||^{2p-2}||\nabla|u^\varepsilon|^{\frac{\beta+1}{2}}||^2ds$$
$$\leq C(t)(\mathbb{E}||\nabla h||^{2p}+1). \tag{3.39}$$

By Theorem 4.1 in [13], we have $1-\frac{1}{2\theta}\geq 0$ and $\alpha-\frac{\theta}{2}\geq 0$ for any $\theta>0$. Then we can get the above estimate easily for $\alpha\geq\frac{1}{4}$ as $\beta=3$. This completes the proof of Lemma 3.4. □

LEMMA 3.5. *Suppose that the Assumption 2.1 and Assumption 2.2 and $h\in H^2$ hold. For any $N>0$, we define*

$$\tau_N^\varepsilon=T\wedge\inf\{t\geq 0:||\nabla u^\varepsilon(t)||^2>N\}\wedge\inf\{t\geq 0:\int_0^t||\Delta u^\varepsilon(s)||^2ds>N\}, \tag{3.40}$$

here, we set $\inf\{\emptyset\}=\infty$. *We deduce for $3<\beta<5$ with any $\alpha>0$ and $\alpha\geq\frac{1}{4}$ as $\beta=3$,*

$$\sup_{\varepsilon\leq\varepsilon_0}\mathbb{E}(\sup_{0\leq s\leq\tau_N^\varepsilon}||\Delta u^\varepsilon(s)||^2+\int_0^{\tau_N^\varepsilon}||u^\varepsilon(s)||_{H^3}^2ds)<\infty. \tag{3.41}$$

Proof. We apply Itô's formula to $||\Delta u^\varepsilon(t)||^2$ for $t\in[0,T]$,

$$\begin{aligned}
||\Delta u^\varepsilon(t)||^2=&||\Delta h||^2-2\int_0^t||u^\varepsilon(s)||_{H^3}^2ds-2\int_0^t\langle B(u^\varepsilon(s)),u^\varepsilon(s)\rangle_{H^2}ds\\
&-2\int_0^t\langle g(u^\varepsilon(s)),u^\varepsilon(s)\rangle_{H^2}ds+2\int_0^t\int_{\mathbb{R}_0}\langle\sigma^\varepsilon(u^\varepsilon(s-),z),u^\varepsilon(s-)\rangle_{H^2}\tilde{N}(dzds)\\
&+\int_0^t\int_{\mathbb{R}_0}||\Delta\sigma^\varepsilon(u^\varepsilon(s-),z)||^2N(dzds)\\
=&||\Delta h||^2-2\int_0^t||u^\varepsilon(s)||_{H^3}^2ds+\sum_{i=11}^{14}I_i(t).
\end{aligned} \tag{3.42}$$

For $I_{11}(t)$, by using the Sobolev embedding, we get

$$\begin{aligned}
I_{11}(t)\leq&2|\int_0^t\int_{\mathbb{T}^3}\Delta(u^\varepsilon\nabla u^\varepsilon)\Delta u^\varepsilon dxds|\\
\leq&\frac{1}{2}\int_0^t||u^\varepsilon(s)||_{H^3}^2ds+C\int_0^t||\nabla(u^\varepsilon\nabla u^\varepsilon)||^2ds\\
\leq&\frac{1}{2}\int_0^t||u^\varepsilon(s)||_{H^3}^2ds+C\int_0^t(||u^\varepsilon||_\infty^2||\Delta u^\varepsilon||^2+||\nabla u^\varepsilon||_3^2||\nabla u^\varepsilon||_6^2)ds\\
\leq&\frac{1}{2}\int_0^t||u^\varepsilon(s)||_{H^3}^2ds+C\int_0^t||\Delta u^\varepsilon(s)||^4ds.
\end{aligned} \tag{3.43}$$

For $I_{12}(t)$, similarly, we get for $3<\beta<5$ with any $\alpha>0$ and $\alpha\geq\frac{1}{4}$ as $\beta=3$,

$$\begin{aligned}
I_{12}(t)\leq&2\alpha|\int_0^t\int_{\mathbb{T}^3}\Delta(|u^\varepsilon|^{\beta-1}u^\varepsilon)\Delta u^\varepsilon dxds|\\
\leq&\frac{1}{2}\int_0^t||u^\varepsilon(s)||_{H^3}^2ds+C\int_0^t||\nabla(|u^\varepsilon|^{\beta-1}u^\varepsilon)||^2ds\\
\leq&\frac{1}{2}\int_0^t||u^\varepsilon(s)||_{H^3}^2ds+C\int_0^t||u^\varepsilon||_{3(\beta-1)}^{2(\beta-1)}||\nabla u^\varepsilon||_6^2ds\\
\leq&\frac{1}{2}\int_0^t||u^\varepsilon(s)||_{H^3}^2ds+C\int_0^t||\nabla u^\varepsilon||^{\frac{8}{\beta-1}}||u^\varepsilon||_{3(\beta+1)}^{\frac{2(\beta-3)}{\beta-1}(\beta+1)}||\Delta u^\varepsilon||^2ds\\
\leq&\frac{1}{2}\int_0^t||u^\varepsilon(s)||_{H^3}^2ds+C\int_0^t(||\nabla u^\varepsilon||^{\frac{8}{5-\beta}}+||u^\varepsilon||_{3(\beta+1)}^{\beta+1})||\Delta u^\varepsilon||^2ds
\end{aligned}$$

$$\leq \frac{1}{2}\int_0^t ||u^\varepsilon(s)||_{H^3}^2 ds + C\int_0^t (||\nabla u^\varepsilon||^{\frac{8}{5-\beta}} + ||\nabla |u^\varepsilon|^{\frac{\beta+1}{2}}||^2)||\Delta u^\varepsilon||^2 ds. \tag{3.44}$$

Putting (3.43) and (3.44) into (3.42), we get

$$\begin{aligned}||\Delta u^\varepsilon(t)||^2 + \int_0^t ||u^\varepsilon(s)||_{H^3}^2 ds \leq ||\Delta h||^2 + C\int_0^t (||\nabla u^\varepsilon||^{\frac{8}{5-\beta}} + ||\nabla |u^\varepsilon|^{\frac{\beta+1}{2}}||^2 \\ + ||\Delta u^\varepsilon||^2)||\Delta u^\varepsilon||^2 ds + I_{13}(t) + I_{14}(t).\end{aligned} \tag{3.45}$$

Taking the supremum on $[0,\tau_N^\varepsilon]$ and applying the Gronwall inequality, then we get

$$\begin{aligned}&\sup_{0\leq t\leq \tau_N^\varepsilon} ||\Delta u^\varepsilon(t)||^2 + \int_0^{\tau_N^\varepsilon} ||u^\varepsilon(s)||_{H^3}^2 ds \\ &\leq [||\Delta h||^2 + \sup_{0\leq t\leq \tau_N^\varepsilon} I_{13}(t) + \sup_{0\leq t\leq \tau_N^\varepsilon} I_{14}(t)] \times e^{C\int_0^t (||\nabla u^\varepsilon||^{\frac{8}{5-\beta}} + ||\nabla |u^\varepsilon|^{\frac{\beta+1}{2}}||^2 + ||\Delta u^\varepsilon||^2) ds} \\ &\leq [||\Delta h||^2 + \sup_{0\leq t\leq \tau_N^\varepsilon} I_{13}(t) + \sup_{0\leq t\leq \tau_N^\varepsilon} I_{14}(t)] e^{C(T)}.\end{aligned} \tag{3.46}$$

