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NONLOCAL APPROACHES FOR MULTILANE TRAFFIC MODELS*

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Abstract. We present a multilane traffic model based on balance laws, where the nonlocal source term is used to describe the lane changing rate. The modelling framework includes the consideration of local and nonlocal flux functions. Based on a Godunov-type numerical scheme, we provide BV estimates and a discrete entropy inequality. Together with the $\mathbf{L^1}$-contractivity property, we prove existence and uniqueness of weak solutions. Numerical examples show the nonlocal impact compared to local flux functions and local sources.

Keywords. Nonlocal balance laws; multilane traffic flow; Godunov scheme.

AMS subject classifications. 35L65; 90B20; 65M12.

1. Introduction

The progress in autonomous driving brings new challenges for the modelling of traffic flow. Classical approaches such as the well-established Lighthill-Whitham-Richards (LWR) model [26, 27] have been recently extended to include more information on the surrounding traffic, see for example [3, 15, 18, 21, 28]. Therefore, we distinguish between *local* traffic flow models governed by conservation laws, where the fundamental diagram gives the relation between flux and density, and *nonlocal* models with flux functions depending on an integral evaluation of the density or velocity, through a convolution product. In case of autonomous vehicles the nonlocal models allow for an interpretation as the connection radius.

Nonlocal traffic flow models have been introduced in [3] and since then have been studied regarding existence and well-posedness, e.g. [7, 15, 21], numerical schemes [3, 4, 14, 15, 18], convergence to local conservation laws, e.g. [23] (even if this question is still an open research problem), microscopic modelling approaches [5, 10, 17, 28], second-order models [5], multi-class models [8], time delay models [22] and network formulations [6, 10].

The aim of this paper is to study a multilane model with local and nonlocal fluxes combined with a source term that also incorporates a nonlocality. Here, the nonlocal source term describes the lane changing rate depending on a (nonlinear) evaluation of the velocity. In this context, we refer to [11], where a nonlocal source term is used to describe the lane change. However, the modelling of our source term is inspired by [20]. We would also like to mention that a similar multilane model with nonlocal flux and source has been recently introduced in [2]. Therein, well-posedness and uniqueness are proven based on Banach's fixed-point theorem, using the method of characteristics. In contrast to the contributions [2,11,20], we do not investigate the model on the continuous description and present a Godunov-type numerical scheme instead that can be used to

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show existence of approximate solutions. Uniqueness of solutions follows from a suitable application of Kružkov doubling of variables technique.

From a modelling point of view, the key differences to [2] are that the nonlocal terms in the flux and in the source do not have necessarily the same kernels. More precisely, in [2] only forward looking and decreasing kernels can be considered within the convolution product. It seems that our approach is more flexible, since also back- and forward looking kernels can be considered.

The paper is organised as follows: In Section 2 we present the model with local flux and nonlocal source, while in Section 3 the Godunov-type discretization is addressed to show the existence of a solution to the model as the limit of a sequence of approximate solutions. The uniqueness result is then discussed in Section 4. The extension to the model with nonlocal flux and source is given in Section 5, with particular focus on differences to the model with local flux only. In Section 6 a collection of numerical experiments is carried out.

2. A multilane model with nonlocal source term

In [20] the authors exploit the traditional LWR model to study vehicular traffic on a road with multiple lanes. The key feature of the model in [20] is that drivers tend to change to a neighbouring lane proportionally to the difference in the (local) velocity between the lanes.

However, as it is already well known, the use of nonlocal terms may lead to other dynamical behaviour, see e.g. [3]. In this paper, we aim to extend the multilane model [20] to account for a *nonlocal* evaluation of the velocity influencing the lane changing rate. The idea is that at position x the flow between neighbouring lanes is governed by the difference in the velocity evaluated on the average density *around* position x, e.g. on the interval $[x-\nu, x+\nu]$, $\nu>0$. This modelling hypothesis is motivated by a feature typical of drivers' behaviour: when driving on a multilane road, at the moment of changing lane, the driver checks what is happening behind and in front of him/her, both on his/her lane and on the neighbouring one(s).

Recall the model introduced in [20] for a road with M lanes:

$$\begin{cases} \partial_t \rho_j + \partial_x (\rho_j v_j(\rho_j)) = S_{j-1}(\rho_{j-1}, \rho_1) - S_j(\rho_j, \rho_{j+1}) & j = 1, \dots, M, \\ \rho_j(0,x) = \rho_{o,j}(x) & j = 1, \dots, M, \end{cases}$$

with

$$S_j(\rho_j, \rho_{j+1}) = K\,(v_{j+1}(\rho_{j+1}) - v_j(\rho_j)) \begin{cases} \rho_j & \text{if } v_{j+1}(\rho_{j+1}) \geq v_j(\rho_j), \\ \rho_{j+1} & \text{if } v_{j+1}(\rho_{j+1}) < v_j(\rho_j), \end{cases}$$

and the boundary conditions

$$S_0(\rho_0, \rho_1) = S_M(\rho_M, \rho_{M+1}) = 0,$$

K being a dimensional constant $(1/m)$. The modelling idea behind the term $S_j(\rho_j, \rho_{j+1})$ lies in the assumption that drivers prefer to be in the faster lane, and that the lane changing rate is proportional to the difference in the (local) velocity.

In contrast, our modelling approach accounts for a *nonlocal* evaluation of the velocity influencing the lane changing rate. Therefore, we introduce a kernel function $w_\nu \in \mathbf{C^0}([-\nu, \nu], \mathbb{R}_+)$, with $\nu > 0$ and $\int_{\mathbb{R}} w_\nu(x)\,\mathrm{d}x = 1$, and define the flow from lane j to lane $j+1$ as follows: for $j=1,\ldots,M-1$

$$\begin{aligned} &S_j(\rho_j,\rho_{j+1},R_j,R_{j+1}) \\ &= K\left(v_{j+1}(R_{j+1})-v_j(R_j)\right)\begin{cases}\rho_j\,(1-\rho_{j+1}) & \text{if } v_{j+1}(R_{j+1})\geq v_j(R_j),\\ \rho_{j+1}\,(1-\rho_j) & \text{if } v_{j+1}(R_{j+1})<v_j(R_j),\end{cases} \quad (2.1)\\ &= K\left[(v_{j+1}(R_{j+1})-v_j(R_j))^+\rho_j\,(1-\rho_{j+1})-(v_{j+1}(R_{j+1})-v_j(R_j))^-\rho_{j+1}\,(1-\rho_j)\right],\\ &\text{with}\quad R_j=R_j(t,x)=(\rho_j(t)*w_\nu)(x), \quad (2.2)\end{aligned}$$

where $(s)^+=\max\{s,0\}$, $(s)^-=-\min\{s,0\}$. Conversely, the flow from lane $j+1$ to lane j equals $-S_j(\rho_j,\rho_{j+1},R_j,R_{j+1})$. Here, K is still a dimensional constant $(1/m)$. For simplicity, in the following, time and space are scaled so that $K=1$. The model we study is thus

$$\begin{cases}\partial_t\rho_j+\partial_x\left(\rho_j\,v_j(\rho_j)\right)=S_{j-1}(\rho_{j-1},\rho_j,R_{j-1},R_j)-S_j(\rho_j,\rho_{j+1},R_j,R_{j+1}) & j=1,\ldots,M,\\ \rho_j(0,x)=\rho_{o,j}(x) & j=1,\ldots,M,\end{cases} \quad (2.3)$$

with boundary conditions

$$S_0(\rho_0,\rho_1,R_0,R_1)=S_M(\rho_M,\rho_{M+1},R_M,R_{M+1})=0. \quad (2.4)$$

The meaning of the source term defined by (2.1) is the following: Similar to the model studied in [20], the lane changing rate is proportional to the difference in the velocity between two adjacent lanes, but the velocities are now evaluated *nonlocally*, i.e. in a neighbourhood of the *current* position. Moreover, this rate is now proportional also to the density in the receiving lane, meaning that, if that lane is crowded, only a few vehicles can actually change lane. We remark that including this latter factor allows to prove the invariance of the set $[0,1]^M$ for model (2.3), see Section 3.1. We emphasize that this is not necessary for the *local* model [20].

REMARK 2.1. To further support the necessity of this additional factor in the source term, consider the *nonlocal* problem (2.3) where the *nonlocal* sources (2.1) are directly derived from the *local* model in [20], and thus are

$$S_j(\rho_j,\rho_{j+1},R_j,R_{j+1})=K\left(v_{j+1}(R_{j+1})-v_j(R_j)\right)\begin{cases}\rho_j & \text{if } v_{j+1}(R_{j+1})\geq v_j(R_j),\\ \rho_{j+1} & \text{if } v_{j+1}(R_{j+1})<v_j(R_j),\end{cases} \quad (2.5)$$

where as usual we can set $K=1$. Assume that there are $M=2$ lanes and choose the initial datum as follows:

$$\rho_{o,1}(x)=0.5, \qquad\qquad \rho_{o,2}(x)=1-H(x).$$

As for the kernel function, choose $w_\nu(x)=\frac{1}{\nu}\chi_{[0,\nu]}(x)$. In this situation, it can be easily computed that the source term (2.5) evaluated at $t=0+$ and $x=-\nu/2$ is positive, meaning that there is a flow from lane 1 to lane 2. However, at $x=-\nu/2$ lane 2 is fully congested: from the modelling point of view we want to avoid such a situation, aiming thus at the invariance of the set $[0,1]^M$.

Furthermore, it is possible to generalise the source term defined in (2.1). We consider model (2.3) with the following assumption on the map S_j.

ASSUMPTION 2.1. *Every map S_j, $j=1,\dots,M-1$, is Lipschitz continuous in each argument, with Lipschitz constant $\mathcal{K}$, and satisfies the following bounds*

$$\begin{aligned} S_j(\rho_j,0,R_j,R_{j+1}) &\geq 0, & S_j(\rho_j,1,R_j,R_{j+1}) &\leq 0, \\ S_j(0,\rho_{j+1},R_j,R_{j+1}) &\leq 0, & S_j(1,\rho_{j+1},R_j,R_{j+1}) &\geq 0. \end{aligned} \tag{2.6}$$

In addition, the map S_j is nondecreasing in the first and third variables and nonincreasing in the second and fourth variables and the boundary conditions (2.4) *hold.*

The conditions (2.6) forbid vehicles' flow to fully congested lanes and allows flow to empty lanes.

REMARK 2.2. The map S_j in (2.1) satisfies the Assumption 2.1. The exact Lipschitz constant and a proof of the Lipschitz continuity are detailed in the appendix, see Lemma A.1.

In the next step, we define a *weak solution* to (2.3) and present the key result of this paper for the existence and uniqueness of the solution. As in [20], we assume that the velocity functions v_i are strictly decreasing, positive and scaled such that $v_i(1)=0$, $i=1,\dots,M$. For simplicity, space and time are scaled so that $K=1$. We assume that each map $f_j(u)=u v_j(u)$ admits a unique global maximum in the interval $[0,1]$ attained at $u=\vartheta_j$.

DEFINITION 2.1. *Let $\rho_{o,j}\in(\mathbf{L^1}\cap\mathbf{BV})(\mathbb{R};[0,1])$, for $j=1,\dots,M$. We say that $\rho_j\in\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};[0,1]))$, with $\rho_j(t,\cdot)\in\mathbf{BV}(\mathbb{R};[0,1])$ for $t\in[0,T]$, is a* weak solution *to* (2.3) *with initial datum $\rho_{o,j}$ if for any $\varphi\in\mathbf{C^1_c}([0,T[\times\mathbb{R};\mathbb{R})$ and for all $j=1,\dots,M$*

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}}(\rho_j\,\partial_t\varphi+\rho_j v_j(\rho_j)\,\partial_x\varphi+(S_{j-1}(\rho_{j-1},\rho_j,R_{j-1},R_j)-S_j(\rho_j,\rho_{j+1},R_j,R_{j+1}))\,\varphi)\mathrm{d}x\,\mathrm{d}t \\ &\qquad+\int_{\mathbb{R}}\rho_{o,j}\,\varphi(0,x)\,\mathrm{d}x=0, \end{aligned}$$

with $R_j=R_j(t,x)=(\rho_j(t)*w_\nu)(x)$. The solution ρ_j is an entropy solution *if for any $\varphi\in\mathbf{C^1_c}([0,T[\times\mathbb{R};\mathbb{R}_+)$, for all convex entropy-entropy flux pairs (η,q) and for all $j=1,\dots,M$*

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}}(\eta(\rho_j)\,\partial_t\varphi+q(\rho_j)\,\partial_x\varphi)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}}\eta(\rho_{o,j})\,\varphi(0,x)\,\mathrm{d}x \\ &\geq\int_0^T\int_{\mathbb{R}}\eta'(\rho_j)\,(S_j(\rho_j,\rho_{j+1},R_j,R_{j+1})-S_{j-1}(\rho_{j-1},\rho_j,R_{j-1},R_j))\,\varphi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

In the following it will be convenient to use the notation $\boldsymbol{\rho}=(\rho_1,\dots,\rho_M)$ to denote the vector of component ρ_j, $j=1,\dots,M$. The initial datum to problem (2.3) is then $\boldsymbol{\rho}_o$.

