COMMUN. MATH. SCI.
Vol. 20, No. 2, pp. 327–357

NECESSARY CONDITIONS FOR BLOW-UP SOLUTIONS TO THE RESTRICTED EULER–POISSON EQUATIONS*

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Abstract. In this work, we study the behavior of blow-up solutions to the multidimensional restricted Euler–Poisson equations which are the localized version of the full Euler–Poisson system. We provide necessary conditions for the existence of finite-time blow-up solutions in terms of the initial data, and describe the asymptotic behavior of the solutions near blow-up times. We also identify a rich set of the initial data which yields global bounded solutions.

Keywords. Restricted Euler–Poisson dynamics; Blow-up solutions; Asymptotic behaviors.

AMS subject classifications. 34C11; 35Q35.

1. Introduction and main results

In this paper, we consider the following ordinary differential equation (ODE) system

$$\lambda_i' = -\lambda_i^2 + \frac{k}{n}(\rho - c_b), \quad i = 1,2,\cdots,n, \quad t>0, \tag{1.1a}$$

$$\rho' = -\rho\lambda, \quad \lambda = \sum_{i=1}^{n} \lambda_i, \tag{1.1b}$$

$$\rho(0) = \rho_0 > 0, \quad \lambda_i(0) = \lambda_{i,0}, \tag{1.1c}$$

where $'$ is the derivative in time t, k, c_b are positive parameters, and $n \geq 2$ is an integer. This system proposed in [15] is a localized version of the Euler–Poisson equations, hence called the restricted Euler–Poisson (REP) system in the literature. We assume that the initial data for λ_i are real and satisfy the order condition

$$\lambda_{1,0} = \cdots = \lambda_{J,0} < \lambda_{J+1,0} \leq \cdots \leq \lambda_{n,0}. \tag{1.1d}$$

Here, we introduce a quantity $1 \leq J \leq n$ with which we characterize the number of the initial λ_i coinciding with $\lambda_{1,0}$. The order of λ_i's is known to be preserved (see [14, 15] and Lemma 2.1). The purpose of this work is to identify necessary conditions for the existence of blow-up solutions to this REP system, and study the detailed solution behavior near the blow-up time.

To understand the physical meaning of each term, we recall the full Euler–Poisson equations for the velocity field $\mathbf{u}$ and local density ρ,

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = k \nabla \Delta^{-1}(\rho - c_b), x \in \mathbb{R}^n, t > 0, \tag{1.2a}$$

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.2b}$$

where the constant k represents a repulsive ($k>0$) or attractive ($k<0$) force, and c_b denotes the background state. This system (1.2) describes the dynamic behavior of several important physical flows, including those for semi-conductors, plasma physics,

*Received: October 29, 2019; Accepted (in revised form): June 30, 2021. Communicated by Shi Jin.
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and the collapse of stars (see [2,3,8,11,18,19]). The existence and behaviors of solutions for (1.2) and related problems have been extensively studied under various assumptions; see, e.g., [5,6,9,12,21] and references therein. In particular, in [15], Liu and Tadmor introduced the method of spectral dynamics, which serves as a powerful tool to study dynamics of the velocity gradient $M = \nabla \mathbf{u}$ along particle paths. Indeed, (1.2) can be converted into

$$M' = -M^2 + k\mathcal{R}[\rho - c_b],$$
$$\rho' = -\rho \operatorname{tr} M,$$

where $'$ is the convective derivative, $\partial_t + \mathbf{u}\cdot\nabla$, and $\mathcal{R}$ is the Riesz matrix operator,

$$\mathcal{R}[f] := \nabla \otimes \nabla \Delta^{-1}[f].$$

It is the global nature of the Riesz matrix, $\mathcal{R}[\rho - c_b]$, which makes the issue of regularity for Euler–Poisson equations such an intricate question to solve, both analytically and numerically. In this paper we focus on the REP equation for M which was proposed in [15] by restricting attention to the local isotropic trace, $\frac{k}{n}(\rho - c_b)I_{n\times n}$, of the global coupling term $k\mathcal{R}[\rho - c_b]$, i.e.,

$$M' = -M^2 + \frac{k}{n}(\rho - c_b)I_{n\times n}, \tag{1.3a}$$
$$\rho' = -\rho \operatorname{tr} M. \tag{1.3b}$$

This is a matrix Ricatti equation for the $n\times n$ matrix M, coupled with the density equation, which should mimic the dynamics of $(\rho, \nabla \mathbf{u})$ in the full Euler–Poisson equations. The REP system (1.1) for the eigenvalue λ_i of M follows from (1.3). We note that the REP system [15] is to the full Euler–Poisson equations what the restricted Euler (RE) model is to the full Euler equations, while the RE system is known to be useful in understanding the local topology of the Euler dynamics; we refer the reader to [1,4,7,22].

The existence of a critical threshold phenomenon associated with this 2D REP model with zero background, $c_b = 0$, was first identified in [15]. A precise description of the critical threshold for the 2D REP system, with both zero and nonzero background charges, was given in [16]. These results have been extended to multi-dimensional REP equations by Lee and Liu in [14]. Lee identified upper-thresholds for finite time blow-up solutions to an improved REP equation in two dimensions ([13]). It is worth mentioning that critical thresholds for restricted Euler equations were studied in [17] and [20].

In this work, we attempt to advance our understanding of the critical threshold phenomenon by providing necessary conditions for the existence of finite-time blow-up solutions to the REP system (1.1). Our results thus provide a complement to the existing results in [13,15,16] for REP systems.

In order to see the subtleness of the problem, we recall that a movable essential singularity cannot be achieved for a first-order scalar differential equation $u' = F(t,u)$, as long as F is a rational function of u with coefficients that are algebraic functions of t ([10]). However, this is not the case for the system of equations considered here. In other words, the singularity types of solutions λ and ρ of (1.1) are not known a priori. This is one of the main difficulties with this problem, because we cannot simply utilize some balance equations to analyze the behavior of solutions near a singular point. To overcome this difficulty, we transform the Riccati-type equations (1.1a) into second-order linear differential equations for

$$u_i(t) = e^{\int_0^t \lambda_i(s)ds}.$$

By analyzing the general solution to the second-order differential equation, we are able to reveal the behavior of u_i, which also provides information on the behavior of λ_i'. Indeed, we can characterize the asymptotic behaviors of λ_i and ρ near the blow-up time by the gap of the initial data $\lambda_{i,0}$ together with ρ_0.

The quantity J defined in (1.1d) is critical in terms of different solution behaviors of λ_i. We state our main results in the following.

THEOREM 1.1. *Suppose that the maximum interval of existence for* (1.1) *is* $[0,t_B)$ *for some* $0<t_B<\infty$. *Then*

$$1\le J\le \frac{n}{2} \quad \textit{and} \tag{1.4}$$

$$t_B\ge \frac{1}{\omega}\arctan\left(\frac{\lambda_{1,0}}{\omega}\right)+\frac{\pi}{2\omega}, \quad \omega:=\sqrt{kc_b/n}. \tag{1.5}$$

Moreover,

$$\lim_{t\to t_B^-}\lambda_i(t)=-\infty, \quad 1\le i\le J, \tag{1.6a}$$

$$\lim_{t\to t_B^-}\lambda_i(t)=\infty, \quad J<i\le n, \tag{1.6b}$$

$$\lim_{t\to t_B^-}\rho(t)=\infty, \tag{1.6c}$$

and also

$$\lim_{t\to t_B^-}\lambda_1(t)e^{\int_0^t\lambda_1(s)+\lambda_n(s)ds}=-p, \tag{1.7a}$$

$$\lim_{t\to t_B^-}\lambda_n(t)e^{\int_0^t\lambda_1(s)+\lambda_n(s)ds}=q, \tag{1.7b}$$

for some $0\le q\le p$.

REMARK 1.1. An interesting feature of the behavior of λ_i is that λ_i diverge to $-\infty$ if and only if $\lambda_{i,0}=\lambda_{1,0}$ and all the other λ_i diverge to $+\infty$. Moreover, J cannot exceed $n/2$ and in the case of $J\ge 3$, J is strictly smaller than $n/2$; see Theorem 1.2. The limits in (1.7) indicate how λ_i are connected through p and q, which are mainly characterized by the gap of the initial data $\lambda_{i,0}$ together with ρ_0.

Our second theorem gives the detailed blow-up rates of solutions. We note that $\lambda_i(t)=\lambda_1(t)$ for $1\le i\le J$ (see Lemma 2.1).

THEOREM 1.2. *Under the hypothesis in Theorem 1.1, the blow-up rates of singular solutions depend on the size of* J, *and can be made more precise as follows:*

(i) *If* $J=1$, *then* $n\ge 2$ *and*

$$\lim_{t\to t_B^-}(t_B-t)\lambda_1(t)=-1,$$
$$\lambda_i(t)=\mathcal{O}(|\ln(t_B-t)|) \ \textit{as}\ t\to t_B^-, \quad 2\le i\le n,$$
$$\rho(t)=\mathcal{O}\left(\frac{1}{t_B-t}\right) \ \textit{as}\ t\to t_B^-.$$

(ii) *If* $J=2$, *then* $n\ge 4$ *and one of the following cases must hold:*

(a) If $n \geq 5$, then

$$\lim_{t\to t_B^-}(t_B-t)\lambda_1(t)=-1,$$
$$\lim_{t\to t_B^-}(t_B-t)\lambda_i(t)=0,\ \lim_{t\to t_B^-}\int_0^t\lambda_i(s)ds=\infty,\quad 3\leq i\leq n,$$
$$\rho(t)=o\left(\frac{1}{(t_B-t)^2}\right)\ \text{as } t\to t_B^-.$$

(b) If $n=4$ and $(\lambda_{1,0}-\lambda_{3,0})(\lambda_{1,0}-\lambda_{4,0})=:A_0>k\rho_0$, then

$$\lim_{t\to t_B^-}(t_B-t)\lambda_1(t)=-\frac{1}{2}-\frac{1}{2}\sqrt{\frac{A_0}{A_0-k\rho_0}},$$
$$\lim_{t\to t_B^-}(t_B-t)\lambda_i(t)=-\frac{1}{2}+\frac{1}{2}\sqrt{\frac{A_0}{A_0-k\rho_0}},\quad i=3,4,$$
$$\rho(t)=\mathcal{O}\Big(\frac{1}{(t_B-t)^2}\Big)\ \text{as } t\to t_B^-.$$

(c) If $n=4$ and $A_0=k\rho_0$, then there exists $C>0$ such that

$$\lim_{t\to t_B^-}(t_B-t)^2\lambda_1(t)=-C,$$
$$\lim_{t\to t_B^-}(t_B-t)^2\lambda_i(t)=C,\quad i=3,4,$$
$$\lim_{t\to t_B^-}(t_B-t)^4\rho(t)=\frac{k}{4}C^2.$$

(iii) If $J\geq 3$, then $n>2J$ and there exists $C>1$ such that

$$\lim_{t\to t_B^-}(t_B-t)\lambda_1(t)=-C,$$
$$\lim_{t\to t_B^-}(t_B-t)\lambda_i(t)=C-1,\quad J+1\leq i\leq n,$$
$$\rho(t)=\mathcal{O}\Big(\frac{1}{(t_B-t)^2}\Big)\ \text{as } t\to t_B^-.$$

Remark 1.2. Note that for each J specified in different cases, n has to be in a certain range so as to fulfill the requirement that the maximum interval of existence for (1.1) be finite.