Taking the expectation and using Burkholder-Davis-Gundy inequality and Assumption 2.2, we get

$$\begin{aligned}&\mathbb{E} \sup_{0\leq s\leq \tau_N^\varepsilon} |I_{13}(s)| \\ &\leq 2\mathbb{E} \sup_{0\leq s\leq \tau_N^\varepsilon} ||\Delta u^\varepsilon(s)|| [\int_0^{\tau_N^\varepsilon}\int_{\mathbb{R}_0} ||\Delta\sigma^\varepsilon(u^\varepsilon(s-),z)||^2 \nu(dz) ds]^{\frac{1}{2}} \\ &\leq C[\mathbb{E} \sup_{0\leq s\leq \tau_N^\varepsilon} ||\Delta u^\varepsilon(s)||^2]^{\frac{1}{2}} [\mathbb{E}\int_0^{\tau_N^\varepsilon}\int_{\mathbb{R}_0} ||\Delta\sigma^\varepsilon(u^\varepsilon(s-),z)||^2 \nu(dz) ds]^{\frac{1}{2}} \\ &\leq \frac{1}{2}\mathbb{E} \sup_{0\leq s\leq \tau_N^\varepsilon} ||\Delta u^\varepsilon(s)||^2 + C\mathbb{E}\int_0^{\tau_N^\varepsilon} ||\Delta u^\varepsilon(s)||^2 ds.\end{aligned} \tag{3.47}$$

By using the Assumption 2.2, we get

$$\begin{aligned}\mathbb{E} \sup_{0\leq s\leq \tau_N^\varepsilon} |I_{14}(s)| &\leq C\mathbb{E}\int_0^{\tau_N^\varepsilon}\int_{\mathbb{R}_0} ||\Delta\sigma^\varepsilon(u^\varepsilon(s-),z)||^2 \nu(dz) ds \\ &\leq C\mathbb{E}\int_0^{\tau_N^\varepsilon} ||\Delta u^\varepsilon(s)||^2 ds.\end{aligned} \tag{3.48}$$

Putting (3.47) and (3.48) into (3.46), we get

$$\mathbb{E}(\sup_{0\leq s\leq \tau_N^\varepsilon} ||\Delta u^\varepsilon(s)||^2 + \int_0^{\tau_N^\varepsilon} ||u^\varepsilon(s)||_{H^3}^2 ds) \leq C(T,N)(1+||\Delta h||^2) < \infty. \tag{3.49}$$

This completes the proof of Lemma 3.5. □

LEMMA 3.6. *Suppose that the Assumption 2.1 and Assumption 2.2 and $h\in H^2$ hold and $3<\beta<5$ with any $\alpha>0$ and $\alpha\geq \frac{1}{4}$ as $\beta=3$. Then the family $\{u^\varepsilon, \varepsilon\leq\varepsilon_0\}$ is tight on the space $D([0,T],V)$.*

Proof. Inspired by Aldou's tightness criterion [1], it suffices to prove that:

(i) there exists a positive constant L_η such that for every $0<\eta<1$,

$$\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(\sup_{0\le t\le T}||\Delta u^\varepsilon(t)||>L_\eta)<\eta; \tag{3.50}$$

(ii) let $\zeta^\varepsilon+\delta=T\wedge(\zeta^\varepsilon+\delta)$ and for any $\eta>0$. The stopping time $0\le\zeta^\varepsilon\le T$ with respect to the natural filtration generated by $\{u^\varepsilon(s),s\le t\}$,

$$\lim_{\delta\to 0}\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(||\nabla(u^\varepsilon(\zeta^\varepsilon+\delta)-u^\varepsilon(\zeta^\varepsilon))||>\eta)=0. \tag{3.51}$$

For (i), by using Lemma 3.1-Lemma 3.5, we have

$$\begin{aligned}\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(\tau_N^\varepsilon<T)&\le\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(\int_0^T||\Delta u^\varepsilon(s)||^2ds>N)+\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(\sup_{0\le s\le T}||\nabla u^\varepsilon(s)||^2>N)\\&\le\frac{1}{N}\sup_{\varepsilon\le\varepsilon_0}\mathbb{E}\int_0^T||\Delta u^\varepsilon(s)||^2ds+\frac{1}{N}\sup_{\varepsilon\le\varepsilon_0}\mathbb{E}\sup_{0\le s\le T}||\nabla u^\varepsilon(s)||^2\\&\le\frac{C}{N}.\end{aligned} \tag{3.52}$$

By virtue of (3.41), we deduce that for any $L>0$,

$$\begin{aligned}&\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(\sup_{0\le t\le T}||\Delta u^\varepsilon(t)||>L)\\&\le\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(\sup_{0\le t\le T}||\Delta u^\varepsilon(t)||>L,\tau_N^\varepsilon=T)+\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(\tau_N^\varepsilon<T)\\&\le\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(\sup_{0\le t\le\tau_N^\varepsilon}||\Delta u^\varepsilon(t)||>L)+\frac{C}{N}\\&\le\frac{1}{L^2}\sup_{\varepsilon\le\varepsilon_0}\mathbb{E}\sup_{0\le t\le\tau_N^\varepsilon}||\Delta u^\varepsilon(t)||^2+\frac{C}{N}\\&\le\frac{C_N}{L^2}+\frac{C}{N}.\end{aligned} \tag{3.53}$$

For any $\eta>0$, choosing large enough $N>0$ and $L>0$ such that $\frac{C_N}{L^2}+\frac{C}{N}<\eta$. This completes the proof of (i).

For (ii), we deduce that for every $\eta>0$,

$$\begin{aligned}&\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(||\nabla(u^\varepsilon(\zeta^\varepsilon+\delta)-u^\varepsilon(\zeta^\varepsilon))||>\eta)\\&\le\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(||\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}Au^\varepsilon(s)ds||_V>\frac{\eta}{4})\\&\quad+\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(||\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}B(u^\varepsilon(s))ds||_V>\frac{\eta}{4})\\&\quad+\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(||\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}g(u^\varepsilon(s))ds||_V>\frac{\eta}{4})\\&\quad+\sup_{\varepsilon\le\varepsilon_0}\mathbb{P}(||\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}\int_{\mathbb{R}_0}\sigma^\varepsilon(u^\varepsilon(s-),z)\tilde{N}(dzds)||_V>\frac{\eta}{4})\\&=J_1+J_2+J_3+J_4.\end{aligned} \tag{3.54}$$

For J_1, by (3.41) and (3.52), Hölder's inequality and Chebyshev's inequality, then we have for $N>0$,

$$\begin{aligned}
J_1 &\leq \sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\delta\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}||Au^\varepsilon(s)||_V^2ds>\frac{\eta^2}{16})\\
&\leq \sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\delta\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}||Au^\varepsilon(s)||_V^2ds>\frac{\eta^2}{16},\tau_N^\varepsilon=T)+\sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\tau_N^\varepsilon<T)\\
&\leq \frac{16}{\eta^2}\delta\sup_{\varepsilon\leq\varepsilon_0}\mathbb{E}\int_0^{\tau_N^\varepsilon}||u^\varepsilon(s)||_{H^3}^2ds+\frac{C}{N}\\
&\leq \frac{C_N\delta}{\eta^2}+\frac{C}{N}.
\end{aligned}\tag{3.55}$$

Inspired by [24], we also get

$$\begin{aligned}
||B(u^\varepsilon(s))||_V^2 &\leq C(||u^\varepsilon(s)||_\infty^2||\Delta u^\varepsilon(s)||^2+||\nabla u^\varepsilon(s)||_4^4)\\
&\leq C||\nabla u^\varepsilon(s)||||\Delta u^\varepsilon(s)||^3.
\end{aligned}$$