THEOREM 2.1. *Let $\boldsymbol{\rho}_o\in(\mathbf{L^1}\cap\mathbf{BV})(\mathbb{R};[0,1]^M)$ and let Assumption 2.1 hold. Then, for all $T>0$, problem* (2.3) *has a unique solution $\boldsymbol{\rho}\in\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};[0,1]^M))$ in the sense of Definition 2.1. Moreover, the following estimates hold: for any $t\in[0,T]$*

$$\begin{aligned} &\|\boldsymbol{\rho}(t)\|_{\mathbf{L^1}(\mathbb{R})}=\sum_{j=1}^M\|\rho_j(t)\|_{\mathbf{L^1}(\mathbb{R})}=\|\boldsymbol{\rho}_o\|_{\mathbf{L^1}(\mathbb{R})}, \\ &\text{for } j=1,\dots,M:\quad 0\leq\rho_j(t,x)\leq 1, \\ &\sum_{j=1}^M\mathrm{TV}(\rho_j(t))\leq\sum_{j=1}^M\mathrm{TV}(\rho_{j,o}). \end{aligned}$$

Existence of solutions to problem (2.3) is ensured by the convergence of a sequence of approximate solutions, constructed through a Godunov scheme, see Section 3.5. Uniqueness follows from the $\mathbf{L^1}$-contractivity property for the whole solution to (2.3), see Section 4. The $\mathbf{L^1}$ and $\mathbf{L^\infty}$ bounds follow from the convergence of the scheme, while the total variation bound is a consequence of the $\mathbf{L^1}$-contractivity, see Corollary 4.1.

Additionally, we can prove that, as the size ν of the support of the kernel function tends to zero, the nonlocal problem (2.3) converges to its local version, obtained by replacing in the source terms the nonlocal operators $R_j(t,x)$ with $\rho_j(t,x)$ for $j=1,\dots,M$.

Corollary 2.1. *Let $\boldsymbol{\rho}_o \in (\mathbf{L^1}\cap\mathbf{BV})(\mathbb{R};[0,1]^M)$ and let Assumption 2.1 hold. Then, for all $T>0$ and $\nu\to 0$, problem (2.3) has a unique solution $\boldsymbol{\rho}\in\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};[0,1]^M))$ in the sense of Definition 2.1 in which we replace the nonlocal operators $R_j(t,x)$ with $\rho_j(t,x)$ for $j=1,\dots,M$.*

The proof is deferred to Section 3.5. Note that the latter Corollary states that as $\nu\to 0$ the solution to problem (2.3) converges to the unique solution of the local problem.

3. Numerical discretization: a Godunov-type scheme

To prove several properties of the model (2.3) and in particular Theorem 2.1, we introduce a uniform space mesh of width Δx and a time step Δt, subject to a CFL condition to be detailed later on. For any $k\in\mathbb{Z}$ denote the centre of the k-th cell by x_k and its interfaces by $x_{k\pm 1/2}$:

$$x_k = \left(k+\frac{1}{2}\right)\Delta x, \qquad\qquad x_{k-1/2} = k\,\Delta x.$$

Set $N_T=\lfloor T/\Delta t\rfloor$ and define the time mesh as $t^n = n\Delta t$, $n=0,\dots,N_T$. Set $\lambda=\Delta t/\Delta x$. The initial data are approximated as follows: for $j=1,\dots,M$ and $k\in\mathbb{Z}$,

$$\rho^0_{j,k} = \frac{1}{\Delta x}\int_{x_{k-1/2}}^{x_{k+1/2}} \rho_{o,j}(x)\,\mathrm{d}x\,.$$

We construct an approximate solution $\boldsymbol{\rho}_\Delta$ to (2.3) as follows: for $j=1,\dots,M$ set

$$\rho_{j,\Delta}(t,x)=\rho^n_{j,k} \quad \text{for} \quad \begin{cases} t\in[t^n,t^{n+1}[,\\ x\in[x_{k-1/2},x_{k+1/2}],\end{cases} \quad \text{with} \quad \begin{array}{l} n=0,\dots,N_T-1,\\ k\in\mathbb{Z}.\end{array} \tag{3.1}$$

The approximate solution $\boldsymbol{\rho}_\Delta$ is obtained via a Godunov-type scheme together with operator splitting, to account for the source terms, see Algorithm 3.1.

Algorithm 3.1.

$$F_j(u,w)=\min\{f_j(\min\{u,\vartheta_j\}),f_j(\max\{w,\vartheta_j\})\} \quad j=1,\dots,M \tag{3.2}$$

for $n=0,\dots,N_T-1$:

for $j=1,\dots,M$ *and* $k\in\mathbb{Z}$:

$$\rho^{n+1/2}_{j,k} = \rho^n_{j,k} - \lambda\left[F_j(\rho^n_{j,k},\rho^n_{j,k+1}) - F_j(\rho^n_{j,k-1},\rho^n_{j,k})\right] \tag{3.3}$$

$$\begin{aligned}\rho^{n+1}_{j,k} = {} & \rho^{n+1/2}_{j,k} + \Delta t\, S_{j-1}(\rho^{n+1/2}_{j-1,k},\rho^{n+1/2}_{j,k},R^{n+1/2}_{j-1,k},R^{n+1/2}_{j,k})\\ & - \Delta t\, S_j(\rho^{n+1/2}_{j,k},\rho^{n+1/2}_{j+1,k},R^{n+1/2}_{j,k},R^{n+1/2}_{j+1,k})\end{aligned} \tag{3.4}$$

Above, $R^{n+1/2}_{j,k}$, for $j=1,\dots,M$, $k\in\mathbb{Z}$ and $n=0,\dots,N_T-1$, denotes the discrete convolution operator, which is defined in the lemma below.

LEMMA 3.1. *Let* $w_\nu \in \mathbf{C^0}([-\nu,\nu];\mathbb{R}_+)$ *be such that* $\int_{\mathbb{R}} w_\nu = 1$. *Define the set*

$$\mathcal{H} = \left\{ h \in \mathbb{Z} : \left\lfloor \frac{\inf \operatorname{spt} w_\nu}{\Delta x} \right\rfloor \leq h \leq \left\lfloor \frac{\sup \operatorname{spt} w_\nu}{\Delta x} \right\rfloor - 1 \right\} \tag{3.5}$$

and for all $h \in \mathcal{H}$ *set*

$$\gamma_h := \int_{x_{h-1/2}}^{x_{h+1/2}} w_\nu(y)\,\mathrm{d}y.$$

Given $r(x) = r_k \chi_{[x_{k-1/2}, x_{k+1/2}]}(x)$, *with* $r_k \in [0,1]$ *and* $k \in \mathbb{Z}$, *the discrete convolution operator defined for all* $k \in \mathbb{Z}$ *as*

$$R_k = \sum_{h \in \mathcal{H}} \gamma_h r_{k+h+1} \tag{3.6}$$

satisfies the following properties:

$$R_k \in [0,1] \text{ for all } k \in \mathbb{Z}, \tag{3.7}$$

$$\sum_{k \in \mathbb{Z}} |R_{k+1} - R_k| \leq \sum_{k \in \mathbb{Z}} |r_{k+1} - r_k|. \tag{3.8}$$

Given $\tilde{r}(x) = \tilde{r}_k \chi_{[x_{k-1/2}, x_{k+1/2}]}(x)$, *with* $\tilde{r}_k \in [0,1]$ *and* $k \in \mathbb{Z}$, *and* $\tilde{R}_k$ *defined according to* (3.6), *then*

$$\sum_{k \in \mathbb{Z}} \left| R_k - \tilde{R}_k \right| \leq \sum_{k \in \mathbb{Z}} |r_k - \tilde{r}_k|. \tag{3.9}$$

Proof. It can be immediately seen that $\gamma_h \in [0,1]$ for all $h \in \mathcal{H}$, due to the properties of w_ν. Hence, for all $k \in \mathbb{Z}$, we clearly have $R_k \geq 0$ and

$$R_k = \sum_{h \in \mathcal{H}} \gamma_h r_{k+h+1} \leq \sum_{h \in \mathcal{H}} \gamma_h = \int_{\operatorname{spt} w_\nu} w_\nu(y)\,\mathrm{d}y = 1,$$

since each $r_k \in [0,1]$.

Pass now to (3.8): rearranging the indexes yields

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |R_{k+1} - R_k| &\leq \sum_{k \in \mathbb{Z}} \sum_{h \in \mathcal{H}} \gamma_h |r_{k+h+2} - r_{k+h+1}| \\ &= \left(\sum_{h \in \mathcal{H}} \gamma_h \right) \sum_{k \in \mathbb{Z}} |r_{k+1} - r_k| = \sum_{k \in \mathbb{Z}} |r_{k+1} - r_k|. \end{aligned}$$

The proof of (3.9) is entirely analogous. ◻

REMARK 3.1. According to the support of the kernel function w_ν, the discrete convolution operator defined by (3.6) has one of the following two forms:

- **Forward looking kernel:**
  if $\operatorname{supp} w_\nu \subseteq [0,\nu]$, then $\mathcal{H} = \left[0, \lfloor \frac{\nu}{\Delta x} \rfloor - 1\right]$, so that

$$R_{j,k}^{n+1/2} = \sum_{h=0}^{\lfloor \nu/\Delta x \rfloor - 1} \gamma_h \rho_{j,k+h+1}^{n+1/2}. \tag{3.10}$$

- **Back- and forward looking kernel:**
  if $\operatorname{supp} w_\nu \subseteq [-\nu,\nu]$, $\mathcal{H} = \left[-\lfloor \frac{\nu}{\Delta x}\rfloor, \lfloor \frac{\nu}{\Delta x}\rfloor - 1\right]$, so that

$$R_{j,k}^{n+1/2} = \sum_{h=-\lfloor \nu/\Delta x\rfloor}^{\lfloor \nu/\Delta x\rfloor -1} \gamma_h \, \rho_{j,k+h+1}^{n+1/2}. \tag{3.11}$$

3.1. Invariance of the set $[0,1]^M$. Under a suitable CFL condition, if each component of the initial datum takes values in the interval $[0,1]$, then also the components of the approximate solution constructed via Algorithm 3.1 attain values in the same interval $[0,1]$: the set $[0,1]^M$ is thus invariant for problem (2.3).

LEMMA 3.2. *Let $\boldsymbol{\rho}_o \in \mathbf{L}^\infty(\mathbb{R};[0,1]^M)$ and let Assumption 2.1 hold. Assume that*

$$\lambda \leq \frac{1}{\max\{2\mathcal{K},\mathcal{V}\}}, \tag{3.12}$$

where

$$\mathcal{V} = \|\boldsymbol{v}\|_{\mathbf{C}^1([0,1];\mathbb{R}^M)} = V_{\max} + V'_{\max}, \tag{3.13}$$

$$V_{\max} = \|\boldsymbol{v}\|_{\mathbf{C}^0([0,1];\mathbb{R}^M)} = \max_{j=1,\dots,M} \|v_j\|_{\mathbf{L}^\infty([0,1];\mathbb{R})}, \tag{3.14}$$

$$V'_{\max} = \|\boldsymbol{v}'\|_{\mathbf{C}^0([0,1];\mathbb{R}^M)} = \max_{j=1,\dots,M} \|v'_j\|_{\mathbf{L}^\infty([0,1];\mathbb{R})}. \tag{3.15}$$

Then, for all $t>0$ and $x \in \mathbb{R}$, the piece-wise constant approximate solution $\boldsymbol{\rho}_\Delta$ constructed through Algorithm 3.1 attains value in the set $[0,1]^M$, i.e.

$$0 \leq \rho_{j,\Delta}(t,x) \leq 1 \qquad \textit{for all } j = 1,\dots,M.$$

Proof. The proof is done by induction and follows the idea of the proof of [16, Lemma 2.2]. Assume that $\boldsymbol{\rho}_\Delta(t,x) \in [0,1]^M$ for all $x \in \mathbb{R}$ and $t < t^{n+1}$. In particular, $0 \leq \rho_{j,k}^n \leq 1$ for $j = 1,\dots,M$ and all $k \in \mathbb{Z}$. Consider the convective step (3.3) of Algorithm 3.1: Under the CFL condition $\lambda\mathcal{V} \leq 1$ the Godunov-type scheme preserves the invariance of the set $[0,1]^M$, i.e. $0 \leq \rho_{j,k}^{n+1/2} \leq 1$ for $j = 1,\dots,M$ and all $k \in \mathbb{Z}$, see [13, Proposition 3.1(b) eq. (1.2)].

Now focus on the relaxation step (3.4), taking care of the source term. Without loss of generality, we may fix $j = 2$, so to have contributions from both the preceding and the subsequent lanes. Omitting the index $n+1/2$ to improve readability, by (3.3) and (2.1), we get

$$\rho_{2,k}^{n+1} = \rho_{2,k} + \Delta t \, S_1(\rho_{1,k},\rho_{2,k},R_{1,k},R_{2,k}) - \Delta t \, S_2(\rho_{2,k},\rho_{3,k},R_{2,k},R_{3,k}). \tag{3.16}$$

Using Assumption 2.1, we obtain

$$\begin{aligned} \rho_{2,k}^{n+1} &\leq \rho_{2,k} + \Delta t \left(S_1(\rho_{1,k},\rho_{2,k},R_{1,k},R_{2,k}) - (S_1(\rho_{1,k},1,R_{1,k},R_{2,k})\right) \\ &\qquad - \Delta t \left(S_2(\rho_{2,k},\rho_{3,k},R_{2,k},R_{3,k}) - S_2(1,\rho_{3,k},R_{2,k},R_{3,k})\right) \\ &\leq \rho_{2,k} + 2\Delta t \mathcal{K}(1-\rho_{2,k}) \\ &\leq 1, \end{aligned}$$

thanks to the Lipschitz continuity of S_j, the CFL condition (3.12) and the fact that $\Delta x < 1$.