In contrast to the finite-time breakdown, the multi-dimensional REP equations admit a large class of global bounded solutions. Our results are summarized below.

Theorem 1.3. *If*

$$J>\frac{n}{2},$$

or

$$J\geq 3\quad\text{and}\quad J=\frac{n}{2},$$

then (1.1) has a global bounded solution.

This has improved upon some global existence results in [14], in particular Theorem 2.3 (corresponding to $n=3, J=3$) and Theorem 2.9 (corresponding to $J=n$) therein.

The remainder of this paper is organized as follows. In Section 2, we first show that no highly oscillating solution exists (see (2.2) for the definition of a highly oscillating solution). That is, we show that ρ and $|\lambda_i|$ diverge to ∞ for some i when (1.1) admits a finite-time blow-up solution. We also provide a proof of (1.5). In Section 3, we transform (1.1a) to a second-order linear differential equation, and demonstrate the solution behaviors of (1.7). To prove Theorem 1.2 using (1.7), we consider the subcases

$$p>q \quad \text{and} \quad p=q.$$

We study the case with $p>q$ in Section 4. Here, we show that the coefficients of leading singular order terms for λ_1 and λ_n can be represented as $-p/(p-q)$ and $q/(p-q)$, respectively. We also conclude that this case yields (i), (a) and (b) of (ii), or (iii) in Theorem 1.2. The last section deals with the case where $p=q$, which implies (c) of (ii) in Theorem 1.2. The main difficulty in this case lies in that the leading singular terms of $-\lambda_1$ and λ_n are the same. For this reason, we have examined the second singular terms. We also provide explicit solutions to the REP system (1.1) assuming that $\lambda_{3,0}=\lambda_{4,0}$. We remark that (1.4) and (1.6) follow from Theorem 1.2.

Notation 1.1. *Throughout the paper we write*

$$f(x)=\mathcal{O}(g(x)) \quad \textit{as } x\to x_0^-,$$

if there are $M,\delta>0$ such that

$$|f(x)|\le M|g(x)| \quad \textit{for all } x_0-x<\delta.$$

Similarly, we write

$$f(x)=o(g(x)) \quad \textit{as } x\to x_0^-,$$

if for any $\varepsilon>0$ there is $\delta>0$ such that

$$|f(x)|\le \varepsilon|g(x)| \quad \textit{for all } x_0-x<\delta.$$

2. Non-oscillating solutions

Suppose that $[0,t_B)$ is the maximum interval of existence of solutions to an ordinary differential equation (ODE) $u'=F(t,u)$. Then, either

$$\lim_{t\to t_B^-}|u(t)|=\infty, \tag{2.1}$$

or

$$0<\limsup_{t\to t_B^-}u(t)-\liminf_{t\to t_B^-}u(t). \tag{2.2}$$

Here, we define $\infty-\infty=0$. We say that a solution blows up at a finite time if it satisfies (2.1), and is oscillating at a finite time if it satisfies (2.2). Note that $u(t)$ satisfying (2.1) may be oscillating in the standard sense, i.e., $u(t)\to\infty$ but $u'(t)\not>0$ (or $u(t)\to-\infty$ but $u'(t)\not<0$).

For the REP system (1.1), we can prove that there exist no finite-time oscillating solutions. More precisely, the following proposition holds.

PROPOSITION 2.1. *Suppose that the maximum interval of existence for* (1.1) *is* $[0,t_B)$ *for some* $0<t_B<\infty$. *Then, it holds for some* i *that*

$$\lim_{t\to t_B^-}|\lambda_i(t)|=\infty. \tag{2.3}$$

Proof. If λ_i is assumed to be an oscillating solution of type (2.2), then there exists a sequence of disjoint intervals $(a_m,b_m)\subset(0,t_B)$ on which λ_i is decreasing and

$$\lim_{m\to\infty}(b_m-a_m)=0, \tag{2.4}$$

$$\lim_{m\to\infty}(\lambda_i(b_m)-\lambda_i(a_m))<0, \tag{2.5}$$

$$\lim_{m\to\infty}(|\lambda_i(b_m)|+|\lambda_i(a_m)|)<\infty. \tag{2.6}$$

Note that if (2.6) fails, then one can conclude that $\lim_{t\to t_B}|\lambda_i(t)|=\infty$, although λ_i is oscillating in the standard sense.

From (1.1b)

$$\rho=\rho_0 e^{-\int_0^t\lambda(s)ds}>0.$$

It follows from (1.1a) that

$$\lambda_i'\geq-\lambda_i^2-\omega^2,\quad \omega:=\sqrt{\frac{kc_b}{n}}.$$

That is,

$$\frac{\lambda_i'}{\lambda_i^2+\omega^2}\geq-1. \tag{2.7}$$

Upon integration over (a_m,b_m), this implies that

$$\frac{1}{\omega}\arctan\left(\frac{\lambda_i(b_m)}{\omega}\right)-\frac{1}{\omega}\arctan\left(\frac{\lambda_i(a_m)}{\omega}\right)\geq-(b_m-a_m).$$

Owing to the conditions (2.5) and (2.6), the left-hand side is strictly less than 0, while the right-hand side converges to 0. Thus, there exist no oscillating solutions of type (2.2) for λ_i. ∎

The order-preserving property of λ_i is well known (see [14, 15]). Indeed, it follows from (1.1a) that

$$(\lambda_i-\lambda_j)'=-(\lambda_i+\lambda_j)(\lambda_i-\lambda_j), \tag{2.8}$$

and this yields the following lemma.

LEMMA 2.1. *For any* $t>0$, *the solutions* λ_i *of* (1.1) *satisfy*

$$\lambda_1(t)=\cdots=\lambda_J(t)<\lambda_{J+1}(t)\leq\cdots\leq\lambda_n(t).$$

Proposition 2.1 states that $\lim\limits_{t\to t_B^-}|\lambda_i(t)|=\infty$ for some i. Then, one can conclude that

$$\lim_{t\to t_B^-}\lambda_1(t)=-\infty,$$

owing to Lemma 2.1 concerning order preservation. In fact, if we assume that there exists no λ_i diverging to $-\infty$, then λ_j tends to $+\infty$ for some j as $t\to t_B^-$, so does $\lambda=\sum_{i=1}^n \lambda_i$. Thus,

$$\min_{0\le t\le t_B} \lambda(t) > -\infty,$$

and

$$\begin{aligned}\lambda_j'(t) &= -\lambda_j^2(t) + \frac{k\rho_0}{n} e^{-\int_0^t \lambda(s)ds} - \omega^2 \\ &\le \frac{k\rho_0}{n} e^{-t_B \min\limits_{0\le t\le t_B} \lambda(t)} - \omega^2 < \infty,\end{aligned}$$

which contradicts the fact that $\lambda_j \to \infty$ as $t\to t_B^-$. Thus, it must hold that $\lim_{t\to t_B^-} \lambda_i(t) = -\infty$ for some i. Owing to the order preservation in Lemma 2.1, the following proposition holds.

Proposition 2.2. *Suppose that the maximum interval of existence for* (1.1) *is* $[0,t_B)$ *for some* $0<t_B<\infty$. *Then, there exist* $1\le J_1\le J_2\le n$ *such that*

$$\lim_{t\to t_B^-} \lambda_i(t) = -\infty, \quad 1\le i\le J_1, \tag{2.9}$$

$$\lim_{t\to t_B^-} \lambda_i(t) = \infty, \quad J_2 < i\le n. \tag{2.10}$$

We remark that there exists no λ_i satisfying (2.10) in the case that $J_2=n$. However, (2.9) indicates that

$$\lim_{t\to t_B^-} \lambda_1 = -\infty.$$

The estimation (1.5) of t_B also follows immediately. Integrating (2.7) for $i=1$ over $(0,t)$ yields that

$$\arctan\left(\frac{\lambda_1(t)}{\omega}\right) > \arctan\left(\frac{\lambda_{1,0}}{\omega}\right) - \omega t.$$

Sending t to t_B implies that

$$-\frac{\pi}{2} \ge \arctan\left(\frac{\lambda_{1,0}}{\omega}\right) - \omega t_B.$$

Thus, we obtain (1.5) in Theorem 1.1.

Theorem 2.1. *Suppose that the maximum interval of existence for* (1.1) *is* $[0,t_B)$ *for some* $0<t_B<\infty$. *Then,*

$$t_B \ge \frac{1}{\omega}\arctan\left(\frac{\lambda_{1,0}}{\omega}\right) + \frac{\pi}{2\omega}.$$

Next, we turn our attention to density ρ. Here, we show that $\rho\notin L^1(0,t_B)$ through a contradiction argument. Assuming that $\rho\in L^1(0,t_B)$, we find that $J_2=n$, because integrating (1.1a) gives that for $i=1,2,\cdots,n$

$$\lambda_i(t) - \lambda_{i,0} = -\int_0^t \lambda_i^2(s)ds + \frac{k}{n}\int_0^t (\rho(s) - c_b)dx < \infty.$$

It follows that $J_1 = n$ or λ_i is finite for $J_1 < i \le n$. Thus, there exists $f(t)$ such that

$$\rho(t) = \rho_0 e^{-\int_0^t \lambda(s)ds} = f(t) e^{-\sum_{i=1}^{J_1} \int_0^t \lambda_i(s)ds}, \quad 0 < f(t) < \infty. \tag{2.11}$$

Now, let $1 \le i \le J_1$. Then, there exists $t_1 \in (0, t_B)$ such that

$$\lambda_i(t) < 0, \quad t_1 < t < t_B,$$

and $\tilde{\lambda}_i := \lambda_i - \omega$ satisfies

$$\tilde{\lambda}_i' = \lambda_i' > -\lambda_i^2 - \omega^2 > -\tilde{\lambda}_i^2.$$

We then deduce that

$$\tilde{\lambda}_i < -\frac{1}{t_B - t}, \quad t_1 < t < t_B.$$

Thus, for some constant $K > 0$ it holds that

$$-\sum_{i=1}^{J_1} \int_0^t \lambda_i(s)ds > \ln(K(t_B - t)^{-J_1}), \quad t_1 < t < t_B.$$

Substituting the inequality into (2.11) yields

$$\rho(t) = f(t) e^{-\sum_{i=1}^{J_1} \int_0^t \lambda_i(s)ds} > \frac{K f(t)}{(t_B - t)^{J_1}}, \quad t_1 < t < t_B.$$

In Proposition 2.2, we have shown $J_1 \ge 1$, which contradicts the assumption that $\rho \in L^1(0, t_B)$. We summarize this result as follows.