For J_2, we get

$$\begin{aligned}
J_2 &\leq \sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}||\Delta u^\varepsilon(s)||^{\frac{3}{2}}||\nabla u^\varepsilon(s)||^{\frac{1}{2}}ds>\frac{\eta}{4C})\\
&\leq \sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}||\Delta u^\varepsilon(s)||^{\frac{3}{2}}||\nabla u^\varepsilon(s)||^{\frac{1}{2}}ds>\frac{\eta}{4C},\tau_N^\varepsilon=T)+\sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\tau_N^\varepsilon<T)\\
&\leq \sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\int_{\zeta^\varepsilon}^{(\zeta^\varepsilon+\delta)\wedge\tau_N^\varepsilon}||\Delta u^\varepsilon(s)||^{\frac{3}{2}}||\nabla u^\varepsilon(s)||^{\frac{1}{2}}ds>\frac{\eta}{4C})+\frac{C}{N}\\
&\leq \frac{4C}{\eta}\sup_{\varepsilon\leq\varepsilon_0}[(\mathbb{E}\int_{\zeta^\varepsilon}^{(\zeta^\varepsilon+\delta)\wedge\tau_N^\varepsilon}||\nabla u^\varepsilon(s)||^2ds)^{\frac{1}{4}}(\mathbb{E}\int_{\zeta^\varepsilon}^{(\zeta^\varepsilon+\delta)\wedge\tau_N^\varepsilon}||\Delta u^\varepsilon(s)||^2ds)^{\frac{3}{4}}]+\frac{C}{N}\\
&\leq \frac{C}{\eta}\delta\sup_{\varepsilon\leq\varepsilon_0}(\mathbb{E}\sup_{0\leq s\leq T}||\nabla u^\varepsilon(s)||^2)^{\frac{1}{4}}\times\sup_{\varepsilon\leq\varepsilon_0}(\mathbb{E}\sup_{0\leq s\leq\tau_N^\varepsilon}||\Delta u^\varepsilon(s)||^2)^{\frac{3}{4}}+\frac{C}{N}\\
&\leq \frac{C_N}{\eta}\delta+\frac{C}{N}.
\end{aligned}$$

For J_3, we deduce

$$\begin{aligned}
J_3 &\leq \sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}||u^\varepsilon(s)||_\infty^{\beta-1}||\nabla u^\varepsilon(s)||ds>\frac{\eta}{4C})\\
&\leq \sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}||\nabla u^\varepsilon(s)||^{\frac{\beta+1}{2}}||\Delta u^\varepsilon(s)||^{\frac{\beta-1}{2}}ds>\frac{\eta}{4C})\\
&\leq \sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}||\nabla u^\varepsilon(s)||^{\frac{\beta+1}{2}}||\Delta u^\varepsilon(s)||^{\frac{\beta-1}{2}}ds>\frac{\eta}{4C},\tau_N^\varepsilon=T)+\sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\tau_N^\varepsilon<T)\\
&\leq \sup_{\varepsilon\leq\varepsilon_0}\mathbb{P}(\int_{\zeta^\varepsilon}^{(\zeta^\varepsilon+\delta)\wedge\tau_N^\varepsilon}||\nabla u^\varepsilon(s)||^{\frac{\beta+1}{2}}||\Delta u^\varepsilon(s)||^{\frac{\beta-1}{2}}ds>\frac{\eta}{4C})+\frac{C}{N}\\
&\leq \frac{4C}{\eta}\sup_{\varepsilon\leq\varepsilon_0}(\mathbb{E}\int_{\zeta^\varepsilon}^{(\zeta^\varepsilon+\delta)\wedge\tau_N^\varepsilon}||\nabla u^\varepsilon(s)||^{\frac{2(\beta+1)}{5-\beta}}ds)^{\frac{5-\beta}{4}}(\mathbb{E}\int_{\zeta^\varepsilon}^{(\zeta^\varepsilon+\delta)\wedge\tau_N^\varepsilon}||\Delta u^\varepsilon(s)||^2ds)^{\frac{\beta-1}{4}}
\end{aligned}$$

$$\leq \frac{C}{\eta}\delta \sup_{\varepsilon\leq\varepsilon_0}(\mathbb{E}\sup_{0\leq s\leq T}||\nabla u^\varepsilon(s)||^{\frac{2(\beta+1)}{5-\beta}})^{\frac{5-\beta}{4}} \times \sup_{\varepsilon\leq\varepsilon_0}(\mathbb{E}\sup_{0\leq s\leq\tau_N^\varepsilon}||\Delta u^\varepsilon(s)||^2)^{\frac{\beta-1}{4}} + \frac{C}{N}$$
$$\leq \frac{C\delta}{\eta} + \frac{C}{N}. \tag{3.56}$$

For J_4, by the Assumption 2.1, then we have

$$\begin{aligned} J_4 &\leq \frac{16}{\eta^2}\sup_{\varepsilon\leq\varepsilon_0}\mathbb{E}||\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}\int_{\mathbb{R}_0}\sigma^\varepsilon(u^\varepsilon(s-),z)\tilde{N}(dzds)||_V^2 \\ &\leq \frac{C}{\eta^2}\sup_{\varepsilon\leq\varepsilon_0}\mathbb{E}\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}\int_{\mathbb{R}_0}||\sigma^\varepsilon(u^\varepsilon(s-),z)||_V^2\nu(dz)ds \\ &\leq \frac{C}{\eta^2}\sup_{\varepsilon\leq\varepsilon_0}\mathbb{E}\int_{\zeta^\varepsilon}^{\zeta^\varepsilon+\delta}||\nabla u^\varepsilon(s)||^2ds \\ &\leq \frac{C}{\eta^2}\delta. \end{aligned} \tag{3.57}$$

Putting (3.55)-(3.57) into (3.54), let $\delta\to 0$ and $N\to\infty$, then we get (3.51). This completes the proof of (ii). □

3.1. The weak convergence. Let μ_ε and μ be the laws of u^ε and u on the spaces $D([0,T],V)$ and $C([0,T],V)$.

THEOREM 3.1. *Let the Assumption 2.1 and Assumption 2.2, (H.3) and $h\in H^2$ hold and $3<\beta<5$ with any $\alpha>0$ and $\alpha\geq\frac{1}{4}$ as $\beta=3$. For any $T>0$, μ_ε converges weakly to μ, for $\varepsilon\to 0$, on the space $D([0,T],V)$ equipped with the Skorohod topology.*

Proof. By virtue of Lemma 3.6, the family $\{\mu_\varepsilon,\varepsilon\leq\varepsilon_0\}$ is tight in $D([0,T],V)$. Let μ_0 be the weak limit of any convergent subsequence $\{\mu_{\varepsilon_n}\}$. We need to prove $\mu=\mu_0$. We first need to prove the following three steps. In step 1, we will prove that μ_0 is supported on the space $C([0,T],V)$. We get that for $\eta>0$ and $N>0$,

$$\begin{aligned} &\mathbb{P}(\sup_{0<t\leq T}||u^\varepsilon(t)-u^\varepsilon(t-)||_V>\eta) \\ \leq&\mathbb{P}(\sup_{0<t\leq T}\sup_{z\in\mathbb{R}_0}||\sigma^\varepsilon(u^\varepsilon(t-),z)||_V>\eta) \\ \leq&\mathbb{P}(\sup_{0<t\leq T}\sup_{z\in\mathbb{R}_0}||\sigma^\varepsilon(u^\varepsilon(t-),z)||_V>\eta,\sup_{0\leq t\leq T}||\nabla u^\varepsilon(t)||\leq N)+\mathbb{P}(\sup_{0\leq t\leq T}||\nabla u^\varepsilon(t)||>N) \\ \leq&\mathbb{P}(\sup_{||\nabla x||\leq N}\sup_{z\in\mathbb{R}_0}||\sigma^\varepsilon(x,z)||_V>\eta)+\frac{1}{N^2}\sup_{\varepsilon\leq\varepsilon_0}\mathbb{E}\sup_{0\leq t\leq T}||\nabla u^\varepsilon(t)||^2. \end{aligned}$$

Applying Lemma 3.3 and (2.4) and let $\varepsilon\to 0$ and $N\to\infty$, then we get

$$\sup_{0<t\leq T}||u^\varepsilon(t)-u^\varepsilon(t-)||_V\to 0 \quad \text{in probability as} \quad \varepsilon\to 0.$$

Thence, by the Theorem 13.4 in [2], we get that μ_0 is supported on the space $C([0,T],V)$. Then we get $\mu_{\varepsilon_n}\to\mu_0$.