The positivity can be be obtained similarly, starting from (3.16) and exploiting Assumption 2.1:

$$\begin{aligned}\rho_{2,k}^{n+1} &\geq \rho_{2,k} + \Delta t\left(S_1(\rho_{1,k},\rho_{2,k},R_{1,k},R_{2,k}) - (S_1(\rho_{1,k},0,R_{1,k},R_{2,k})\right) \\ &\quad - \Delta t\left(S_2(\rho_{2,k},\rho_{3,k},R_{2,k},R_{3,k}) - S_2(0,\rho_{3,k},R_{2,k},R_{3,k})\right) \\ &\geq \rho_{2,k} - 2\Delta t\mathcal{K}\rho_{2,k} \\ &\geq 0.\end{aligned}$$

The proof is completed. ∎

3.2. Conservation of total mass. When considering an initial datum $\boldsymbol{\rho}_o$ with finite total mass, that is $\sum_{j=1}^{M}\|\rho_{o,j}\|_{\mathbf{L}^1(\mathbb{R})} < +\infty$, it is possible to prove that the corresponding solution preserves this norm. Clearly, because of lane changing, the $\mathbf{L}^1$-norm is not preserved in each lane, but only in the whole.

LEMMA 3.3. *Let $\boldsymbol{\rho}_o \in \mathbf{L}^1(\mathbb{R};[0,1]^M)$. Under the CFL condition (3.12), the piecewise constant approximate solution $\boldsymbol{\rho}_\Delta$ constructed through Algorithm 3.1 preserves the $\mathbf{L}^1$-norm, in the sense that for all $t>0$*

$$\|\boldsymbol{\rho}_\Delta(t)\|_{\mathbf{L}^1(\mathbb{R})} = \sum_{j=1}^{M}\|\rho_{j,\Delta}(t)\|_{\mathbf{L}^1(\mathbb{R})} = \sum_{j=1}^{M}\|\rho_{o,j}\|_{\mathbf{L}^1(\mathbb{R})} = \|\boldsymbol{\rho}_o\|_{\mathbf{L}^1(\mathbb{R})}.$$

Proof. The proof is done by induction. Since the Godunov-type scheme (3.3) is conservative [25, Chapter 13], we have

$$\sum_{j=1}^{M}\left\|\rho_j^{n+1/2}\right\|_{\mathbf{L}^1(\mathbb{R})} = \sum_{j=1}^{M}\|\rho_{o,j}\|_{\mathbf{L}^1(\mathbb{R})}.$$

The positivity of $\boldsymbol{\rho}_\Delta$ and the fact that the source terms sum up to 0 when considering the relaxation step in (3.4) yields the thesis:

$$\sum_{j=1}^{M}\|\rho_j^{n+1}\|_{\mathbf{L}^1(\mathbb{R})} = \sum_{j=1}^{M}\left\|\rho_j^{n+1/2}\right\|_{\mathbf{L}^1(\mathbb{R})} = \sum_{j=1}^{M}\|\rho_{o,j}\|_{\mathbf{L}^1(\mathbb{R})}.$$

∎

3.3. BV estimates. The Lipschitz continuity of the source term is one of the key ingredients in order to prove the following total variation bound on the numerical approximation.

PROPOSITION 3.1 (**BV estimate in space**). *Let $\boldsymbol{\rho}_o \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R};[0,1]^M)$. Assume that the CFL condition (3.12) and let Assumption 2.1 hold. Then, for $n=0,\ldots,N_T-1$ the following estimate holds*

$$\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}|\rho_{j,k+1}^n - \rho_{j,k}^n| \leq e^{8t^n\mathcal{K}}\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}|\rho_{j,k+1}^0 - \rho_{j,k}^0| = e^{8t^n\mathcal{K}}\sum_{j=1}^{M}\mathrm{TV}(\rho_j^0). \tag{3.17}$$

Proof. By (3.4), for $j=1,\ldots,M$ we have

$$\begin{aligned}&\rho_{j,k+1}^{n+1} - \rho_{j,k}^{n+1} = \rho_{j,k+1}^{n+1/2} - \rho_{j,k}^{n+1/2} \\ &+ \Delta t\left[S_{j-1}\left(\rho_{j-1,k+1}^{n+1/2},\rho_{j,k+1}^{n+1/2},R_{j-1,k+1}^{n+1/2},R_{j,k+1}^{n+1/2}\right) - S_{j-1}\left(\rho_{j-1,k}^{n+1/2},\rho_{j,k}^{n+1/2},R_{j-1,k}^{n+1/2},R_{j,k}^{n+1/2}\right)\right]\end{aligned}$$

$$-\Delta t\left[S_j\left(\rho_{j,k+1}^{n+1/2},\rho_{j+1,k+1}^{n+1/2},R_{j,k+1}^{n+1/2},R_{j+1,k+1}^{n+1/2}\right)-S_j\left(\rho_{j,k}^{n+1/2},\rho_{j+1,k}^{n+1/2},R_{j,k}^{n+1/2},R_{j+1,k}^{n+1/2}\right)\right].$$

By the Lipschitz continuity of the maps in the source term, see Assumption 2.1, and the properties of the discrete convolution operator, see Lemma 3.1, we obtain

$$\begin{aligned}
&\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}\left|\rho_{j,k+1}^{n+1}-\rho_{j,k}^{n+1}\right|\\
&\leq \sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}(1+4\Delta t\mathcal{K})\left|\rho_{j,k+1}^{n+1/2}-\rho_{j,k}^{n+1/2}\right|+4\Delta t\mathcal{K}\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}\left|R_{j,k+1}^{n+1/2}-R_{j,k}^{n+1/2}\right|\\
&\leq (1+8\Delta t\mathcal{K})\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}\left|\rho_{j,k+1}^{n+1/2}-\rho_{j,k}^{n+1/2}\right|.
\end{aligned}$$

Since the Godunov scheme used in (3.2) is total variation diminishing [13, Proposition 3.1(d)], we get

$$\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}\left|\rho_{j,k+1}^{n+1}-\rho_{j,k}^{n+1}\right|\leq(1+8\Delta t\mathcal{K})\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}\left|\rho_{j,k+1}^{n}-\rho_{j,k}^{n}\right|\leq e^{8\Delta t\mathcal{K}}\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}\left|\rho_{j,k+1}^{n}-\rho_{j,k}^{n}\right|, \tag{3.18}$$

which, when applied recursively, yields the thesis. ∎

Proposition 3.2. *Let $\boldsymbol{\rho}_o\in(\mathbf{L^1}\cap\mathbf{BV})(\mathbb{R};[0,1]^M)$. Assume that the CFL condition (3.12) and Assumption 2.1 hold. Then, for $n=0,\dots,N_T-1$,*

$$\Delta x\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}\left|\rho_{j,k}^{n+1}-\rho_{j,k}^{n}\right|\leq 2\Delta t\left(2\mathcal{K}\|\boldsymbol{\rho}_o\|_{\mathbf{L^1}(\mathbb{R})}+\mathcal{V}e^{8t^n\mathcal{K}}\sum_{j=1}^{M}\mathrm{TV}\left(\rho_j^0\right)\right), \tag{3.19}$$

with $\mathcal{V}$ as in (3.13).

Proof. Observe that

$$\left|\rho_{j,k}^{n+1}-\rho_{j,k}^{n}\right|\leq\left|\rho_{j,k}^{n+1}-\rho_{j,k}^{n+1/2}\right|+\left|\rho_{j,k}^{n+1/2}-\rho_{j,k}^{n}\right|.$$

We then separately estimate each term on the right-hand side of the inequality above.

By the relaxation step (3.4) we have

$$\begin{aligned}
&\left|\rho_{j,k}^{n+1}-\rho_{j,k}^{n+1/2}\right|\\
&=\Delta t\left|S_{j-1}\left(\rho_{j-1,k}^{n+1/2},\rho_{j,k}^{n+1/2},R_{j-1,k}^{n+1/2},R_{j,k}^{n+1/2}\right)-S_j\left(\rho_{j,k}^{n+1/2},\rho_{j+1,k}^{n+1/2},R_{j,k}^{n+1/2},R_{j+1,k}^{n+1/2}\right)\right|.
\end{aligned}$$

Due to (2.6), it is easy to see that the numerical source term S_j satisfies $S_j(0,0,R_{j,k}^{n+1/2},R_{j+1,k}^{n+1/2})=0$. Thus, by the Lipschitz continuity of the map S_j and the positivity of each $\rho_{j,k}^{n+1/2}$,

$$\left|\rho_{j,k}^{n+1}-\rho_{j,k}^{n+1/2}\right|\leq\Delta t\mathcal{K}\left(\rho_{j-1,k}^{n+1/2}+2\rho_{j-1,k}^{n+1/2}+\rho_{j-1,k}^{n+1/2}\right). \tag{3.20}$$

By the convective step (3.3), since the numerical flux defined in (3.2) is Lipschitz continuous in both arguments with Lipschitz constant $\mathcal{V}$ (3.13), we have

$$\left|\rho_{j,k}^{n+1/2}-\rho_{j,k}^{n}\right|=\lambda\left|F_j\left(\rho_{j,k}^{n},\rho_{j,k+1}^{n}\right)-F_j\left(\rho_{j,k-1}^{n},\rho_{j,k}^{n}\right)\right|$$

$$\leq \lambda \mathcal{V} \left(\left| \rho_{j,k}^n - \rho_{j,k-1}^n \right| + \left| \rho_{j,k+1}^n - \rho_{j,k}^n \right| \right). \tag{3.21}$$

Collecting together (3.20) and (3.21) and exploiting Lemma 3.3 and Proposition 3.1 yields

$$\begin{aligned} \Delta x \sum_{j=1}^{M} \sum_{k\in\mathbb{Z}} \left| \rho_{j,k}^{n+1} - \rho_{j,k}^n \right| &\leq \Delta t \mathcal{K} \sum_{j=1}^{M} \sum_{k\in\mathbb{Z}} 4 \left\| \rho_j^{n+1/2} \right\|_{\mathbf{L^1}(\mathbb{R})} + 2 \Delta t \mathcal{V} \sum_{j=1}^{M} \sum_{k\in\mathbb{Z}} \left| \rho_{j,k}^n - \rho_{j,k-1}^n \right| \\ &\leq 2 \Delta t \left(2\mathcal{K} \|\boldsymbol{\rho}_o\|_{\mathbf{L^1}(\mathbb{R})} + \mathcal{V} e^{8 t^n \mathcal{K}} \sum_{j=1}^{M} \mathrm{TV}(\rho_j^0) \right). \end{aligned}$$

□

Using the estimates provided by Propositions 3.1 and 3.2, we obtain the following **BV** estimate in space and time.

COROLLARY 3.1 (**BV estimate in space and time**). *Let $\boldsymbol{\rho}_o \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,1]^M)$. Assume that the CFL condition (3.12) holds. Then, for all $n = 1,\dots,N_T$, the following estimate holds*

$$\begin{aligned} &\sum_{m=0}^{n-1} \sum_{j=1}^{M} \sum_{k\in\mathbb{Z}} \left(\Delta t \left| \rho_{j,k+1}^m - \rho_{j,k}^m \right| + \Delta x \left| \rho_{j,k}^{m+1} - \rho_{j,k}^m \right| \right) \\ &\leq n \Delta t e^{8 t^n \mathcal{K}} \left((2\mathcal{V}+1) \sum_{j=1}^{M} \mathrm{TV}(\rho_j^0) + 4\mathcal{K} \sum_{j=1}^{M} \|\rho_{o,j}\|_{\mathbf{L^1}(\mathbb{R})} \right). \end{aligned}$$

3.4. Discrete entropy inequality. We derive a discrete entropy inequality for the approximate solution $\boldsymbol{\rho}_\Delta$ constructed through Algorithm 3.1. The proof is entirely similar to [16, Lemma 2.7], with the simplification that now the flux does not depend on the spatial variable.

Define, for each $c \in [0,1]$ and $j = 1,\dots,M$, the Kružkov numerical entropy flux as

$$\mathscr{F}_j^c(u,w) = F_j(u \vee c, w \vee c) - F_j(u \wedge c, w \wedge c),$$

where $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$.

LEMMA 3.4. *Let $\boldsymbol{\rho}_o \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,1]^M)$. Assume that the CFL condition (3.12) holds. Then, the approximate solution $\boldsymbol{\rho}_\Delta$ constructed by Algorithm 3.1 satisfies the following discrete entropy inequality: for all $j = 1,\dots,M$, for $k \in \mathbb{Z}$, for $n = 0,\dots,N_T - 1$ and for any $c \in [0,1]$,*

$$\begin{aligned} &\left| \rho_{j,k}^{n+1} - c \right| - \left| \rho_{j,k}^n - c \right| + \lambda \left(\mathscr{F}_j^c \left(\rho_{j,k}^n, \rho_{j,k+1}^n \right) - \mathscr{F}_j^c \left(\rho_{j,k-1}^n, \rho_{j,k}^n \right) \right) \\ &\quad - \Delta t \,\mathrm{sgn}\left(\rho_{j,k}^{n+1} - c \right) \left(S_{j-1}\left(\rho_{j-1,k}^{n+1/2}, \rho_{j,k}^{n+1/2}, R_{j-1,k}^{n+1/2}, R_{j,k}^{n+1/2} \right) \right. \\ &\qquad \left. - S_j\left(\rho_{j,k}^{n+1/2}, \rho_{j+1,k}^{n+1/2}, R_{j,k}^{n+1/2}, R_{j+1,k}^{n+1/2} \right) \right) \leq 0. \end{aligned} \tag{3.22}$$

3.5. Convergence. The results obtained in the preceding sections, namely Lemma 3.2 for the invariance of the set $[0,1]^M$ and Corollary 3.1 for the total variation bound in space and time, allow to apply Helly's compactness theorem, which ensures the existence of a subsequence of $\boldsymbol{\rho}_\Delta$ converging in $\mathbf{L^1}$ to a function $\boldsymbol{\rho} \in \mathbf{L}^\infty([0,T] \times \mathbb{R};[0,1]^M)$, with the additional property of preserving the initial mass, that is $\|\boldsymbol{\rho}(t)\|_{\mathbf{L}^1(\mathbb{R})} = \|\boldsymbol{\rho}_o\|_{\mathbf{L}^1(\mathbb{R})}$ for $t \in [0,T]$. Moreover, Proposition 3.2 and in particular formula (3.19), imply that $\boldsymbol{\rho} \in \mathbf{C}^0([0,T];\mathbf{L}^1(\mathbb{R};[0,1]^M))$.