PROPOSITION 2.3. *Suppose that the maximum interval of existence for* (1.1) *is* $[0, t_B)$ *for some* $0 < t_B < \infty$. *Then,*

$$\lim_{t \to t_B^-} \int_0^t \rho(s)ds = \infty. \tag{2.12}$$

3. Transformed equations

Although Proposition 2.2 and Proposition 2.3 state that for some i, λ_i and $\int_0^t \rho(s)ds$ diverge as $t \to t_B^-$, respectively, they do not illuminate the behaviors of λ_i' and ρ near t_B, which are essential for analyzing solution singularities. To go further, we transform the Riccati-type equation (1.1a) to a second-order linear differential equation by defining

$$u_i(t) = e^{\int_0^t \lambda_i(s)ds}. \tag{3.1}$$

This gives

$$\lambda_i = \frac{u_i'}{u_i},$$
$$u_i(0) = 1, \quad u_i'(0) = \lambda_{i,0}, \tag{3.2}$$

and

$$\rho(t) = \rho_0 e^{-\sum_{i=1}^{n} \int_0^t \lambda_i(s)ds} = \rho_0 \prod_{i=1}^{n} \frac{1}{u_i(t)}. \tag{3.3}$$

Equation (1.1a) is also transformed to

$$u_i'' - \frac{k}{n}(\rho - c_b)u_i = 0, \tag{3.4}$$

or (recall that $\omega^2 = kc_b/n$)

$$u_i'' + \omega^2 u_i = \frac{k}{n}\rho_0 \prod_{\substack{m=1\\ m\neq i}}^{n} \frac{1}{u_m} =: g_i. \tag{3.5}$$

The general solution of (3.5) is thus given by

$$u_i(t) = c_1 \sin\omega t + c_2 \cos\omega t + \frac{1}{\omega}\int_0^t g_i(s)\sin\omega(t-s)ds. \tag{3.6}$$

We proceed to observe the behavior of u_i near t_B. Let $1 \le i \le J_1$, i.e.,

$$\lambda_i = \frac{u_i'}{u_i} \to -\infty \quad \text{as } t \to t_B^-, \tag{3.7}$$

then the positivity of u_i implies that $u_i' < 0$, and thus u_i converges. Let $-\alpha_i \le 0$ be the least upper bound of u_i'. Then, for any $\varepsilon > 0$ there exists $t_0 \in (0, t_B)$ such that

$$-\alpha_i - \varepsilon < u_i'(t_0) \le -\alpha_i.$$

On the other hand, it follows from (3.4) that for any $t_0 < t < t_B$,

$$u_i'(t) - u_i'(t_0) = \int_{t_0}^t \frac{k}{n}(\rho(s) - c_b)u_i(s)ds. \tag{3.8}$$

Owing to Proposition 2.3 and the convergence of u_i, there exists $t_1 \in (t_0, t_B)$ such that

$$u_i'(t) - u_i'(t_0) = \int_{t_0}^t \frac{k}{n}(\rho(s) - c_b)u_i(s)ds > 0, \quad t_1 \le t < t_B$$

and

$$-\alpha_i - \varepsilon < u_i'(t_0) < u_i'(t) \le -\alpha_i, \quad t_0 \le t_1 \le t < t_B.$$

This implies that

$$\lim_{t \to t_B^-} u_i'(t) = -\alpha_i, \tag{3.9}$$

and thus u_i converges to 0 as t approaches t_B, satisfying (3.7). We may extend the interval of existence and obtain the boundary conditions:

$$u_i(t_B) = 0, \quad u_i'(t_B) = -\alpha_i. \tag{3.10}$$

It follows from (3.6) that

$$u_i(t) = \frac{1}{\omega}\left(\int_{t_B}^t g_i(s)\sin\omega(t-s)ds - \alpha_i \sin\omega(t - t_B)\right). \tag{3.11}$$

In (3.8), we observe that

$$g_i = \frac{k}{n}\rho_0 \prod_{\substack{m=1\\m\neq i}}^{n} \frac{1}{u_m} = \frac{k}{n}\rho u_i \in L^1(0,t_B). \tag{3.12}$$

Next we consider the case that $J_2 < j \leq n$; that is,

$$\lambda_j = \frac{u_j'}{u_j} \to \infty. \tag{3.13}$$

Because $u_j > 0$, u_j' must be positive in a neighborhood of t_B, which implies that u_j is increasing near t_B. Thus,

$$\text{either } \lim_{t\to t_B^-} u_j(t) = \infty \quad \text{or } \lim_{t\to t_B^-} u_j(t) = \beta_j \tag{3.14}$$

for some $\beta_j > 0$. In either case, u_j' must diverge to ∞, owing to (3.13).

Thanks to the behavior of u_i near t_B, we obtain the following lemma.

LEMMA 3.1. *Suppose that the maximum interval of existence for (1.1) is $[0,t_B)$ for some $0 < t_B < \infty$. Then, for $1 \leq J_1 \leq J_2 \leq n$ defined in Proposition 2.2*

$$\lambda_{i,0} = \lambda_{j,0}, \quad 1 \leq i,j \leq J_1,$$
$$\lambda_i(t) = \lambda_j(t), \quad 1 \leq i,j \leq J_1,$$

and

$$\lim_{t\to t_B^-} \frac{\lambda_i(t)}{\lambda_j(t)} = 1, \quad J_2 < i,j \leq n,$$
$$\lim_{t\to t_B^-} \frac{u_j(t)}{u_n(t)} = \frac{\lambda_{j,0} - \lambda_{1,0}}{\lambda_{n,0} - \lambda_{1,0}}, \quad J_2 < j < n.$$

Proof. We employ Abel's identity for (3.4) together with the initial conditions (3.2) to obtain

$$u_i'(t)u_j(t) - u_i(t)u_j'(t) = \lambda_{i,0} - \lambda_{j,0}, \quad 0 \leq t < t_B. \tag{3.15}$$

Let $1 \leq i,j \leq J_1$. Because $u_i(t_B) = u_j(t_B) = 0$ and $u_i'(t_B), u_j'(t_B)$ are bounded, the left-hand side of (3.15) vanishes at $t = t_B$, and thus $\lambda_{i,0} = \lambda_{j,0}$, as desired.

We rewrite (3.15) as

$$\lambda_i(t) - \lambda_j(t) = \frac{u_i'(t)}{u_i(t)} - \frac{u_j'(t)}{u_j(t)} = \frac{\lambda_{i,0} - \lambda_{j,0}}{u_i(t)u_j(t)}. \tag{3.16}$$

This yields

$$\lim_{t\to t_B^-} \frac{\lambda_i(t)}{\lambda_j(t)} = 1, \quad J_2 < i,j \leq n,$$

because $1/(u_iu_j)$ converges for $J_2 < i,j \leq n$.

From (3.15) we observe that u_1 and u_n are linearly independent solutions of (3.4). Then, for $J_2 < j < n$ we can represent u_j as a linear combination of u_1 and u_n. Further using the initial conditions (3.2), we obtain

$$u_j = \frac{\lambda_{n,0} - \lambda_{j,0}}{\lambda_{n,0} - \lambda_{1,0}} u_1 + \frac{\lambda_{j,0} - \lambda_{1,0}}{\lambda_{n,0} - \lambda_{1,0}} u_n. \tag{3.17}$$

On the other hand, the behaviors of u_1 and u_j near t_B in (3.10) and (3.14) imply that

$$\lim_{t \to t_B^-} u_1(t)/u_j(t) = 0,$$

and it follows that

$$\lim_{t \to t_B^-} \frac{u_j(t)}{u_n(t)} = \frac{\lambda_{j,0} - \lambda_{1,0}}{\lambda_{n,0} - \lambda_{1,0}}, \quad J_2 < j < n.$$

□

Further, we are able to show that $J_1 = J_2 = J$. That is, there don't exist bounded λ_i. More precisely, we have

Theorem 3.1. *Suppose that the maximum interval of existence for* (1.1) *is* $[0, t_B)$ *for some* $0 < t_B < \infty$. *Then,*

$$1 \le J < n$$

and

$$\lim_{t \to t_B^-} \lambda_i(t) = \begin{cases} -\infty, & 1 \le i \le J, \\ \infty, & J < i \le n. \end{cases}$$

Proof. From Lemma 3.1 it follows that $J = J_1$ and also $J = J_1 < n$; otherwise, all $\lambda_{i,0}$ would be identical, and this implies the existence of a global solution (see Theorem 2.9 in [14]).

Now we show $J_1 = J_2$ by a contradiction argument. Indeed, if it is assumed that $J_1 < J_2$, then there exists $|\lambda_i| < \infty$ for $J_1 < i \le J_2$. It follows that for all $0 < t < t_B$,

$$\int_0^t \rho(s) ds = \int_0^t \left[c_b + \frac{n}{k} (\lambda_i'(s) + \lambda_i^2(s)) \right] ds < \infty,$$

which contradicts Proposition 2.3. □

Theorem 3.1 implies that for $i = 1, \cdots, J$ and $j = J+1, \cdots, n$, $u_i' = u_1' < 0$ and $u_j' > 0$ in a neighborhood of t_B. Because $u_1, u_j > 0$, we observe from (3.15) that $u_1' u_j$ and $u_1 u_j'$ should be bounded in $[0, t_B]$. Furthermore, it follows from (3.16) that $u_1 u_j$ converges to 0.

Corollary 3.1. *Let* t_B *and* J *be as in Theorem 3.1, and* u_j *as in* (3.1). *Then, for any* $J < j \le n$,

$$|u_1'(t) u_j(t)| < \infty, \quad 0 \le t < t_B,$$
$$|u_1(t) u_j'(t)| < \infty, \quad 0 \le t < t_B,$$

and

$$\lim_{t \to t_B^-} (u_1 u_j)(t) = 0. \tag{3.18}$$

Now, we may divide (3.10) and (3.14) into the following cases, assuming that $J < j \leq n$:

$$u_1'(t_B) = -\alpha_1 < 0 \quad \text{and} \quad \lim_{t \to t_B^-} u_j(t) = \infty, \tag{3.19}$$

$$u_1'(t_B) = 0 \quad \text{and} \quad \lim_{t \to t_B^-} u_j(t) = \beta_j > 0, \tag{3.20}$$

$$u_1'(t_B) = -\alpha_1 < 0 \quad \text{and} \quad \lim_{t \to t_B^-} u_j(t) = \beta_j > 0,$$

$$u_1'(t_B) = 0 \quad \text{and} \quad \lim_{t \to t_B^-} u_j(t) = \infty.$$

However, (3.19) and (3.20) cannot occur. Indeed, (3.19) contradicts the boundedness of $u_1'u_j$ in Corollary 3.1. If (3.20) is assumed, then $u_1'u_j \to 0$, and thus $u_1u_j' \to -\lambda_{1,0} + \lambda_{j,0} > 0$ as t approaches t_B. It follows that, in a neighborhood of t_B,

$$(u_1u_j)' > 0.$$

This also contradicts Corollary 3.1, owing to (3.18) and the fact that $u_1u_j > 0$. Thus, we have the following proposition.