In step 2, let $f(x)=((x,l_k))((x,l_j))$, $x\in V$, for any $k,j\in\mathbb{N}$. Then ∇f and f'' are defined by, respectively,

$$\nabla f(x)=((x,l_j))l_k+((x,l_k))l_j, \tag{3.58}$$
$$f''(x)=l_j\otimes l_k+l_k\otimes l_j. \tag{3.59}$$

Let

$$\begin{aligned} L^\varepsilon f(x) = & -((A\nabla f(x),x)) - ((B(x),\nabla f(x))) - ((g(x),\nabla f(x))) \\ & + \int_{\mathbb{R}_0} [f(x+\sigma^\varepsilon(x,z)) - f(x) - ((\nabla f(x),\sigma^\varepsilon(x,z)))]\nu(dz), \end{aligned} \tag{3.60}$$

$$Lf(x) = -((A\nabla f(x),x)) - ((B(x),\nabla f(x))) - ((g(x),\nabla f(x))) + \frac{1}{2}((f''(x)\sigma(x),\sigma(x))). \tag{3.61}$$

Applying the Itô's formula, we get

$$\begin{aligned} & f(u^\varepsilon(t)) - f(h) - \int_0^t L^\varepsilon f(u^\varepsilon(s))ds \\ = & \int_0^t \int_{\mathbb{R}_0} [f(u^\varepsilon(s-) + \sigma^\varepsilon(u^\varepsilon(s-),z)) - f(u^\varepsilon(s-))]\tilde{N}(dzds) \end{aligned} \tag{3.62}$$

is a martingale. Let $X_t(\omega) = \omega(t), \omega \in D([0,T],V)$ be the coordinate process in $D([0,T],V)$. Applying the martingale property, we have that for any $0 \le s_0 < s_1 < \cdots < s_n \le s < t$ and $f_0, f_1, \cdots, f_n \in C_b(V)$,

$$\mathbb{E}^{\mu_\varepsilon}[(f(X_t) - f(X_s) - \int_s^t L^\varepsilon f(X_r)dr) f_0(X_{s_0}) \cdots f_n(X_{s_n})] = 0. \tag{3.63}$$

Let

$$G_\varepsilon(x) = |\int_{\mathbb{R}_0} ((\sigma^\varepsilon(x,z),l_k))((\sigma^\varepsilon(x,z),l_j))\nu(dz) - ((\sigma(x),l_k))((\sigma(x),l_j))|, \quad x \in V. \tag{3.64}$$

Applying (3.60) and (3.61), then we get

$$|L^\varepsilon f(X_r) - Lf(X_r)| = G_\varepsilon(X_r). \tag{3.65}$$

We next claim that

$$\lim_{n\to\infty} \mathbb{E}^{\mu_{\varepsilon_n}}[\int_s^t |L^{\varepsilon_n} f(X_r) - Lf(X_r)|dr] = 0. \tag{3.66}$$

Then, we have

$$\mathbb{E}^{\mu_{\varepsilon_n}}[\int_s^t |L^{\varepsilon_n} f(X_r) - Lf(X_r)|dr] = \int_s^t \mathbb{E} G_{\varepsilon_n}(u^{\varepsilon_n}(r))dr. \tag{3.67}$$

Since

$$\sup_{\varepsilon \le \varepsilon_0} G_\varepsilon(x) \le C(1 + ||\nabla x||^2), \tag{3.68}$$

applying the dominated convergence theorem, we need to prove that for any $r \in [0,T]$,

$$\lim_{n\to\infty} EG_{\varepsilon_n}(u^{\varepsilon_n}(r)) = 0. \tag{3.69}$$

By the Skorohod's representation theorem and μ_{ε_n} converging weakly to μ_0, we assume that $u^{\varepsilon_n}(r)$ converges to a $V-$valued random variable u^0. Then we get that $\{||\nabla u^{\varepsilon_n}(r)||^2\}_{n\ge 1}$ is uniformly integrable. Moreover, we have $u^0 \in L^2(\Omega, V)$ and

$$\lim_{n\to\infty} \mathbb{E}||u^{\varepsilon_n}(r) - u^0||_V^2 = 0. \tag{3.70}$$

Applying the dominated convergence theorem and (3.68) and (2.5), we deduce

$$\lim_{n\to\infty}\mathbb{E}G_{\varepsilon_n}(u^0)=0. \tag{3.71}$$

Then, we will next prove

$$\lim_{n\to\infty}\mathbb{E}|G_{\varepsilon_n}(u^{\varepsilon_n}(r))-G_{\varepsilon_n}(u^0)|=0. \tag{3.72}$$

Then we deduce

$$\begin{aligned}
&\mathbb{E}|G_{\varepsilon_n}(u^{\varepsilon_n}(r))-G_{\varepsilon_n}(u^0)|\\
\le&\mathbb{E}|\int_{\mathbb{R}_0}((\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z),l_k))((\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z),l_j))\nu(dz)\\
&-\int_{\mathbb{R}_0}((\sigma^{\varepsilon_n}(u^0,z),l_k))((\sigma^{\varepsilon_n}(u^0,z),l_j))\nu(dz)|\\
&+\mathbb{E}|((\sigma(u^{\varepsilon_n}(r)),l_k))((\sigma(u^{\varepsilon_n}(r)),l_j))-((\sigma(u^0),l_k))((\sigma(u^0),l_j))|\\
=&I_1+I_2.
\end{aligned} \tag{3.73}$$

For I_1, by the Assumption 2.1, we get

$$\begin{aligned}
I_1\le&\mathbb{E}\int_{\mathbb{R}_0}|((\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z),l_k))((\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z)-\sigma^{\varepsilon_n}(u^0,z),l_j))|\nu(dz)\\
&+\mathbb{E}\int_{\mathbb{R}_0}|((\sigma^{\varepsilon_n}(u^0,z),l_j))((\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z)-\sigma^{\varepsilon_n}(u^0,z),l_k))|\nu(dz)\\
\le&C[\mathbb{E}\int_{\mathbb{R}_0}||\nabla\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z)||^2\nu(dz)]^{\frac{1}{2}}[\mathbb{E}\int_{\mathbb{R}_0}||\nabla(\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z)-\sigma^{\varepsilon_n}(u^0,z))||^2\nu(dz)]^{\frac{1}{2}}\\
&+[\mathbb{E}\int_{\mathbb{R}_0}||\nabla\sigma^{\varepsilon_n}(u^0,z)||^2\nu(dz)]^{\frac{1}{2}}[\mathbb{E}\int_{\mathbb{R}_0}||\nabla(\sigma^{\varepsilon_n}(u^{\varepsilon_n}(r),z)-\sigma^{\varepsilon_n}(u^0,z))||^2\nu(dz)]^{\frac{1}{2}}\\
\le&C[(1+\mathbb{E}||\nabla u^0||^2)^{\frac{1}{2}}+\sup_{\varepsilon_n}(1+\mathbb{E}||\nabla u^{\varepsilon_n}(r)||^2)^{\frac{1}{2}}](\mathbb{E}||\nabla(u^{\varepsilon_n}(r)-u^0)||^2)^{\frac{1}{2}}.
\end{aligned} \tag{3.74}$$

By (3.17) and (3.70), we have $I_1\to 0$. Similarly, we have $I_2\to 0$. Hence, we prove (3.72). Then, (3.66) holds.