The limit function $\boldsymbol{\rho}$ is a solution to problem (2.3) in the sense of Definition 2.1. Indeed, the weak formulation, i.e. the integral equality in the first part of Definition 2.1, follows from a Lax–Wendroff-type calculation [25, Theorem 12.1], and the presence of the source terms does not add any difficulty in the proof.

Concerning the entropy inequality in the second part of Definition 2.1, rather standard computations starting from the discrete entropy inequality in Lemma 3.4 yield the desired result.

Proof. **(Proof of Corollary 2.1.)** Since the maximum principle and the **BV** estimates are uniform in ν, the existence of solutions in the limit $\nu \to 0$ is ensured. Therefore, we start from the entropy inequality in Definition 2.1 and add and subtract

$$\int_0^T \int_{\mathbb{R}} \eta'(\rho_j)\left(S_j(\rho_j,\rho_{j+1},\rho_j,\rho_{j+1}) - S_{j-1}(\rho_{j-1},\rho_j,\rho_{j-1},\rho_j)\right)\varphi \,\mathrm{d}x\,\mathrm{d}t.$$

We need to show that

$$\int_0^T \int_{\mathbb{R}} \eta'(\rho_j)(A_j + A_{j-1})\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \overset{\nu\to 0}{\longrightarrow} 0,$$

where

$$A_j := S_j(\rho_j,\rho_{j+1},R_j,R_{j+1}) - S_j(\rho_j,\rho_{j+1},\rho_j,\rho_{j+1}).$$

By the Lispchitz continuity of the source term we directly obtain

$$\int_0^T \int_{\mathbb{R}} |\eta'(\rho_j)||A_j|\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \le\ 2\mathcal{K}\|\varphi\|\,\|\eta'\| \int_0^T \int_{\mathbb{R}} (|R_j-\rho_j| + |R_{j+1}-\rho_{j+1}|)\,\mathrm{d}x\,\mathrm{d}t.$$

To improve readability, we deal only with the first difference. Exploiting the definition of the convolution product (2.2), we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} |R_j-\rho_j|\,\mathrm{d}x\,\mathrm{d}t &\le \int_0^T \int_{\mathbb{R}}\int_{\mathbb{R}} |\rho_j(t,y)-\rho_j(t,x)|w_\nu(y-x)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T \int_{\mathbb{R}} w_\nu(y)\int_{\mathbb{R}} |\rho_j(t,y+x)-\rho_j(t,x)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.\end{aligned}$$

Now, by [12, Lemma 2.4] and the total variation bound (3.17), which we denote by C to improve readability, we get

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} |R_j-\rho_j|\,\mathrm{d}x\,\mathrm{d}t &\le C\int_0^T \int_{\mathbb{R}} w_\nu(y)|y|\,\mathrm{d}y\,\mathrm{d}t\\ &\le CT\max\{|\inf\operatorname{spt} w_\nu|, |\sup\operatorname{spt} w_\nu|\}\int_{\mathbb{R}} w_\nu(y)\,\mathrm{d}y\\ &= CT\max\{|\inf\operatorname{spt} w_\nu|, |\sup\operatorname{spt} w_\nu|\}.\end{aligned}$$

The last term goes to zero as $\nu \to 0$. The remaining terms can be treated analogously such that we obtain

$$\int_0^T \int_{\mathbb{R}} (A_j + A_{j-1})\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \to 0 \qquad \text{as } \nu \to 0.$$

This yields the existence of solutions.

Although the local model differs from that in [20] in the source term, the proof of the uniqueness of the solution is entirely the same. ∎

4. Uniqueness of solutions: $\mathbf{L^1}$ contractivity

As for the *local* model [20], the special form of the source terms implies the $\mathbf{L^1}$-contractivity of the solution to (2.3). In particular, this result guarantees uniqueness of solutions to problem (2.3).

THEOREM 4.1. *Let $\boldsymbol{\rho}$ and $\boldsymbol{\pi}$ be two solutions to problem (2.3) in the sense of Definition 2.1, with initial data $\boldsymbol{\rho}_o, \boldsymbol{\pi}_o \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,1]^M)$, respectively. Let Assumption 2.1 hold. Then, for a.e. $t \in [0,T]$,*

$$\sum_{j=1}^{M} \|\rho_j(t) - \pi_j(t)\|_{\mathbf{L^1}(\mathbb{R})} \le \sum_{j=1}^{M} \|\rho_{j,o} - \pi_{j,o}\|_{\mathbf{L^1}(\mathbb{R})}. \tag{4.1}$$

Proof. The proof follows the idea of [20, Theorem 3.3], with the main difference that now the source terms are *nonlocal* functions of the solution. We recall the proof briefly for completeness, focusing mainly on those parts where the nonlocality comes in.

Kružkov doubling of variables technique, together with the fact that $\boldsymbol{\rho}$ and $\boldsymbol{\pi}$ are solutions to (2.3), yields, for $\tau \in [0,T]$ and for any $j = 1,\dots,M$,

$$\begin{aligned}\int_{\mathbb{R}} (\rho_j(\tau) - \pi_j(\tau))^+ \, \mathrm{d}x \le & \int_{\mathbb{R}} (\rho_j(0) - \pi_j(0))^+ \, \mathrm{d}x \\ & + \int_0^\tau \int_{\mathbb{R}} H(\rho_j - \pi_j)\left(\mathcal{S}(\boldsymbol{\rho},\boldsymbol{R},j) - \mathcal{S}(\boldsymbol{\pi},\boldsymbol{P},j)\right) \mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.2}$$

where H is the Heaviside function,

$$\begin{aligned}\mathcal{S}(\boldsymbol{u},\boldsymbol{U},j) &= S_{j-1}(u_{j-1},u_j,U_{j-1},U_j) - S_j(u_j,u_{j+1},U_j,U_{j+1}), \\ U_j(t,x) &= (u_j(t) * w_\nu)(x),\end{aligned}$$

and we denote by $\boldsymbol{U}$ the vector of components U_j, $j = 1,\dots,M$. Due to Assumption 2.1, $S_j(u,w,U,W)$ is nondecreasing in the first and third variables and nonincreasing in the second and fourth variables, thus $\partial_u S_j, \partial_U S_j \ge 0$ and $\partial_w S_j, \partial_W S_j \le 0$. Hence, if $\rho_j > \pi_j$, clearly $R_j > P_j$ and moreover

$$\begin{aligned}& \mathcal{S}(\boldsymbol{\rho},\boldsymbol{R},j) - \mathcal{S}(\boldsymbol{\pi},\boldsymbol{P},j) \\ = {} & S_{j-1}(\rho_{j-1},\rho_j,R_{j-1},R_j) - S_{j-1}(\pi_{j-1},\pi_j,P_{j-1},P_j) \\ & - S_j(\rho_j,\rho_{j+1},R_j,R_{j+1}) + S_j(\pi_j,\pi_{j+1},P_j,P_{j+1}) \\ \le {} & S_{j-1}(\rho_{j-1},\pi_j,R_{j-1},P_j) - S_{j-1}(\pi_{j-1},\pi_j,P_{j-1},P_j) \\ & - S_j(\rho_j,\rho_{j+1},R_j,R_{j+1}) + S_j(\rho_j,\pi_{j+1},R_j,P_{j+1}) \\ = {} & \partial_u S_{j-1}(\sigma_{j-1},\pi_j,R_{j-1},P_j)\,(\rho_{j-1} - \pi_{j-1}) + \partial_U S_{j-1}(\pi_{j-1},\pi_j,T_{j-1},P_j)\,(R_{j-1} - P_{j-1}) \\ & - \partial_w S_j(\rho_j,\sigma_{j+1},R_j,R_{j+1})\,(\rho_{j+1} - \pi_{j+1}) - \partial_W S_j(\rho_j,\sigma_{j+1},R_j,T_{j+1})\,(R_{j+1} - P_{j+1}) \\ \le {} & \mathcal{K}\Big((\rho_{j-1} - \pi_{j-1})^+ + (R_{j-1} - P_{j-1})^+ + (\rho_{j+1} - \pi_{j+1})^+ + (R_{j+1} - P_{j+1})^+\Big),\end{aligned}$$

where $\sigma_{j\pm1}$ lies in the interval between $\rho_{j\pm1}$ and $\pi_{j\pm1}$, $T_{j\pm1}$ lies in the interval between $R_{j\pm1}$ and $P_{j\pm1}$. Therefore

$$\sum_{j=1}^{M} H(\rho_j - \pi_j)\left(\mathcal{S}(\boldsymbol{\rho},\boldsymbol{R},j) - \mathcal{S}(\boldsymbol{\pi},\boldsymbol{P},j)\right) \le 2\mathcal{K} \sum_{j=1}^{M} (\rho_j - \pi_j)^+ + 2\mathcal{K} \sum_{j=1}^{M} (R_j - P_j)^+. \tag{4.3}$$

Observe that $\int_{\mathbb{R}}(g*w_\nu)^+(x)\,\mathrm{d}x=\int_{\mathbb{R}}g^+(x)\,\mathrm{d}x$, thus, due to (3.6), when integrating (4.3) over $\mathbb{R}$ we obtain

$$\sum_{j=1}^{M}\int_{\mathbb{R}}H(\rho_j-\pi_j)\left(\mathcal{S}(\boldsymbol{\rho},\boldsymbol{R},j)-\mathcal{S}(\boldsymbol{\pi},\boldsymbol{P},j)\right)\mathrm{d}x\leq 4\mathcal{K}\sum_{j=1}^{M}\int_{\mathbb{R}}(\rho_j-\pi_j)^+\,\mathrm{d}x. \tag{4.4}$$

Define

$$\Theta(t)=\sum_{j=1}^{M}\int_{\mathbb{R}}(\rho_j-\pi_j)^+\,\mathrm{d}x,$$

so that, collecting together (4.2) and (4.4), we get

$$\Theta(\tau)\leq\Theta(0)+4\mathcal{K}\int_0^\tau\Theta(t)\,\mathrm{d}t.$$

Gronwall's inequality yields $\Theta(t)\leq e^{4\mathcal{K}t}\Theta(0)$. If $\Theta(0)=0$, that is $\rho_{o,j}(x)\leq\pi_{o,j}(x)$ a.e. in $\mathbb{R}$ for all j, then $\Theta(t)=0$ for $t>0$, that is $\rho_j(t,x)\leq\pi_j(t,x)$ a.e. in $\mathbb{R}$ for all j.

The proof of $\mathbf{L^1}$-contractivity is concluded by an application of the Crandall–Tartar lemma [19, Lemma 2.13]. ∎

Following [20, Corollary 3.4], the $\mathbf{L^1}$-contractivity of the solution proved in Theorem 4.1 guarantees that the solution to problem (2.3) satisfies some *a priori* estimates.

Corollary 4.1. *Let $\boldsymbol{\rho}$ be a solution to problem* (2.3) *in the sense of Definition 2.1, with initial datum $\boldsymbol{\rho}_o\in(\mathbf{L^1}\cap\mathbf{BV})(\mathbb{R};[0,1]^M)$. Then,*

$$\sum_{j=1}^{M-1}\|\rho_{j+1}(t)-\rho_j(t)\|_{\mathbf{L^1}(\mathbb{R})}\leq\sum_{j=1}^{M-1}\|\rho_{j+1,o}-\rho_{j,o}\|_{\mathbf{L^1}(\mathbb{R})}, \tag{4.5}$$

$$\sum_{j=1}^{M}\mathrm{TV}(\rho_j(t))\leq\sum_{j=1}^{M}\mathrm{TV}(\rho_{j,o}), \tag{4.6}$$

$$\sum_{j=1}^{M}\|\rho_j(t+h)-\rho_j(t)\|_{\mathbf{L^1}(\mathbb{R})}\leq\sum_{j=1}^{M}\|\rho_j(h)-\rho_{j,o}\|_{\mathbf{L^1}(\mathbb{R})},\quad h\in\mathbb{R}. \tag{4.7}$$

The proof relies solely on (4.1), together with the enforced boundary conditions $\rho_0(t,x)=\rho_1(t,x)$, $v_0(u)=v_1(u)$, $\rho_{M+1}(t,x)=\rho_M(t,x)$, $v_{M+1}(u)=v_M(u)$.

Notice that Corollary 4.1 provides better estimates than those coming from the approximate solution built in Section 3. Compare in particular (4.6) to the total variation in space provided by (3.17).