PROPOSITION 3.1. *Suppose that the maximum interval of existence for* (1.1) *is* $[0, t_B)$ *for some* $0 < t_B < \infty$. *Define* u_j *as* (3.1). *If* $J < j \leq n$, *then either*

$$u_1'(t_B) = -\alpha_1 \ \text{and} \ \lim_{t \to t_B^-} u_j(t) = \beta_j \ \text{for some} \ \alpha_1, \beta_j > 0, \tag{3.21}$$

or

$$u_1'(t_B) = 0 \ \text{and} \ \lim_{t \to t_B^-} u_j(t) = \infty \tag{3.22}$$

must hold.

Next, we demonstrate the convergence of $u_1'u_j$ and u_1u_j' for $J < j \leq n$. If (3.21) holds in Proposition 3.1, then the convergence follows from (3.15). In the case of (3.22), we show the convergence through several lemmas.

LEMMA 3.2. *Under the hypothesis of Proposition 3.1, for any* $t \in [0, t_B)$

$$\left| \int_0^t u_1'(s)u_j'(s) + g_{1j}(s)ds \right| < \infty,$$

where

$$g_{ij} := \frac{k}{n} \rho u_i u_j.$$

Proof. From (3.5), we deduce that

$$u_1''u_j + \omega^2 u_1 u_j = g_{1j}.$$

Integrating this equation over $[0, t]$ yields

$$u_1'(t)u_j(t) - \lambda_{1,0} + \int_0^t \omega^2 u_1(s)u_j(s)ds = \int_0^t u_1'(s)u_j'(s) + g_{1j}(s)ds. \tag{3.23}$$

Then, the lemma follows from Corollary 3.1. ∎

LEMMA 3.3. *Under the hypothesis of Proposition 3.1,*

$$\int_0^t u_1'(s)u_j'(s)+g_{1j}(s)ds$$

converges as $t\to t_B^-$.

Proof. If (3.21) holds in Proposition 3.1, then the lemma immediately follows from (3.23) and (3.18).

In the case of (3.22), from (3.11) and (3.6) we have that

$$\begin{aligned} u_1'(s) &= \int_{t_B}^s g_1(\tau)\cos\omega(s-\tau)d\tau, \\ u_j'(s) &= \lambda_{j0}\cos\omega s-\omega\sin\omega s+\int_0^s g_j(\tau)\cos\omega(s-\tau)d\tau, \end{aligned}$$

and

$$\begin{aligned} \int_0^t u_1'(s)u_j'(s)ds &= \int_0^t\left[\int_{t_B}^s g_1(x)\cos\omega(s-x)dx\,(\lambda_{j0}\cos\omega s-\omega\sin\omega s)\right]ds \\ &\quad+\int_0^t\left[\int_{t_B}^s g_1(x)\cos\omega(s-x)dx\int_0^s g_j(y)\cos\omega(s-y)dy\right]ds \\ &=: I+II. \end{aligned}$$

We notice that $\frac{d}{dt}I\to 0$ as $t\to t_B^-$, because $g_1\in L^1(0,t_B)$. It follows that I also converges. Thus, it suffices to show that

$$h(t):=II+\int_0^t g_{1j}(s)ds$$

converges.

Changing the order of integration yields

$$II=\int_0^t\int_{t_B}^y g_1(x)g_j(y)\left(\frac{1}{2\omega}\sin\omega(x-y)+\frac{x-y}{2}\cos\omega(x-y)\right)dxdy$$

and the integral representation of u_1, (3.11), yields

$$\begin{aligned} \int_0^t g_{1j}(y)dy &= \int_0^t g_j(y)u_1(y)dy \\ &= \int_0^t\int_{t_B}^y g_j(y)g_1(x)\frac{1}{\omega}\sin\omega(y-x)dxdy. \end{aligned}$$

We combine the two equations to obtain

$$h(t)=\frac{1}{2\omega}\int_0^t\int_{t_B}^y g_1(x)g_j(y)\left[\omega(x-y)\cos\omega(x-y)-\sin\omega(x-y)\right]dxdy.$$

Now, take $0<t_0<t_B$ such that

$$\omega(t_B-t_0)<\frac{\pi}{2}.$$

Then, for $t_0 \le t < t_B$,

$$h(t) = \frac{1}{2\omega}\int_{t_0}^{t}\int_{t_B}^{y} g_1(x)g_j(y)\left[\omega(x-y)\cos\omega(x-y) - \sin\omega(x-y)\right]dxdy + h(t_0)$$

is a decreasing function, as the integrand $h'(t)$ is negative over the domain (t_0, t_B). Furthermore, we observe from Lemma 3.2 and the convergence of I that

$$h(t) = \left(\int_0^t u_1'(s)u_j'(s)ds + \int_0^t g_{1j}(s)ds\right) - I$$

is bounded. It follows that $h(t)$ converges as $t \to t_B^-$, as desired. ∎

We proceed to show the convergence of $u_1'u_j$ and u_1u_j', which gives (1.7) in Theorem 1.1.

Theorem 3.2. *Suppose that the maximum interval of existence for* (1.1) *is* $[0, t_B)$ *for some* $0 < t_B < \infty$. *Define* u_j *as* (3.1). *If* $J < j \le n$, *then there exist* $0 \le q_j \le p_j$ *such that*

$$\lim_{t \to t_B^-} u_1'(t)u_j(t) = -p_j, \quad \lim_{t \to t_B^-} u_1(t)u_j'(t) = q_j$$

Proof. The convergence of $u_1'u_j$ follows from Lemma 3.3 together with (3.23), and the convergence of u_1u_j' follows from (3.15).

Clearly, $p_j, q_j \ge 0$ and $p_j + q_j = -(\lambda_{1,0} - \lambda_{j,0})$, by (3.15). Furthermore, one can show that $0 \le q_j \le p_j$. Suppose that $p_j < q_j$. Then, there exists $t_1 \in (0, t_B)$ such that if $t_1 < t < t_B$, then

$$\frac{\lambda_j(t)}{-\lambda_1(t)} > 1$$

and

$$\lambda_j'(t) = -\lambda_j^2(t) + \frac{k}{n}(\rho(t) - c_b) < -\lambda_1^2(t) + \frac{k}{n}(\rho(t) - c_b) = \lambda_1'(t).$$

Integration over $[t_1, t]$ yields

$$\lambda_j(t) - \lambda_j(t_1) < \lambda_1(t) - \lambda_1(t_1)$$

which contradicts the fact that $\lambda_1 \to -\infty$ and $\lambda_j \to +\infty$. ∎

From now on, we let p and q denote p_n and q_n, respectively. Then from Theorem 3.2 either

$$p > q$$

or

$$p = q$$

must hold. We investigate the solution behaviors stated in Theorem 1.2 by considering these cases in the following two sections. Indeed, we obtain (c) of (ii) in Theorem 1.2 by assuming that $p = q$, and all the other cases follow from $p > q$.

4. The case $p > q$

In this section, we describe the behaviors of blow-up solutions of (1.1) assuming that

$$p > q.$$

We first state a technical lemma.

LEMMA 4.1. *Suppose that a function $R(t)$ defined in $[0, t_B)$ satisfies*

$$(t_B - t)R(t) \to 0 \quad \text{as } t \to t_B^-.$$

Then,

$$\lim_{t \to t_B^-} (t_B - t) \int_0^t R^2(s)ds = 0,$$
$$\lim_{t \to t_B^-} (t_B - t) \int_0^t \frac{1}{t_B - s} R(s)ds = 0.$$

Furthermore, for any $0 < \varepsilon < 1$ there exists $M > 0$ such that

$$\frac{(t_B - t)^\varepsilon}{M} < e^{-\int_0^t R(s)ds} < \frac{M}{(t_B - t)^\varepsilon}. \tag{4.1}$$

Proof. The first two limits follow from L'Hôpital's rule. Let $0 < \varepsilon < 1$. Then, because $\lim_{s \to t_B^-} (t_B - s)R(s) = 0$, there exists $t_1 \in (0, t_B)$ such that for all $t_1 < s < t_B$,

$$(t_B - s)|R(s)| < \varepsilon.$$

Then, for $t > t_1$,

$$\begin{aligned} \left| \int_0^t R(s)ds \right| &\le \int_0^t (t_B - s)|R(s)| \frac{1}{t_B - s} ds \\ &< \varepsilon \int_{t_1}^t \frac{1}{t_B - s} ds + \int_0^{t_1} |R(s)|ds \\ &\le -\varepsilon \ln(t_B - t) + C, \end{aligned}$$

for some constant C that is independent of t. With $M = e^C$ it follows that

$$\frac{(t_B - t)^\varepsilon}{M} < e^{-\int_0^t R(s)ds} < \frac{M}{(t_B - t)^\varepsilon}.$$

□

Because of (3.18) and Theorem 3.2, we set $(u_1 u_n)(t_B) = 0$ and $(u_1 u_n)'(t_B) = -p + q < 0$. Then, for some $\eta(t)$ such that

$$\eta(t_B) = 0, \quad \eta'(t_B) = 0, \tag{4.2}$$

it holds that

$$(u_1 u_n)(t) = (p - q)(t_B - t) + \eta(t).$$

It follows that

$$\lambda_1(t)-\lambda_n(t)=\frac{\lambda_{1,0}-\lambda_{n,0}}{(u_1u_n)(t)}=\frac{\lambda_{1,0}-\lambda_{n,0}}{(p-q)(t_B-t)+\eta(t)}=\frac{-p-q}{(p-q)(t_B-t)+\eta(t)},$$
$$\lambda_1(t)+\lambda_n(t)=\frac{(u_1u_n)'(t)}{u_1u_n(t)}=\frac{-(p-q)+\eta'(t)}{(p-q)(t_B-t)+\eta(t)}.$$

Hence,

$$\lambda_1(t)=\frac{-p+\eta'(t)/2}{(p-q)(t_B-t)+\eta(t)},$$
$$\lambda_n(t)=\frac{q+\eta'(t)/2}{(p-q)(t_B-t)+\eta(t)}.$$

Owing to (4.2) we have the following forms:

$$\lambda_1(t)=\frac{-p}{p-q}\frac{1}{t_B-t}+R_1(t),$$
$$\lambda_n(t)=\frac{q}{p-q}\frac{1}{t_B-t}+R_n(t),$$

where $R_j(t)$ $(j=1,n)$ satisfies $\lim_{t\to t_B^-}R_j(t)(t_B-t)=0$.