We will show that

$$M_{k,j}(t)=f(X_t)-f(h)-\int_0^t Lf(X_r)dr \tag{3.75}$$

is a martingale under μ_0. Similarly, we will prove that

$$\mathbb{E}^{\mu_0}[(f(X_t)-f(X_s)-\int_s^t Lf(X_r)dr)f_0(X_{s_0})\dots f_n(X_{s_n})]=0. \tag{3.76}$$

Inspired by the Theorem 1.6.8 in [8] and $|f(x)|\le C||\nabla x||^2$, we deduce that

$$\mathbb{E}^{\mu_0}[f(X_t)f_0(X_{s_0})\cdots f_n(X_{s_n})]=\lim_{n\to\infty}\mathbb{E}^{\mu_{\varepsilon_n}}[f(X_t)f_0(X_{s_0})\cdots f_n(X_{s_n})]. \tag{3.77}$$

Since $l_j\in H^3$, we have

$$-((\nabla f(x),Ax))-((B(x),\nabla f(x)))+\frac{1}{2}((f''(x)\sigma(x),\sigma(x)))\le C(||\nabla x||^2+||\nabla x||^3)$$

$$\leq C(1+||\nabla x||^3). \tag{3.78}$$

Next, we obtain for $3<\beta<5$ with any $\alpha>0$ and $\alpha\geq\frac{1}{4}$ as $\beta=3$,

$$\begin{aligned}-((g(x),\nabla f(x)))=&-((g(x),l_k))((x,l_j))-((g(x),l_j))((x,l_k))\\ &\leq C||x||^{\beta}_{\frac{6\beta}{5}}||\Delta l_k||_6\cdot||\nabla x||||\nabla l_j||+C||x||^{\beta}_{\frac{6\beta}{5}}||\Delta l_j||_6\cdot||\nabla x||||l_k||\\ &\leq C(||\nabla x||^{2\beta}+||\nabla x||^2).\end{aligned} \tag{3.79}$$

Hence, $Lf(x)$ is a continuous function on V and

$$|Lf(x)|\leq C(1+||\nabla x||^2+||\nabla x||^3+||\nabla x||^{2\beta}). \tag{3.80}$$

For any $r\in[s,t]$, we get

$$\mathbb{E}^{\mu_0}[(Lf(X_r))f_0(X_{s_0})\cdots f_n(X_{s_n})]=\lim_{n\to\infty}\mathbb{E}^{\mu_{\varepsilon_n}}[(Lf(X_r))f_0(X_{s_0})\cdots f_n(X_{s_n})]. \tag{3.81}$$

Applying the Fubini theorem and dominated convergence theorem, then we have

$$\begin{aligned}&\mathbb{E}^{\mu_0}[(\int_s^t Lf(X_r)dr)f_0(X_{s_0})\cdots f_n(X_{s_n})]\\ &=\lim_{n\to\infty}\mathbb{E}^{\mu_{\varepsilon_n}}[(\int_s^t Lf(X_r)dr)f_0(X_{s_0})\cdots f_n(X_{s_n})].\end{aligned} \tag{3.82}$$

By (3.63), (3.66), (3.77) and (3.82), then we get

$$\begin{aligned}&\mathbb{E}^{\mu_0}[(f(X_t)-f(X_s)-\int_s^t Lf(X_r)dr)f_0(X_{s_0})\cdots f_n(X_{s_n})]\\ &=\lim_{n\to\infty}\mathbb{E}^{\mu_{\varepsilon_n}}[(f(X_t)-f(X_s)-\int_s^t Lf(X_r)dr)f_0(X_{s_0})\cdots f_n(X_{s_n})]\\ &=\lim_{n\to\infty}\mathbb{E}^{\mu_{\varepsilon_n}}[(f(X_t)-f(X_s)-\int_s^t L^{\varepsilon_n}f(X_r)dr)f_0(X_{s_0})\cdots f_n(X_{s_n})]\\ &=0.\end{aligned} \tag{3.83}$$

Therefore, $M_{k,j}(t)$ is defined by (3.75) and is a martingale under μ_0.

Let $m(x)=((x,l_k)),x\in V,k\in\mathbb{N}$. Similarly, we prove

$$\begin{aligned}M_k(t)=&m(X_t)-m(h)-\int_0^t Lm(X_r)dr\\ &=((X_t,l_k))-((h,l_k))+\int_0^t((Al_k,X_s))ds+\int_0^t((B(X_s),l_k))ds\\ &+\int_0^t((g(X_s),l_k))ds\end{aligned} \tag{3.84}$$

is a martingale under μ_0.

In step 3, we will prove that μ_0 is the law of a weak solution of (2.6). Applying (3.75) and (3.84), by Itô's formula, we have

$$<M_k,M_j>(t)=\int_0^t((\sigma(X_s),l_k))((\sigma(X_s),l_j))ds, \tag{3.85}$$

here, $<M_k,M_j>$ represents the sharp bracket of the two martingales. By [8], then there exists a probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ with a filtration $\mathbb{F}'$ such that the extensional

$$(\Omega\times\Omega',\mathcal{F}\times\mathcal{F}',\mathbb{F}\times\mathbb{F}',\mu_0\times\mathbb{P}') \tag{3.86}$$

of $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. Then there exists a one-dimensional Brownian motion $W(t),t\geq 0$ such that

$$M_k(t)=\int_0^t((\sigma(X_s),l_k))dW(s). \tag{3.87}$$

Meanwhile, we have for any $k\geq 1$,

$$\begin{aligned}((X_t,l_k))-((h,l_k))=&-\int_0^t((Al_k,X_s))ds-\int_0^t((B(X_s),l_k))ds\\&-\int_0^t((g(X_s),l_k))ds+\int_0^t((\sigma(X_s),l_k))dW(s).\end{aligned} \tag{3.88}$$

We get that $\{X_t,t\geq 0\}$ is a solution to (2.6) under μ_0. Applying the uniqueness of (2.6), we get $\mu=\mu_0$. This completes the proof of Theorem 3.1. □

Let μ_ε and μ be the laws of u^ε and u on the spaces $D([0,T],H)$ and $C([0,T],H)$.

THEOREM 3.2. *Let the Assumption 2.1, (H.1), (H.2) and $h\in V$ hold and $3<\beta<5$ with any $\alpha>0$ and $\alpha\geq\frac{1}{4}$ as $\beta=3$. For any $T>0$, μ_ε converges weakly to μ, for $\varepsilon\to 0$, on the space $D([0,T],H)$ equipped with the Skorohod topology.*

Proof. Let $h^n,\sigma_n(u),\sigma_n^\varepsilon(u,z)$ be the corresponding orthogonal projections of h, $\sigma(u)$, $\sigma^\varepsilon(u,z)$ into the n-dimensional space $\mathrm{span}\{e_1,\cdots,e_n\}$, respectively. For any $n\in\mathbb{N}$, $\{\sigma_n^\varepsilon\}_{\varepsilon\leq\varepsilon_0}$ satisfies the Assumption 2.1, (H.1) and (H.2). For any $u,u_1,u_2\in H$, there exists $C>0$ independent of n such that

$$\sup_{n\in\mathbb{N}}||\sigma_n(u)||^2+\sup_{n\in\mathbb{N},\varepsilon\leq\varepsilon_0}\int_{\mathbb{R}_0}||\sigma_n^\varepsilon(u,z)||^2\nu(dz)\leq C(1+||u||^2), \tag{3.89}$$

$$\sup_{n\in\mathbb{N}}||\sigma_n(u_1)-\sigma_n(u_2)||^2+\sup_{n\in\mathbb{N},\varepsilon\leq\varepsilon_0}\int_{\mathbb{R}_0}||\sigma_n^\varepsilon(u_1,z)-\sigma_n^\varepsilon(u_2,z)||^2\nu(dz)\leq C||u_1-u_2||^2. \tag{3.90}$$