5. A multilane model with nonlocal flux and nonlocal source term

In the following, we consider a modification of problem (2.3) assuming additionally a *nonlocal* velocity in the flux function. In particular, the treatment of the nonlocal flux in each lane is inspired by [3]. The problem under consideration reads

$$\begin{cases}\partial_t\rho_j+\partial_x\left(\rho_j v_j(\rho_j*w_\iota)\right)=S_{j-1}(\rho_{j-1},\rho_j,R_{j-1},R_j)-S_j(\rho_j,\rho_{j+1},R_j,R_{j+1}) & j=1,\dots,M,\\ \rho_j(0,x)=\rho_{o,j}(x) & j=1,\dots,M.\end{cases} \tag{5.1}$$

In order to have a well defined model, we only consider kernel functions such that $\operatorname{supp}w_\iota\subseteq[0,\iota]$, meaning that drivers adapt their speed to the downstream traffic. In addition, we assume that the kernel $w_\iota \in \mathbf{C^1}([0,\iota];\mathbb{R}_+)$ is non-increasing, i.e. $w_\iota' \leq 0$, and, as usual $\int_{\mathbb{R}} w_\iota = 1$. The convolution product is thus defined as

$$R_j^\iota = R_j^\iota(t,x) = (\rho(t) * w_\iota)(x) := \int_x^{x+\iota} w_\iota(y-x)\rho(t,y)\,\mathrm{d}y. \tag{5.2}$$

We remark that the additional assumptions on the kernel w_ι in the flux are not needed for the kernel w_ν in the source term, see Section 2. Moreover, when considering both nonlocal flux and nonlocal source, we underline that the kernels may differ. In the following, we denote the convolution products in the source by R_j^ν (2.2) and those in the flux by R_j^ι (5.2), to emphasize the different kernels.

We underline that the kernel function w_ν appearing in the source can look either only forward or both back- and forward, differently from the kernel function w_ι appearing in the flux, which is assumed to be only forward-looking. As already mentioned in the introduction, these are the key points in which the proposed model (5.1) differs from the approach presented in [2]. Therein, the uniqueness is only shown for the same nonlocality in the flux and source term, such that both have to be forward looking with the same non-increasing kernel and the same nonlocal range. So the model (5.1) provides more flexibility in terms of modelling. However, using the same non-increasing, forward looking kernel and nonlocal range, the model (5.1) fits into the framework proposed in [2, Definition 1.1, Assumption 2.2, Assumption 3.1]. We also note that in [2] the authors use a different technique to show existence and uniqueness of solutions, which enables them to prove uniqueness without an entropy condition.

We proceed as in Section 3: We construct a sequence of approximate solutions to problem (5.1) and prove its convergence. The approximate solution $\boldsymbol{\rho}_\Delta$ is defined as in (3.1) and it is constructed as in Algorithm 3.1, substituting the numerical flux in (3.2) by

$$F_j(\rho_{j,k}^n, R_{j,k}^{\iota,n}) = v_j(R_{j,k}^{\iota,n})\,\rho_{j,k}^n, \tag{5.3}$$

and the convective step (3.3) by

$$\rho_{j,k}^{n+1/2} = \rho_{j,k}^n - \lambda\left[F_j(\rho_{j,k}^n, R_{j,k}^{\iota,n}) - F_j(\rho_{j,k-1}^n, R_{j,k-1}^{\iota,n}),\right] \tag{5.4}$$

where $R_{j,k}^{\iota,n}$ is computed as in (3.6), and in particular as in (3.10), with w_ι instead of w_ν. Due to the definition of the kernel w_ι, notice that the case (3.11) does not apply to the present setting. Accordingly, we rename the discrete convolution appearing in the source, defined by (3.6), as $R_{j,k}^{\nu,n+1/2}$. The choice of the numerical flux (5.3) follows from [8, 15].

We report below the definition of solution to problem (5.1), analogous to Definition 2.1, and then recall the main results, analogous to those in Section 3. Only those parts of the proofs which are substantially different will be reported.

Definition 5.1. *Let $\rho_{o,j} \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,1])$, for $j = 1,\ldots,M$. We say that $\rho_j \in \mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};[0,1]))$, with $\rho_j(t,\cdot) \in \mathbf{BV}(\mathbb{R};[0,1])$ for $t \in [0,T]$, is a* weak solution *to (5.1) with initial datum $\rho_{o,j}$ if for any $\varphi \in \mathbf{C_c^1}([0,T[\times\mathbb{R};\mathbb{R})$ and for all $j = 1,\ldots,M$*

$$\int_0^T\!\!\int_{\mathbb{R}} \left(\rho_j\,\partial_t\varphi + \rho_j\,V_j\,\partial_x\varphi + \left(S_{j-1}(\rho_{j-1},\rho_j,R_{j-1}^\nu,R_j^\nu) - S_j(\rho_j,\rho_{j+1},R_j^\nu,R_{j+1}^\nu)\right)\varphi\right)\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\mathbb{R}} \rho_{o,j}\,\varphi(0,x)\,\mathrm{d}x = 0,$$

where $V_j(t,x) = v_j((\rho_j(t) * w_\iota)(x))$, S_j *is as in* (2.1) *and* $R_j^\nu = R_j^\nu(t,x) = (\rho_j(t) * w_\nu)(x)$. *The solution* ρ_j *is an* entropy solution *if for any* $\varphi \in \mathbf{C_c^1}([0,T[\times\mathbb{R};\mathbb{R}_+)$, *for all* $\kappa \in \mathbb{R}$ *and for all* $j = 1,\dots,M$

$$\int_0^T \int_{\mathbb{R}} (|\rho_j - \kappa| \partial_t \varphi + |\rho_j - \kappa| V_j \, \partial_x \varphi) \, \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}} |\rho_{o,j} - \kappa| \varphi(0,x) \, \mathrm{d}x$$
$$\geq \int_0^T \int_{\mathbb{R}} \mathrm{sgn}(\rho_j - \kappa)(S_j(\rho_j, \rho_{j+1}, R_j, R_{j+1}) - S_{j-1}(\rho_{j-1}, \rho_j, R_{j-1}, R_j) + \kappa \partial_x V_j) \varphi \, \mathrm{d}x \, \mathrm{d}t.$$

In the following, whenever we refer to the *modified Algorithm* we mean Algorithm 3.1 with (3.2) and (3.3) substituted by (5.3) and (5.4), respectively. All the approximate solutions appearing in the results below are constructed via this modified Algorithm.

LEMMA 5.1. *Let* $\boldsymbol{\rho}_o \in \mathbf{L}^\infty(\mathbb{R};[0,1]^M)$. *Assume that the CFL condition* (3.12) *and Assumption 2.1 hold. Then, for all* $t > 0$ *and* $x \in \mathbb{R}$, *the piece-wise constant approximate solution* $\boldsymbol{\rho}_\Delta$ *constructed through the modified Algorithm attains a value in the set* $[0,1]^M$, *i.e.*

$$0 \leq \rho_{j,\Delta}(t,x) \leq 1 \qquad \textit{for all } j = 1,\dots,M.$$

Proof. Since the CFL condition (3.12) is more restrictive than that necessary for the convergence of the Godunov-type scheme, see [15, Theorem 3.1], the convective step (3.3) still preserves the invariance of the set $[0,1]^M$ and the rest of the proof of Lemma 3.2 can be applied. □

Lemma 3.3 still holds, since the modified Algorithm preserves the $\mathbf{L^1}$-norm.

PROPOSITION 5.1 (**BV estimate in space**). *Let* $\boldsymbol{\rho}_o \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,1]^M)$. *Assume that the CFL condition* (3.12) *and Assumption 2.1 hold. Then, for* $n = 0,\dots,N_T - 1$ *the following estimate holds*

$$\sum_{j=1}^M \sum_{k \in \mathbb{Z}} |\rho_{j,k+1}^n - \rho_{j,k}^n| \leq e^{t^n (8\mathcal{K} + w_\iota(0)\mathcal{V})} \sum_{j=1}^M \mathrm{TV}(\rho_j^0). \tag{5.5}$$

Proof. The proof of Proposition 3.1 can be easily adapted. We just have to replace estimate (3.18), involving the convective step, since the scheme with the new numerical flux (5.3) is not total variation diminishing. Following [15, Theorem 3.2] we obtain

$$\sum_{j=1}^M \sum_{k \in \mathbb{Z}} \left| \rho_{j,k+1}^{n+1} - \rho_{j,k}^{n+1} \right| \leq (1 + 8\Delta t \mathcal{K})(1 + \Delta t \, w_\iota(0)\mathcal{V}) \sum_{j=1}^M \sum_{k \in \mathbb{Z}} |\rho_{j,k+1}^n - \rho_{j,k}^n|$$
$$\leq e^{\Delta t (8\mathcal{K} + w_\iota(0)\mathcal{V})} \sum_{j=1}^M \sum_{k \in \mathbb{Z}} |\rho_{j,k+1}^n - \rho_{j,k}^n|,$$

which, when applied recursively, yields the thesis. □

PROPOSITION 5.2. *Let* $\boldsymbol{\rho}_o \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,1]^M)$. *Assume that the CFL condition* (3.12) *and Assumption 2.1 hold. Then, for* $n = 0,\dots,N_T - 1$,

$$\Delta x \sum_{j=1}^M \sum_{k \in \mathbb{Z}} \left| \rho_{j,k}^{n+1} - \rho_{j,k}^n \right| \leq 2\Delta t \left(2\mathcal{K} \|\boldsymbol{\rho}_o\|_{\mathbf{L^1}(\mathbb{R})} + \mathcal{V} e^{t^n (8\mathcal{K} + w_\iota(0)\mathcal{V})} \sum_{j=1}^M \mathrm{TV}(\rho_j^0) \right), \tag{5.6}$$

with $\mathcal{V}$ *as in* (3.13) *and* $V_{\max}$ *as in* (3.14).

Proof. Observe that

$$\left|\rho_{j,k}^{n+1}-\rho_{j,k}^{n}\right|\leq\left|\rho_{j,k}^{n+1}-\rho_{j,k}^{n+1/2}\right|+\left|\rho_{j,k}^{n+1/2}-\rho_{j,k}^{n}\right|.$$

We then estimate each term on the right-hand side separately.

By the relaxation step (3.4) and Assumption 2.1 we have

$$\begin{aligned}&\left|\rho_{j,k}^{n+1}-\rho_{j,k}^{n+1/2}\right|\\ &=\Delta t\left|S_{j-1}\left(\rho_{j-1,k}^{n+1/2},\rho_{j,k}^{n+1/2},R_{j-1,k}^{n+1/2},R_{j,k}^{n+1/2}\right)-S_{j}\left(\rho_{j,k}^{n+1/2},\rho_{j+1,k}^{n+1/2},R_{j,k}^{n+1/2},R_{j+1,k}^{n+1/2}\right)\right|\\ &\leq\Delta t\mathcal{K}\left(\rho_{j-1,k}^{n+1/2}+2\rho_{j,k}^{n+1/2}+\rho_{j+1,k}^{n+1/2}\right).\end{aligned}$$

Therefore, thanks to Lemma 3.3

$$\Delta x\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}\left|\rho_{j,k}^{n+1}-\rho_{j,k}^{n+1/2}\right|\leq\Delta t\mathcal{K}\sum_{j=1}^{M}4\left\|\rho_{j}^{n+1/2}\right\|_{\mathbf{L^1}(\mathbb{R})}=4\Delta t\mathcal{K}\sum_{j=1}^{M}\|\rho_{j,o}\|_{\mathbf{L^1}(\mathbb{R})}. \tag{5.7}$$

Exploiting the modified convective step (5.4), since the numerical flux defined in (5.3) is Lipschitz continuous in both variables with Lipschitz constant $\mathcal{V}$ (3.13), we have

$$\begin{aligned}\left|\rho_{j,k}^{n+1/2}-\rho_{j,k}^{n}\right|&=\lambda\left|F_j\left(\rho_{j,k}^{n},R_{j,k}^{\iota,n}\right)-F_j\left(\rho_{j,k-1}^{n},R_{j,k-1}^{\iota,n}\right)\right|\\ &\leq\lambda\mathcal{V}\left(\left|\rho_{j,k}^{n}-\rho_{j,k-1}^{n}\right|+\left|R_{j,k}^{\iota,n}-R_{j,k-1}^{\iota,n}\right|\right).\end{aligned}$$

Hence, using also (3.8) and the total variation bound provided by Proposition 5.1, we get

$$\begin{aligned}&\Delta x\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}\left|\rho_{j,k}^{n+1/2}-\rho_{j,k}^{n}\right|\\ &\leq 2\Delta t\mathcal{V}\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}\left|\rho_{j,k}^{n}-\rho_{j,k-1}^{n}\right|\leq 2\Delta t\mathcal{V}e^{t^n(8\mathcal{K}+w_\iota(0)\mathcal{V})}\sum_{j=1}^{M}\mathrm{TV}(\rho_j^0).\end{aligned} \tag{5.8}$$

Collecting together (5.7) and (5.8) yields the thesis

$$\Delta x\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}}\left|\rho_{j,k}^{n+1}-\rho_{j,k}^{n}\right|\leq 2\Delta t\left(2\mathcal{K}\|\boldsymbol{\rho}_o\|_{\mathbf{L^1}(\mathbb{R})}+\mathcal{V}e^{t^n(8\mathcal{K}+w_\iota(0)\mathcal{V})}\sum_{j=1}^{M}\mathrm{TV}(\rho_j^0)\right).$$

∎

Proceeding as in Corollary 3.1, combining the results of Proposition 5.1 and Proposition 5.2 we obtain a **BV** estimate in space and time.