Let

$$\lambda_j(t)=\frac{\xi_j}{t_B-t}+R_j \quad (j=1,n),$$

with

$$\xi_1:=\frac{-p}{p-q},\quad \xi_n:=\frac{q}{p-q}. \tag{4.3}$$

Substituting this into the main equation (1.1a) yields

$$R_j'(t)=-\frac{\xi_j^2+\xi_j}{(t_B-t)^2}-R_j^2(t)-\frac{2\xi_j}{t_B-t}R_j(t)+\frac{k\rho_0}{n}e^{-\int_0^t\lambda(s)ds}-\omega^2. \tag{4.4}$$

Integrating over $(0,t)$ and multiplying by (t_B-t) give

$$\begin{aligned}(t_B-t)R_j(t)=&-(\xi_j^2+\xi_j)-(t_B-t)\int_0^t\left[R_j^2(\tau)+\frac{2\xi_j}{t_B-\tau}R_j(\tau)\right]d\tau\\&+\frac{k\rho_0}{n}(t_B-t)\int_0^t e^{-\int_0^\tau\lambda(s)ds}d\tau\\&+(t_B-t)\left[\frac{\xi_j^2+\xi_j}{t_B}-\omega^2t+R_j(0)\right].\end{aligned}$$

Because $(t_B-t)\int_0^t\left[R_j^2(\tau)+2\xi_jR_j(\tau)/(t_B-\tau)\right]d\tau$ converges to 0 as $t\to t_B^-$ by Lemma 4.1, we obtain the following quadratic equation for ξ:

$$\xi^2+\xi-\frac{k\rho_0}{n}\lim_{t\to t_B^-}(t_B-t)\int_0^t e^{-\int_0^\tau\lambda(s)ds}d\tau=0. \tag{4.5}$$

Here, $\xi=\xi_1,\xi_n$, for which the limit in (4.5) must exist.

Owing to Lemma 3.1 together with Theorem 3.1, we have that

$$\lambda_i(t)=\frac{\xi_1}{t_B-t}+R_i(t),\quad R_i(t)=R_1(t),\quad 1\le i\le J, \tag{4.6}$$

$$\lambda_i(t)=\frac{\xi_n}{t_B-t}+R_i(t),\quad J<i\le n, \tag{4.7}$$

where $\lim_{t\to t_B^-}(t_B-t)R_i(t)=0$ for all $1\le i\le n$. It follows that

$$\lambda(t)=\frac{-pJ+q(n-J)}{p-q}\frac{1}{t_B-t}+\sum_{i=1}^n R_i(t)=\frac{\gamma}{t_B-t}+R(t), \tag{4.8}$$

where

$$\gamma:=\frac{-pJ+q(n-J)}{p-q}, \tag{4.9}$$

$$R(t):=\sum_{i=1}^n R_i(t),\quad \lim_{t\to t_B^-}(t_B-t)R(t)=0.$$

Now, we evaluate the limit in (4.5) as follows. Note that

$$\int_0^t e^{-\int_0^\tau \lambda(s)ds}d\tau=t_B^{-\gamma}\int_0^t (t_B-\tau)^\gamma e^{-\int_0^\tau R(s)ds}d\tau. \tag{4.10}$$

If follows from (4.1) that for any $0<\varepsilon<1$, there exists $M>0$ such that

$$\frac{t_B-t}{M}\int_0^t (t_B-\tau)^{\gamma+\varepsilon}d\tau<(t_B-t)\int_0^t (t_B-\tau)^\gamma e^{-\int_0^\tau R(s)ds}d\tau<M(t_B-t)\int_0^t (t_B-\tau)^{\gamma-\varepsilon}d\tau.$$

Assume that $\gamma+2<0$. Then, the lower bound

$$\frac{-1}{M(\gamma+1+\varepsilon)}\left[(t_B-t)^{\gamma+2+\varepsilon}-t_B^{\gamma+1+\varepsilon}(t_B-t)\right]\to+\infty \text{ as } t\to t_B^-$$

by taking ε sufficiently small so that $\gamma+2+\varepsilon<0$. This is not the case, as the limit in (4.5) must converge, as previously mentioned. On the other hand, $\gamma+2>0$ implies that the upper bound

$$\frac{-M}{\gamma+1-\varepsilon}\left[(t_B-t)^{\gamma+2-\varepsilon}-t_B^{\gamma+1-\varepsilon}(t_B-t)\right]\to 0 \text{ as } t\to t_B^-$$

by taking ε such that $\gamma+2-\varepsilon>0$ and $\gamma-\varepsilon\neq-1$. This ensures that

$$\xi^2+\xi=0.$$

It follows that

$$\xi_1=\frac{-p}{p-q}=-1,\quad \xi_n=\frac{q}{p-q}=0.$$

Substituting $q=0$ into (4.9) together with $\gamma+2>0$ then yields

$$J=1.$$

Now, consider the case that $\gamma+2=0$. We first claim that

$$\lim_{t\to t_B^-}(t_B-t)\int_0^t (t_B-\tau)^{-2}e^{-\int_0^\tau R(s)ds}d\tau \tag{4.11a}$$

$$=\lim_{t\to t_B^-} e^{-\int_0^t R(s)ds}. \tag{4.11b}$$

We remark that, in general, the convergence of (4.11a), which we have already verified, does not guarantee the convergence of (4.11b), because (4.11a) may converge for an oscillating divergent $\int_0^t R(s)ds$. However, the decay property of R can eliminate this case. By integration by parts,

$$\begin{aligned}&\lim_{t\to t_B^-}(t_B-t)\int_0^t (t_B-\tau)^{-2}e^{-\int_0^\tau R(s)ds}d\tau\\ &=\lim_{t\to t_B^-}\left[e^{-\int_0^t R(s)ds}-\frac{t_B-t}{t_B}+(t_B-t)\int_0^t (t_B-\tau)^{-1}e^{-\int_0^\tau R(s)ds}R(\tau)d\tau\right].\end{aligned} \tag{4.12}$$

Recall that $(t_B-t)R(t)\to 0$ as $t\to t_B^-$. Then, there exists $t_1\in(0,t_B)$ such that

$$(t_B-t)|R(t)|<1,\quad t_1<t<t_B,$$

and

$$\left|(t_B-t)\int_0^t (t_B-\tau)^{-1}e^{-\int_0^\tau R(s)ds}R(\tau)d\tau\right| \tag{4.13}$$

$$\leq (t_B-t)\int_0^{t_1}(t_B-\tau)^{-1}e^{-\int_0^\tau R(s)ds}|R(\tau)|d\tau+(t_B-t)\int_{t_1}^t (t_B-\tau)^{-2}e^{-\int_0^\tau R(s)ds}d\tau. \tag{4.14}$$

Because (4.11a) converges, the second term in (4.14) converges, and thus (4.13) converges as $t\to t_B^-$. The convergence of $\exp\left(-\int_0^{t_B} R(s)ds\right)$ follows from (4.12). Now, apply L'Hôpital's rule to obtain (4.11).

Thus, the case with $\gamma+2=0$ may be considered as either

$$\lim_{t\to t_B^-}(t_B-t)\int_0^t (t_B-\tau)^{\gamma}e^{-\int_0^\tau R(s)ds}d\tau=\lim_{t\to t_B^-}e^{-\int_0^t R(s)ds}=0, \tag{4.15}$$

or

$$\lim_{t\to t_B^-}(t_B-t)\int_0^t (t_B-\tau)^{\gamma}e^{-\int_0^\tau R(s)ds}d\tau=\lim_{t\to t_B^-}e^{-\int_0^t R(s)ds}=R_0>0. \tag{4.16}$$

For the case that (4.15), a similar argument as that in the case for $\gamma+2<0$ yields

$$\xi_1=\frac{-p}{p-q}=-1,\quad \xi_n=\frac{q}{p-q}=0,$$

and

$$J=2.$$

Furthermore, (4.15) implies that

$$\lim_{t\to t_B^-}\int_0^t R(s)ds=\infty.$$

For the case that (4.16), we deduce from (4.5) and (4.10) that

$$\xi^2+\xi-\frac{k\rho_0 t_B^2 R_0}{n}=0,$$

and from (4.9) that

$$p(J-2)=q(n-J-2).$$

We divide this into two cases, by taking into account $p>q$:

$$J=2,\ n=4 \quad \text{or}$$
$$J\geq 3,\ n>2J$$

In summary, we have the following:

THEOREM 4.1. *Suppose that $[0,t_B)$ be the maximum interval of existence for (1.1). Define u_i as (3.1), and let*

$$\lim_{t\to t_B^-} u_1'(t)u_n(t)=-p, \quad \lim_{t\to t_B^-} u_1(t)u_n'(t)=q.$$

If $p>q$, then λ_i $(i=1,2,\cdots,n)$ and λ can be represented by (4.6), (4.7), and (4.8). Moreover, one of the following must hold, where $\xi=\xi_1,\ \xi_n$:

(1) *$J=1$ and*

$$\xi^2+\xi=0.$$

(2-a) *$J=2$, $\lim_{t\to t_B^-}\int_0^t R(s)ds=\infty$, and*

$$\xi^2+\xi=0.$$

(2-b) *$J=2$, $n=4$, $\lim_{t\to t_B^-}\exp(-\int_0^t R(s)ds)=R_0>0$, and*

$$\xi^2+\xi-\frac{k\rho_0 t_B^2 R_0}{4}=0.$$

(3) *$J\geq 3$, $n>2J$, $\lim_{t\to t_B^-}\exp(-\int_0^t R(s)ds)=R_0>0$, and*

$$\xi^2+\xi-\frac{k\rho_0 t_B^2 R_0}{n}=0.$$

Furthermore, these cases imply (i), (a), (b) of (ii), and (iii) in Theorem 1.2, respectively.

The remainder of the proof of Theorem 4.1 demonstrates the relations between the cases in Theorem 4.1 and in Theorem 1.2.

Assuming **(1)**, we immediately have the following representation of λ_i:

$$\lambda_i(t)=\begin{cases}\dfrac{-1}{t_B-t}+R_1(t), & i=1,\\ R_i(t), & 2\le i\le n,\end{cases}$$

$$\lambda(t)=\frac{-1}{t_B-t}+R(t),\quad R(t)=\sum_{i=1}^{n}R_i(t).$$

Although $\lim_{t\to t_B^-}(t_B-t)R_i(t)=0$ for $i=1,2,\cdots,n$, we require the integrability of R_i to obtain (i) in Theorem 1.2. Indeed, this is the case.