Let $u^{n,\varepsilon},u^n$ denote the solution of the three-dimensional stochastic Navier-Stokes equations with damping:

$$\begin{aligned}u^{n,\varepsilon}(t)=h^n&-\int_0^t Au^{n,\varepsilon}(s)ds-\int_0^t B(u^{n,\varepsilon}(s))ds-\int_0^t g_n(u^{n,\varepsilon}(s))ds\\&+\int_0^t\int_{\mathbb{R}_0}\sigma_n^\varepsilon(u^{n,\varepsilon}(s-),z)\widetilde{N}(dzds)\end{aligned} \tag{3.91}$$

and

$$u^n(t)=h^n-\int_0^t Au^n(s)ds-\int_0^t B(u^n(s))ds-\int_0^t g_n(u^n(s))ds+\int_0^t\sigma_n(u^n(s))dW(s). \tag{3.92}$$

Applying the above Theorem 3.1, we get for any $n\in\mathbb{N}$,

$$u^{n,\varepsilon}\to u^n,\quad \text{as in distribution} \tag{3.93}$$

on the space $D([0,T],H)$. Applying the Lemma 3.3 and Lemma 3.4, we have

$$\sup_{n\in\mathbb{N},\varepsilon\leq\varepsilon_0}\mathbb{E}(\sup_{0\leq t\leq T}||\nabla u^{n,\varepsilon}(t)||^{2p}+\int_0^T||\nabla u^{n,\varepsilon}(s)||^{2p-2}||\Delta u^{n,\varepsilon}(s)||^2ds$$
$$+\int_0^T||\nabla u^{n,\varepsilon}(s)||^{2p-2}\int_D|u^{n,\varepsilon}|^{\beta-1}|\nabla u^{n,\varepsilon}|^2dxds$$
$$+\int_0^T||\nabla u^{n,\varepsilon}(s)||^{2p-2}||\nabla|u^{n,\varepsilon}|^{\frac{\beta+1}{2}}||^2ds)<\infty, \tag{3.94}$$

and

$$\sup_{n\in\mathbb{N}}\mathbb{E}(\sup_{0\leq t\leq T}||\nabla u^{n}(t)||^{2p}+\int_0^T||\nabla u^{n}(s)||^{2p-2}||\Delta u^{n}(s)||^2ds$$
$$+\int_0^T||\nabla u^{n}(s)||^{2p-2}\int_D|u^{n}|^{\beta-1}|\nabla u^{n}|^2dxds$$
$$+\int_0^T||\nabla u^{n}(s)||^{2p-2}||\nabla|u^{n}|^{\frac{\beta+1}{2}}||^2ds)<\infty. \tag{3.95}$$

We will prove that for each $\delta>0$,

$$\lim_{n\to\infty}\mathbb{P}(\sup_{0\leq t\leq T}||u^n(t)-u(t)||>\delta)=0, \tag{3.96}$$
$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\mathbb{P}(\sup_{0\leq t\leq T}||u^{n,\varepsilon}(t)-u^{\varepsilon}(t)||>\delta)=0. \tag{3.97}$$

Similarity, we need to prove (3.97). By Itô's formula, we get

$$e^{-\gamma\int_0^t||\nabla u^{\varepsilon}(\rho)||^4d\rho}||u^{n,\varepsilon}(t)-u^{\varepsilon}(t)||^2$$
$$=||h^n-h||^2-\gamma\int_0^t e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||u^{n,\varepsilon}(s)-u^{\varepsilon}(s)||^2||\nabla u^{\varepsilon}(s)||^4ds$$
$$-2\int_0^t e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}\langle A(u^{n,\varepsilon}(s)-u^{\varepsilon}(s)),u^{n,\varepsilon}(s)-u^{\varepsilon}(s)\rangle ds$$
$$-2\int_0^t e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}\langle B(u^{n,\varepsilon}(s))-B(u^{\varepsilon}(s)),u^{n,\varepsilon}(s)-u^{\varepsilon}(s)\rangle ds$$
$$-2\int_0^t e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}\langle g_n(u^{n,\varepsilon}(s))-g(u^{\varepsilon}(s)),u^{n,\varepsilon}(s)-u^{\varepsilon}(s)\rangle ds$$
$$+2\int_0^t\int_{\mathbb{R}_0}e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}$$
$$\times(\sigma_n^{\varepsilon}(u^{n,\varepsilon}(s-),z)-\sigma^{\varepsilon}(u^{\varepsilon}(s-),z),u^{n,\varepsilon}(s-)-u^{\varepsilon}(s-))\widetilde{N}(dzds)$$
$$+\int_0^t\int_{\mathbb{R}_0}e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||\sigma_n^{\varepsilon}(u^{n,\varepsilon}(s-),z)-\sigma^{\varepsilon}(u^{\varepsilon}(s-),z)||^2N(dzds)$$
$$=\sum_{k=1}^{7}J_k^{n,\varepsilon}(t). \tag{3.98}$$

Inspired by [15], we get for $\varepsilon_1>0$

$$2|\langle B(u^{n,\varepsilon}(s))-B(u^{\varepsilon}(s)),u^{n,\varepsilon}(s)-u^{\varepsilon}(s)\rangle|$$

$$\begin{aligned}&\leq C||u^{n,\varepsilon}(s)-u^{\varepsilon}(s)||^{\frac{1}{2}}||\nabla(u^{n,\varepsilon}(s)-u^{\varepsilon}(s))||^{\frac{3}{2}}||\nabla u^{\varepsilon}(s)||\\&\leq \varepsilon_1||\nabla(u^{n,\varepsilon}(s)-u^{\varepsilon}(s))||^2+C_{\varepsilon_1}||u^{n,\varepsilon}(s)-u^{\varepsilon}(s)||^2||\nabla u^{\varepsilon}(s)||^4,\end{aligned} \tag{3.99}$$

and

$$\langle g_n(u^{n,\varepsilon}(s))-g(u^{\varepsilon}(s)),u^{n,\varepsilon}(s)-u^{\varepsilon}(s)\rangle\geq 0. \tag{3.100}$$

Let $\gamma=C_{\varepsilon_1}$, we have

$$\sum_{k=2}^{5}J_k^{n,\varepsilon}(t)\leq\int_0^t e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}(\varepsilon_1-2)||\nabla(u^{n,\varepsilon}(s)-u^{\varepsilon}(s))||^2ds. \tag{3.101}$$

Let us choose ε_1 small enough such that

$$\sum_{k=2}^{5}J_k^{n,\varepsilon}(t)\leq-\int_0^t e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||\nabla(u^{n,\varepsilon}(s)-u^{\varepsilon}(s))||^2ds. \tag{3.102}$$