Analogous to Section 3.4, a discrete entropy inequality could also be derived in the case of nonlocal flux, see [1, Proposition 2.8]. Indeed, combining [15, Theorem 3.4] for the nonlocal flux and Lemma 3.4 for the treatment of the source terms, we get the following result.

LEMMA 5.2. *Let $\boldsymbol{\rho} \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,1]^M)$. Let the CFL condition (3.12) hold. Then the approximate solution $\boldsymbol{\rho}_\Delta$ constructed through the modified Algorithm satisfies the following discrete entropy inequality: for all $j=1,\dots,M$, for $k\in\mathbb{Z}$, for $n=0,\dots,N_T-1$ and for any $c\in[0,1]$*

$$\begin{aligned}&\left|\rho_{j,k}^{n+1}-c\right|-\left|\rho_{j,k}^{n}-c\right|+\lambda\left(\mathscr{F}_j^c\left(\rho_{j,k}^n\right)-\mathscr{F}_j^c\left(\rho_{j,k-1}^n\right)\right)\\&\quad-\Delta t\,\mathrm{sgn}\left(\rho_{j,k}^{n+1}-c\right)\left(S_{j-1}\left(\rho_{j-1,k}^{n+1/2},\rho_{j,k}^{n+1/2},R_{j-1,k}^{\nu,n+1/2},R_{j,k}^{\nu,n+1/2}\right)\right.\\&\quad\left.-S_j\left(\rho_{j,k}^{n+1/2},\rho_{j+1,k}^{n+1/2},R_{j,k}^{\nu,n+1/2},R_{j+1,k}^{\nu,n+1/2}\right)\right)\\&\quad+\lambda\,\mathrm{sgn}\left(\rho_{j,k}^{n+1}-c\right)c\left(v_j\left(R_{j,k+1}^{\iota,n}\right)-v_j\left(R_{j,k}^{\iota,n}\right)\right)\le 0,\end{aligned}$$

where $R_{j,k}^{\nu,n+1/2}$ and $R_{j,k}^{\iota,n}$ are defined according to (3.6) and

$$\mathscr{F}_j^c(u)=G_j(u\vee c)-G_j(u\wedge c),\qquad \text{with } G_j(\rho_{j,k}^n)=\rho_{j,k}^n\,v_j\left(R_{j,k}^{\iota,n}\right).$$

The results described in Section 3.5 hold analogously for the modified Algorithm, given the bounds obtained in the present section: this ensures the existence of solutions to (5.1).

Uniqueness of solution follows from the Lipschitz continuous dependence of the solution on the initial data. Different from Theorem 4.1, in the case of *nonlocal* flux function the solution is not contractive in $\mathbf{L^1}$.

THEOREM 5.1. *Let $\boldsymbol{\rho}$ and $\boldsymbol{\pi}$ be two solutions to problem (5.1) in the sense of Definition 5.1, with initial data $\boldsymbol{\rho}_o,\boldsymbol{\pi}_o\in(\mathbf{L^1}\cap\mathbf{BV})(\mathbb{R};[0,1]^M)$ respectively. Assume $v\in\mathbf{C^2}([0,1],\mathbb{R})$ and let Assumption 2.1 hold. Then, for a.e. $t\in[0,T]$,*

$$\sum_{j=1}^M\|\rho_j(t)-\pi_j(t)\|_{\mathbf{L^1}(\mathbb{R})}\le e^{\mathcal{C}t}\sum_{j=1}^M\|\rho_{j,o}-\pi_{j,o}\|_{\mathbf{L^1}(\mathbb{R})},$$

with $\mathcal{C}$ defined as in (5.16).

Proof. The doubling of variables technique [24] allows to get the following estimate, see [9, Lemma 4] for the treatment of the nonlocal flux, while the source terms are treated similarly to [20, Theorem 3.3]: any $j=1,\dots,M$,

$$\begin{aligned}\int_{\mathbb{R}}|\rho_j(\tau,x)-\pi_j(\tau,x)|\,\mathrm{d}x\le{}&\int_{\mathbb{R}}|\rho_j(0,x)-\pi_j(0,x)|\,\mathrm{d}x\\&+\int_0^\tau\int_{\mathbb{R}}|\mathcal{S}(\boldsymbol{\rho},\boldsymbol{R}^\nu,j)-\mathcal{S}(\boldsymbol{\pi},\boldsymbol{P}^\nu,j)|\,\mathrm{d}x\,\mathrm{d}t &(5.9)\\&+\int_0^\tau\int_{\mathbb{R}}\left|v_j(R_j^\iota)-v_j(P_j^\iota)\right||\partial_x\rho_j(t,x)|\,\mathrm{d}x\,\mathrm{d}t &(5.10)\\&+\int_0^\tau\int_{\mathbb{R}}\left|\partial_x v_j(R_j^\iota)-\partial_x v_j(P_j^\iota)\right||\rho_j(t,x)|\,\mathrm{d}x\,\mathrm{d}t, &(5.11)\end{aligned}$$

where, for the source terms we use the notation introduced in the proof of Theorem 4.1, while for the kernel we refer to (2.2), emphasizing which kernel, w_ι or w_ν, is used. We remark that $\partial_x\rho$ should be understood in the sense of measures.

To bound the term in (5.9), exploit the Lipschitz continuity of the map S_j (2.1) in the source term:

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}} |\mathcal{S}(\boldsymbol{\rho},\boldsymbol{R}^\nu,j)-\mathcal{S}(\boldsymbol{\pi},\boldsymbol{P}^\nu,j)|\,\mathrm{d}x\,\mathrm{d}t\\
&\le \mathcal{K}\int_0^\tau \Big(\|\rho_{j-1}(t)-\pi_{j-1}(t)\|_{\mathbf{L^1}(\mathbb{R})}+2\|\rho_j(t)-\pi_j(t)\|_{\mathbf{L^1}(\mathbb{R})}+\|\rho_{j+1}(t)-\pi_{j+1}(t)\|_{\mathbf{L^1}(\mathbb{R})}\\
&\quad+\left\|R^\nu_{j-1}(t)-P^\nu_{j-1}(t)\right\|_{\mathbf{L^1}(\mathbb{R})}+2\left\|R^\nu_j(t)-P^\nu_j(t)\right\|_{\mathbf{L^1}(\mathbb{R})}+\left\|R^\nu_{j+1}(t)-P^\nu_{j+1}(t)\right\|_{\mathbf{L^1}(\mathbb{R})}\Big)\,\mathrm{d}t.
\end{aligned}$$

Observe that for each $j=1,\ldots,M$

$$\left\|R^\nu_j(t)-P^\nu_j(t)\right\|_{\mathbf{L^1}(\mathbb{R})}\le\|\rho_j(t)-\pi_j(t)\|_{\mathbf{L^1}(\mathbb{R})},$$

since $\int_{\mathbb{R}} w_\nu = 1$. Therefore

$$\sum_{j=1}^M [(5.9)] \le 4\mathcal{K}\sum_{j=1}^M \int_0^\tau \|\rho_j(t)-\pi_j(t)\|_{\mathbf{L^1}(\mathbb{R})}\,\mathrm{d}t. \tag{5.12}$$

Concerning (5.10), note that

$$\left|v_j(R^\iota_j)-v_j(P^\iota_j)\right|\le w_\iota(0)\left\|v_j'\right\|_{\mathbf{L^\infty}([0,1];\mathbb{R})}\|\rho_j(t)-\pi_j(t)\|_{\mathbf{L^1}(\mathbb{R})},$$

thus

$$\begin{aligned}
\sum_{j=1}^M [(5.10)] &\le w_\iota(0)\,V'_{\max}\sum_{j=1}^M\int_0^\tau \mathrm{TV}(\rho_j(t))\,\|\rho_j(t)-\pi_j(t)\|_{\mathbf{L^1}(\mathbb{R})}\,\mathrm{d}t\\
&\le w_\iota(0)\,V'_{\max}\left(\sum_{j=1}^M \sup_{t\in[0,\tau]}\mathrm{TV}(\rho_j(t))\right)\left(\sum_{j=1}^M\int_0^\tau \|\rho_j(t)-\pi_j(t)\|_{\mathbf{L^1}(\mathbb{R})}\,\mathrm{d}t\right).
\end{aligned} \tag{5.13}$$

Pass now to (5.11). Observe first that

$$\begin{aligned}
\left|\partial_x R^\iota_j(t,x)\right| &= |\partial_x(\rho_j(t)*w_\iota)(x)|\\
&=\left|-\int_x^{x+\iota}\rho_j(t,y)\,w_\iota'(x-y)\,\mathrm{d}y+\rho_j(t,x+\iota)\,w_\iota(\iota)-\rho_j(t,x)\,w_\iota(0)\right|\\
&\le\left|\int_0^\iota \rho_j(t,u+x)\,w_\iota'(u)\,\mathrm{d}u\right|+\|\rho_j(t)\|_{\mathbf{L^\infty}(\mathbb{R})}(w_\iota(\iota)+w_\iota(0))\\
&\le \|\rho_j(t)\|_{\mathbf{L^\infty}(\mathbb{R})}\left(\int_0^\iota |w_\iota'(u)|\,\mathrm{d}u+w_\iota(\iota)+w_\iota(0)\right)\\
&= \|\rho_j(t)\|_{\mathbf{L^\infty}(\mathbb{R})}\left(-\int_0^\iota w_\iota'(u)\,\mathrm{d}u+w_\iota(\iota)+w_\iota(0)\right)\\
&= 2\,w_\iota(0)\,\|\rho_j(t)\|_{\mathbf{L^\infty}(\mathbb{R})},
\end{aligned}$$

since the kernel w_ι is such that $w_\iota'\le 0$. Hence,

$$\begin{aligned}
&\left|\partial_x v_j(R^\iota_j)-\partial_x v_j(P^\iota_j)\right|\\
&\le \left|v_j'(R^\iota_j)-v_j'(P_j)\right|\left|\partial_x R^\iota_j\right|+\left|v_j'(P_j)\right|\left|\partial_x R^\iota_j-\partial_x P^\iota_j\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\|v_j''\right\|_{\mathbf{L}^\infty([0,1])} \left|R_j^\iota - P_j^\iota\right| 2\, w_\iota(0) \left\|\rho_j(t)\right\|_{\mathbf{L}^\infty(\mathbb{R})} \\
&+ \left\|v_j'\right\|_{\mathbf{L}^\infty([0,1])} \left| \int_x^{x+\iota} (\pi_j - \rho_j)(t,y)\, w_\iota'(x-y)\,\mathrm{d}y + (\rho_j - \pi_j)(t,x+\iota)\, w_\iota(\iota) - (\rho_j - \pi_j)(t,x)\, w_\iota(0) \right| \\
&\leq \left(2\,(w_\iota(0))^2 \left\|v_j''\right\|_{\mathbf{L}^\infty([0,1])} + \left\|v_j'\right\|_{\mathbf{L}^\infty([0,1])} \left\|w_\iota'\right\|_{\mathbf{L}^\infty([0,\iota])} \right) \left\|\rho_j(t) - \pi_j(t)\right\|_{\mathbf{L}^1(\mathbb{R})} \\
&\quad + w_\iota(0) \left\|v_j'\right\|_{\mathbf{L}^\infty([0,1])} \left(|(\rho_j - \pi_j)|(t,x+\iota) + |(\rho_j - \pi_j)|(t,x) \right).
\end{aligned}
$$

Therefore, since the total mass is conserved and $\rho_j(t,x) \in [0,1]$ for all $j = 1, \ldots, M$, $t \in [0,\tau]$ and $x \in \mathbb{R}$ by Lemma 5.1,

$$
\begin{aligned}
\sum_{j=1}^M [(5.11)] \leq\ & \left(2\,(w_\iota(0))^2\, V''_{\max} + V'_{\max} \left\|w_\iota'\right\|_{\mathbf{L}^\infty([0,\iota])} \right) \left\|\boldsymbol{\rho}_o\right\|_{\mathbf{L}^1(\mathbb{R})} \sum_{j=1}^M \int_0^\tau \left\|\rho_j(t) - \pi_j(t)\right\|_{\mathbf{L}^1(\mathbb{R})} \mathrm{d}t \\
& + 2\, w_\iota(0)\, V'_{\max} \sum_{j=1}^M \int_0^\tau \left\|\rho_j(t) - \pi_j(t)\right\|_{\mathbf{L}^1(\mathbb{R})} \mathrm{d}t, \qquad (5.14)
\end{aligned}
$$

where

$$
V''_{\max} = \left\|\boldsymbol{v}''\right\|_{\mathbf{C^0}([0,1];\mathbb{R}^M)} = \max_{j=1,\ldots,M} \left\|v_j''\right\|_{\mathbf{L}^\infty([0,1];\mathbb{R})}.
$$

Collecting together (5.12), (5.13) and (5.14) we obtain

$$
\int_{\mathbb{R}} |\rho_j(\tau,x) - \pi_j(\tau,x)|\,\mathrm{d}x \leq \int_{\mathbb{R}} |\rho_j(0,x) - \pi_j(0,x)|\,\mathrm{d}x + \mathcal{C} \sum_{j=1}^M \int_0^\tau \left\|\rho_j(t) - \pi_j(t)\right\|_{\mathbf{L}^1(\mathbb{R})} \mathrm{d}t, \qquad (5.15)
$$

where

$$
\begin{aligned}
\mathcal{C} =\ & 4\mathcal{K} + 2\, w_\iota(0)\, V'_{\max} + w_\iota(0)\, V'_{\max} \left(\sum_{j=1}^M \sup_{t \in [0,\tau]} \mathrm{TV}(\rho_j(t)) \right) \\
& + \left(2\,(w_\iota(0))^2\, V''_{\max} + V'_{\max} \left\|w_\iota'\right\|_{\mathbf{L}^\infty([0,\iota])} \right) \left\|\boldsymbol{\rho}_o\right\|_{\mathbf{L}^1(\mathbb{R})}. \qquad (5.16)
\end{aligned}
$$

An application of Gronwall Lemma to (5.15) yields the desired result. □

The following theorem, analogous to Theorem 2.1, collects the main result on problem (5.1), as well as some *a priori* estimates on its solution.