LEMMA 4.2. *Assuming **(1)** in Theorem 4.1,*

$$\lambda_i\in L^1(0,t_B),\quad i=2,\cdots,n,$$

and

$$\int_0^{t_B}R_1(s)ds=C.$$

Proof. Let $i=2,3,\cdots,n$. Then, we deduce that

$$\begin{aligned}\lambda_i(t)&=-\int_0^t\lambda_i^2(s)ds+\frac{k\rho_0t_B}{n}\int_0^t\frac{1}{t_B-s}e^{-\int_0^sR(\tau)d\tau}ds-\frac{kc_b}{n}t+\lambda_{i,0} &(4.17)\\ &\le\int_0^t\frac{1}{t_B-s}e^{-\int_0^sR(\tau)d\tau}ds+\lambda_{i,0}. &(4.18)\end{aligned}$$

Multiplying by $(t_B-t)^{1/2}$ yields

$$(t_B-t)^{1/2}\lambda_i(t)\le(t_B-t)^{1/2}\int_0^t\frac{1}{t_B-s}e^{-\int_0^sR(\tau)d\tau}ds+(t_B-t)^{1/2}\lambda_{i,0}. \tag{4.19}$$

Now, we take $\varepsilon=1/3$ in (4.1) to obtain

$$e^{-\int_0^sR(\tau)d\tau}\le\frac{M}{(t_B-s)^{1/3}}.$$

Then, we observe that the right-hand side of (4.19) converges to 0. Thus,

$$\lim_{t\to t_B^-}(t_B-t)^{1/2}\lambda_i(t)=0,$$

because $\lambda_i>0$ near t_B. This implies that

$$\lambda_i\in L^1(0,t_B),\quad i=2,3,\cdots,n. \tag{4.20}$$

To demonstrate the convergence of $\int_0^tR_1(s)ds$, we deduce from (2.8) that

$$(t_B-t)(\lambda_1(t)-\lambda_n(t))=(\lambda_{1,0}-\lambda_{n,0})t_Be^{-\int_0^tR_1(s)+\lambda_n(s)ds}. \tag{4.21}$$

Because the left-hand side converges to -1 assuming **(1)**, there exists a constant C_1 such that

$$\int_0^{t_B}R_1(s)+\lambda_n(s)ds=C_1, \tag{4.22}$$

and thus (4.20), $\lambda_n \in L^1(0,t_B)$, yields

$$\int_0^{t_B} R_1(s)ds = C.$$

□

Lemma 4.2 enhances the estimate (4.1) as

$$0 < \lim_{t\to t_B^-} e^{-\int_0^t R(s)ds} < \infty.$$

Immediately, we obtain

$$\rho(t) = \mathcal{O}\Big(\frac{1}{t_B - t}\Big) \quad \text{as } t \to t_B^-.$$

Furthermore, it follows from (4.18) that λ_i is at most $\mathcal{O}(\ln(t_B - t))$ for $i = 2,3,\cdots,n$. Then, $\lambda_i \in L^2(0,t_B)$, and applying (4.17) again yields

$$\lambda_i(t) = \mathcal{O}(\ln(t_B - t)) \quad i = 2,3,\cdots,n.$$

This shows that **(1)** implies (i) in Theorem 1.2.

In the case of **(2-a)** in Theorem 4.1,

$$\lambda_i(t) = \begin{cases} \dfrac{-1}{t_B - t} + R_i(t), & i = 1,2, \\ R_i(t), & 3 \le i \le n, \end{cases}$$

$$\lambda(t) = \frac{-2}{t_B - t} + R(t), \quad R(t) = \sum_{i=1}^n R_i(t), \quad R_1(t) = R_2(t).$$

Now, let $3 \le i \le n$. Then, similar to the derivation of (4.22), we have that

$$\int_0^{t_B} R_1(s) + \lambda_i(s)ds = C_i. \tag{4.23}$$

If $\int_0^t \lambda_i(s)ds$ is assumed to converge, then $\int_0^t R_1(s)ds$, and thus $\int_0^t R(s)ds$ converges, which does not belong to **(2-a)**. Taking into account $\lambda_i \to \infty$, we must have

$$\lim_{t\to t_B^-} \int_0^t \lambda_i(s)ds = \infty, \quad i = 3,4,\cdots,n.$$

Then, (4.23) yields

$$\lim_{t\to t_B^-} \int_0^t R_1(s)ds = -\infty. \tag{4.24}$$

Summing (4.23) over $i = 3,4,\cdots,n$ yields that for some constant C,

$$\int_0^{t_B} R(s) + (n-4)R_1(s)ds = C. \tag{4.25}$$

Because $\lim_{t\to t_B^-} \int_0^t R(s)ds = \infty$ in **(2-a)**, we have that

$$n \ge 5,$$

and it follows that

$$\rho(t) = o\left(\frac{1}{(t_B - t)^2}\right) \text{ as } t \to t_B^-.$$

Hence, we conclude that **(2-a)** in Theorem 4.1 implies (a) of (ii) in Theorem 1.2.

Now, we consider the case of **(2-b)**. Because the solutions to the characteristic equation (4.5) are $\xi_1 = -p/(p-q)$ and $\xi_4 = q/(p-q)$, it follows that

$$\frac{pq}{(p-q)^2} = \frac{k\rho_0 t_B^2 R_0}{4} \tag{4.26}$$

and

$$\lambda_i(t) = \begin{cases} \dfrac{\xi_1}{t_B - t} + R_i(t), & i = 1,2, \\ \dfrac{\xi_4}{t_B - t} + R_i(t), & i = 3,4, \end{cases}$$

$$\lambda(t) = \frac{-2}{t_B - t} + R(t), \quad R(t) = \sum_{i=1}^{4} R_i(t), \quad R_1(t) = R_2(t).$$

Note that the representation of λ follows from $\xi_1 + \xi_4 = -1$, and the representation of λ_3 (i.e., $\xi_3 = \xi_4$) follows from Lemma 3.1. Because $\lim_{t \to t_B^-} \exp(-\int_0^t R(s)ds) = R_0 > 0$, we immediately obtain that

$$\rho(t) = \mathcal{O}\left(\frac{1}{(t_B - t)^2}\right) \quad \text{as } t \to t_B^-.$$

Similar to (4.21), we deduce that

$$(t_B - t)(\lambda_1(t) - \lambda_i(t)) = (\lambda_{1,0} - \lambda_{i,0}) t_B e^{-\int_0^t R_1(s) + R_i(s) ds}, \quad i = 3,4.$$

Sending $t \to t_B^-$ and multiplying the two equations for $i = 3,4$ yield that

$$\frac{(p+q)^2}{(p-q)^2} = A_0 t_B^2 R_0. \tag{4.27}$$

Recall that

$$A_0 := (\lambda_{1,0} - \lambda_{3,0})(\lambda_{1,0} - \lambda_{4,0}).$$

Then, we combine (4.26) and (4.27) to obtain

$$(p-q)^2 = 4\left(\frac{A_0}{k\rho_0} - 1\right) pq.$$

Thus, it must hold that

$$A_0 > k\rho_0. \tag{4.28}$$

Furthermore, we obtain representations of ξ_1 and ξ_4 in terms of the given parameters. Indeed, we have

$$\xi_1 = -\frac{1}{2} - \frac{1}{2}\sqrt{\frac{A_0}{A_0 - k\rho_0}},$$

$$\xi_3 = \xi_4 = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{A_0}{A_0 - k\rho_0}},$$

as described in (b) of (ii) in Theorem 1.2.

In the case of **(3)** in Theorem 4.1, we have that

$$\lim_{t\to t_B^-}(t_B - t)\lambda_1(t) = \frac{-p}{p-q},$$
$$\lim_{t\to t_B^-}(t_B - t)\lambda_i(t) = \frac{q}{p-q}, \quad J+1 \le i \le n.$$

The behavior of ρ,

$$\rho(t) = \mathcal{O}\left(\frac{1}{(t_B - t)^2}\right) \quad \text{as } t \to t_B^-,$$

follows from $(-pJ + q(n-J))/(p-q) = -2$ and $\exp(-\int_0^{t_B} R(s)ds) = R_0$. This shows that **(3)** implies (iii) in Theorem 1.2.

5. The case $p = q$

In this section, we investigate the blow-up solution behaviors when

$$p = q\left(= \frac{\lambda_{n,0} - \lambda_{1,0}}{2}\right). \tag{5.1}$$

As previously noted, understanding the behaviors of λ_i' near t_B is essential. One technique to achieve this is to compare the behaviors of λ_i^2 and ρ from (1.1a). However, the main difficulty lies in the fact that the condition (5.1) implies that the leading singular terms of $\int \lambda_i^2$ and $k/n\int\rho$ are the same. Indeed, integrating (1.1a) yields

$$\lambda_1(t) - \lambda_{1,0} = -\int_0^t \lambda_1^2(s)ds + \frac{k}{n}\int_0^t \rho(s) - \omega^2 ds \to -\infty,$$
$$\lambda_n(t) - \lambda_{n,0} = -\int_0^t \lambda_n^2(s)ds + \frac{k}{n}\int_0^t \rho(s) - \omega^2 ds \to +\infty,$$

implying that in a neighborhood of t_B,

$$\int_0^t \lambda_n^2(s)ds < \int_0^t \rho(s)ds < \int_0^t \lambda_1^2(s)ds. \tag{5.2}$$

However, the condition (5.1) yields

$$\lim_{t\to t_B^-}\frac{\lambda_1(t)}{\lambda_n(t)} = \lim_{t\to t_B^-}\frac{u_1'(t)u_n(t)}{u_1(t)u_n'(t)} = -1, \tag{5.3}$$

which indicates that the leading singular terms of all integrals in (5.2) are the same. For this reason, we study the case of (5.1) by examining the second singular terms of $\int\lambda_i^2$ and $\int\rho$. We remark that one cannot compare λ_i^2 and $k\rho/n$ directly as Proposition 2.3 demonstrates the behavior of $\int\rho$ rather than ρ. Furthermore, we notice that the case (5.1) occurs only in the case of (3.22) in Proposition 3.1. Indeed, (3.21) implies that

$$\lim_{t\to t_B}\int_0^t \lambda_n(s)ds = \lim_{t\to t_B}\int_0^t \frac{u_n'(s)}{u_n(s)}ds = \ln\beta_n < \infty.$$

Assuming (5.1), we have observed (5.3), which implies that $\lambda_1 \in L^1(0,t_B)$. Thus, $\lambda_i \in L^1(0,t_B)$ for all i, and thus ρ is bounded. This contradicts Proposition 2.3. More precisely, $\lambda_i \in L^1(0,t_B)$ $(i > J)$, which is a necessary and sufficient condition for the convergence of u_i $(i > J)$ or (3.21) in Proposition 3.1, only holds in (i) in Theorem 1.2. That is, (i) is equivalent to (3.21), and all other cases in Theorem 1.2 are associated with (3.22) in Proposition 3.1.