For $J_6^{n,\varepsilon}(t)$, applying Burkhölder's-Davis-Gundy inequality, we have

$$\begin{aligned}&\mathbb{E}\sup_{0\leq s\leq t}|J_6^{n,\varepsilon}(s)|\\&\leq C\mathbb{E}[\int_0^t\int_{\mathbb{R}_0}e^{-2\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||\sigma_n^{\varepsilon}(u^{n,\varepsilon}(s),z)-\sigma^{\varepsilon}(u^{\varepsilon}(s),z)||^2\\&\quad||u^{n,\varepsilon}(s)-u^{\varepsilon}(s)||^2\nu(dz)ds]^{\frac{1}{2}}\\&\leq\frac{1}{2}\mathbb{E}\sup_{0\leq s\leq t}e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||u^{n,\varepsilon}(s)-u^{\varepsilon}(s)||^2\\&\quad+C\mathbb{E}\int_0^t\int_{\mathbb{R}_0}e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||\sigma_n^{\varepsilon}(u^{n,\varepsilon}(s),z)-\sigma^{\varepsilon}(u^{\varepsilon}(s),z)||^2\nu(dz)ds\\&\leq\frac{1}{2}\mathbb{E}\sup_{0\leq s\leq t}e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||u^{n,\varepsilon}(s)-u^{\varepsilon}(s)||^2\\&\quad+C\mathbb{E}\int_0^t e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||u^{n,\varepsilon}(s)-u^{\varepsilon}(s)||^2ds\\&\quad+C\mathbb{E}\int_0^t\int_{\mathbb{R}_0}e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||\sigma_n^{\varepsilon}(u^{n,\varepsilon}(s),z)-\sigma^{\varepsilon}(u^{n,\varepsilon}(s),z)||^2\nu(dz)ds,\end{aligned} \tag{3.103}$$

here, the property of σ^{ε} was used. Similarly, we also get

$$\begin{aligned}\mathbb{E}\sup_{0\leq s\leq t}|J_7^{n,\varepsilon}(s)|\leq& C\mathbb{E}\int_0^t e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||u^{n,\varepsilon}(s)-u^{\varepsilon}(s)||^2ds\\&+2\mathbb{E}\int_0^t\int_{\mathbb{R}_0}e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||\sigma_n^{\varepsilon}(u^{n,\varepsilon}(s),z)-\sigma^{\varepsilon}(u^{n,\varepsilon}(s),z)||^2\nu(dz)ds.\end{aligned} \tag{3.104}$$

By (3.98), (3.102), (3.103) and (3.104), we get

$$\mathbb{E}\sup_{0\leq s\leq t}e^{-\gamma\int_0^s||\nabla u^{\varepsilon}(\rho)||^4d\rho}||u^{n,\varepsilon}(s)-u^{\varepsilon}(s)||^2$$

$$+2\mathbb{E}\int_0^t e^{-\gamma\int_0^s||\nabla u^\varepsilon(\rho)||^4d\rho}||\nabla(u^{n,\varepsilon}(s)-u^\varepsilon(s))||^2ds$$
$$\leq 2\mathbb{E}||h^n-h||^2+C\mathbb{E}\int_0^t e^{-\gamma\int_0^s||\nabla u^\varepsilon(\rho)||^4d\rho}||u^{n,\varepsilon}(s)-u^\varepsilon(s)||^2ds$$
$$+C\mathbb{E}\int_0^t\int_{\mathbb{R}_0} e^{-\gamma\int_0^s||\nabla u^\varepsilon(\rho)||^4d\rho}||\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z)-\sigma^\varepsilon(u^{n,\varepsilon}(s),z)||^2\nu(dz)ds. \quad (3.105)$$

By the Gronwall's inequality, we have

$$\mathbb{E}\sup_{0\leq s\leq t} e^{-\gamma\int_0^s||\nabla u^\varepsilon(\rho)||^4d\rho}||u^{n,\varepsilon}(s)-u^\varepsilon(s)||^2+\mathbb{E}\int_0^t e^{-\gamma\int_0^s||\nabla u^\varepsilon(\rho)||^4d\rho}||\nabla(u^{n,\varepsilon}(s)-u^\varepsilon(s))||^2ds$$
$$\leq C(T)\times[\mathbb{E}||h^n-h||^2+\mathbb{E}\int_0^t\int_{\mathbb{R}_0}||\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z)-\sigma^\varepsilon(u^{n,\varepsilon}(s),z)||^2\nu(dz)ds]. \quad (3.106)$$

We will prove that

$$\lim_{n\to+\infty}\lim_{\varepsilon\to0}\mathbb{E}\int_0^T\int_{\mathbb{R}_0}||\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z)-\sigma^\varepsilon(u^{n,\varepsilon}(s),z)||^2\nu(dz)ds=0. \quad (3.107)$$

The above equality (3.107) is proved, we will get the following equality

$$\lim_{n\to+\infty}\lim_{\varepsilon\to0}\mathbb{E}\sup_{0\leq s\leq T} e^{-\gamma\int_0^s||\nabla u^\varepsilon(\rho)||^4d\rho}||u^{n,\varepsilon}(s)-u^\varepsilon(s)||^2=0. \quad (3.108)$$

Let

$$G_n^\varepsilon(x)=\int_{\mathbb{R}_0}||\sigma_n^\varepsilon(x,z)-\sigma^\varepsilon(x,z)||^2\nu(dz), \quad (3.109)$$

we have

$$\sup_{n\in\mathbb{N},\varepsilon\leq\varepsilon_0} G_n^\varepsilon(x)\leq C(1+||x||^2). \quad (3.110)$$

We also need to prove the following equality

$$\lim_{n\to\infty}\lim_{\varepsilon\to0}\mathbb{E}G_n^\varepsilon(u^{n,\varepsilon}(s))=0. \quad (3.111)$$

Now, we want to prove the following three equalities.

$$\lim_{\varepsilon\to0}\mathbb{E}G_n^\varepsilon(u^{n,\varepsilon}(s))=\lim_{\varepsilon\to0}\mathbb{E}G_n^\varepsilon(u^n(s)),\quad \forall n\in\mathbb{N}, \quad (3.112)$$
$$\lim_{n\to\infty}\lim_{\varepsilon\to0}\mathbb{E}G_n^\varepsilon(u^n(s))=\lim_{n\to\infty}\lim_{\varepsilon\to0}\mathbb{E}G_n^\varepsilon(u(s)), \quad (3.113)$$
$$\lim_{n\to\infty}\lim_{\varepsilon\to0}\mathbb{E}G_n^\varepsilon(u(s))=0. \quad (3.114)$$

To prove (3.112), since u^n is continuous, by (3.93), we have that for any $n\in\mathbb{N}$, $s\in[0,T]$,

$$u^{n,\varepsilon}(s)\longrightarrow u^n(s)\quad\text{as}\quad \varepsilon\to0, \quad (3.115)$$

in distribution. We set $||u^{n,\varepsilon}(s)-u^n(s)||\to0$ a.s. as $\varepsilon\to0$ by using the Skorohod's respresentation theorem. By virtue of (3.94), $\{||u^{n,\varepsilon}(s)||^2\}_{\varepsilon\leq\varepsilon_0}$ is uniformly integrable. Furthermore, we have

$$\lim_{\varepsilon\to0}\mathbb{E}||u^{n,\varepsilon}(s)-u^n(s)||^2=0. \quad (3.116)$$

In other words, we have

$$\begin{aligned}
&\mathbb{E}|G_n^\varepsilon(u^{n,\varepsilon}(s))-G_n^\varepsilon(u^n(s))|\\
\le&\mathbb{E}\int_{\mathbb{R}_0}(||\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z)-\sigma^\varepsilon(u^{n,\varepsilon}(s),z)||^2-||\sigma_n^\varepsilon(u^n(s),z)-\sigma^\varepsilon(u^n(s),z)||^2)\nu(dz)\\
\le&[2\mathbb{E}\int_{\mathbb{R}_0}(||\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z)-\sigma_n^\varepsilon(u^n(s),z)||^2+||\sigma^\varepsilon(u^{n,\varepsilon}(s),z)-\sigma^\varepsilon(u^n(s),z)||^2)\nu(dz)]^{\frac12}\\
&\times[8\mathbb{E}\int_{\mathbb{R}_0}(||\sigma_n^\varepsilon(u^{n,\varepsilon}(s),z)||^2+||\sigma_n^\varepsilon(u^n(s),z)||^2+||\sigma^\varepsilon(u^{n,\varepsilon}(s),z)||^2+||\sigma^\varepsilon(u^n(s),z)||^2)\nu(dz)]^{\frac12}\\
=&I_1^\varepsilon\times I_2^\varepsilon. \qquad (3.117)
\end{aligned}$$