THEOREM 5.2. *Let* $\boldsymbol{\rho}_o \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,1]^M)$. *Assume* $v \in \mathbf{C^2}([0,1],\mathbb{R})$ *and let Assumption 2.1 hold. Then, for all* $T > 0$, *problem (5.1) has a unique solution* $\boldsymbol{\rho} \in \mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};[0,1]^M))$ *in the sense of Definition 5.1. Moreover, the following estimates hold: for any* $t \in [0,T]$

$$
\begin{aligned}
&\left\|\boldsymbol{\rho}(t)\right\|_{\mathbf{L^1}(\mathbb{R})} = \sum_{j=1}^M \left\|\rho_j(t)\right\|_{\mathbf{L^1}(\mathbb{R})} = \left\|\boldsymbol{\rho}_o\right\|_{\mathbf{L^1}(\mathbb{R})}, \\
&\textit{for } j = 1, \ldots, M: \quad 0 \leq \rho_j(t,x) \leq 1, \\
&\sum_{j=1}^M \mathrm{TV}(\rho_j(t)) \leq e^{t(8\mathcal{K} + w_\iota(0)\mathcal{V})} \sum_{j=1}^M \mathrm{TV}(\rho_{j,o}).
\end{aligned}
$$

Also for problem (5.1), it is possible to prove the convergence to a version with *local* source terms, following the same steps as in Corollary 2.1.

COROLLARY 5.1. *Let $\boldsymbol{\rho}_o \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};[0,1]^M)$ and let Assumption 2.1 hold. Then, for all $T>0$, $\iota>0$ and $\nu \to 0$, problem (5.1) has a unique solution $\boldsymbol{\rho} \in \mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};[0,1]^M))$ in the sense of Definition 5.1, in which we replace the nonlocal operators $R_j^\nu(t,x)$ with $\rho_j(t,x)$ for $j=1,\dots,M$.*

6. Numerical experiments

We now present some numerical examples. We divide this section in two parts: In the first part, we discuss an example with local flux and *nonlocal* source, as in (2.3), while in the second part we focus on *nonlocal* flux and source, as in (5.1). For simplicity, we restrict ourselves to only two lanes, i.e. $M=2$, and use the source defined in (2.1) with scaling parameter $K=1$.

6.1. Local flux and nonlocal source: results for model (2.3).

In the first example we choose the following velocity functions

$$v_1(\rho) = v_2(\rho) = 1-\rho,$$

and the initial data

$$\rho_{1,o}(x) = 0.5 \quad \text{and} \quad \rho_{2,o}(x) = \begin{cases} 0, & \text{if } x \in [1,2] \cup [2,3], \\ 0.5, & \text{else.} \end{cases}$$

We consider the space interval $[0,5]$ with periodic boundary conditions. Figure 6.1 displays the density profiles with three different source terms at times $T=0.5$ (left column), $T=1.5$ (middle column) and $T=4$ (right column). We use the nonlocal source term (2.1) with both (3.10) and (3.11), and constant kernel, namely $w_\nu(x) = 1/\nu$ for (3.10) with $\nu=1.5$, and $w_\nu(x)=1/(2\nu)$ for (3.11) with $\nu = 0.75$. Therefore, both nonlocal models have an interaction range equal to 1.5. To emphasize the influence of the nonlocality, we also include a local version of the source term (2.1) with

$$R_j = \rho_j(t,x). \tag{6.1}$$

Notice that such a local version differs from that used in [20]: Here the lane changing rate is also proportional to the density in the receiving lane. In the simulations, we consider $\Delta x = 0.01$ and Δt given by an adaptive version of the CFL condition (3.12), where $\mathcal{V}$ is computed at each time step using finite differences for the derivative of v_j.

As can be seen in Figure 6.1, different source terms already give rise to significant differences for rather small times. We can observe two effects of the nonlocal source terms: On one hand, mass is transported from lane 1 to lane 2 in the regions of higher density on lane 2 as the free space ahead is anticipated. In the local model with source (6.1) no lane change is present. On the other hand, in regions of low density on lane 2 more mass is transported to lane 2 in the local model. This is due to the fact that the nonlocal models are aware of the higher density in front, so that a lane change becomes less favourable. Both effects are even stronger in the nonlocal model with forward looking kernel (3.10) than in the one with back- and forward looking kernel (3.11). After a few time steps, all models result in a saw-tooth like profile, even though the back ends of the areas with high density are located further downstream (respectively upstream) on lane 2 (respectively lane 1) in the nonlocal models than in the local one.

These observations are also supported by the evolution of the $\mathbf{L^1}$-norm over time, see Figure 6.2. We can see that the nonlocal models transport more slowly the density from lane 1 to lane 2. In addition, the model with the forward looking kernel (3.10) is even slower than the one with (3.11).

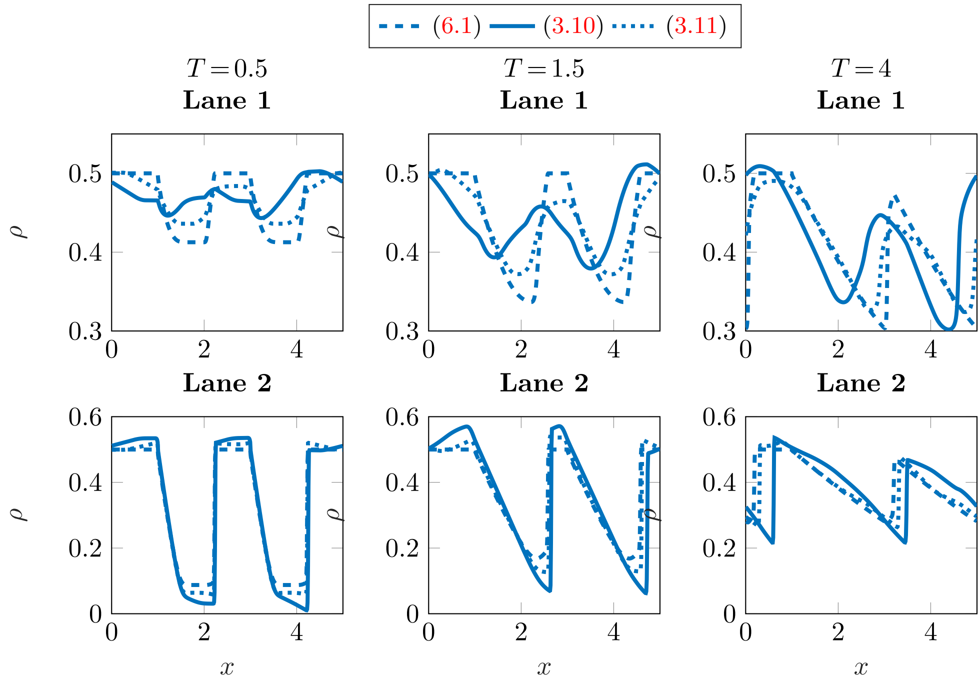

FIG. 6.1. *Density profiles on each lane at* $T = 0.75$ *(left column),* $T = 1.5$ *(middle column) and* $T = 4$ *(right column). The source term* (2.1) *is computed with* (6.1) *(dashed line),* (3.10) *with* $w_\nu(x) = 1/\nu$ *and* $\nu = 1.5$ *(solid line) and* (3.11) *with* $w_\nu(x) = 1/(2\nu)$ *and* $\nu = 0.75$ *(dotted line) on lane 1 (top row) and lane 2 (bottom row).*

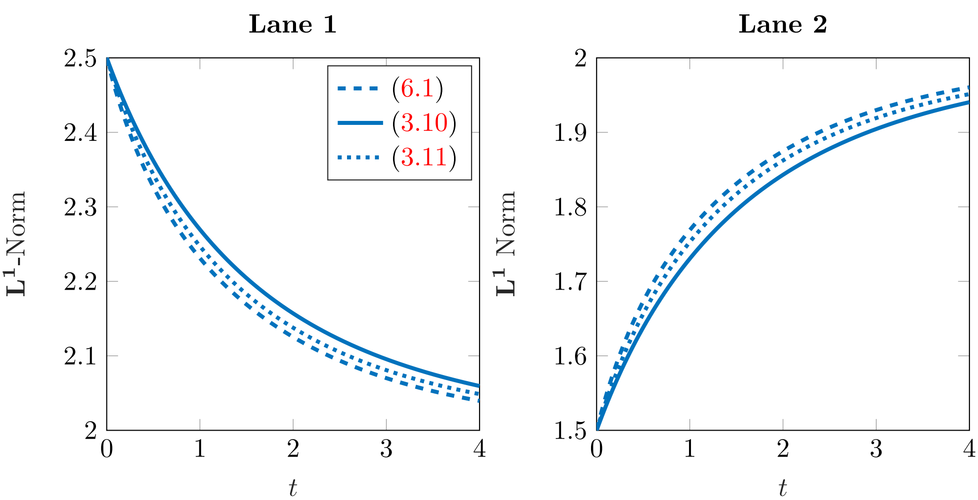

FIG. 6.2. *Evolution of* $\mathbf{L^1}$*-norm over time for the different source terms: Lane 1 (left) and Lane 2 (right). The dashed line represents the source term in* (2.1) *with* (6.1)*, the continuous line refers to* (3.10) *and the dotted line to* (3.11)*.*

6.2. Nonlocal flux and nonlocal source: results for model (5.1). In the following we consider model (5.1) including a nonlocality in the source with parameter ν and a nonlocality in the flux with parameter ι. We focus on the following two lanes example, inspired by [2]: The velocity function is the same on both lanes and given by

$$v_1(\rho) = v_2(\rho) = 1 - \rho^2, \tag{6.2}$$

and the initial condition is given by

$$\rho_{1,o}(x) = q\left(2x - \frac{1}{2}\right) \quad \text{and} \quad \rho_{2,o}(x) = q(x) \tag{6.3}$$

with

$$q(x) = 4x^2(1-x)^2 \chi_{(0,1)}(x),$$

χ_A being the characteristic function of the set A.

Model (5.1) fits into the model framework proposed in [2], if we consider $\nu = \iota$, the same kernel functions for the source and the flux and a forward looking nonlocal term as in (3.10). Therefore we consider, if not stated otherwise, the parameters $\iota = \nu = 0.5$ and the kernels

$$w_\iota(x) = 2\frac{\iota - x}{\iota^2}, \qquad w_\nu(x) = 2\frac{\nu - x}{\nu^2}. \tag{6.4}$$

Figure 6.3 compares models (2.3) and (5.1) and clearly shows the impact of the nonlocal flux. For both models the same nonlocal term (3.10) is used. Because of the nonlocal transport, the solutions display completely different dynamics, mainly due to the high nonlocal range. Indeed, the density does not decrease at the front part of its support on each lane since the vehicles just behind the leading ones anticipate the free space ahead, so that the average density is lower than in the local case.

As already mentioned, the model introduced in [2] has the same nonlocal term, i.e. the same kernel and nonlocal range, both in the source and in the flux. On the other hand, the model (5.1) presented in this paper has more flexibility since it is able to deal with different types of nonlocality in the flux and in the source, the latter being independent of the forward nonlocal term. Therefore, we now focus on varying the nonlocality in the source term.

First of all, we observe that the nonlocal range in the flux and in the source term do not necessarily have to be equal: If a driver wants to overtake a car and thus starts to accelerate, getting ready to change lane, he/she might look further ahead when performing a lane change than if he/she keeps on driving in the same lane. In Figure 6.4, we display the solutions to (5.1) with initial datum (6.3), velocities (6.2), kernels (6.4), $\iota = 0.5$ and varying the nonlocal range in the source, thus varying the parameter ν. Due to the initial condition, the main influence of the different parameters ν in the source term can be seen at the back of the support of the density in lane 1: the smaller the range ν, the smaller the average density (and thus the larger the velocity on lane 1), the more vehicles move from lane 2 to lane 1. An analogous situation happens at the front of the support of the density in lane 2. To sum up, the greater the nonlocal range ν, the lesser the effect of the source term: When ν is large, cars get a better awareness of the actual free space ahead so that lane changing may be evaluated as not necessary.