We define η as

$$\frac{u_1'}{u_1} + \frac{u_n'}{u_n} = -2\eta. \tag{5.4}$$

Because $\lim_{t\to t_B^-}(u_1u_n)(t) = 0$ in Corollary 3.1 and $\lim_{t\to t_B^-}(u_1u_n)'(t) = 0$ from the condition (5.1), η satisfies

$$\lim_{t\to t_B^-} \eta(t)(u_1u_n)(t) = -\lim_{t\to t_B^-} \frac{(u_1u_n)'(t)}{2} = 0, \tag{5.5}$$

$$\lim_{t\to t_B^-} \int_0^t \eta(s)ds = -\lim_{t\to t_B^-} \frac{\ln((u_1u_n)(t))}{2} = \infty. \tag{5.6}$$

We remark that the behavior of η near t_B is not clear at this point, owing to the highly oscillating type (2.2).

Recall (3.15) or that for all $t \in (0,t_B)$,

$$\frac{u_1'}{u_1} - \frac{u_n'}{u_n} = -2\frac{p}{u_1u_n}. \tag{5.7}$$

Then, together with (5.4), we have that

$$\lambda_1 = \frac{u_1'}{u_1} = -\frac{p}{u_1u_n} - \eta, \tag{5.8}$$

$$\lambda_n = \frac{u_n'}{u_n} = \frac{p}{u_1u_n} - \eta. \tag{5.9}$$

Substituting these representations into the main equation (1.1a) yields

$$\lambda_1' = -\lambda_1^2 + \frac{k}{n}\rho - \omega^2 = -\left[\left(\frac{p}{u_1u_n}\right)^2 + 2\frac{p\eta}{u_1u_n} + \eta^2\right] + \frac{k}{n}\rho - \omega^2, \tag{5.10}$$

$$\lambda_n' = -\lambda_1^2 + \frac{k}{n}\rho - \omega^2 = -\left[\left(\frac{p}{u_1u_n}\right)^2 - 2\frac{p\eta}{u_1u_n} + \eta^2\right] + \frac{k}{n}\rho - \omega^2. \tag{5.11}$$

Owing to the property of η in (5.5), we obtain

$$\lim_{t\to t_B^-} \frac{\displaystyle\int_0^t \frac{p\eta(s)}{(u_1u_n)(s)}ds}{\displaystyle\int_0^t \left(\frac{p}{(u_1u_n)(s)}\right)^2 ds} = 0.$$

Thus, the leading singular term of $\int_0^t \lambda_i^2(s)ds$ $(i = 1,n)$ is

$$\int_0^t \left(\frac{p}{(u_1u_n)(s)}\right)^2 ds,$$

and this should be the same as the leading singular term of $k/n\int_0^t \rho(s)ds$, otherwise the integrations of (5.10) and (5.11) yield that $\lambda_1\lambda_n > 0$ near t_B. Now, we define δ as

$$\int_0^t \left(\frac{p}{(u_1u_n)(s)}\right)^2 ds + \delta(t) := \int_0^t \frac{k}{n}\rho(s) - \omega^2 ds, \tag{5.12}$$

satisfying

$$\lim_{t\to t_B^-} \frac{\delta(t)}{\displaystyle\int_0^t \left(\frac{p}{(u_1u_n)(s)}\right)^2 ds} = 0, \quad \delta(0) = 0. \tag{5.13}$$

It follows from (5.10) and (5.11) that

$$\lambda_1(t) - \lambda_{1,0} = \int_0^t \left(-2\frac{p\eta(s)}{(u_1u_n)(s)} - \eta^2(s)\right) ds + \delta(t), \tag{5.14}$$

$$\lambda_n(t) - \lambda_{n,0} = \int_0^t \left(2\frac{p\eta(s)}{(u_1u_n)(s)} - \eta^2(s)\right) ds + \delta(t). \tag{5.15}$$

Now, we present a technical lemma. In Corollary 3.1, we showed that $u_1u_n \to 0$ as t tends to t_B. Thus, one may expect that for some $\theta > 1$, $u_1u_n^\theta$ converges to a nonzero constant by assuming (3.22). However, this is not the case, at least when $p = q$.

LEMMA 5.1. *Assume the hypothesis of Theorem 4.1, and suppose that*

$$p = q.$$

Then, for any $\theta \le 1$,

$$\lim_{t\to t_B^-} (u_1u_n^\theta)(t) = 0.$$

For $\theta > 1$, if the convergence of $u_1u_n^\theta$ is assumed, then

$$\lim_{t\to t_B^-} (u_1u_n^\theta)(t) = 0.$$

Proof. Recall that $p = q$ only occurs in (3.22), i.e., $u_n \to \infty$. Then, it clearly holds that $\lim_{t\to t_B^-}(u_1u_n^\theta)(t) = 0$ for $\theta \le 1$, because $\lim_{t\to t_B^-}(u_1u_n)(t) = 0$ in Corollary 3.1.

Let $\theta > 1$ and

$$\lim_{t\to t_B^-} (u_1u_n^\theta)(t) = C.$$

We deduce from (3.15) that

$$\left(\frac{u_i(t)}{u_j(t)}\right)' = \frac{\lambda_{i,0} - \lambda_{j,0}}{u_j^2(t)},$$

$$u_i(t) = u_j(t) + u_j(t)(\lambda_{i,0} - \lambda_{j,0}) \int_0^t \frac{1}{u_j^2(s)} ds.$$

Multiplying by u_n^θ in the equation for $i = 1, j = n$ yields

$$(u_1u_n^\theta)(t) = u_n^{\theta+1}(t)\left(1 + (\lambda_{1,0} - \lambda_{n,0}) \int_0^t \frac{1}{u_n^2(s)} ds\right).$$

Then, $1+(\lambda_{10}-\lambda_{n,0})\int_0^{t_B} 1/u_n^2(s)ds=0$, because the left-hand side converges to C, and $u_n^{\theta+1}\to\infty$. Apply L'Hôpital's rule to the right-hand side, to yield

$$\begin{aligned}\lim_{t\to t_B^-}(u_1u_n^\theta)(t) &= \lim_{t\to t_B^-}\frac{\lambda_{1,0}-\lambda_{n,0}}{-\theta-1}\frac{u_n^\theta(t)}{u_n'(t)}\\ &= \lim_{t\to t_B^-}\frac{\lambda_{1,0}-\lambda_{n,0}}{-\theta-1}\frac{(u_n^\theta u_1)(t)}{(u_n' u_1)(t)}.\end{aligned}$$

Because the final limit exists,

$$C=\frac{2p}{\theta+1}\frac{C}{p}=\frac{2C}{\theta+1}.$$

Then, as we assumed that $\theta>1$, it follows that $C=0$, as desired. ∎

In the following theorem, we claim that the case of $p=q$ implies (c) of (ii) in Theorem 1.2.

THEOREM 5.1. *Suppose that* $[0,t_B)$ *be the maximum interval of existence for* (1.1). *Define* u_i *as* (3.1), *and let*

$$\lim_{t\to t_B^-}u_1'(t)u_n(t)=-p,\qquad \lim_{t\to t_B^-}u_1(t)u_n'(t)=q.$$

If

$$p=q,$$

then $J=2$, $n=4$, *and*

$$(\lambda_{1,0}-\lambda_{3,0})(\lambda_{1,0}-\lambda_{4,0})=:A_0=k\rho_0.$$

Moreover, this implies (c) of (ii) in Theorem 1.2.

Proof. Recall (3.3)

$$\rho=\rho_0\prod_{i=1}^n\frac{1}{u_i}.$$

From (5.12) and (5.13),

$$\lim_{t\to t_B^-}\frac{\int_0^t(u_1u_n)^{-2}ds}{\int_0^t\prod_{i=1}^n u_i^{-1}ds}=\frac{k\rho_0}{np^2}.$$

We apply Cauchy's mean value theorem, to obtain

$$\frac{\int_0^t(u_1u_n)^{-2}ds}{\int_0^t\prod_{i=1}^n u_i^{-1}ds}\,\frac{1-\frac{\int_0^{t_1}(u_1u_n)^{-2}ds}{\int_0^t(u_1u_n)^{-2}ds}}{1-\frac{\int_0^{t_1}\prod_{i=1}^n u_i^{-1}ds}{\int_0^t\prod_{i=1}^n u_i^{-1}ds}}=\frac{\prod_{i=1}^n u_i(\tau)}{u_1^2(\tau)u_n^2(\tau)}$$

for some $t_1<\tau<t$. Owing to the convergence of the left-hand side as $t\to t_B^-$, we can construct a sequence $\{\tau_l\}_{l=1}^\infty$ converging to t_B such that

$$\lim_{l\to\infty}\frac{\prod_{i=1}^n u_i(\tau_l)}{u_1^2(\tau_l)u_n^2(\tau_l)}=\frac{k\rho_0}{np^2}.$$

It follows from Lemma 3.1 that

$$\lim_{l\to\infty} u_1^{J-2}(\tau_l)u_n^{n-J-2}(\tau_l)=\frac{k\rho_0}{np^2}\prod_{j=J+1}^{n-1}\frac{\lambda_{n,0}-\lambda_{1,0}}{\lambda_{j,0}-\lambda_{1,0}}. \tag{5.16}$$

If it is assumed that $J\neq 2$, then

$$\lim_{l\to\infty} u_1(\tau_l)u_n^{\frac{n-J-2}{J-2}}(\tau_l)=\left(\frac{k\rho_0}{np^2}\prod_{j=J+1}^{n-1}\frac{\lambda_{n,0}-\lambda_{1,0}}{\lambda_{j,0}-\lambda_{1,0}}\right)^{\frac{1}{J-2}}. \tag{5.17}$$

If it is additionally assumed that $(n-J-2)/(J-2)\leq 1$, then Lemma 5.1 implies that

$$\left(\frac{k\rho_0}{np^2}\prod_{j=J+1}^{n-1}\frac{\lambda_{n,0}-\lambda_{1,0}}{\lambda_{j,0}-\lambda_{1,0}}\right)^{\frac{1}{J-2}}=0,$$

which is not possible. The assumption that $(n-J-2)/(J-2)>1$ also yields a contradiction. Indeed, under this assumption one can show that $u_1u_n^{(n-J-2)/(J-2)}$ is an increasing function in a neighborhood of t_B, by showing that for any $\varepsilon>0$ there exists $t_1\in(0,t_B)$ such that

$$(u_1u_n^{1+\varepsilon})'(t)>0,\qquad t_1<t<t_B \tag{5.18}$$

in the case of $p=q$. Then, together with (5.17) we have

$$\lim_{t\to t_B^-} u_1(t)u_n^{\frac{n-J-2}{J-2}}(t)=\left(\frac{k\rho_0}{np^2}\prod_{j=J+1}^{n-1}\frac{\lambda_{n0}-\lambda_{10}}{\lambda_{j0}-\lambda_{10}}\right)^{\frac{1}{J-2}}.$$

Now, we apply Lemma 5.1 to obtain

$$\left(\frac{k\rho_0}{np^2}\prod_{j=J+1}^{n-1}\frac{\lambda_{n,0}-\lambda_{1,0}}{\lambda_{j,0}-\lambda_{1,0}}\right)^{\frac{1}{J-2}}=0,$$

which is also not possible.