By the above Assumption 2.1, (3.89)-(3.90), (3.94)-(3.95) and (3.116), we have

$$|I_1^\varepsilon|^2\le C\mathbb{E}||u^{n,\varepsilon}(s)-u^n(s)||^2\to0,\quad \text{as}\quad \varepsilon\to0. \qquad (3.118)$$

$$\sup_{\varepsilon\le\varepsilon_0}|I_2^\varepsilon|^2\le C\sup_{n\in\mathbb{N},\varepsilon\le\varepsilon_0}\mathbb{E}(1+||u^{n,\varepsilon}(s)||^2+||u^n(s)||^2)<\infty. \qquad (3.119)$$

By (3.118)-(3.119), we get (3.112). Similarly, by (3.96), we deduce that

$$\lim_{n\to\infty}\sup_{\varepsilon\le\varepsilon_0}\mathbb{E}|G_n^\varepsilon(u^n(s))-G_n^\varepsilon(u(s))|=0. \qquad (3.120)$$

Then, (3.113) is proved. To prove (3.114), by (2.2) and (2.3), we deduce for $x\in H$,

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{\varepsilon\to0}\int_{\mathbb{R}_0}||\sigma_n^\varepsilon(x,z)-\sigma^\varepsilon(x,z)||^2\nu(dz)\\
=&\lim_{n\to\infty}\lim_{\varepsilon\to0}[\int_{\mathbb{R}_0}||\sigma_n^\varepsilon(x,z)||^2\nu(dz)-\int_{\mathbb{R}_0}||\sigma^\varepsilon(x,z)||^2\nu(dz)]\\
=&\lim_{n\to\infty}||\sigma_n(x)||^2-||\sigma(x)||^2=0. \qquad (3.121)
\end{aligned}$$

By (3.110), (3.121) and the dominated convergence theorem, (3.114) holds. Then (3.108) is proved.

To prove (3.97), by virtue of the Lemmas 3.3-3.4, there exists a $M_1>0$ such that for any $\delta_1>0$,

$$\begin{aligned}
&\sup_{n\in\mathbb{N},\varepsilon\le\varepsilon_0}\mathbb{P}(\sup_{0\le t\le T}||u^{n,\varepsilon}(t)-u^\varepsilon(t)||>\delta,\int_0^T||\nabla u^\varepsilon(s)||^4ds>M_1)\\
\le&\sup_{n\in\mathbb{N},\varepsilon\le\varepsilon_0}\mathbb{P}(\int_0^T||\nabla u^\varepsilon(s)||^4ds>M_1)\le\delta_1. \qquad (3.122)
\end{aligned}$$

Meanwhile, applying (3.108), we get

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{\varepsilon\to0}\mathbb{P}(\sup_{0\le t\le T}||u^{n,\varepsilon}(t)-u^\varepsilon(t)||>\delta,\int_0^T||\nabla u^\varepsilon(s)||^4ds\le M_1)\\
\le&\lim_{n\to\infty}\lim_{\varepsilon\to0}\mathbb{P}(\sup_{0\le s\le T}e^{-\gamma\int_0^s||\nabla u^\varepsilon(\rho)||^4d\rho}||u^{n,\varepsilon}(s)-u^\varepsilon(s)||^2\ge e^{-\gamma M_1}\delta^2)\\
\le&e^{\gamma M_1}\frac{1}{\delta^2}\lim_{n\to\infty}\lim_{\varepsilon\to0}\mathbb{E}\sup_{0\le s\le T}e^{-\gamma\int_0^s||\nabla u^\varepsilon(\rho)||^4d\rho}||u^{n,\varepsilon}(s)-u^\varepsilon(s)||^2=0. \qquad (3.123)
\end{aligned}$$

By (3.122) and (3.123), we deduce

$$\lim_{n\to\infty}\lim_{\varepsilon\to0}\mathbb{P}(\sup_{0\le t\le T}||u^{n,\varepsilon}(t)-u^\varepsilon(t)||>\delta)\le\delta_1. \qquad (3.124)$$

Because δ_1 is arbitrary, (3.97) holds.

Finally, we will show that μ^ε converges weakly to μ. Let μ_n^ε, μ_n be the laws of $u^{n.\varepsilon}$ and u^n on the space $S := D([0,T],H)$, respectively. Assume that G is a given bounded, uniformly continuous function on the space S. We have that for each $n \geq 1$,

$$\begin{aligned}
&\int_S G(w)\mu^\varepsilon(dw) - \int_S G(w)\mu(dw)\\
=&\int_S G(w)\mu^\varepsilon(dw) - \int_S G(w)\mu_n^\varepsilon(dw) + \int_S G(w)\mu_n^\varepsilon(dw)\\
&-\int_S G(w)\mu_n(dw) + \int_S G(w)\mu_n(dw) - \int_S G(w)\mu(dw)\\
=&\mathbb{E}[G(u^\varepsilon) - G(u^{n,\varepsilon})] + (\int_S G(w)\mu_n^\varepsilon(dw) - \int_S G(w)\mu_n(dw))\\
&+\mathbb{E}[G(u^n) - G(u)]. \qquad (3.125)
\end{aligned}$$

Then there exists a positive constant $\delta_1 > 0$ such that for $n \geq 1$ and $\varepsilon > 0$,

$$|\mathbb{E}[G(u^\varepsilon) - G(u^{n,\varepsilon}); \sup_{0\leq s\leq T} ||u^{n,\varepsilon}(s) - u^\varepsilon(s)|| \leq \delta_1]| \leq \frac{\delta}{4}. \qquad (3.126)$$

By virtue of (3.96)-(3.97), there exist positive constants n_1 and ε_{n_1} such that

$$\begin{aligned}
&\sup_{\varepsilon\leq\varepsilon_{n_1}} |\mathbb{E}[G(u^\varepsilon) - G(u^{n_1,\varepsilon}); \sup_{0\leq s\leq T} ||u^{n_1,\varepsilon}(s) - u^\varepsilon(s)|| > \delta_1]|\\
\leq& C \sup_{\varepsilon\leq\varepsilon_{n_1}} \mathbb{P}(\sup_{0\leq s\leq T} ||u^{n_1,\varepsilon}(s) - u^\varepsilon(s)|| > \delta_1) \leq \frac{\delta}{4}, \qquad (3.127)
\end{aligned}$$

and

$$|\mathbb{E}[G(u^{n_1}) - G(u)]| \leq \frac{\delta}{4}. \qquad (3.128)$$

In other words, applying (3.93), there exists a positive constant ε_1 such that for any $\varepsilon \leq \varepsilon_1$,

$$|\int_S G(w)\mu_{n_1}^\varepsilon(dw) - \int_S G(w)\mu_{n_1}(dw)| \leq \frac{\delta}{4}. \qquad (3.129)$$

By (3.125)-(3.129), we deduce that for any $\varepsilon \leq \min\{\varepsilon_{n_1}, \varepsilon_1\}$,

$$|\int_S G(w)\mu^\varepsilon(dw) - \int_S G(w)\mu(dw)| \leq \delta. \qquad (3.130)$$

Because $\delta > 0$ is small enough, we get

$$\lim_{\varepsilon\to 0}\int_S G(w)\mu^\varepsilon(dw) = \int_S G(w)\mu(dw). \qquad (3.131)$$

This completes the proof of Theorem 3.2. ∎

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