The second reasonable aspect to keep in mind when performing a lane change is to take into account also the backward traffic, both in the present lane and in the target

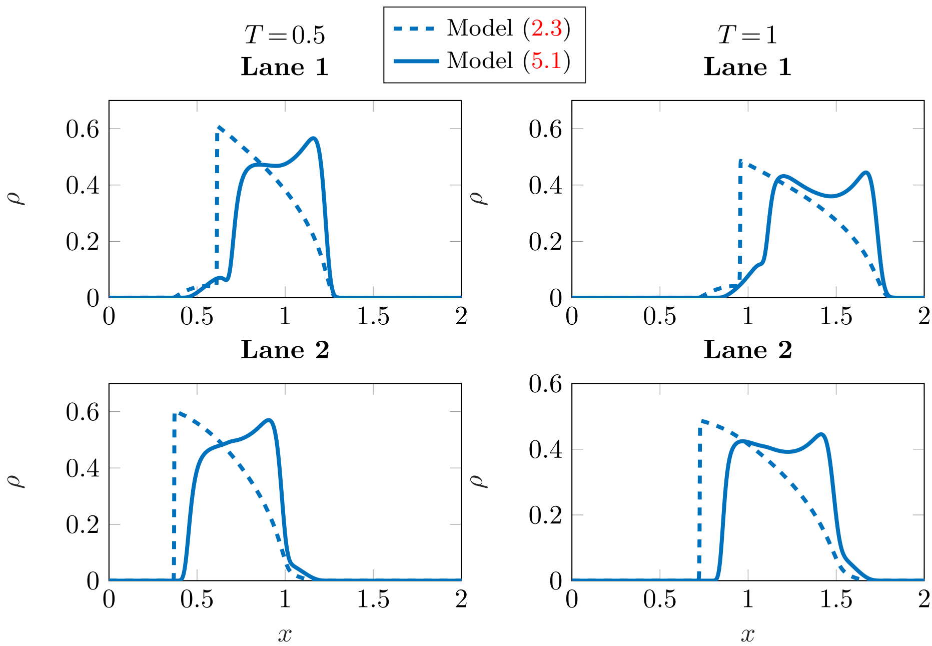

FIG. 6.3. *Density profiles at $T=0.5$ (left) and $T=1$ (right) for the local flux model (2.3) and the nonlocal flux model (5.1), both with the nonlocal source term using (3.10) with (6.4). Velocity functions as in (6.2) and initial datum as in (6.3).*

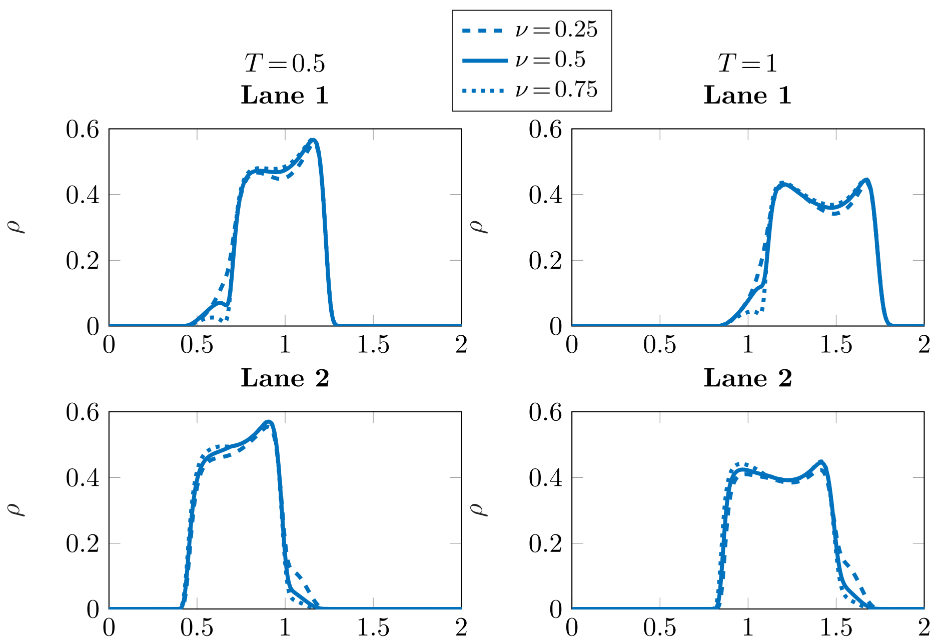

FIG. 6.4. *Density profiles at $T=0.5$ (left) and $T=1$ (right) for model (5.1) with forward looking kernel (3.10) with (6.4), $\iota=0.5$ and $\nu=0.25, 0.5, 0.75$. Velocity functions as in (6.2) and initial datum as in (6.3).*

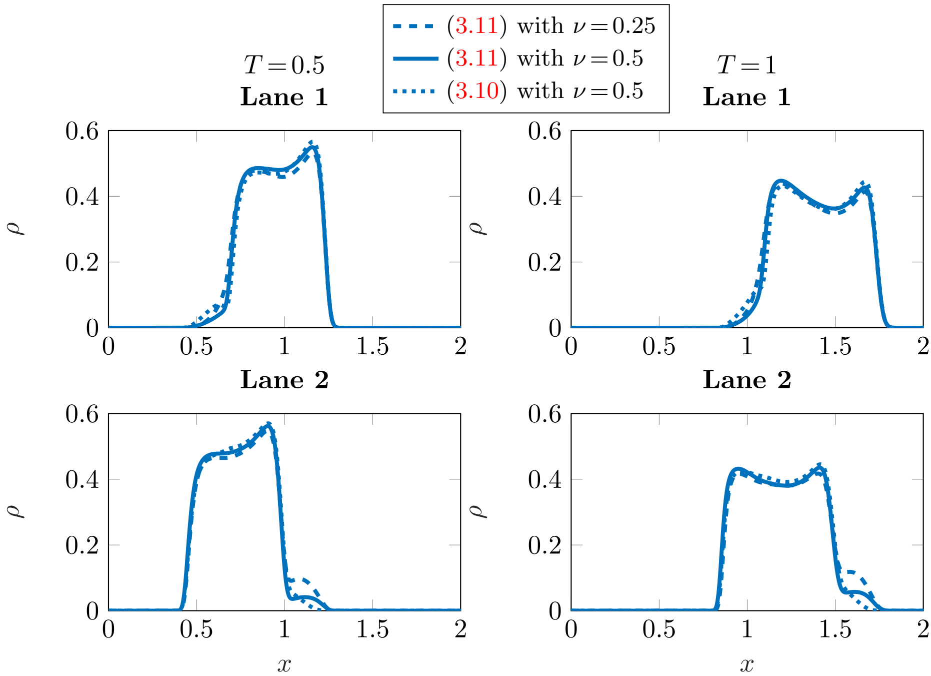

FIG. 6.5. *Density profiles for the model* (5.1) *at* $T = 0.5$ *(left column) and* $T = 1$ *(right column), velocities* (6.2)*, initial datum* (6.3)*,* $\iota = 0.5$*,* w_ι *as in* (6.4)*. Concerning* w_ν *and the parameter* ν*: dashed lines represent case (a), solid lines case (b), dotted lines case (c).*

lane. This can be done by considering model (5.1) with back- and forward looking kernel (3.11) in the source term. For this example, we consider a linear symmetric kernel, i.e.

$$w_\nu(x) = \frac{\nu - |x|}{\nu^2}. \tag{6.5}$$

Figure 6.5 considers the solutions to model (5.1) with velocities (6.2), initial datum (6.3), $\iota = 0.5$, w_ι as in (6.4) and the following choices of w_ν and ν:

(a) the back- and forward looking kernel (3.11)–(6.5) with $\nu = 0.25$, to have the same nonlocal influence as in the flux;

(b) the back- and forward looking kernel (3.11)–(6.5) with $\nu = 0.5$, to have the same look ahead parameter as in the flux;

(c) the forward looking kernel (3.10)–(6.3) with $\nu = 0.5$, exactly as in the flux.

In cases (a) and (b) with the nonlocal term of type (3.11), more mass is transported from lane 1 to lane 2, especially in the front part of the support of the density of lane 2, even though the leading part of lane 1 is aware of the density on lane 2. In addition, more mass is transported with smaller nonlocal range due to similar effects as already described above. In contrast, more mass seems to be transported from the rear part of lane 2 to lane 1 when the nonlocal term with forward looking kernel (3.10) is used. This may be due to the fact that for the back- and forward looking kernel (3.11) the nonlocal velocities on both roads depend on free space and density, but for the forward looking kernel (3.11) the velocity of the second lane does not include some free space and lane changing becomes favourable.

Conclusion. Inspired by the models presented in [2] and [20], we have introduced a multilane traffic model that allows for nonlocality in the source and in the flux term. For both approaches we have shown existence and uniqueness of solutions. Based on a Godunov-type discretization, we also present a numerical study comparing the influence of the nonlocality and different kernels. Future works include the consideration of the continuum limit for infinitely many lanes and comparisons to real data.

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Appendix. Lipschitz continuity of the source term (2.1). Here, we prove the Lipschitz continuity of the source term (2.1) for each of its argument.

LEMMA A.1. *For all $j=1,\dots,M$, the map S_j defined in (2.1) is Lipschitz continuous for each argument with Lipschitz constant*

$$\mathcal{K}=\max\{V_{\max},2V'_{\max}\}, \tag{A.1}$$

where $V_{\max}$ and $V'_{\max}$ are defined in (3.14) and (3.15), respectively.

Proof. For $j\in\{1,\dots,M-1\}$ we have

$$\begin{aligned}
&\left|S_j(\rho_j,\rho_{j+1},R_j,R_{j+1})-S_j(\tilde\rho_j,\tilde\rho_{j+1},\tilde R_j,\tilde R_{j+1})\right|\\
\le\ &|S_j(\rho_j,\rho_{j+1},R_j,R_{j+1})-S_j(\tilde\rho_j,\rho_{j+1},R_j,R_{j+1})| &\text{(A.2)}\\
&+|S_j(\tilde\rho_j,\rho_{j+1},R_j,R_{j+1})-S_j(\tilde\rho_j,\tilde\rho_{j+1},R_j,R_{j+1})| &\text{(A.3)}\\
&+\left|S_j(\tilde\rho_j,\tilde\rho_{j+1},R_j,R_{j+1})-S_j(\tilde\rho_j,\tilde\rho_{j+1},\tilde R_j,R_{j+1})\right| &\text{(A.4)}\\
&+\left|S_j(\tilde\rho_j,\tilde\rho_{j+1},\tilde R_j,R_{j+1})-S_j(\tilde\rho_j,\tilde\rho_{j+1},\tilde R_j,\tilde R_{j+1})\right|. &\text{(A.5)}
\end{aligned}$$

By the definition of the source term (2.1) we have

$$\begin{aligned}
[\text{(A.2)}]&=\left|(v_{j+1}(R_{j+1})-v_j(R_j))^+(1-\rho_{j+1})(\rho_j-\tilde\rho_j)-(v_{j+1}(R_{j+1})-v_j(R_j))^-\rho_{j+1}(\tilde\rho_j-\rho_j)\right|\\
&\le V_{\max}|\rho_j-\tilde\rho_j|,\\
[\text{(A.3)}]&\le V_{\max}|\rho_{j+1}-\tilde\rho_{j+1}|.
\end{aligned}$$

Pass now to (A.4):

$$\begin{aligned}
[\text{(A.4)}]=\ &\Big|\Big((v_{j+1}(R_{j+1})-v_j(R_j))^+-(v_{j+1}(R_{j+1})-v_j(\tilde R_j))^+\Big)\tilde\rho_j(1-\tilde\rho_{j+1})\\
&-\Big((v_{j+1}(R_{j+1})-v_j(R_j))^--(v_{j+1}(R_{j+1})-v_j(\tilde R_j))^-\Big)\tilde\rho_{j+1}(1-\tilde\rho_j)\Big|.
\end{aligned}$$

We distinguish the following cases:

	$v_{j+1}(R_{j+1})\ge v_j(R_j)$	$v_j(R_{j+1})<v_j(R_j)$
$v_{j+1}(R_{j+1})\ge v_j(\tilde R_j)$	**Case A**	**Case B**
$v_{j+1}(R_{j+1})<v_j(\tilde R_j)$	**Case C**	**Case D**

We analyse in detail cases A and B, the others being entirely similar.

Case A. We have

$$[(\text{A.4})] = \left| \left(v_j(\tilde{R}_j) - v_j(R_j) \right) \tilde{\rho}_j (1 - \tilde{\rho}_{j+1}) \right| \leq V'_{\max} \left| R_j - \tilde{R}_j \right|.$$

Case B. Add and subtract $(v_{j+1}(R_{j+1}) - v_j(\tilde{R}_j))\tilde{\rho}_{j+1}(1-\tilde{\rho}_j)$ inside the absolute value in (A.4) to obtain

$$\begin{aligned}
[(\text{A.4})] = & \left| (v_j(\tilde{R}_j) - v_j(R_j))\tilde{\rho}_{j+1}(1-\tilde{\rho}_j) + (v_{j+1}(R_{j+1}) - v_j(\tilde{R}_j))(\tilde{\rho}_{j+1}(1-\tilde{\rho}_j) - \tilde{\rho}_j(1-\tilde{\rho}_{j+1})) \right| \\
& \leq V'_{\max} \left| R_j - \tilde{R}_j \right| + (v_{j+1}(R_{j+1}) - v_j(\tilde{R}_j)) \\
& < V'_{\max} \left| R_j - \tilde{R}_j \right| + (v_j(R_j) - v_j(\tilde{R}_j)) \\
& \leq 2V'_{\max} \left| R_j - \tilde{R}_j \right|,
\end{aligned}$$

since $v_{j+1}(R_{j+1}) < v_j(R_j)$ and $|\tilde{\rho}_{j+1} - \tilde{\rho}_j| \leq 1$, with $\tilde{\rho}_j, \tilde{\rho}_{j+1} \in [0,1]$.

Cases D and C are treated similarly to Case A and Case B, respectively. Therefore we have

$$[(\text{A.4})] \leq 2V'_{\max} \left| R_j - \tilde{R}_j \right|.$$

The term (A.5) is treated analogously to (A.4), leading to

$$[(\text{A.5})] \leq 2V'_{\max} \left| R_{j+1} - \tilde{R}_{j+1} \right|.$$

The proof is completed. ∎

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