Hence, $J=2$. Because the case of $p=q$ is corresponds to (3.22), we must have that $n=4$. Moreover, substituting (5.1) into (5.16) with $J=2$ and $n=4$ yields

$$k\rho_0=(\lambda_{4,0}-\lambda_{1,0})(\lambda_{3,0}-\lambda_{1,0}), \tag{5.19}$$

as desired.

It remains to verify the solution behaviors described in (c) of (ii) in Theorem 1.2. We first state a lemma describing the behavior of δ near t_B.

Lemma 5.2. *Under the hypothesis of Theorem 5.1, δ defined in (5.12) satisfies*

$$\lim_{t\to t_B^-}\delta'(t)=-\omega^2.$$

Proof. We have shown that $J=2$ and $n=4$ when $p=q$. Thus, we deduce from (5.12) and (3.3) that

$$\delta'=\frac{1}{u_1^2u_4}\left(\frac{k\rho_0}{4}\frac{1}{u_3}-\frac{p^2}{u_4}\right)-\omega^2.$$

Using the representation in (3.17),

$$u_3 = \frac{\lambda_{4,0} - \lambda_{3,0}}{\lambda_{4,0} - \lambda_{1,0}} u_1 + \frac{\lambda_{3,0} - \lambda_{1,0}}{\lambda_{4,0} - \lambda_{1,0}} u_4,$$

we have that

$$\delta' = \frac{-p^2(\lambda_{4,0} - \lambda_{3,0})}{(\lambda_{4,0} - \lambda_{3,0})u_1 + (\lambda_{3,0} - \lambda_{1,0})u_4} \frac{1}{u_1 u_4^2} - \omega^2. \tag{5.20}$$

The condition $p = q$ implies that in a neighborhood of t_B,

$$\lim_{t \to t_B^-} (u_1 u_4^2)'(t) > 0,$$

as mentioned in (5.18). Thus, $1/(u_1 u_4^2)$ is a decreasing function near t_B and converges. Moreover, in the case of (3.22) we have that

$$\frac{-p^2(\lambda_{4,0} - \lambda_{3,0})}{(\lambda_{4,0} - \lambda_{3,0})u_1 + (\lambda_{3,0} - \lambda_{10})u_4} \to 0.$$

Hence, we conclude that

$$\lim_{t \to t_B^-} \delta'(t) = -\omega^2.$$

□

Proof. **(Continued Proof of Theorem 5.1.)** Substituting (5.8) and (5.9) into (5.14) and (5.15) yields

$$-\frac{p}{(u_1 u_4)(t)} - \eta(t) - \lambda_{1,0} = \int_0^t \Big(-2\frac{p\eta(s)}{(u_1 u_4)(s)} - \eta^2(s) \Big) ds + \delta(t),$$
$$\frac{p}{(u_1 u_4)(t)} - \eta(t) - \lambda_{4,0} = \int_0^t \Big(2\frac{p\eta(s)}{(u_1 u_4)(s)} - \eta^2(s) \Big) ds + \delta(t).$$

We deduce that

$$\eta(t) = \int_0^t \eta^2(s) ds - \delta(t) - \frac{\lambda_{1,0} + \lambda_{4,0}}{2}.$$

Notice that the integral equation, together with (5.6) and Lemma 5.2, yields

$$\eta(t) \to \infty, \quad \text{as } t \to t_B^-. \tag{5.21}$$

The integral equation can be rewritten as

$$\eta' = \eta^2 - \delta', \quad \eta(0) = -\frac{\lambda_{1,0} + \lambda_{4,0}}{2}. \tag{5.22}$$

Then, for t sufficiently close to t_B so that

$$0 < \int_t^{t_B} \frac{\eta^2(s) - \delta'(s)}{\eta^2(s) + 1} ds < \pi,$$

we have that

$$\arctan(\eta(\tau)) - \arctan(\eta(t)) = \int_t^{\tau} \frac{\eta^2(s) - \delta'(s)}{\eta^2(s) + 1} ds, \quad t < \tau < t_B.$$

Now, send $\tau \to t_B^-$ to obtain

$$\eta(t) = \cot\left(\int_t^{t_B} \frac{\eta^2(s) - \delta'(s)}{\eta^2(s)+1} ds\right).$$

Owing to Lemma 5.2 and (5.21), we have

$$\lim_{t\to t_B^-} (t_B - t)\eta(t) = 1$$

and for some σ,

$$\eta(t) = \frac{1}{t_B - t} + \sigma(t), \quad \sigma(t) = o(t_B - t). \tag{5.23}$$

Moreover, one can show that σ is integrable, i.e.,

$$\left|\int_0^{t_B} \sigma(s) ds\right| < \infty. \tag{5.24}$$

Indeed, substituting (5.23) into (5.22) yields

$$\sigma'(t) = \sigma^2(t) + \frac{2\sigma(t)}{t_B - t} - \delta'(t),$$

and for $t_1 < t < t_B$

$$(t_B - t)\sigma(t) - (t_B - t_1)\sigma(t_1) = \int_{t_1}^t (t_B - s)\sigma^2(s) ds + \int_{t_1}^t \sigma(s) ds - \int_{t_1}^t (t_B - s)\delta'(s) ds.$$

If $\int_{t_1}^t (t_B - s)\sigma^2(s) ds$ were unbounded, then $\int_{t_1}^t \sigma(s) ds \to -\infty$ as $t \to t_B^-$, as the left-hand side and $\delta'(t)$ converge. However, this yields a contradiction, because

$$\frac{\int_{t_1}^t (t_B - s)\sigma^2(s) ds}{\int_{t_1}^t \sigma(s) ds} \to 0.$$

Thus, for t_1 sufficiently close to t_B,

$$\begin{aligned} \Big|(t_B - t)\sigma(t) - (t_B - t_1)\sigma(t_1)\Big| &= \left|\int_{t_1}^t (t_B - s)\sigma^2(s) ds + \int_{t_1}^t \sigma(s) ds - \int_{t_1}^t (t_B - s)\delta'(s) ds\right| \\ &> \int_{t_1}^t (t_B - s)\sigma^2(s) ds \to \infty, \quad \text{as } t \to t_B^-. \end{aligned}$$

Hence, $\int_{t_1}^t (t_B - s)\sigma^2(s) ds$ converges. Thus we have (5.24).

We now estimate λ_i and ρ from the representation of η. Integrating (5.4) together with (5.23) yields

$$(u_1 u_4)(t) = \frac{(t_B - t)^2}{t_B^2 e^{2\int_0^t \sigma(s) ds}}.$$

Substituting this representation and (5.23) into (5.8), (5.9), and (5.12) yields

$$\lambda_1(t) = -\frac{p t_B^2 e^{2\int_0^t \sigma(s) ds}}{(t_B - t)^2} - \frac{1}{(t_B - t)} - \sigma(t).$$

$$\lambda_4(t) = \frac{pt_B^2 e^{2\int_0^t \sigma(s)ds}}{(t_B - t)^2} - \frac{1}{(t_B - t)} - \sigma(t),$$

and

$$\rho(t) = \frac{4p^2 t_B^4 e^{4\int_0^t \sigma(s)ds}}{k(t_B - t)^4} + \frac{4\delta'(t)}{k} + c_b.$$

These representations, together with (5.24), imply (c) of (ii) in Theorem 1.2. ∎

We close this section by providing a specific example with $p = q$.

EXAMPLE 5.1. *Recall that*

$$\omega = \sqrt{\frac{kc_b}{4}}, \quad p = q = \frac{\lambda_{4,0} - \lambda_{1,0}}{2}.$$

Let $\lambda_{3,0} = \lambda_{4,0}$. *Then, we have that*

$$k\rho_0 = 1, \quad \delta'(t) = -\omega^2,$$

from (5.19) *and* (5.20), *respectively. Furthermore, we can obtain an explicit formula for* η *by solving* (5.22)*:*

$$\eta(t) = \omega \tan\left(\omega t - \arctan\left(\frac{\lambda_{1,0} + \lambda_{4,0}}{2\omega}\right)\right).$$

The maximum interval of existence follows from the domain of η*:*

$$t_B = \frac{\pi/2 + \arctan\left(\frac{\lambda_{1,0}+\lambda_{4,0}}{2\omega}\right)}{\omega}.$$

Then, integrating (5.4) *yields*

$$(u_1 u_4)(t) = \left(\left(\frac{\lambda_{1,0} + \lambda_{4,0}}{2\omega}\right)^2 + 1\right)\cos^2\left(\omega t - \arctan\left(\frac{\lambda_{1,0} + \lambda_{4,0}}{2\omega}\right)\right).$$

Finally, we deduce from (5.8), (5.9), *and* (5.12) *that*

$$\begin{aligned}
\lambda_1 = \lambda_1 = &- \frac{p}{\left(\frac{\lambda_{1,0}+\lambda_{4,0}}{2\omega}\right)^2 + 1} \sec^2\left(\omega t - \arctan\left(\frac{\lambda_{1,0} + \lambda_{4,0}}{2\omega}\right)\right) \\
&- \omega \tan\left(\omega t - \arctan\left(\frac{\lambda_{1,0} + \lambda_{4,0}}{2\omega}\right)\right), \\
\lambda_3 = \lambda_4 = &\frac{p}{\left(\frac{\lambda_{1,0}+\lambda_{4,0}}{2\omega}\right)^2 + 1} \sec^2\left(\omega t - \arctan\left(\frac{\lambda_{1,0} + \lambda_{4,0}}{2\omega}\right)\right) \\
&- \omega \tan\left(\omega t - \arctan\left(\frac{\lambda_{1,0} + \lambda_{4,0}}{2\omega}\right)\right), \\
\rho = &\frac{\rho_0}{\left(\left(\frac{\lambda_{1,0}+\lambda_{4,0}}{2\omega}\right)^2 + 1\right)^2} \sec^4\left(\omega t - \arctan\left(\frac{\lambda_{1,0} + \lambda_{4,0}}{2\omega}\right)\right).
\end{aligned}$$

Acknowledgments. Liu was partially supported by the National Science Foundation under Grant DMS1812666. Shin was supported by newly appointed professor research fund of Hanbat National University and the National Research Foundation under Grant NRF-2017R1E1A1A03070498.

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